Cohomology and Coquasi-bialgebras in the category of Yetter-Drinfeld modules

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COHOMOLOGY AND COQUASI-BIALGEBRAS IN THE CATEGORY OF
YETTER-DRINFELD MODULES

IVÁN ANGIONO, ALESSANDRO ARDIZZONI, AND CLAUDIA MENINI

ABSTRACT. We prove that a finite-dimensional Hopf algebra with the dual Chevalley Property
over a field of characteristic zero is quasi-isomorphic to a Radford-Majid bosonization whenever
the third Hochschild cohomology group in the category of Yetter-Drinfeld modules of its diagram
with coefficients in the base field vanishes. Moreover we show that this vanishing occurs in
meaningful examples where the diagram is a Nichols algebra.

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INTRODUCTION

Let $A$ be a finite-dimensional Hopf algebra over a field $k$ of characteristic zero such that the
coradical $H$ of $A$ is a sub-Hopf algebra (i.e. $A$ has the dual Chevalley Property). Denote by
$D(A)$ the diagram of $A$. The main aim of this paper (see Theorem 5.6) is to prove that, if the
third Hochschild cohomology group in $H^3_{YD}$ of the algebra $D(A)$ with coefficients in $k$ vanishes, in
symbols $H^3_{YD}(D(A),k) = 0$, then $A$ is quasi-isomorphic to the Radford-Majid bosonization $E\# H$
of some connected bialgebra $E$ in $H^1_{YD}$ with $\text{gr}(E) \cong D(A)$ as bialgebras in $H^1_{YD}$.

The paper is organized as follows. Let $H$ be a Hopf algebra over a field $k$. In Section 1 we
investigate the properties of coalgebras with multiplication and unit in the category $H^1_{YD}$ (in
particular of coquasi-bialgebras) and their associated graded coalgebra. The main result of this
section, Theorem 1.5, establishes that the associated graded coalgebra $\text{gr}Q$ of a connected coquasi-
bialgebra in $H^1_{YD}$ is a connected bialgebra in $H^1_{YD}$.

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In Section 2 we study the deformation of coquasi-bialgebras in $H_H^YD$ by means of gauge transformations. In Proposition 2.5 we investigate its behaviour with respect to bosonization while in Proposition 2.6 with respect to the associated graded coalgebra.

In Section 3 we consider the associated graded coalgebra in case the Hopf algebra $H$ is semisimple and cosemisimple (e.g. $H$ is finite-dimensional cosemisimple over a field of characteristic zero). In particular, in Theorem 3.2, we prove that a f.d. connected coquasi-bialgebra $H$ is equivalent to a connected bialgebra in $H^YD$ whenever $H^3_{YD}(\text{gr} Q, k) = 0$. This result is inspired by [23, Proposition 2.3].

In Section 4 we focus on the link between $H^3_{YD}(B, k)$ and the invariants of $H^n(B, k)$, where $B$ is a bialgebra in $H^YD(B, k)$. In particular, in Proposition 4.7 we show that $H^3_{YD}(B, k)$ is isomorphic to $H^n(B, k)^{D(H)}$, which is a subspace of $H^n(B, k)^H \cong H^n(B\#H, k)$, see Corollary 4.3.

Section 5 is devoted to the proof of the main result of the paper, the aforementioned Theorem 5.4.

In Section 6 we provide examples where $H^3_{YD}(B, k) = 0$ in case $B$ is the Nichols algebra $B(V)$ of a Yetter-Drinfeld module $V$. In particular we show that that $H^3_{YD}(B(V), k)$ can be zero although $H^3(B(V)\#H, k)$ is non-trivial.

Preliminaries

Given a category $C$ and objects $M, N \in C$, the notation $\mathcal{C}(M, N)$ stands for the set of morphisms in $C$. This notation will be mainly applied to the case $C$ is the category of vector space $\text{Vec}_k$ over a field $k$ or $C$ is the category of Yetter-Drinfeld modules $H^YD$ over a Hopf algebra $H$. The set of natural numbers including $0$ is denoted by $\mathbb{N}_0$ while $\mathbb{N}$ denotes the same set without $0$.

1. Yetter-Drinfeld

Definition 1.1. Let $C$ be a coalgebra. Denote by $C_n$ the $n$-th term of the coradical filtration of $C$ and set $C_{-1} := 0$. For every $x \in C$, we set

$$|x| := \min \{i \in \mathbb{N}_0 : x \in C_i\} \quad \text{and} \quad \varpi := x + C_{|x|-1}.$$ 

Note that, for $x = 0$, we have $|x| = 0$. One can define the associated graded coalgebra

$$\text{gr}
C := \bigoplus_{i \in \mathbb{N}_0} \frac{C_i}{C_{i-1}}$$

with structure given, for every $x \in C$, by

$$(1) \quad \Delta_{\text{gr} C}(\varpi) = \sum_{0 \leq i \leq |x|} (x_1 + C_{i-1}) \otimes (x_2 + C_{|x|-i-1})$$

$$(2) \quad \varepsilon_{\text{gr} C}(\varpi) = \delta_{|x|, 0} C (x).$$

1.2. For every $i \in \mathbb{N}_0$, take a basis $\{\overline{x^{i,j}} : j \in B_i\}$ of the $k$-module $C_i/C_{i-1}$ with $\overline{x^{i,j}} \neq \overline{x^{i,l}}$ for $j \neq l$ and

$$|x^{i,j}| = i.$$ 

Then $\{x^{i,j} : 0 \leq i \leq n, j \in B_i\}$ is a basis of $C_n$ and $\{x^{i,j} : i \in \mathbb{N}_0, j \in B_i\}$ is a basis of $C$. Assume that $C$ has a distinguished grouplike element $1 = 1_C \neq 0$ and take $i > 0$. If $\varepsilon (x^{i,j}) \neq 0$ then we have that

$$\overline{x^{i,j} - \varepsilon(x^{i,j})} 1 = \overline{x^{i,j}}$$

so that we can take $x^{i,j} - \varepsilon (x^{i,j}) 1$ in place of $x^{i,j}$. In other words we can assume

$$(3) \quad \varepsilon (x^{i,j}) = 0, \quad \text{for every } i > 0, j \in B_i.$$ 

It is well-known there is a $k$-linear isomorphism $\varphi : C \rightarrow \text{gr} C$ defined on the basis by $\varphi (x^{i,j}) := \overline{x^{i,j}}$.

We compute

$$\varepsilon_{\text{gr} C} \varphi (x^{i,j}) = \varepsilon_{\text{gr} C} (\overline{x^{i,j}}) \mathbb{1} \delta_{i,0} \varepsilon (x^{0,j}) \mathbb{1} \varepsilon (x^{i,j}).$$
Hence we obtain
\begin{equation}
\varepsilon_{\text{gr}C} \circ \varphi = \varepsilon.
\end{equation}

Let $H$ be a Hopf algebra. A **coalgebra with multiplication and unit** in $\mathcal{H}_H^{\text{YD}}$ is a datum $(Q, m, u, \Delta, \varepsilon)$ where $(Q, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{H}_H^{\text{YD}}$, $m : Q \otimes Q \to Q$ is a coalgebra morphism in $\mathcal{H}_H^{\text{YD}}$ called multiplication (which may fail to be associative) and $u : \k \to Q$ is a coalgebra morphism in $\mathcal{H}_H^{\text{YD}}$ called unit. In this case we set $1_Q := u(1_k)$.

Note that, for every $h \in H$, $k \in \k$, we have
\begin{align}
(5) \quad h1_Q &= hu(1_k) = u(h1_k) = u(\varepsilon_H(h)1_k) = \varepsilon_H(h)u(1_k) = \varepsilon_H(h)1_Q, \\
(6) \quad (1_Q)_{-1} \otimes (1_Q)_0 &= (u(1_k))_{-1} \otimes (u(1_k))_0 = (1_k)_{-1} \otimes u((1_k)_0) = 1_H \otimes u(1_k) = 1_H \otimes 1_Q.
\end{align}

**Proposition 1.3.** Let $H$ be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon)$ be a coalgebra with multiplication and unit in $\mathcal{H}_H^{\text{YD}}$. If $Q_0$ is a subcoalgebra of $Q$ in $\mathcal{H}_H^{\text{YD}}$ such that $Q_0 \cdot Q_0 \subseteq Q_0$, then $Q_n$ is a subcoalgebra of $Q$ in $\mathcal{H}_H^{\text{YD}}$ for every $n \in \mathbb{N}_0$. Moreover $Q_a \cdot Q_b \subseteq Q_{a+b}$ for every $a, b \in \mathbb{N}_0$ and the graded coalgebra $\text{gr}Q$, associated with the coradical filtration of $Q$, is a coalgebra with multiplication and unit in $\mathcal{H}_H^{\text{YD}}$ with respect to the usual coalgebra structure and with multiplication and unit defined by
\begin{align}
(7) \quad m_{\text{gr}Q}((x + Q_{a-1}) \otimes (y + Q_{b-1})) :&= xy + Q_{a+b-1}, \\
u_{\text{gr}Q}(k) :&= k1_Q + Q_{-1}.
\end{align}

**Proof.** The coalgebra structure of $Q$ induces a coalgebra structure on $\text{gr}Q$. Since $Q_0$ is a subcoalgebra of $Q$ in $\mathcal{H}_H^{\text{YD}}$ and, for $n \geq 1$, one has $Q_n = Q_{n-1} \wedge Q_0$, then inductively one proves that $Q_n$ is a subcoalgebra of $Q$ in $\mathcal{H}_H^{\text{YD}}$. As a consequence one gets that $\text{gr}Q$ is a coalgebra in $\mathcal{H}_H^{\text{YD}}$ (this construction can be performed in the setting of monoidal categories under suitable assumptions, see e.g. [AM Theorem 2.10]). Let us prove that $\text{gr}Q$ inherits also a multiplication and unit. Let us check that $Q_a \cdot Q_b \subseteq Q_{a+b}$ for every $a, b \in \mathbb{N}_0$. We proceed by induction on $n = a + b$. If $n = 0$ there is nothing to prove. Let $n \geq 1$ and assume that $Q_i \cdot Q_j \subseteq Q_{i+j}$ for every $i, j \in \mathbb{N}_0$ such that $0 \leq i + j \leq n - 1$. Let $a, b \in \mathbb{N}_0$ be such that $n = a + b$. Since $\Delta(Q_a) \subseteq \sum_{i=0}^a Q_i \otimes Q_{a-i}$ and $c_{Q_a}(Q_a \otimes Q_a) \subseteq Q_a \otimes Q_a$, where $c_{Q_a}$ denotes the braiding in $\mathcal{H}_H^{\text{YD}}$, using the compatibility condition between $\Delta$ and $m$, one easily gets that $\Delta(Q_a \cdot Q_b) \subseteq Q_{a+b-1} \otimes Q + Q \otimes Q_a$. Therefore $Q_a \cdot Q_b \subseteq Q_{a+b}$. This property implies we have a well-defined map in $\mathcal{H}_H^{\text{YD}}$
\begin{equation}
m^{a,b}_Q : \frac{Q_a}{Q_{a-1}} \otimes \frac{Q_b}{Q_{b-1}} \to \frac{Q_{a+b}}{Q_{a+b-1}}
\end{equation}
defined, for $x \in Q_a$ and $y \in Q_b$, by $[\Box]$. It can be seen as the graded component of a morphism in $\mathcal{H}_H^{\text{YD}}$ that we denote by $m_{\text{gr}Q} : \text{gr}Q \otimes \text{gr}Q \to \text{gr}Q$. Let us check that $m_{\text{gr}Q}$ is a coalgebra morphism in $\mathcal{H}_H^{\text{YD}}$. Consider a basis of $Q$ with terms of the form $x^i j$ as in $[\square]$. Hence we can write the comultiplication in the form
\begin{equation}
\Delta(x^a) = \sum_{s+t \leq a} \sum_{l,m} \eta^{a,u}_{s,t,l,m} x^{s,l} \otimes x^{t,m}.
\end{equation}
Now, using $[\Box]$, one gets that
\begin{equation}
\Delta_{\text{gr}Q}(\bar{x^{a,u}}) = \sum_{0 \leq i \leq a} \sum_{l,m} \eta^{a,u}_{i,a-i,l,m} \bar{x^{i,l}} \otimes \bar{x^{a-i,m}}.
\end{equation}
Using that $\Delta_{\text{gr}Q} \otimes \Delta_{\text{gr}Q} = (\text{gr}Q \otimes \text{gr}Q \otimes \text{gr}Q)(\Delta_{\text{gr}Q} \otimes \Delta_{\text{gr}Q})$ and $[\Box]$, it is straightforward to check that $(m_{\text{gr}Q} \otimes m_{\text{gr}Q}) \Delta_{\text{gr}Q} \otimes \Delta_{\text{gr}Q} (\bar{x^{a,u}} \otimes \bar{x^{b,v}}) = \Delta_{\text{gr}Q} m_{\text{gr}Q} (\bar{x^{a,u}} \otimes \bar{x^{b,v}})$. Moreover, since $\varepsilon_{\text{gr}Q} \otimes \varepsilon_{\text{gr}Q} = \varepsilon_{\text{gr}Q} \otimes \varepsilon_{\text{gr}Q}$, we get that $\varepsilon_{\text{gr}Q} m_{\text{gr}Q} (\bar{x^{a,u}} \otimes \bar{x^{b,v}}) = \varepsilon_{\text{gr}Q} \otimes \varepsilon_{\text{gr}Q} (\bar{x^{a,u}} \otimes \bar{x^{b,v}})$.

This proves that $m_{\text{gr}Q}$ is a coalgebra morphism in $\mathcal{H}_H^{\text{YD}}$.

The fact that $u_{\text{gr}Q} : \k \to \text{gr}Q$, defined by $u_{\text{gr}Q}(k) := k1_Q + Q_{-1}$ is a coalgebra morphism in $\mathcal{H}_H^{\text{YD}}$ easily follows by means of $[\Box]$ and $[\square]$. \[\square\]
DEFINITION 1.4 ([ABM, Definition 5.2]). Let $H$ be a Hopf algebra. Recall that a coquasi-bialgebra $(Q, m, u, \Delta, \varepsilon, \alpha)$ in the pre-braided monoidal category $\mathcal{H}_{m}^\mathcal{YD}$ is a coalgebra $(Q, \Delta, \varepsilon)$ together with coalgebra homomorphisms $m : Q \otimes Q \to Q$ and $u : k \to Q$ in $\mathcal{H}_{m}^\mathcal{YD}$ and a convolution invertible element $\alpha \in \mathcal{H}_{m}^\mathcal{YD}(Q^{\otimes 3}, k)$ (braided reassociator) such that

\begin{align*}
(9) \quad & \alpha (Q \otimes Q \otimes m) * \alpha (m \otimes Q \otimes Q) = (\varepsilon \otimes \alpha) * \alpha (Q \otimes m \otimes Q) * (\alpha \otimes \varepsilon), \\
(10) \quad & \alpha (Q \otimes u \otimes Q) = \alpha (u \otimes Q \otimes Q) = \alpha (Q \otimes Q \otimes u) = \varepsilon_{Q \otimes Q}, \\
(11) \quad & m (Q \otimes m) * \alpha = \alpha * m (m \otimes Q), \\
(12) \quad & m (u \otimes Q) = \text{Id}_{Q} = m (Q \otimes u).
\end{align*}

Here $*$ denotes the convolution product, where $Q^{\otimes 3}$ is the tensor product of coalgebras in $\mathcal{H}_{m}^\mathcal{YD}$ whence it depends on the braiding of this category. Note that in [10] any of the three equalities such as $\alpha (u \otimes Q \otimes Q) = \varepsilon_{Q \otimes Q}$ implies that $\alpha$ is unital.

THEOREM 1.5. Let $H$ be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a connected coquasi-bialgebra in $\mathcal{H}_{m}^\mathcal{YD}$. Then $\text{gr}Q$ is a connected bialgebra in $\mathcal{H}_{m}^\mathcal{YD}$.

**Proof.** By Proposition 1.3, we know that $\text{gr}Q$ is a coalgebra with multiplication and unit in $\mathcal{H}_{m}^\mathcal{YD}$. We have to check that the multiplication is associative and unitary.

Given two coalgebras $D, E$ in $\mathcal{H}_{m}^\mathcal{YD}$ endowed with coquasibialgebra filtration $(D_{(n)})_{n \in \mathbb{N}_{0}}$ and $(E_{(n)})_{n \in \mathbb{N}_{0}}$ in $\mathcal{H}_{m}^\mathcal{YD}$ such that $D_{(0)}$ and $E_{(0)}$ are one-dimensional, let us check that $C_{(n)} := \sum_{0 \leq i \leq n} D_{(i)} \otimes E_{(n-i)}$ gives a coalgebra filtration on $C := D \otimes E$ in $\mathcal{H}_{m}^\mathcal{YD}$. First note that the coquasibialgebra structure of $C$ is connected.

Thus, we have

\[
\Delta_{C} (C_{(n)}) = (D \otimes c_{D,E} \otimes E) (\Delta_{D} \otimes \Delta_{E}) \left( \sum_{i=0}^{n} D_{(i)} \otimes E_{(n-i)} \right)
\]

\[
\subseteq (D \otimes c_{D,E} \otimes E) \left( \sum_{i=0}^{n} \sum_{a=0}^{i} \sum_{b=0}^{n-i} D_{(a)} \otimes D_{(i-a)} \otimes E_{(b)} \otimes E_{(n-i-b)} \right)
\]

\[
\subseteq \sum_{i=0}^{n} \sum_{a=0}^{i} \sum_{b=0}^{n-i} D_{(a)} \otimes c_{D,E} (D_{(i-a)} \otimes E_{(b)}) \otimes E_{(n-i-b)}
\]

\[
\subseteq \sum_{i=0}^{n} \sum_{a=0}^{i} \sum_{b=0}^{n-i} D_{(a)} \otimes c_{D_{(i-a)},E_{(b)}} (D_{(i-a)} \otimes E_{(b)}) \otimes E_{(n-i-b)}
\]

\[
\subseteq \sum_{w=0}^{n} \sum_{a+b=n-i} D_{(a)} \otimes E_{(b)} \otimes D_{(i-a)} \otimes E_{(n-i-b)}
\]

Moreover, by [Sw], Proposition 11.1.1], we have that the coradical of $C$ is contained in $D_{(0)} \otimes E_{(0)}$ and hence it is one-dimensional.

This argument can be used to produce a coquasibialgebra filtration on $C := Q \otimes Q \otimes Q$ using as a filtration on $Q$ the coradical filtration. Let $n > 0$ and let $w \in C_{(n)} = \sum_{i+j+k \leq n} Q^{i} Q^{j} Q^{k}$. By [AMS], Lemma 3.69, we have that

\[
\Delta_{C} (w) - w \otimes (1Q)^{\otimes 3} - (1Q)^{\otimes 3} \otimes w \in C_{(n-1)} \otimes C_{(n-1)}.
\]

Thus we get

\[
w_{1} \otimes w_{2} \otimes w_{3} - \Delta_{C} (w) \otimes (1Q)^{\otimes 3} - \Delta_{C} ((1Q)^{\otimes 3}) \otimes w \in \Delta_{C} (C_{(n-1)}) \otimes C_{(n-1)}
\]

and hence, tensoring the first relation by $(1Q)^{\otimes 3}$ on the right and adding it to the second one, we get

\[
w_{1} \otimes w_{2} \otimes w_{3} - w \otimes (1Q)^{\otimes 3} \otimes (1Q)^{\otimes 3} - (1Q)^{\otimes 3} \otimes w \otimes (1Q)^{\otimes 3} - (1Q)^{\otimes 6} \otimes w \in C_{(n-1)} \otimes C_{(n-1)} \otimes C_{(n-1)}.
\]

For shortness, we set $\nu_{n} (z) := m (Q \otimes m) (z) + Q_{n-1}$ for every $z \in C$. Thus, by applying to the last displayed relation $C_{(n-1)} \otimes m (Q \otimes m) \otimes C_{(n-1)}$ and factoring out the middle term by $Q_{n-1}$,
we get
\[
\left[ w_1 \otimes \nu_n (w_2) \otimes w_3 - w \otimes \nu_n \left( (1_Q)_{\otimes 3} \right) \otimes \left( (1_Q)_{\otimes 3} \right) + \right. \\
\left. - \left( (1_Q)_{\otimes 3} \otimes \nu_n (w) \otimes (1_Q)_{\otimes 3} \right) - (1_Q)_{\otimes 3} \otimes \nu_n \left( (1_Q)_{\otimes 3} \right) \otimes w \right]
\in C_{(n-1)} \otimes \left( \frac{\nu_n (C_{(n-1)})}{Q_{n-1}} \right) \otimes C_{(n-1)} \subseteq C_{(n-1)} \otimes \frac{Q_{n-1}}{Q_{n-1}} \otimes C_{(n-1)} = 0.
\]

Thus we can express the first term with respect to the remaining ones as follows
\[
w_1 \otimes \nu_n (w_2) \otimes w_3 = w \otimes \nu_n \left( (1_Q)_{\otimes 3} \right) \otimes (1_Q)_{\otimes 3} + (1_Q)_{\otimes 3} \otimes \nu_n \left( (1_Q)_{\otimes 3} \right) \otimes w.
\]

Let \( Q \) be a Hopf algebra and let \( \nu \in C_{(n)} \) be a coquasi-bialgebra in \( \mathcal{H}_c \mathcal{D} \)

\[
\text{Then } x, y, z \in Q. \text{ Then } x \otimes y \otimes z \in C_{(x|+y|+z|)} \text{ so that}
\]
\[
(x \cdot y) \cdot z = (x + Q_{|x|-1}) \cdot (y + Q_{|y|-1}) \cdot (z + Q_{|z|-1})
\]
\[
= ((xy) + Q_{|x|+|y|-1}) \cdot (z + Q_{|z|-1})
\]
\[
= (xy) + Q_{|x|+|y|+|z|-1}
\]
\[
= \omega^{-1} ((x \otimes y \otimes z)_{1}) \nu_{|x|+|y|+|z|} ((x \otimes y \otimes z)_{2}) (x \otimes y \otimes z) \cdot (1Q \otimes 1Q \otimes 1Q)
\]
\[
= x \cdot Q_{|x|+|y|+|z|-1} = x \cdot (y \cdot z) = x \cdot y \otimes z.
\]

Therefore the multiplication is associative. It is also unitary as
\[
\forall x \in Q, \quad H_{\mathcal{D}} = (x + Q_{|x|-1}) \cdot (Q_{|x|-1}) = x \cdot 1Q + Q_{|x|-1} = x + Q_{|x|-1} = x.
\]

\[\text{and similarly } \forall x \in Q, \quad \overline{x} \cdot \overline{x} = \overline{x} \quad \forall x \in Q. \]

2. GAUGE DEFORMATION

**Definition 2.1.** Let \( H \) be a Hopf algebra and let \((Q, m, u, \Delta, \varepsilon, \omega)\) be a coquasi-bialgebra in \( \mathcal{H}_c \mathcal{D} \).

A **gauge transformation** for \( Q \) is a morphism \( \gamma : Q \otimes Q \to k \) in \( \mathcal{H}_c \mathcal{D} \) which is convolution invertible in \( \mathcal{H}_c \mathcal{D} \) and which is also unitary on both entries.

**Remark 2.2.** For \( \gamma \) as above, let us check that \( \gamma^{-1} \) is unitary whence a gauge transformation too.

First note that for all \( x \in Q \), by means of (6) and (7), one gets
\[
(1Q \otimes x)_{1} \otimes (1Q \otimes x)_{2} = 1Q \otimes x_{1} \otimes 1Q \otimes x_{2}
\]
\[
(x \otimes 1Q)_{1} \otimes (x \otimes 1Q)_{2} = x_{1} \otimes 1Q \otimes x_{2} \otimes 1Q
\]

Thus
\[
\gamma^{-1} (1Q \otimes x) = \gamma^{-1} (1Q \otimes x_{1}) \varepsilon (x_{2}) = \gamma^{-1} (1Q \otimes x_{1}) \gamma (1Q \otimes x_{2}) = (\gamma^{-1} \ast \gamma) (1Q \otimes x) = \varepsilon (x)
\]

and similarly \( \gamma^{-1} (x \otimes 1Q) = \varepsilon (x). \)

**Lemma 2.3.** Let \( H \) be a Hopf algebra and let \( C \) be a coalgebra in \( \mathcal{H}_c \mathcal{D} \). Given a map \( \gamma \in \mathcal{H}_c \mathcal{D} (C, k) \), we have that \( \gamma \) is convolution invertible in \( \mathcal{H}_c \mathcal{D} (C, k) \) if and only if it is convolution invertible in \( \mathbf{Vec}_{k} (C, k) \). Moreover the inverse is the same.
Let $H$ be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in $H \mathcal{YD}$. Let $\gamma : Q \otimes Q \rightarrow k$ be a gauge transformation in $H \mathcal{YD}$. Then

$$Q^\gamma := (Q, m^\gamma, u, \Delta, \varepsilon, \omega^\gamma)$$

is a coquasi-bialgebra in $H \mathcal{YD}$, where

$$m^\gamma := \gamma * m * \gamma^{-1}$$

$$\omega^\gamma := (\varepsilon \otimes \gamma) * \gamma (Q \otimes m) * \omega * \gamma^{-1} (m \otimes Q) * (\gamma^{-1} \otimes \varepsilon).$$

Proof. The proof is analogue to [4, Proposition XV.3.2] in its dual version. We include some details for the reader’s sake. Note that $Q^\gamma$ has the same underlying coalgebra of $Q$ which is a coalgebra in $H \mathcal{YD}$. The unit is also the same and hence it is a coalgebra map in $H \mathcal{YD}$. Since $m^\gamma$ is the convolution product of morphisms in $H \mathcal{YD}$, it results that $m^\gamma$ is in $H \mathcal{YD}$ as well.

Since $m$ is a coalgebra map in $H \mathcal{YD}$ and $\gamma$ is convolution invertible with convolution inverse $\gamma^{-1}$, it follows that $m^\gamma$ is a coalgebra map in $H \mathcal{YD}$.

By means of [4] and [4], one gets that $m^\gamma (1_Q \otimes x) = x = m^\gamma (x \otimes 1_Q)$.

Let us consider now $\omega^\gamma$. Since it is the convolution product of morphisms in $H \mathcal{YD}$, it results that $\omega^\gamma$ is in $H \mathcal{YD}$ as well.

Let us check that $\omega^\gamma$ is unitary. Consider the map $\alpha_2 : Q \otimes Q \rightarrow Q \otimes Q \otimes Q$ defined by $\alpha_2 (x \otimes y) = x \otimes 1_Q \otimes y$. The equalities [4] and [4] yield

$$\begin{align*}
(\alpha_2 (x \otimes y))_1 \otimes (\alpha_2 (x \otimes y))_2 &= \alpha_2 (x_1 \otimes (x_2)_1 \otimes y_1) \otimes (x_2)_0 \otimes \alpha_2 (x_2)_1 \otimes y_2) \\
&= \alpha_2 (x \otimes y)_1 \otimes \alpha_2 (x \otimes y)_2
\end{align*}$$

so that $\alpha_2$ is comultiplicative.

Thus

$$\omega^\gamma \alpha_2 := (\varepsilon \otimes \gamma) \alpha_2 * \gamma (Q \otimes m) \alpha_2 * \omega \alpha_2 * \gamma^{-1} (m \otimes Q) \alpha_2 * (\gamma^{-1} \otimes \varepsilon) \alpha_2$$

and computing the factors of this convolution products one gets

$$\begin{align*}
(\varepsilon \otimes \gamma) \alpha_2 &= \varepsilon \otimes \varepsilon, \\
(\gamma (Q \otimes m)) \alpha_2 &= \gamma, \\
\omega \alpha_2 &= \varepsilon \otimes \varepsilon, \\
\gamma^{-1} (m \otimes Q) \alpha_2 &= \gamma^{-1}, \\
(\gamma^{-1} \otimes \varepsilon) \alpha_2 &= \varepsilon \otimes \varepsilon
\end{align*}$$

and hence $\omega^\gamma \alpha_2 = \gamma * \gamma^{-1} = \varepsilon \otimes \varepsilon$, which means that $\omega^\gamma (x \otimes 1_Q \otimes y) = \varepsilon (x) \varepsilon (y)$ for every $x, y \in Q$.

Similarly, considering $\alpha_1 : Q \otimes Q \rightarrow Q \otimes Q \otimes Q$ defined by $\alpha_2 (x \otimes y) = 1_Q \otimes x \otimes y$, one proves that

$$\omega^\gamma (1_Q \otimes x \otimes y) = \varepsilon (x) \varepsilon (y).$$

A symmetric argument shows that $\omega^\gamma (x \otimes y \otimes 1_Q) = \varepsilon (x) \varepsilon (y)$.

Note that, by Lemma 2.3, $\omega^\gamma$ is convolution invertible in $H \mathcal{YD} (D, k)$ as it is convolution invertible in $\text{Vec}_k (D, k)$.

Let us check that the multiplication is quasi-associative. By [4, Lemma 2.10 formula (2.7)], we have

$$m^\gamma (Q \otimes \gamma * m * \gamma^{-1}) = (\varepsilon \otimes \gamma) * m^\gamma (Q \otimes m) * (\varepsilon \otimes \gamma^{-1}),$$
\[(\varepsilon \otimes \gamma^{-1}) \ast (\varepsilon \otimes \gamma) = \varepsilon \otimes (\gamma^{-1} \ast \gamma) = \varepsilon \otimes \varepsilon \otimes \varepsilon,\]
\[m^{-1}(m^{-1} \otimes Q) = m^{-1}(\gamma \ast m \ast \gamma^{-1} \otimes Q) = (\gamma \otimes \varepsilon) \ast (m \ast \gamma^{-1} \otimes Q) = (\gamma \otimes \varepsilon) \ast m^{-1} (m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon),\]
\[(\gamma^{-1} \otimes \varepsilon) \ast (\gamma \otimes \varepsilon) = ((\gamma^{-1} \ast \gamma) \otimes \varepsilon) = \varepsilon \otimes \varepsilon \otimes \varepsilon.\]

By using these equalities one obtains
\[m^{-1}(Q \otimes m^{-1}) \ast \omega^{-1} = (\varepsilon \otimes \gamma) \ast (Q \otimes m) \ast m(Q \otimes m) \ast \omega \ast (\gamma^{-1} \otimes m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon),\]
\[\omega \ast m^{-1}(m^{-1} \otimes Q) = (\varepsilon \otimes \gamma) \ast (Q \otimes m) \ast (m \ast (m \otimes \omega) \ast (m \ast \gamma^{-1} \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon)),\]
so that \[\omega \ast m^{-1}(m^{-1} \otimes Q) = m^{-1}(Q \otimes m^{-1}) \ast \omega^{-1} .\]

It remains to check that \(\omega^{-1}\) is a reassociator. By [ABM, Lemma 2.10 formula (2.7)], we have
\[\omega^{\gamma}(Q \otimes Q \otimes \gamma \ast m \ast \gamma^{-1}) = (\varepsilon \otimes \varepsilon \otimes \varepsilon) \ast \omega^{\gamma}(Q \otimes Q \otimes m) \ast (\varepsilon \otimes \varepsilon \otimes \gamma^{-1}),\]
\[\omega^{\gamma}(\gamma \ast m \ast \gamma^{-1} \otimes Q \otimes Q) = (\gamma \otimes \varepsilon \otimes \varepsilon) \ast \omega^{\gamma}(m \otimes Q \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon \otimes \varepsilon),\]
\[(\gamma \otimes \varepsilon \otimes \varepsilon) \ast (\varepsilon \otimes \varepsilon \otimes \gamma) = \gamma \otimes \gamma = (\varepsilon \otimes \varepsilon \otimes \gamma) \ast (\gamma \otimes \varepsilon \otimes \varepsilon).\]

By using these equalities one obtains
\[\omega^{\gamma}(Q \otimes Q \otimes m^{-1}) \ast \omega^{\gamma}(m^{-1} \otimes Q \otimes Q) = \left(\varepsilon \otimes \omega^{\gamma}\right) \ast \omega^{\gamma}(Q \otimes m^{-1} \otimes Q) \ast (\omega^{-1} \otimes \varepsilon).\]

Therefore
\[\omega^{\gamma}(Q \otimes Q \otimes m^{-1}) \ast \omega^{\gamma}(m^{-1} \otimes Q \otimes Q) = (\varepsilon \otimes \omega^{\gamma}) \ast \omega^{\gamma}(Q \otimes m^{-1} \otimes Q) \ast (\omega^{-1} \otimes \varepsilon).\]

\[\square\]

In analogy to the case of Hopf algebras, one can define the bosonization \(E \# H\) of a coquasibialgebra in \(H^YD\) by a Hopf algebra \(H\), see [ABM, Definition 5.4] for further details on the structure. The following result was originally stated for \(E\) a Hopf algebra. Yorck Sommerhäuser suggested the present more general form which investigates the behaviour of the bosonization under a suitable gauge transformation.

**Proposition 2.5.** Let \(H\) be a Hopf algebra and let \((E, m, u, \Delta, \varepsilon, \omega)\) be a coquasibialgebra in \(H^YD\). Let \(\gamma : E \otimes E \rightarrow k\) be a gauge transformation in \(H^YD\). Set
\[\Gamma : (E \# H) \otimes (E \# H) \rightarrow k : (x \# h) \otimes (x' \# h') \mapsto \gamma(x \otimes h x') \varepsilon_H(h').\]
Then \(\Gamma\) is a gauge transformation and \((E \# H)^\Gamma = E^{-1} \# H\) as ordinary coquasibialgebras.

**Proof.** By [ABM, Lemma 2.15 and what follows], we have that \(\Gamma\) is convolution invertible \(H\)-bilinear and \(H\)-balanced. Moreover \(\Gamma^{-1}((x \# h) \otimes (x' \# h')) = \gamma^{-1}(x \otimes h x') \varepsilon_H(h')\). If \(\alpha : (E \# H) \otimes (E \# H) \rightarrow E \# H\) is \(H\)-bilinear and \(H\)-balanced, it is easy to check that \(\Gamma \ast \alpha \ast \Gamma^{-1}\) is \(H\)-bilinear and \(H\)-balanced too.

In particular, since
\[m_{E \# H}((x \# h) \otimes (x' \# h')) = m(x \otimes h_1 x') \otimes h_2 h',\]
we have that \(m_{E \# H}\) is \(H\)-bilinear and \(H\)-balanced where \(E \# H\) carries the left \(H\)-diagonal action and the right regular action over \(H\).
Thus $m_{(E\#H)^c} = \Gamma * m_{E\#H} * \Gamma^{-1}$ is $H$-bilinear and $H$-balanced. Moreover, since $E^\gamma$ is also a coquasi-bialgebra in $\mathcal{H}\mathcal{Y}\mathcal{D}$ we have that $m_{E^\gamma \# H} : (E\#H) \otimes (E\#H) \to E\#H$ is $H$-bilinear and $H$-balanced too.

Therefore, in order to check that $m_{(E\#H)^c} = m_{E^\gamma \# H}$, it suffices to prove that they coincide on elements of the form $(x\#1_H) \otimes (x'\#1_H)$.

Let us consider the multiplication

$$m_{(E\#H)^c} ((x\#1_H) \otimes (x'\#1_H))$$

$$= (\Gamma * m_{E\#H} * \Gamma^{-1}) ((x\#1_H) \otimes (x'\#1_H))$$

$$= \Gamma ((x\#1_H) \otimes (x'\#1_H)) \cdot m_{E\#H} ((x\#1_H) \otimes (x'\#1_H)) \cdot \Gamma^{-1} ((x\#1_H) \otimes (x'\#1_H)).$$

Now, from

$$\Delta_{E\#H} (x\#h) = \sum (x^{(1)} \# x^{(2)} \langle \cdot \rangle_{1_H}) \otimes (x^{(2)} \langle \cdot \rangle_{0} \# h_{2})$$

we get

$$(x\#1_H) \otimes (x\#1_H) \otimes (x\#1_H)$$

$$= \sum (x^{(1)} \# x^{(2)} \langle \cdot \rangle_{1_H}) \otimes (x^{(2)} \langle \cdot \rangle_{0} \# x^{(3)} \langle \cdot \rangle_{1_H}) \otimes (x^{(3)} \langle \cdot \rangle_{0} \# 1_H)$$

so that

$$m_{(E\#H)^c} ((x\#1_H) \otimes (x'\#1_H))$$

$$= \Gamma ((x\#1_H) \otimes (x'\#1_H)) \cdot m_{E\#H} ((x\#1_H) \otimes (x'\#1_H)) \cdot \Gamma^{-1} ((x\#1_H) \otimes (x'\#1_H))$$

$$= \sum \gamma \left( x^{(1)} \otimes x^{(2)} \langle \cdot \rangle_{1_H} \otimes x^{(3)} \langle \cdot \rangle_{1_H} \right) \otimes m_{E\#H} ((x\#1_H) \otimes (x'\#1_H)) \cdot m_{(E\#H)^c} ((x\#1_H) \otimes (x'\#1_H))$$

$$= \sum \gamma \left( x^{(1)} \otimes x^{(2)} \langle \cdot \rangle_{1_H} \otimes x^{(3)} \langle \cdot \rangle_{1_H} \right) \otimes \gamma^{-1} \left( x^{(3)} \langle \cdot \rangle_{0} \otimes x^{(3)} \langle \cdot \rangle_{0} \right) \otimes 1_H.$$
3.69], we have that \( \Delta_k \) is a coquasi-bialgebra in \( \mathbb{R} \) and hence in \( \mathbb{R} \). Finally let us check that \( \omega_{E^\# H} \) coincides with \( E^\# H \) and hence with \( E^\# H \). Let \( Q \) be a Hopf algebra and let \( Q_m,u,\Delta,\varepsilon,\omega \) be a connected coquasi-bialgebra in \( \mathbb{R} \). Let \( \gamma : Q \otimes Q \rightarrow \mathbb{k} \) be a gauge transformation in \( \mathbb{R} \). Then \( \text{gr} (Q^\gamma) \) and \( \text{gr} (Q) \) coincide as bialgebras in \( \mathbb{R} \).
Let \( x, y \in Q \). We compute
\[
\mathcal{T} \cdot \gamma \overline{y} = (x + Q_{[x][-1]}) \cdot \gamma (y + Q_{[y][-1]})
\]
\[
= (x \cdot \gamma y) + Q_{[x]+[y][-1]}
\]
\[
= \gamma ((x \otimes y) \sigma) m ((x \otimes y) \sigma) \gamma^{-1} ((x \otimes y) \sigma) + Q_{[x]+[y][-1]}
\]
\[
= \gamma ((x \otimes y) \sigma) \left[ m ((x \otimes y) \sigma) + Q_{[x]+[y][-1]} \right] \gamma^{-1} ((x \otimes y) \sigma)
\]
\[
\gamma \left( (1_Q)^{\otimes 2} \right) (m (x \otimes y) + Q_{[x]+[y][-1]} \gamma^{-1} ((1_Q)^{\otimes 2})
\]
\[
= m (x \otimes y) + Q_{[x]+[y][-1]} = (x \cdot y) + Q_{[x]+[y][-1]} = \mathcal{T} \cdot \overline{y}.
\]
Note that \( Q^\gamma \) and \( Q \) have the same unit so that \( \text{gr}Q \) and \( \text{gr}Q^\gamma \) have.

\[ \square \]

3. (CO)SEMISIMPLE CASE

Assume \( H \) is a semisimple and cosemisimple Hopf algebra (e.g. \( H \) is finite-dimensional cosemisimple over a field of characteristic zero). Note that \( H \) is then separable (see e.g. [SI, Corollary 3.7] or [AMS, Theorem 2.34]) whence finite-dimensional. Let \( (Q, m, u, \Delta, \varepsilon) \) be a f.d. coalgebra with multiplication and unit in \( \underline{H} \mathcal{YD} \). Assume that the coradical \( Q_0 \) is a subcoalgebra of \( Q \) in \( \underline{H} \mathcal{YD} \) such that \( Q_0 \cdot Q_0 \subseteq Q_0 \). Let \( y^{n,i} \) with \( 1 \leq i \leq \dim (Q_n/Q_{n-1}) \) be a basis for \( Q_n/Q_{n-1} \). Consider, for every \( n > 0 \), the exact sequence in \( \underline{H} \mathcal{YD} \) given by
\[
0 \rightarrow Q_{n-1} \xrightarrow{s_n} Q_n \xrightarrow{\pi_n} Q_{n-1} \rightarrow 0
\]
Now, since \( H \) is semisimple and cosemisimple, by [Rn2, Proposition 7] the Drinfeld double \( D(H) \) is semisimple. By a result essentially due to Majid (see [M, Proposition 10.4.19]) and by [R1, Proposition 6]), we get that the category \( \underline{H} \mathcal{YD} \cong D(H) \mathcal{M} \) is a semisimple category. Therefore \( \pi_n \) cosimplifies i.e. there is a morphism \( \sigma_n : (Q_n/Q_{n-1}) \rightarrow Q_n \) in \( \underline{H} \mathcal{YD} \) such that \( \pi_n \sigma_n = \Id \). Let \( u_n : k \rightarrow Q_n \) be the restriction of the unit \( u : k \rightarrow Q \) and let \( \varepsilon_n = \varepsilon_{|Q_n} : Q_n \rightarrow k \) be the counit of the subcoalgebra \( Q_n \). Set
\[
\sigma'_n := \sigma_n - u_n \circ \varepsilon_n \circ \sigma_n
\]
This is a morphism in \( \underline{H} \mathcal{YD} \). Moreover
\[
\pi_n \circ \sigma'_n = \pi_n \cdot \sigma_n - \pi_n \circ u_n \circ \varepsilon_n \circ \sigma_n = \id_{Q_n/Q_{n-1}} - 0 = \id_{Q_n/Q_{n-1}},
\]
\[
\varepsilon_n \circ \sigma'_n = \varepsilon_n \circ \sigma_n - \varepsilon_n \circ u_n \circ \varepsilon_n = \varepsilon_n \circ \sigma_n - \varepsilon_n = 0.
\]
Therefore, without loss of generality we can assume that \( \varepsilon_n \circ \sigma_n = 0 \). A standard argument on split short exact sequences shows that there exists a morphism \( p_n : Q_n \rightarrow Q_{n-1} \) in \( \underline{H} \mathcal{YD} \) such that
\[
s_n p_n + \sigma_n \pi_n = \id_{Q_n}, \quad p_n s_n = \id_{Q_{n-1}}, \quad \text{and} \quad p_n \sigma_n = 0.
\]
We set
\[
x^{n,i} := \sigma_n (y^{n,i}).
\]
Therefore
\[
y^{n,i} = \pi_n \sigma_n \left( y^{n,i} \right) = \pi_n \left( x^{n,i} \right) + Q_{n-1} = x^{n,i}.
\]
These terms \( x^{n,i} \) define a \( k \)-basis for \( Q \). As \( Q \) is finite-dimensional, there exists \( d \in \mathbb{N}_0 \) such that \( Q = Q_d \); we fix \( d \) minimal. For all \( 0 \leq a \leq b \), define the maps
\[
p_{a,b} : Q_b \rightarrow Q_a, \quad p_{a,b} := p_{a+1} \circ p_{a+2} \circ \cdots \circ p_{b-1} \circ p_b.
\]
\[
s_{b,a} : Q_a \rightarrow Q_b, \quad s_{b,a} := s_b \circ s_{b-1} \circ \cdots \circ s_{a+2} \circ s_{a+1}.
\]
Clearly one has
\[
p_{a,b} \circ s_{b,a} = \id_{Q_a}.
\]
Thus, for \( 0 \leq i, a \leq b \) we have
\[
p_{i,b} \circ s_{b,a} = \left\{ \begin{array}{ll}
p_{i,b} \circ s_{b,i} \circ s_{i,a} & i > a \\
p_{i,a} \circ p_{a,b} \circ s_{a,i} & i \leq a
\end{array} \right.
\]
(17)
\[
\varphi (x) := \left( \begin{array}{c}
p_{0,d} (x) + \pi_1 p_{1,d} (x) + \pi_2 p_{2,d} (x) + \cdots + \pi_{d-2} p_{d-2,d} (x) + \pi_{d-1} p_{d-1,d} (x) + \pi_d (x)
\end{array} \right)
\]
where we set
\[ \pi_0 = \text{Id}_{Q_0}, \quad p_{d,d} = \text{Id}_{Q_d}. \]

For \(0 \leq n \leq d\), we have
\[
\varphi (x^{n,i}) = \varphi (s_{d,n} (x^{n,i})) = \varphi (s_{d,n} \sigma_n (y^{n,i})) = \sum_{0 \leq t \leq d} \pi_t p_{t,d} (s_{d,n} \sigma_n (y^{n,i}))
\]
\[
= \sum_{0 \leq t \leq d} \pi_t p_{t,d} (s_{d,n} \sigma_n (y^{n,i})) + \sum_{0 \leq t \leq n} \pi_t p_{t,d} (s_{d,n} \sigma_n (y^{n,i}))
\]
\[
= \sum_{0 \leq t \leq d} \pi_t s_{t,n} \sigma_n (y^{n,i}) + \sum_{0 \leq t \leq n} \pi_t p_{t,n} \sigma_n (y^{n,i}) + \pi_{n} p_{n,d} s_{d,n} \sigma_n (y^{n,i})
\]
\[
= \sum_{0 \leq t \leq d} \pi_t s_{t,n} \sigma_n (y^{n,i}) + \sum_{0 \leq t \leq n} \pi_t p_{t,n} \sigma_n (y^{n,i}) + \pi_{n} \sigma_n (y^{n,i})
\]
\[
= 0 + 0 + y^{n,i} = y^{n,i}.
\]

Hence \(\varphi (x^{n,i}) = y^{n,i}\). Since \(y^{n,i}\) with \(1 \leq i \leq \dim (Q_n/Q_{n-1}) =: d_n\) form a basis for \(Q_n/Q_{n-1}\) we have that
\[
y^{n,i} \in \frac{Q_n}{Q_{n-1}}, \quad (y^{n,i})_0 \in H \otimes \frac{Q_n}{Q_{n-1}}.
\]

Therefore there are \(\chi_{t,i}^n \in H^*\) and \(h_{t,i}^n \in H\) such that
\[
y^{n,i} = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n (h) y^{n,t}, \quad (y^{n,i})_0 = \sum_{1 \leq t \leq d_n} h_{t,i}^n \otimes y^{n,t}.
\]

We have
\[
h (h' y^{n,i}) = \sum_{1 \leq s \leq d_n} \chi_{s,i}^n (h') h y^{n,s} = \sum_{1 \leq s \leq d_n} \chi_{s,i}^n (h') \sum_{1 \leq t \leq d_n} \chi_{t,s}^n (h) y^{n,t}
\]
\[
= \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} \chi_{t,i}^n (h) \chi_{s,i}^n (h') y^{n,t},
\]
and hence
\[
\chi_{t,i}^n (h h') = \sum_{1 \leq s \leq d_n} \chi_{t,s}^n (h) \chi_{s,i}^n (h').
\]

Moreover
\[
y^{n,i} = 1_H y^{n,i} = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n (1_H) y^{n,t}
\]
and hence
\[
\chi_{t,i}^n (1_H) = \delta_{t,i}.
\]

We also have
\[
(y^{n,i})_0 = (y^{n,i})_0 = \sum_{1 \leq s \leq d_n} h_{t,i}^n \otimes (y^{n,s})_0 = \sum_{1 \leq s \leq d_n} h_{t,i}^n \otimes (y^{n,s})_0
\]
\[
= \sum_{1 \leq s \leq d_n} h_{t,i}^n \otimes \sum_{1 \leq t \leq d_n} h_{s,t}^n \otimes y^{n,t}
\]
\[
= \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_{t,i}^n \otimes h_{s,i}^n \otimes y^{n,t},
\]
\[
(y^{n,i})_1 = y^{n,i} = \sum_{1 \leq t \leq d_n} \Delta_H (h_{t,i}^n) \otimes y^{n,t}
\]
so that
\[
\Delta_H (h_{t,i}) = \sum_{1 \leq s \leq d_n} h_{t,i}^n \otimes h_{s,i}^n.
\]

Moreover
\[
y^{n,i} = \varepsilon_H (y^{n,i})_0 = \sum_{1 \leq t \leq d_n} \varepsilon_H (h_{t,i}^n) y^{n,t}
\]
and hence
\[
\varepsilon_H (h_{t,i}^n) = \delta_{t,i}.
\]
Finally
\[(h_1 y^{n,i})_{-1} h_2 \otimes (h_1 y^{n,i})_0 = \sum_{1 \leq s \leq d_n} \chi_{s,i}^{n} (h_1) (y^{n,s})_{-1} h_2 \otimes (y^{n,s})_0 = \sum_{1 \leq s \leq d_n} \chi_{s,i}^{n} (h_1) \sum_{1 \leq t \leq d_n} h^i_{s,t} h_2 \otimes y^{n,t} = \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h^i_{s,t} \chi_{s,i}^{n} (h_1) h_2 \otimes y^{n,t},\]
\[h_1 (y^{n,i})_{-1} \otimes h_2 (y^{n,i})_0 = \sum_{1 \leq s \leq d_n} h_1 h^{n}_{s,i} \otimes h_2 y^{n,s} = \sum_{1 \leq s \leq d_n} h_1 h^{n}_{s,i} \otimes \sum_{1 \leq t \leq d_n} \chi_{i,s}^{n} (h_2) y^{n,t} = \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_1 \chi_{i,s}^{n} (h_2) h^{n}_{s,t} \otimes y^{n,t}.\]
Therefore, we get
\[\sum_{1 \leq s \leq d_n} h^{n}_{s,i} \chi_{s,i}^{n} (h_1) h_2 = \sum_{1 \leq s \leq d_n} h_1 \chi_{i,s}^{n} (h_2) h^{n}_{s,t}.\]
We have
\[h x^{n,i} = h \sigma_n (y^{n,i}) = \sigma_n (h y^{n,i}) = \sigma_n \left( \sum_{1 \leq t \leq d_n} \chi_{i,t}^{n} (h) y^{n,t} \right) = \sum_{1 \leq t \leq d_n} \chi_{i,t}^{n} (h) x^{n,t},\]
\[(x^{n,i})_{-1} \otimes (x^{n,i})_0 = (\sigma_n (y^{n,i}))_{-1} \otimes (\sigma_n (y^{n,i}))_0 = (y^{n,i})_{-1} \otimes \sigma_n ((y^{n,i})_0) = \sum_{1 \leq t \leq d_n} h^{n}_{i,t} \otimes x^{n,t},\]
\[\varepsilon_Q (x^{n,i}) = \varepsilon_n (x^{n,i}) = \varepsilon_n \sigma_n (y^{n,i}) = 0 \text{ for } n > 0.\]
If \(Q\) is connected, then \(d_0 = 1\) so we may assume \(y^{0,0} := 1_Q + Q_{-1}\). Since \(\pi_0 = \text{Id}_{Q_0}\) we get
\[\sigma_0 = \text{Id}_{Q_0} \circ \sigma_0 = \pi_0 \circ \sigma_0 = \text{Id}_{Q_0}\]
and hence
\[x^{0,0} = \sigma_0 (y^{0,0}) = \sigma_0 (1_Q + Q_{-1}) = 1_Q.\]
Since, by Proposition 3, \(Q_a \cdot Q_{a'} \subseteq Q_{a+a'}\) for every \(a, a' \in \mathbb{N}_0\), we can write the product of two elements of the basis in the form
\[(19)\]
\[x^{a,l}_a x^{a',l'}_a = \sum_{u \leq a + a'} \sum_{v} \mu^{a,l,a',l'}_u x^{u,v}.\]
We compute
\[\overline{x^{a,l} \cdot x^{a',l'}} = (x^{a,l} + Q_{a-1}) (x^{a',l'} + Q_{a'-1}) = (x^{a,l} x^{a',l'}) + Q_{a+a'-1}
\]
\[ \partial^1 (f) = f \otimes \varepsilon_A - f m_A + \varepsilon_A \otimes f, \]
\[ \partial^2 (f) = f \otimes \varepsilon_A - f (A \otimes m_A) + f (m_A \otimes A) - \varepsilon_A \otimes f, \]
\[ \partial^3 (f) = f \otimes \varepsilon_A - f (A \otimes A \otimes m_A) + f (A \otimes m_A \otimes A) - f (m_A \otimes A \otimes A) + \varepsilon_A \otimes f. \]

For every \( n \geq 1 \) denote by
\[ Z^n_{YD} (A, \mathfrak{k}) := \ker (\partial^n), \quad B^n_{YD} (A, \mathfrak{k}) := \text{Im} (\partial^{n-1}) \quad \text{and} \quad H^n_{YD} (A, \mathfrak{k}) := \frac{Z^n_{YD} (A, \mathfrak{k})}{B^n_{YD} (A, \mathfrak{k})} \]
the abelian groups of \( n \)-cocycles, of \( n \)-coboundaries and the \( n \)-th Hochschild cohomology group in \( H^n_{YD} \) of the algebra \( A \) with coefficients in \( \mathfrak{k} \). We point out that the construction above works for an arbitrary \( A \)-bimodule \( M \) in \( H^n_{YD} \) instead of \( \mathfrak{k} \).

Next result is inspired by [EG, Proposition 2.3]. Two coquasi-bialgebras \( Q \) and \( Q' \) in \( H^n_{YD} \) will be called gauge equivalent whenever there is some gauge transformation \( \gamma : Q \otimes Q \to \mathfrak{k} \) in \( H^n_{YD} \) such that \( Q \cong Q' \) as coquasi-bialgebras in \( H^n_{YD} \), see Proposition 2.4 for the structure of \( Q' \).

**Theorem 3.2.** Let \( H \) be a semisimple and cosemisimple Hopf algebra and let \( (Q, m, u, \Delta, \varepsilon, \omega) \) be a f.d. connected coquasi-bialgebra in \( H^n_{YD} \). If \( H^n_{YD} (\text{gr} Q, \mathfrak{k}) = 0 \) then \( Q \) is gauge equivalent to a connected bialgebra in \( H^n_{YD} \).

**Proof.** For \( t \in N_0 \), and \( x, y, z \) in the basis of \( Q \), we set
\[ \omega_t (x \otimes y \otimes z) := \delta_{|x|+|y|+|z|, t} \omega (x \otimes y \otimes z). \]

Let us check it defines a morphism \( \omega_t : Q \otimes Q \otimes Q \to \mathfrak{k} \) in \( H^n_{YD} \). It is left \( H \)-linear as, by means of (18), the definition of \( \omega_t \) and the \( H \)-linearity of \( \omega \), we can prove that \( \omega_t (h \left( x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right)) = \varepsilon_H (h) \omega_t (x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''}). \)

Moreover it is left \( H \)-cobilinear as, by means of (18), the definition of \( \omega_t \) and the \( H \)-cobilinearity of \( \omega \), we can prove that
\[ (x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''}) \left( -1 \right) \otimes \omega_t \left( (x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''}) \left( 0 \right) \right) = 1_H \otimes \omega_t \left( x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right). \]

Clearly, for \( x, y, z \in Q \) in the basis, one has
\[ \sum_{t \in N_0} \omega_t (x \otimes y \otimes z) = \sum_{t \in N_0} \delta_{|x|+|y|+|z|, t} \omega (x \otimes y \otimes z) = \omega (x \otimes y \otimes z) \]
so that we can formally write
\[ \omega = \sum_{t \in N_0} \omega_t. \]

Since \( \varepsilon \) is trivial on elements in the basis of strictly positive degree, one gets
\[ \omega_0 = \varepsilon \otimes \varepsilon \otimes \varepsilon. \]

If \( \omega = \omega_0 \), then \( Q \) is a (connected) bialgebra in \( H^n_{YD} \) and the proof is finished. Thus we can assume \( \omega \neq \omega_0 \) and set
\[ s := \min \{ i \in N : \omega_i \neq 0 \}, \]
\[ \overline{s} := \omega_s (\varphi^{-1} \otimes \varphi^{-1} \otimes \varphi^{-1}), \]
\[ Q := \text{gr} Q. \]

Note that \( \overline{s} \) is a morphism in \( H^n_{YD} \) as a composition of morphisms in \( H^n_{YD} \).

Let \( n \in N_0 \), let \( C^4 = Q \otimes Q \otimes Q \otimes Q \) and let \( u \in C^4_{(n)} = \sum_{1 \leq i \leq n} Q_i \otimes Q_j \otimes Q_k \otimes Q_l \).

A direct computation rewriting the cocycle condition using (21) proves that, for every \( n \in N_0 \), and \( u \in C^4_{(n)} \)
\[ \sum_{0 \leq i+j \leq n} \left[ \omega_i (Q \otimes Q \otimes m) \ast \omega_j (m \otimes Q \otimes Q) \right] (u) \]
Next aim is to check that $s \in H^3_{\mathcal{YD}}(\gr Q, k)$ i.e. that

$$\omega_s \left( m_{Q^2} \otimes \mathcal{Q} \otimes \mathcal{Q} \right) + \omega_s \left( \mathcal{Q} \otimes \mathcal{Q} \otimes m_{Q^2} \right) = \left( \varepsilon \otimes \omega_s \right) + \omega_s \left( \mathcal{Q} \otimes m_{Q^2} \otimes \mathcal{Q} \right) + \left( \varepsilon \otimes \omega_s \right).$$

This is achieved by evaluating the two sides of the equality above on $\mathfrak{m} := \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \otimes \mathfrak{t}$ where $x, y, z, t$ are elements in the basis and using (23). If $\mathfrak{m}$ has homogeneous degree greater than $s$, then both terms are zero. Otherwise, i.e. if $\mathfrak{m}$ has homogeneous degree at most $s$, one has

$$\omega_s \left( m_{Q^2} \otimes \mathcal{Q} \otimes \mathcal{Q} \right)(u) = \omega_s \left( m_{Q^2} \otimes \mathcal{Q} \otimes \mathcal{Q} \right)(u)$$

and similarly for the other pieces so that one has to check that

$$\omega_s \left( m_{Q^2} \otimes \mathcal{Q} \otimes \mathcal{Q} \right)(u) + \omega_s \left( \mathcal{Q} \otimes m_{Q^2} \otimes \mathcal{Q} \right)(u) + \omega_s \left( m_{Q^2} \otimes \mathcal{Q} \otimes \mathcal{Q} \right)(u).$$

This equality follows by using (23) and the definition of $s$.

By assumption $H^3_{\mathcal{YD}}(\gr Q, k) = 0$ so that there exists a morphism $\mathfrak{m} : \mathcal{Q} \otimes \mathcal{Q} \rightarrow k$ in $\mathcal{H}_Y \mathcal{D}$ such that

$$\omega_s = \partial^2 \mathfrak{m} = \mathfrak{x} \otimes \varepsilon \mathfrak{Q} - \mathfrak{x} \mathfrak{Q} \otimes \varepsilon \mathfrak{Q} + \mathfrak{m} \mathfrak{Q} \rightarrow \mathfrak{x} \mathfrak{Q} \otimes \varepsilon \mathfrak{Q}.$$

Explicitly, on elements in the basis we get

$$\omega_s \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) = \mathfrak{x} \left( \mathfrak{x} \otimes \mathfrak{y} \right) \varepsilon \mathfrak{z} - \mathfrak{x} \left( \mathfrak{z} \otimes \mathfrak{z} \right) \mathfrak{y} + \mathfrak{x} \left( \mathfrak{y} \otimes \mathfrak{z} \right) \mathfrak{x} - \mathfrak{x} \left( \mathfrak{x} \otimes \mathfrak{z} \right) \mathfrak{y}.\varepsilon \mathfrak{z}.$$

Define $\zeta : \mathcal{Q} \otimes \mathcal{Q} \rightarrow k$ on the basis by setting

$$\zeta \left( \mathfrak{x} \otimes \mathfrak{y} \right) := \delta_{|x|+|y|,s} \mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right).$$

As we have done for $\omega_s$, one can check that $\zeta$ is a morphism in $\mathcal{H}_Y \mathcal{D}$.

Moreover on elements in the basis we get

$$\left( \partial^2 \zeta \right) \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) = \left( \varepsilon \otimes \zeta \right) \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) - \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) \varepsilon \mathfrak{z} + \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) \mathfrak{z} - \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) \mathfrak{z} \varepsilon \mathfrak{z}.$$

By using (23), one gets

$$\zeta \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) = \delta_{|x|+|y|+|z|,s} \mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) \mathfrak{z} \varepsilon \mathfrak{z} \quad \text{and} \quad \zeta \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) = \delta_{|x|+|y|+|z|,s} \mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) \mathfrak{z} \varepsilon \mathfrak{z}.$$

By means of these equalities one gets

$$\left( \partial^2 \zeta \right) \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) = \delta_{|x|+|y|+|z|,s} \left( \partial^2 \mathfrak{m} \right) \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) = \delta_{|x|+|y|+|z|,s} \mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \otimes \mathfrak{z} \right) \mathfrak{z} \varepsilon \mathfrak{z}.$$

Therefore $\partial^2 \zeta = \omega_s$. This means that we can assume that $\mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right) = 0$ for $|x| + |y| \neq s$. Equivalently

$$\mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right) = \delta_{|x|+|y|,s} \mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right) \mathfrak{z} \varepsilon \mathfrak{z} \quad \text{for} \quad x, y \in \text{the basis}.$$

Equivalently

$$\mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right) = \delta_{|x|+|y|,s} \mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right) \mathfrak{z} \varepsilon \mathfrak{z} \quad \text{for} \quad x, y \in \text{the basis}.$$

In particular, one gets

$$\mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right) = \delta_{|x|+|y|,s} \mathfrak{m} \left( \mathfrak{x} \otimes \mathfrak{y} \right) \mathfrak{z} \varepsilon \mathfrak{z} \quad \text{for} \quad x, y \in \text{the basis}.$$

Note also that both $v$ and $\gamma$ are morphisms in $\mathcal{H}_Y \mathcal{D}$ as they are obtained as composition or sum of morphisms in this category. Let us check that $\gamma$ is a gauge transformation on $Q$ in $\mathcal{H}_Y \mathcal{D}$.

Recall that $x^0 \otimes 1_Q = 1_Q$ is in the basis. For $x$ in the basis, we have $\gamma \left( x \otimes 1_Q \right) = \varepsilon \left( x \right) + v \left( x \otimes 1_Q \right).$ Note that

$$\gamma \left( x \otimes 1_Q \right) = \delta_{|x|,s} \varepsilon \left( x \right) \left( x \otimes 1_Q \right) + \delta_{|x|+1_Q,|x|,s} \omega \left( x \otimes 1_Q \otimes 1_Q \right).$$
and in view of (27), the term \( H \) coincide with \( \gamma \). Hence \( \gamma \) is unital. Note that the coalgebra \( C = Q \otimes \mathbb{K} \) is connected as \( Q \) is. Thus, in order to prove that \( \gamma : Q \otimes \mathbb{K} \to \mathbb{K} \) is convolution invertible it suffices to check (see [Ma, Lemma 5.2.10]) that \( \gamma|_{k1Q \otimes k1Q} \) is convolution invertible. But for \( k, k' \in \mathbb{K} \) we have

\[
\gamma(k1Q \otimes k'1Q) = kk'\gamma(1Q \otimes 1Q) = kk' = (\varepsilon \otimes \varepsilon)(k1Q \otimes k'1Q)
\]

Hence \( \gamma|_{k1Q \otimes k1Q} = (\varepsilon \otimes \varepsilon)|_{k1Q \otimes k1Q} \) which is convolution invertible. Thus there is a \( \mathbb{K} \)-linear map \( \gamma^{-1} : Q \otimes Q \to \mathbb{K} \) and such that

\[
\gamma \cdot \gamma^{-1} = \varepsilon \otimes \varepsilon = \gamma^{-1} \cdot \gamma.
\]

Note that, by Lemma 2.3, \( \gamma \in H^2 \mathcal{YD} \) implies \( \gamma^{-1} \in H^2 \mathcal{YD} \). Therefore \( \gamma \) is a gauge transformation in \( H^2 \mathcal{YD} \). By Proposition 2.4, \( Q^\gamma \) is a coquasialgebra in \( H^2 \mathcal{YD} \). By Proposition 2.4, we have that \( grQ^\gamma \) and \( grQ \) coincide as bialgebras in \( H^2 \mathcal{YD} \). Hence \( H^3_{\mathcal{YD}}(grQ^\gamma, k) = H^3_{\mathcal{YD}}(grQ, k) = 0 \). Therefore \( Q^\gamma \) fulfills the same requirement of \( Q \) as in the statement. Let us check that \( (\omega^\gamma)_t = 0 \) for \( 1 \leq t \leq s \) (this will complete the proof by an induction process as \( Q \) is finite-dimensional). Note that the definition of \( \gamma \) and (25) imply

\[
(\omega^\gamma)_x(u) = \delta_{|x|+|y|+|z|,s}(\varepsilon \otimes \varepsilon)(x \otimes y) + \delta_{|x|+|y|+|z|,s}(x \otimes y)
\]

for \( x, y \) in the basis. Let \( C^2 = Q \otimes Q \) and let \( C^{2}_x = \sum_{i+j \leq n} Q_i \otimes Q_j \). For \( u \in C^{2}_x \), we have

\[
(\gamma \cdot (\varepsilon \otimes \varepsilon - v))(u) = (\varepsilon \otimes \varepsilon)(u) - v(u) + v(u) - v(u_1) v(u_2) (\varepsilon \otimes \varepsilon)(u).
\]

Therefore \( (\gamma \cdot (\varepsilon \otimes \varepsilon - v)) \in C^{2}_x \). By uniqueness of the convolution inverse, we deduce

\[
\gamma^{-1}(u) = (\varepsilon \otimes \varepsilon)(u) - v(u), \text{ for } u \in C^{2}_x.
\]

Now, all terms appearing in the last two lines, excepted \( \omega_s \), vanish out of degrees 0 and \( s \) and coincide with \( \varepsilon \otimes \varepsilon \) on degree 0. On the other hand \( \omega_s \) vanishes out of \( s \). Since \( \gamma := (\varepsilon \otimes \varepsilon) + v \) and in view of (27), the term \( \delta_{|x|+|y|+|z|,s} \) forces the following simplification

\[
(\omega^\gamma)_x(u) = \delta_{|x|+|y|+|z|,s} [(\varepsilon \otimes v)(u) + v(Q \otimes m)(u) - v(m \otimes Q)(u) - (\varepsilon \otimes \varepsilon)(u)] + \delta_{|x|+|y|+|z|,s} \omega_s(u).
\]

Now \( \omega_s(u) = \delta_{|x|+|y|+|z|,s} \) while one proves that \( (\varepsilon \otimes v)(u) = \left( \varepsilon \otimes \varepsilon \right)(u) + \delta_{|x|+|y|+|z|,s} v(m \otimes Q)(u) = \delta_{|x|+|y|+|z|,s} \left( \varepsilon \otimes v \right)(m \otimes Q) \) and similarly for the other pieces of the equality.

Thus one gets

\[
(\omega^\gamma)_x(u) = \delta_{|x|+|y|+|z|,s} \left( \varepsilon \otimes v \right)(u) + \delta_{|x|+|y|+|z|,s} v(m \otimes Q)(u) - (\varepsilon \otimes \varepsilon)(u).
\]
\[
= -\delta_{|x|+|y|+|z|,x} \partial^2 \omega + \delta_{|x|+|y|+|z|,x} \omega^2 (\overline{\epsilon}) = 0.
\]
For \(0 \leq t \leq s - 1\), analogously to the above, we compute
\[
(\omega^\gamma)_t (u) = \delta_{|x|+|y|+|z|,x} \omega^\gamma (u)
\]
\[
= \delta_{|x|+|y|+|z|,x} \left[ (\varepsilon \otimes \gamma) \ast (Q \otimes m) \ast \omega \ast \gamma^{-1} (m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon) \right] (u)
\]
\[
= \delta_{|x|+|y|+|z|,x} \left[ (\varepsilon \otimes \gamma) \ast (Q \otimes m) \ast \omega_0 \ast \gamma^{-1} (m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon) \right] (u)
\]
\[
\delta_{|x|+|y|+|z|,x} \left[ (\varepsilon \otimes \gamma) \ast (Q \otimes m) \ast \gamma^{-1} (m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon) \right] (u)
\]
\[
= \delta_{|x|+|y|+|z|,x} \left[ (\varepsilon \otimes \varepsilon) \ast (Q \otimes m) \ast \gamma^{-1} (m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon) \right] (u).
\]

Therefore we can now repeat the argument on \(\omega^\gamma\) instead of \(\omega\). Deforming several times we will get a reassociator, say \(\omega'\), whose first non trivial component \(\omega'_t\), with \(t \neq 0\), exceeds the dimension of \(Q\). In other words \(\omega' = \omega_0\) which is trivial. Hence \(Q\) is gauge equivalent to a connected bialgebra in \(\mathcal{H} \mathcal{Y} \mathcal{D}\).

\[\square\]

4. INVARpANTS

Given a \(k\)-algebra \(A\), we denote by \(H^n (A, -)\) the \(n\)-th right derived functor of \(\text{Hom}_{A,A} (A, -)\) in the category of \(A\)-bimodules. In other words, for every \(A\)-bimodule \(M\), \(H^n (A, M)\) is the Hochschild cohomology group of \(A\) with coefficients in \(M\). Denote by \(Z^n (A, M)\) and \(B^n (A, M)\) the abelian groups of \(n\)-cocycles and of \(n\)-coboundaries respectively.

Let \(H\) be a Hopf algebra, let \(B\) be a left \(H\)-module algebra and let \(M\) be a \(B\#H\)-bimodule, where \(B\#H\) denotes the smash product algebra, see e.g. [Mk, Definition 4.1.3]. Then \(H^n (B, M)\) becomes an \(H\)-bimodule as follows. Its structure of left \(H\)-module is defined, for every \(f \in Z^n (B, M)\) and \(h \in H\), by setting
\[
[f] h := [\chi^n_h (M) (f)]
\]
where, for every \(k \in k, b_1, \ldots, b_n \in B\), we set
\[
\chi^0_h (M) (f) (k) := (1_B \# S (h_1)) f (k) (1_B \# h_2) \quad \text{for } n = 0 \quad \text{while and for } n \geq 1
\]
\[
\chi^h_h (M)(f) (b_1 \otimes b_2 \otimes \cdots \otimes b_n) := (1_B \# S (h_1)) f (h_2 b_1 \otimes h_3 b_2 \otimes \cdots \otimes h_{n+1} b_n) (1_B \# h_{n+2}).
\]

Moreover
\[
(28) \quad \partial^n \ast \chi^h_h (M) = \chi^{h+1}_h (M) \ast \partial^n, \quad \text{for every } n \geq -1,
\]
where \(\partial^n : \text{Hom}_k (B^\otimes n, M) \to \text{Hom}_k (B^\otimes (n+1), M)\) denotes the differential of the usual Hochschild cohomology.

Denote by \(H^n (B, M)^H\) the space of \(H\)-invariant elements of \(H^n (B, M)\).

PROPOSITION 4.1. Let \(H\) be a semisimple Hopf algebra and let \(B\) be a left \(H\)-module algebra. Denote by \(A := B\#H\). Then, for each \(n \in \mathbb{N}_0\) and for every \(A\)-bimodule \(M\)
\[
H^n (B\#H, M) \cong H^n (B, M)^H.
\]

Proof. We will apply [St, Equation (3.6.1)]. To this aim we have to prove first that \(A/B\) is an \(H\)-Galois extension such that \(A\) is flat as left and right \(B\)-module. Now, \(A = B\#H\) for \(\xi : H \otimes H \to B\) defined by \(\xi (x, y) = \varepsilon_H (x) \varepsilon_H (y) 1_A\), cf. [Mo, Definition 7.1.1]. Moreover a direct computation shows that \(\iota : B \to A : b \mapsto b \# 1_H\) is a right \(H\)-extension where \(A\) is regarded as a right \(H\)-comodule algebra via \(\rho : A \to B \otimes H : b \mapsto (b \# h_1) \otimes h_2\). Thus, by [Mo, Proposition 7.2.7], we know that \(\iota : B \to A\) is \(H\)-cleft and hence, by [Mo, Theorem 8.2.4], it is \(H\)-Galois. The \(B\)-bimodule structure of \(A\) is induced by \(\iota\) so that, explicitly, we have
\[
b' (b \# h) = (b' \# 1_H) (b \# h) = b' b \# h,
\]
\[
(b \# h) b' = (b \# h) (b' \# 1_H) = b (h_1 b') \# h_2.
\]
Note that \( A = B \# H \) is flat as a left \( B \)-module as \( H \) is a free \( k \)-module (\( k \) is a field). Now consider the map \( \alpha : H \otimes B \rightarrow A \) defined by setting \( \alpha (h \otimes b) := h_1 b \otimes h_2 \) (note that it is defined as the braiding in \( \text{YD}^H \)). We have

\[
\alpha (h \otimes bb') = h_1 (bb') \otimes h_2 = (h_1b)(h_2b') \otimes h_3 = (h_1b\# h_2)b' = \alpha (h \otimes b)b'
\]

so that \( \alpha \) is right \( B \)-linear where \( H \otimes B \) is regarded as a right module via \((h \# b)b' = h \# bb'\). Now \( H \) is semisimple and hence separable (see [Stu, Corollary 3.7]). Thus \( H \) is finite-dimensional and hence it has bijective antipode \( S_H \). Thus \( \alpha \) is invertible with inverse given by \( \alpha^{-1} (b \# h) := h_2 \otimes S_H^{-1} (h_1) b \). Therefore \( \alpha \) is an isomorphism of right \( B \)-modules and hence \( A \) is flat as a right \( B \)-module as \( H \otimes B \) is.

We have now the hypotheses necessary to apply [Stu] Equation (3.6.1) and obtain

\[
H^n (A, M) \cong \text{Hom}_{-H} (k, H^n (B, M)) = \text{Hom}_k (k, H^n (B, M))^H \cong H^n (B, M)^H.
\]

\[\square\]

Remark 4.2. Proposition [4.2] in the particular case when \( M = k \) and \( B \) is finite-dimensional is [SV, Theorem 2.17]. Note that in the notations therein, one has \( E(B) = \oplus_{n \in \mathbb{N}} E_n (B, k) \) where \( E_n (B, k) = \text{Ext}^n_B (k, k) \cong H^n (B, k) \). The latter isomorphism is [CE, Corollary 4.4, page 170].

Let \( H \) be a Hopf algebra and let \( B \) be a bialgebra in the braided category \( \text{YD}^H \). Denote by \( A := B \# H \) the Radford-Majid bosonization of \( B \) by \( H \), see e.g. [Rn2, Theorem 1]. Note that \( A \) is endowed with an algebra map \( \varepsilon_A : A \rightarrow k \) defined by \( \varepsilon_A (b \# h) = \varepsilon_B (b) \varepsilon_H (h) \) so that we can regard \( k \) as an \( A \)-bimodule via \( \varepsilon_A \). Then we can consider \( H^n (B, k) \) as an \( H \)-bimodule as follows. Its structure of left \( H \)-module is given via \( \varepsilon_H \) and its structure of right \( H \)-module is defined, for every \( f \in \mathcal{Z}^n (B, k) \) and \( h \in H \), by setting

\[
[f] h := [fh],
\]

where \((fh)(z) = f(hz)\), for every \( z \in B^\otimes n \). The latter is the usual right \( H \)-module structure of \( \text{Hom}_k (B^\otimes n, k) \). Indeed, for every \( n \geq -1 \), the vector space \( \text{Hom}_k (B^\otimes n, k) \) is an \( H \)-bimodule with respect to this right \( H \)-module structure and the left one induced by \( \varepsilon_H \).

Corollary 4.3. Let \( H \) be a semisimple Hopf algebra and let \( B \) be a bialgebra in the braided category \( \text{YD}^H \). Set \( A := B \# H \). Then, for each \( n \in \mathbb{N}_0 \)

\[
H^n (B \# H, k) \cong H^n (B, k)^H
\]

and the differential \( \partial^n : \text{Hom}_k (B^\otimes n, k) \rightarrow \text{Hom}_k (B^\otimes (n+1), k) \) of the usual Hochschild cohomology is \( H \)-bilinear.

Proof. In the particular case where \( M = k \), the right module \( H \)-structure used in Proposition [4.2] simplifies as follows. It is defined, for every \( f \in \mathcal{Z}^n (B, k) \) and \( h \in H \), by setting

\[
[f] h := [\chi^n_k (k) (f)]
\]

where, for every \( k \in k, b_1, \ldots, b_n \in B \), we set

\[
\chi^n_k (k) (f) (k) := \varepsilon_H (h) f (k) \quad \text{for} \quad n = 0 \quad \text{while and for} \quad n \geq 1
\]

\[
\chi^n_k (k) (f) (b_1 \otimes b_2 \otimes \cdots \otimes b_n) := f (h_1 b_1 \otimes h_2 b_2 \otimes \cdots \otimes h_n b_n).
\]

More concisely \( \chi^n_k (k) (f) (z) = f (hz) \) for every \( n \in \mathbb{N}_0 \) and \( z \in B^\otimes n \) i.e. \( [fh] := [\chi^n_k (k) (f)] \).

Now consider the differential \( \partial^n : \text{Hom}_k (B^\otimes n, k) \rightarrow \text{Hom}_k (B^\otimes (n+1), k) \) of the usual Hochschild cohomology. Note that for each \( n \in \mathbb{N}_0 \), \( \text{Hom}_k (B^\otimes n, k) \) is regarded as a bimodule over \( H \) using the left \( H \)-module structures of its arguments. By [20], we have

\[
\partial^n \chi^n_k (k) (f) = \chi^{n+1}_k (k) (\partial^n f)
\]

Since \( \chi^n_k (k) (f) = fh \), the last displayed equality becomes \( \partial^n (fh) = \partial^n (f) h \) for every \( n \in \mathbb{N}_0 \). Thus \( \partial^n \) is right \( H \)-linear. Since \( hf = \varepsilon_H (h) f \) for every \( f \in \text{Hom}_k (B^\otimes n, k), h \in H \), we get that \( \partial^n \) is also left \( H \)-linear whence \( H \)-bilinear. \( \square \)
Remark 4.4. Note that, in the context of the proof of [EC] Proposition 5.1, one has
\[ H^3(B(V) \# C[Z_p], C) \cong H^3(B(V), C)^Z. \]
This is a particular case of Corollary 4.3 where \( H = C[Z_p], V \in \mathcal{Y} \) and \( B = B(V) \).

Proposition 4.5. Let \( C \) and \( D \) be abelian categories. Let \( r, \omega : C \to D \) be exact functors such that \( r \) is a subfunctor of \( \omega \) i.e. there is a natural transformation \( \eta : r \to \omega \) which is a monomorphism when evaluated on objects. If \( X \) is a subobject of \( Y \) then \( r(X) = \omega(X) \cap r(Y) \). Moreover, for every morphism \( f : X \to Y \) in \( C \) one has
\[ \ker (r(f)) = (\ker (f)) \cap r(X) = \ker (\omega(f)) \cap r(X), \]
\[ \text{Im}(r(f)) = \text{Im}(\omega(f)) \cap r(Y) = r(\text{Im}(f)). \]
Proof. The proof is similar to [Stn, Proposition 1.7, page 138].

Remark 4.6. From Corollary 4.3 we have
\[ H^n(B, k)^H = \{ f \mid f \in Z^n(B, k), \varepsilon_H(h)[f] = [f] h, \text{ for every } h \in H \} \]
where, for every \( z \in B^\otimes n \), we have
\[ (fh)(z) = f(hz). \]
Note that, for any \( H \)-bimodule \( M \) one has
\[ \text{Hom}_{H,M}(H, M) \cong M^H = \{ m \in M \mid hm = mh, \text{ for every } h \in H \}. \]
Note also that \( H \) is a separable \( k \)-algebra whence it is projective in the category of \( H \)-bimodules. As a consequence \( \text{Hom}_{H,H}(H, -) \cong (-)^H : H\text{-Mod}_H \to \mathcal{M} \) is an exact functor (here \( H\text{-Mod}_H \) is the category of \( H \)-modules and \( \mathcal{M} \) the category of \( k \)-vector spaces). By Proposition 4.3 applied to the case when \( r := (-)^H : H\text{-Mod}_H \to \mathcal{M} \) and \( \omega \) is the forgetful functor, for every morphism \( f : X \to Y \) of \( H \)-bimodules one has
\[ \ker (f^H) = (\ker f) \cap X^H = (\ker f)^H \quad \text{and} \quad \text{Im}(f^H) = (\text{Im} f) \cap Y^H = (\text{Im} f)^H. \]
Still by Corollary 4.3 we know that the differential \( \partial^n : \text{Hom}_k(B^\otimes n, k) \to \text{Hom}_k(B^\otimes (n+1), k) \) of the usual Hochschild cohomology is \( H \)-bilinear. Thus we can apply the argument above to get
\[ \ker \left( (\partial^n)^H \right) = \ker (\partial^n) \cap \text{Hom}_k(B^\otimes n, k)^H = (\ker (\partial^n))^H \quad \text{and} \quad \text{Im} \left( (\partial^{n-1})^H \right) = \text{Im} (\partial^{n-1}) \cap \text{Hom}_k(B^\otimes n, k)^H = (\text{Im} (\partial^{n-1}))^H. \]
Now \( \text{Hom}_k(B^\otimes n, k)^H = \text{Hom}_{H,-}(B^\otimes n, k) \) so that we get
\[ Z^n_{H,\text{-Mod}}(B, k) = Z^n(B, k) \cap \text{Hom}_{H,-}(B^\otimes n, k) = Z^n(B, k)^H \quad \text{and} \quad B^n_{H,\text{-Mod}}(B, k) = B^n(B, k) \cap \text{Hom}_{H,-}(B^\otimes n, k) = B^n(B, k)^H. \]
where \( Z^n_{H,\text{-Mod}}(B, k) \) and \( B^n_{H,\text{-Mod}}(B, k) \) denotes the abelian groups of \( n \)-cocycles, of \( n \)-coboundaries for the cohomology of the algebra \( B \) with coefficients in \( k \) computed in the monoidal category \( H\text{-Mod} \) of left \( H \)-modules. The corresponding \( n \)-th Hochschild cohomology group is
\[ H^n_{H,\text{-Mod}}(B, k) := \frac{Z^n_{H,\text{-Mod}}(B, k)}{B^n_{H,\text{-Mod}}(B, k)} = \frac{Z^n(B, k)^H}{B^n(B, k)^H} \cong \left( \frac{Z^n(B, k)}{B^n(B, k)} \right)^H = H^n(B, k)^H. \]
Denote by \( D(H) \) the Drinfeld double, see e.g. the first structure of [Maj, Theorem 7.1.1].

Proposition 4.7. In the setting of Corollary 4.3 assume that \( H \) is also cosemisimple. Then, for \( n \in \mathbb{N}_0 \)
\[ Z^n_{D}(B, k) = Z^n(B, k)^{D(H)}, \quad B^n_{D}(B, k) = B^n(B, k)^{D(H)} \quad \text{and} \quad H^n_{D}(B, k) \cong H^n(B, k)^{D(H)}. \]
where \( Z^n(B, k) \) and \( B^n(B, k) \) are regarded as \( D(H) \)-subbimodules of \( \text{Hom}_k(B^\otimes n, k) \) whose structure is induced by the left \( D(H) \)-module structures of its arguments.
Moreover $H^n(B,\mathbb{k})^{D(H)}$ is a subspace of $H^n(B,\mathbb{k})^H$.

Proof. For shortness, in this proof, we denote $D(H)$ by $D$. Consider the analogue of the standard complex as in Remark 3.4.1:

\[
\begin{array}{cccc}
H^n_\mathcal{YD}(k,\mathbb{k}) & \xrightarrow{\partial^n} & H^n_\mathcal{YD}(B,\mathbb{k}) & \xrightarrow{\partial^1} \quad H^n_\mathcal{YD}(B^{\otimes 2},\mathbb{k}) & \xrightarrow{\partial^2} & \cdots \\
\end{array}
\]

where $\partial^n$ is induced by the differential $\partial^n : \text{Hom}_k(B^{\otimes n},\mathbb{k}) \to \text{Hom}_k(B^{\otimes (n+1)},\mathbb{k})$ of the ordinary Hochschild cohomology. Now, since $H$ is semisimple, it is finite-dimensional (whence it has bijective antipode) so that, by a result essentially due to Majid (see [Mo, Proposition 10.6.16]) and by [RT, Proposition 6], we get a category isomorphism $H_\mathcal{YD} \cong D_{\mathcal{M}}$. Thus the complex above can be rewritten as follows

\[
\begin{array}{cccc}
\text{Hom}_{D,-}(k,\mathbb{k}) & \xrightarrow{\partial^n} & \text{Hom}_{D,-}(B,\mathbb{k}) & \xrightarrow{\partial^1} \quad \text{Hom}_{D,-}(B^{\otimes 2},\mathbb{k}) & \xrightarrow{\partial^2} & \cdots \\
\end{array}
\]

Now, since, for each $n \in \mathbb{N}_0$, we have $\text{Hom}_{D,-}(B^{\otimes n},\mathbb{k}) = \text{Hom}_k(B^{\otimes n},\mathbb{k})^D$, we obtain the complex

\[
\begin{array}{cccc}
\text{Hom}_k(k,\mathbb{k})^D & \xrightarrow{\partial^n} & \text{Hom}_k(B,\mathbb{k})^D & \xrightarrow{\partial^1} \quad \text{Hom}_k(B^{\otimes 2},\mathbb{k})^D & \xrightarrow{\partial^2} & \cdots \\
\end{array}
\]

We will write $(\partial^n)^D$ instead of $\partial^n$ when we would like to stress that the map considered is the one induced on invariants. Thus we will write equivalently

\[
\begin{array}{cccc}
\text{Hom}_k(k,\mathbb{k})^D & \xrightarrow{(\partial^n)^D} & \text{Hom}_k(B,\mathbb{k})^D & \xrightarrow{(\partial^1)^D} \quad \text{Hom}_k(B^{\otimes 2},\mathbb{k})^D & \xrightarrow{(\partial^2)^D} & \cdots \\
\end{array}
\]

Now, assume $H$ is also cosesimisimple. Since $H$ is both semisimple and cosesimisimple, by [Ra2, Proposition 7] the Hopf algebra $D$ is semisimple as an algebra. Thus, as in Remark 3.4.1, in case of $H$, the functor $(-)^D : D\mathcal{M}_D \to \mathcal{M}$ is exact (here $D\mathcal{M}_D$ is the category of $D$-bimodules and $\mathcal{M}$ the category of $\mathbb{k}$-vector spaces). By Proposition 7.1.6 applied to the case when $r := (\omega)^D : D\mathcal{M}_D \to \mathcal{M}$ and $\omega$ is the forgetful functor, for every morphism $f : X \to Y$ of $D$-bimodules one has

$$\ker (f^D) = \ker (f) \cap X^D = (\ker (f))^D \quad \text{and} \quad \text{Im} (f^D) = \text{Im} (f) \cap Y^D = (\text{Im} (f))^D.$$ 

In particular we get

$$\ker \left( (\partial^n)^D \right) = \ker (\partial^n) \cap \text{Hom}_k \left( B^{\otimes n}, \mathbb{k} \right)^D = \ker (\partial^n)^D \quad \text{and} \quad \text{Im} \left( (\partial^n-1)^D \right) = \text{Im} (\partial^n-1) \cap \text{Hom}_k \left( B^{\otimes n}, \mathbb{k} \right)^D = (\text{Im} (\partial^n-1))^D$$

and hence

$$Z^n_{\mathcal{YD}}(B,\mathbb{k}) = Z^n(B,\mathbb{k}) \cap \text{Hom}_{D,-} \left( B^{\otimes n}, \mathbb{k} \right) = Z^n(B,\mathbb{k})^D \quad \text{and} \quad B^n_{\mathcal{YD}}(B,\mathbb{k}) = B^n(B,\mathbb{k}) \cap \text{Hom}_{D,-} \left( B^{\otimes n}, \mathbb{k} \right) = B^n(B,\mathbb{k})^D$$

Then we obtain

$$H^n_{\mathcal{YD}}(B,\mathbb{k}) = \frac{Z^n_{\mathcal{YD}}(B,\mathbb{k})}{B^n_{\mathcal{YD}}(B,\mathbb{k})} = \frac{Z^n(B,\mathbb{k})^D}{B^n(B,\mathbb{k})^D} \cong H^n(B,\mathbb{k})^D.$$ 

Let us prove the last part of the statement. The correspondence between the left $D$-module structure and the structure of Yetter-Drinfeld module over $H$ is written explicitly in [Ma, Proposition 7.1.6]. In particular $D = H^* \otimes H$ and given $v \in H_\mathcal{YD}$, the two structures are related by the following equality $(f \otimes h) \triangleright v = f \left( (h \triangleright v)_{-1} \right) (h \triangleright v)_0$ for every $f \in H^*$, $h \in H$, $v \in V$. Thus $(\varepsilon_H \otimes h) \triangleright v = h \triangleright v$. Moreover $H$ is a Hopf subalgebra of $D$ via $h \mapsto \varepsilon_H \otimes h$, where $D$ is considered with the first structure of [Ma, Theorem 7.1.1]. Since the $D$-bimodule structure of $H^n(B,\mathbb{k})$ is induced by the one of $\text{Hom}_k \left( B^{\otimes n}, \mathbb{k} \right)$ which comes from the left $D$-module structures of its arguments and similarly for the $H$-bimodule structure of $H^n(B,\mathbb{k})$, we deduce that $H^n(B,\mathbb{k})^D$ is a subspace of $H^n(B,\mathbb{k})^H$. \qed
Example 4.8. In the setting of the proof of [AM3, Theorem 4.1.3], a Nichols algebra $B(V)$ such that $H^3(B(V), k)_{Z_m} = 0$ is considered where $k$ is a field of characteristic zero. By Proposition II.3 applied in the case $H = kZ_m$ and $B = B(V)$, we have that $H^3_{YD}(B(V), k) \cong H^3(B(V), k)^{D(H)}$ is a subspace of $H^3(B(V), k)^H = H^3(B(V), k)^{Z_m} = 0$. Thus we get $H^3_{YD}(B(V), k) = 0$. Therefore, in view of Theorem II.3 if $(q, m, u, \Delta, \varepsilon, \omega)$ is a f.d. connected coquasi-bialgebra in $H^H_{YD}$ such that $grQ \cong B(V)$ (as above) as augmented algebras in $H^H_{YD}$ (the count must be the same in order to have the same Yetter-Drinfeld module structure on $k$), then we can conclude that $Q$ is gauge equivalent to a connected bialgebra in $H^H_{YD}$.

Remark 4.9. Let $A$ be a finite-dimensional coquasi-bialgebra with the dual Chevalley property i.e. the coradical $H$ of $A$ is a coquasi-subbialgebra of $A$ (in particular $H$ is cosemisimple). Assume the coquasi-bialgebra structure of $H$ has trivial reassociator (i.e. it is an ordinary bialgebra) and also assume it has an antipode (i.e. it is a Hopf algebra). Then, by [AR, Corollary 6.4], $grA$ is isomorphic to $R\#H$ as a coquasi-bialgebra, where $R$ is a suitable connected bialgebra in $H^H_{YD}$. Note that $R\#H$ is the usual Radford-Majid bosonization as $H$ has trivial reassociator, see [AR, Definition 5.4]. Hence we can compute

$$H^3(grA, k) = H^3(R\#H, k).$$

Assume further that $H$ is semisimple. Then, by Corollary II.3, we have

$$H^n(R\#H, k) \cong H^n(R, k)^H$$

so that $H^3(grA, k) \cong H^3(R, k)^H$. Thus, if $H^3(R, k)^H = 0$, one gets $H^3(grA, k) = 0$ which is the analogue of the condition II.3. Proposition 2.3 (note that our $A$ is the dual of the one considered therein) which guarantees that $A$ is gauge equivalent to an ordinary Hopf algebra, if $A$ has an a quasi-antipode and $k = \mathbb{C}$. Next we will give another approach to arrive at the same conclusion but just requiring $H^3_{YD}(R, k) = 0$. Note that a priori $H^3_{YD}(R, k) \cong H^3(R, k)^{D(H)}$ is smaller than $H^3(R, k)^H$.

5. Dual Chevalley

The main aim of this section is to prove Theorem II.4. Let $A$ be a Hopf algebra over a field $k$ of characteristic zero such that the coradical $H$ of $A$ is a sub-Hopf algebra (i.e. $A$ has the dual Chevalley Property). Assume $H$ is finite-dimensional so that $H$ is semisimple. By [ABM, Theorem I], there is a gauge transformation $\zeta : A \otimes A \to k$ such that $A^\zeta$ is isomorphic, as a coquasi-bialgebra, to the bosonization $Q\#H$ of a connected coquasi-bialgebra $Q$ in $H^H_{YD}$ by $H$. By construction $\zeta$ is $H$-bilinear and $H$-balanced: this follows from [ABM, Proposition 5.7] (note that gauge transformation $v_B : B \otimes B \to k$, used therein for $B := R\#_\xi H$, is $H$-bilinear and $H$-balanced, as observed in the proof) and the fact that there is an $H$-bilinear Hopf algebra isomorphism $\psi : B \to A$ (see [ABM, Proof of Theorem I, page 36 and Theorem 6.1] which is a consequence of [AMSI, Theorem 3.64]) where $(R, \xi)$ is a suitable connected pre-bialgebra with cocycle in $H^H_{YD}$ (note that $\xi = v_B \circ (\psi^{-1} \otimes \psi^{-1})$): here by connected pre-bialgebra we mean that the coradical $R_0$ of $R$ is $1_R$ (by the properties of $1_R$ this implies that $R_0$ is a subcoalgebra in $H^H_{YD}$ of $R$). Assume that $A$ is finite-dimensional. Then $Q\#H$ and hence $Q$ is finite dimensional.

Thus, by Theorem II.2, if $H^3_{YD}(grQ, k) = 0$, then $Q$ is gauge equivalent to a connected bialgebra in $H^H_{YD}$.

First let us check which condition on $A$ guarantee that $H^3_{YD}(grQ, k) = 0$. Note that by construction $Q = R''$ (see [ABM, Proposition 5.7]) where $v := (\lambda \xi)^{-1}$, the convolution inverse of $\lambda \xi$ and $\lambda : H \to k$ denotes the total integral on $H$. Thus we can rewrite $gr(Q)$ as $gr(R'')$.

Moreover $v_B$ is given by $v_B((r \# h) \otimes (r' \# h')) = v(r \otimes hr') \varepsilon_H(h')$ for every $r, r' \in R, h, h' \in H$. By [AMSt1, Proposition 2.5], $gr(R)$ inherits the pre-bialgebra structure in $H^H_{YD}$ of $R$. This is proved by checking that $R_i \cdot R_j \subseteq R_{i+j}$ for every $i, j \in \mathbb{N}_0$, where $R_i$ denotes the $i$-th term of the coradical filtration of $R$. Moreover $R_1$ is a subcoalgebra of $R$ in $H^H_{YD}$. 


Lemma 5.1. Keep the above hypotheses and notations. Then \( \text{gr} (R^n) \) and \( \text{gr} (R) \) coincide as bialgebras in \( \mathcal{HYD} \) where the structures of \( \text{gr}(R) \) are induced by the ones of \( (R, \xi) \).

Proof. By Theorem 4.1, \( \text{gr} (R^n) = \text{gr} (Q) \) is a connected bialgebra in \( \mathcal{HYD} \).

Note that \( R^n \) and \( R \) coincide as coalgebras in \( \mathcal{HYD} \) so that \( \text{gr} (R^n) \) and \( \text{gr} (R) \) coincide as coalgebras in \( \mathcal{HYD} \). They also have the same unit. It remains to check that their two multiplications coincide too.

Since \( \xi \) is unital, by \( \text{[AMS1, Proposition 4.8]} \), we have that \( v \) is unital and this is equivalent to \( v^{-1} \) unital (see the proof therein).

Let \( C := R \otimes R \). Let \( n > 0 \) and let \( w \in C(n) = \sum_{i+j \leq n} R_i \otimes R_j \). By \( \text{[AMS]} \) Lemma 3.69, we have that

\[
\Delta_C(w) - w \otimes (1_R) \otimes (1_R) \otimes (1_R) \otimes w \in C(n-1) \otimes C(n-1).
\]

Thus we get

\[
w_1 \otimes w_2 \otimes w_3 - \Delta_C(w) \otimes (1_R) \otimes (1_R) \otimes w \in \Delta_C (C(n-1)) \otimes C(n-1)
\]

and hence

\[
w_1 \otimes w_2 \otimes w_3 - w \otimes (1_R) \otimes (1_R) \otimes w \otimes (1_R) \otimes w \in C(n-1) \otimes C(n-1) \otimes C(n-1).
\]

Since \( m(C(n-1)) \subseteq \sum_{i+j \leq n} m(R_i \otimes R_j) \subseteq R_{n-1} \) we get

\[
w_1 \otimes m(w_2) \otimes w_3 - w \otimes 1_R \otimes (1_R) \otimes w \otimes 1_R \otimes w - m(w) \otimes (1_R) \otimes w \in C(n-1) \otimes R_{n-1} \otimes C(n-1)
\]

and hence

\[
w_1 \otimes (m(w_2) + R_{n-1}) \otimes w_3 = (1_R) \otimes (m(w) + R_{n-1}) \otimes (1_R).
\]

Let \( x, y \in R \). We compute

\[
\varpi \cdot y = (x + R_{|x|-1}) \cdot v \cdot (y + R_{|y|-1}) = (x \cdot y) + R_{|x|+|y|-1} = m(x \otimes y) + R_{|x|+|y|-1} = v ((x \otimes y)_1) m ((x \otimes y)_2) v^{-1} \left((x \otimes y)_3 + R_{|x|+|y|-1}\right) = v ((x \otimes y)_1) m ((x \otimes y)_2) + R_{|x|+|y|-1} v^{-1} \left((x \otimes y)_3\right) = v \left((1_R) \otimes 1_R\right) \left(m(x \otimes y) + R_{|x|+|y|-1}\right) v^{-1} \left((1_R) \otimes 1_R\right) = m(x \otimes y) + R_{|x|+|y|-1} = \varpi \cdot y.
\]

The following result is inspired by \( \text{[AMS1, Theorem 3.71]} \).

Lemma 5.2. Let \( H \) be a coassociative Hopf algebra. Let \( C \) be a left \( H \)-comodule coalgebra such that \( C_0 \) is a one-dimensional left \( H \)-comodule subcoalgebra of \( C \). Let \( B = C \# H \) be the smash coproduct of \( C \) by \( H \) i.e. the coalgebra defined by

\[
\Delta_B(c \# h) = \sum \left(c_1 \# (c_2)_{(-1)} h_1\right) \otimes \left((c_2)_{(0)} \# h_2\right),
\]

\[
\varepsilon_B(c \# h) = \varepsilon_C(c) \varepsilon_H(h).
\]

Then, for every \( n \in \mathbb{N}_0 \) we have \( B_n = C_n \# H \).

Proof. Since \( C_0 \) is a subalgebra of \( C \) in \( \mathcal{HR} \) and, for \( n \geq 1 \), one has \( C_n = C_{n-1} \cap C_0 \), then inductively one proves that \( C_n \) is a subalgebra of \( C \) in \( \mathcal{HR} \). Set \( B(n) := C_n \# H \) for every \( n \in \mathbb{N}_0 \). Let us check that \( B(n) = B_n \) by induction on \( n \in \mathbb{N}_0 \).

Let \( n = 0 \). First note \( B = \cup_{m \in \mathbb{N}_0} B(m) \) and, since \( \Delta_C(C_m) \subseteq \sum_{0 \leq i \leq m} C_i \otimes C_m-i \), we also have

\[
\Delta_B(B(m)) = \Delta_B(C_m \# H) \subseteq \sum_{0 \leq i \leq m} \left(C_i \# (C_m-i)_{(-1)} (H)_{1}\right) \otimes \left((C_m-i)_{(0)} \# (H)_{2}\right)
\]

\[
\subseteq \sum_{0 \leq i \leq m} \left(C_i \# (H) \otimes (C_m-i \# (H)) = \sum_{0 \leq i \leq m} B(i) \otimes B(m-i).
\]
Therefore $(B_{(m)})_{m \in \mathbb{N}_0}$ is a coalgebra filtration for $B$ and hence, by [SW] Proposition 11.1.1, we get that $B_{(0)} \supseteq B_0$. Since $C_0$ is one-dimensional, there is a grouplike element $1_C \in C_0$ such that $C_0 = k1_C$. Moreover one checks that $C_0$ is a subcoalgebra of $C$ in $\mathcal{H} \mathcal{M}$ implies $\sum (1_C)(-1) \otimes (1_C)(0) = 1_H \otimes 1_C$.

Let $\sigma : H \to C \otimes H : h \mapsto 1_C \otimes h$ be the canonical injection. We have

\[
\Delta_B \sigma (h) = \Delta_B (1_C \otimes h) = \sum (1_C \otimes (1_C)(-1) h_1) \otimes (1_C)(0) h_2
\]

\[
= \sum (1_C \otimes 1_H h_1) \otimes (1_C \otimes h_2) = \sum \sigma (h_1) \otimes \sigma (h_2) = (\sigma \otimes \sigma) \Delta_H (h),
\]

\[
\varepsilon_B \sigma (h) = \varepsilon_B (1_C \otimes h) = \varepsilon_C (1_C) \varepsilon_H (h) = \varepsilon_H (h)
\]

so that $\sigma$ is a coalgebra map. Since $H$ is cosemisimple and $\sigma$ an injective coalgebra map we deduce that also $\sigma (H) = C_0 \otimes H = B_{(0)}$ is a cosemisimple subcoalgebra of $B$ whence $B_{(0)} \subseteq B_0$.

Let $n > 0$ and assume that $B_i = B_{(i)}$ for $0 \leq i \leq n - 1$. Let $\sum c_i \# h_i \in B_n$. Then

\[
\Delta_B \left( \sum_{i \in I} c_i \# h_i \right) \in B_{n-1} \otimes B + B \otimes B_0 = C_{n-1} \otimes H \otimes C \otimes H + C \otimes H \otimes C_0 \otimes H.
\]

Let $p_n : C \to \frac{C}{C_0}$ be the canonical projection. If we apply $(p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H)$ we get

\[
0 = (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H) \Delta_B \left( \sum_{i \in I} c_i \# h_i \right)
\]

\[
= (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H) \left( \sum_{i \in I} (c_i)_{1} \# ((c_i)(-1) h_i)_{1} \otimes ((c_i)(0) \# (h_i)(2)) \right)
\]

\[
= (p_{n-1} \otimes p_0 \otimes H) \left( \sum_{i \in I} (c_i)_1 \otimes (c_i)_2 \otimes h_i \right) = ((p_{n-1} \otimes p_0) \Delta_C \otimes H) \left( \sum_{i \in I} c_i \# h_i \right).
\]

Thus $\sum c_i \# h_i \in \ker ((p_{n-1} \otimes p_0) \Delta_C \otimes H) = [\ker ((p_{n-1} \otimes p_0) \Delta_C)] \otimes H = C_n \otimes H \supseteq B_{(n)}$. Thus $B_n \subseteq B_{(n)}$.

On the other hand, form $\Delta_C (C_n) \subseteq C_{n-1} \otimes C + C \otimes C_0$ we deduce

\[
\Delta_B \left( B_{(n)} \right) = \Delta_B \left( C_n \otimes H \right)
\]

\[
\leq \sum \left( (C_n)_1 \# ((C_n)(-1) H)_1 \otimes ((C_n)(0) \# H)_2 \right)
\]

\[
\leq \sum \left( C_{n-1} \# (C)(-1) H \otimes (C)(0) \# H \right) + \sum \left( C \# (C)(-1) H \otimes (C)(0) \# H \right)
\]

\[
= B_{(n-1)} \otimes B + B \otimes B_0 = B_{n-1} \otimes B + B \otimes B_0
\]

and hence $B_{(n)} \subseteq B_n$.

**Definition 5.3.** Let $A$ be a Hopf algebra over a field $k$ such that the coradical $H$ of $A$ is a sub-Hopf algebra (i.e. $A$ has the dual Chevalley Property). Set $G := \text{gr} (A)$. There are two canonical Hopf algebra maps

\[
\sigma_G : H \to \text{gr} (A) : h \mapsto h + A_{-1},
\]

\[
\pi_G : \text{gr} (A) \to H : a + A_{n-1} \mapsto a \delta_{n,0}, \quad n \in \mathbb{N}_0.
\]

The diagram of $A$ (see [AS1], page 659) is the vector space

\[
\mathcal{D} (A) := \left\{ d \in \text{gr} (A) \mid \sum d_1 \otimes \pi_G (d_2) = d \otimes 1_H \right\}.
\]

It is a bialgebra in $H_H \mathcal{YD}$ as follows. $\mathcal{D} (A)$ is a subalgebra of $G$. The left $H$-action, the left $H$-coaction of $\mathcal{D} (A)$, the comultiplication and counit are given respectively by

\[
h \cdot d := \sum \sigma_G (h_1) d \sigma_G S (h_2), \quad \rho (d) = \sum \pi_G (d_1) \otimes d_2,
\]
Thus \( \psi \) so that \( G \) and by setting \( H \) compute \( D(A) \).

Although the following result seems to be folklore, we include here its statement for future references.

**Proposition 5.4.** Let \( A \) be a Hopf algebra over a field \( k \) such that the coradical \( H \) of \( A \) is a sub-Hopf algebra. Let \( A' \) be a Hopf algebra over a field \( k \). Let \( f: A' \to A \) be an isomorphism of Hopf algebras. Then \( H' := f^{-1}(H) \cong H \) is the coradical of \( A' \) and it is a sub-Hopf algebra of \( A' \). Thus we can identify \( H' \) with \( H \). Moreover \( f \) induces an isomorphism \( D(f): D(A') \to D(A) \) of bialgebras in \( HYD \).

**Proposition 5.5.** Keep the hypotheses and notations of the beginning of the section. Then \( D(A) \cong D(R\#_\xi H) \cong \text{gr}(R) \) as bialgebras in \( HYD \).

**Proof.** Apply Proposition 5.4 to the canonical isomorphism \( \psi: B := R\#_\xi H \to A \) that we recalled at the beginning of the section to get that \( D(R\#_\xi H) \cong D(A) \). Note that, by \( H \)-linearity we have

\[
\psi(1_R\#h) = \psi((1_R\#1_H)(1_R\#h)) = \psi((1_R\#1_H)h) = \psi(1_R\#1_H)h = h
\]

so that \( \psi(k1_R \otimes H) = H \) and hence \( H' = \psi^{-1}(H) = k1_R \otimes H \) with the notation of Proposition 5.4.

Thus \( B_0 = k1_R \otimes H \cong R \otimes H \otimes H \) and hence we can identify \( B_0 \) with \( H \) via the canonical isomorphism \( H \to R_0 \otimes H : h \mapsto 1_R \otimes h \). Its inverse is \( R_0 \otimes H \to H : r \otimes h \mapsto \varepsilon_R(r)h \). With this identification and by setting \( G := \text{gr}(B) \), we can consider the canonical bialgebra maps

\[
\begin{align*}
\sigma_G: & H \to \text{gr}(B) : h \mapsto 1_R\#h + (R\#_\xi H)_{-1}, \\
\pi_G: & \text{gr}(B) \to H : r\#h + (R\#_\xi H)_{-1} \to \varepsilon_R(r)h\delta_{n,0}, \text{ where } r\#h \in (R\#_\xi H)_n, n \in \mathbb{N}_0.
\end{align*}
\]

Since the underlying coalgebra of \( B \) is exactly the smash coproduct of \( R \) by \( H \) and \( (R, \xi) \) is a connected pre-bialgebra with cocycle in \( HYD \), by Lemma 5.4 we have that \( B_n = R_n \otimes H \). Let us compute \( D := D(B) \). As a vector space it is

\[
D := \left\{ d \in G \mid \sum d_1 \otimes \pi_G(d_2) = d \otimes 1_H \right\}.
\]

By Lemma 2.1, we have that \( D = \bigoplus_{n \in \mathbb{N}_0} D^n \) where \( D^n = D \cap G^n = D \cap B_{n-1} \). Let \( d := \sum_{i \in I} r_i \# h_i \in D^n \) where we can assume \( \sum_{i \in I} r_i \# h_i \in B_n \setminus B_{n-1} \) and, for every \( i \in I \), \( r_i \# h_i \in B_n \) and hence the fact that \( d \) is coinvariant rewrites as

\[
\sum_{i \in I} \left( r_i \# h_i \right)_1 \otimes \pi_G \left( \left( r_i \# h_i \right)_2 \right) = \sum_{i \in I} r_i \# h_i \otimes 1_H.
\]

By definition of \( \pi_G \) and (1), the left-hand side becomes

\[
\sum_{i \in I} \left( r_i \# h_i \right)_1 \otimes \pi_G \left( \left( r_i \# h_i \right)_2 \right) = \sum_{i \in I} \left( (r_i \# (h_i)_1) + B_{n-1} \right) \otimes (h_i)_2
\]

so that (31) becomes

\[
\sum_{i \in I} \left( (r_i \# (h_i)_1) + B_{n-1} \right) \otimes (h_i)_2 = \sum_{i \in I} r_i \# h_i \otimes 1_H = \sum_{i \in I} (r_i \# h_i + B_{n-1}) \otimes 1_H
\]

i.e.

\[
\sum_{i \in I} (r_i \# (h_i)_1) \otimes (h_i)_2 \in \bigoplus_{n \in \mathbb{N}_0} D^n = D \cap G^n = D \cap B_{n-1} \otimes H = R_{n-1} \otimes H \oplus H.
\]

If we apply \( R \otimes \varepsilon_H \otimes H \), we get

\[
\sum_{i \in I} r_i \otimes h_i - \sum_{i \in I} r_i \varepsilon_H (h_i) \otimes 1_H \in R_{n-1} \otimes H = B_{n-1}.
\]

Thus \( \sum_{i \in I} r_i \# h_i = \sum_{i \in I} r_i \# h_i = \sum_{i \in I} (r_i \# h_i + B_{n-1}) = \sum_{i \in I} (r_i \varepsilon_H (h_i) \otimes 1_H) + B_{n-1} \).
Since \( \sum_{i \in I} r_i \# h_i \in B_n \setminus B_{n-1} \) we get that \( \left( \sum_{i \in I} r_i \varepsilon_H (h_i) \right) \otimes 1_H \notin B_{n-1} \) and hence \( \sum_{i \in I} r_i \varepsilon_H (h_i) \notin R_{n-1} \) and we can write
\[
\sum_{i \in I} r_i \# h_i = \left( \sum_{i \in I} r_i \varepsilon_H (h_i) \right) \otimes 1_H.
\]
Therefore we have proved that the map
\[
\varphi_n : \frac{R_n}{R_{n-1}} \to \mathcal{D}^n : r \mapsto r \otimes 1_H,
\]
which is well-defined as \( \mathcal{D}^n = \mathcal{D} \cap G^n = \mathcal{D} \cap \frac{B_n}{R_{n-1}} = \mathcal{D} \cap \frac{R_n \otimes H}{R_{n-1} \otimes H} \), is also surjective.

It is also injective as \( \varphi_n (\overline{r}) = \varphi_n (\overline{\tau}) \) implies \( r \otimes 1_H - s \otimes 1_H \in B_{n-1} = R_{n-1} \otimes H \) and hence, by applying \( R \otimes \varepsilon_H \), we get \( r - s \in R_{n-1} \) i.e. \( \overline{r} = \overline{s} \). Therefore \( \varphi_n \) is an isomorphism such that
\[
\sum_{i \in I} r_i \# h_i = \varphi_n \left( \sum_{i \in I} r_i \varepsilon_H (h_i) \right)
\]
and hence
\[
\varphi_n^{-1} \left( \sum_{i \in I} r_i \# h_i \right) = \sum_{i \in I} r_i \varepsilon_H (h_i).
\]
Clearly this extends to a graded \( \mathbb{k} \)-linear isomorphism
\[
\varphi : \text{gr} (R) \to \mathcal{D}.
\]
Let us check that \( \varphi \) is a morphism in \( H \mathcal{YD} \). First note that, for every \( r \in R_n \), we have
\[
\varphi (r + R_{n-1}) = \delta_{|r|, n} \varphi (r + R_{n-1}) = \delta_{|r|, n} \varphi_n (r + R_{n-1}) = \delta_{|r|, n} \varphi_n (\overline{r})
\]
\[
= \delta_{|r|, n} r \otimes 1_H = \delta_{|r|, n} \left( r \otimes 1_H + (R \# \varepsilon_H)_{n-1} \right) = r \otimes 1_H + (R \# \varepsilon_H)_{n-1}.
\]
Thus
\[
(32) \quad \varphi (r + R_{n-1}) = r \otimes 1_H + (R \# \varepsilon_H)_{n-1}, \text{ for every } r \in R_n.
\]
For every \( r \in R_n \setminus R_{n-1} \), by using (32), it is straightforward to prove that \( h \circ \varphi (\overline{r}) = \varphi (h \overline{r}) \).

Moreover, by applying (1), (30), the definition of \( \pi_G \) and (32), we get that \( \rho \varphi (\overline{r}) = (\mathcal{H} \otimes \varepsilon) \rho (\overline{r}) \).

Let us check that \( \varphi \) is a morphism of bialgebras in \( H \mathcal{YD} \). Fix \( r \in R_n \setminus R_{n-1} \).

Using the definition of \( \Delta_D \), (1), (22), the definition of \( \pi_G \), the definition of \( \sigma_G \), (32) and (1) again, we obtain \( \Delta_D \varphi (\overline{r}) = (\varphi \otimes \varphi) \Delta_{\text{gr}(R)} (\overline{r}) \).

Let us check \( \varphi \) is coroutinary:
\[
\varepsilon_D \varphi (\overline{r}) = \varepsilon_D \varphi (\overline{r}) = \varepsilon_D \left( r \otimes 1_H \right) = \delta_{n, 0} \varepsilon_B (r \otimes 1_H).
\]
Let us check \( \varphi \) is multiplicative. Let \( s \in R_m \setminus R_{m-1} \). Then, by definition of \( \varphi \), of \( m_D \) and of the multiplication of \( R \# \varepsilon_H \), we have that
\[
m_D (\varphi \otimes \varphi) (\overline{s} \otimes \overline{\tau}) = \sum_{0 \leq i \leq m} \left( s^{(1)} (s^{(2)}_{(-1)} r^{(1)}_{(0)} \# \xi (s^{(2)}_{(0)} \otimes r^{(2)}_{(0)})) + (R \# \varepsilon_H)_{m+n-1} \right).
\]
Now write \( s^{(1)} \otimes s^{(2)} = \sum_{0 \leq i \leq m} s_i \otimes s'_i \) for some \( s_i, s'_i \in R_i \) and similarly \( r^{(1)} \otimes r^{(2)} = \sum_{0 \leq j \leq n} r_j \otimes r'_j \) for some \( r_j, r'_j \in R_j \). Then
\[
m_D (\varphi \otimes \varphi) (\overline{s} \otimes \overline{\tau}) = \sum_{0 \leq i \leq m} \sum_{0 \leq j \leq n} \left( s_i \left( (s'_i)_{(-1)} r_j \right) \# \xi \left( (s'_i)_{(0)} \otimes r'_j \right)) + (R \# \varepsilon_H)_{m+n-1} \right).
\]
Finally,
\[
R_0 = \sum_{s_i \otimes s'_i} \sum_{r_j \otimes r'_j} \varepsilon_R (s'_i) \varepsilon_R (r'_j) \varepsilon_R (r_j) \varepsilon_R (r'_j) \varepsilon_R (r_i) \varepsilon_R (r'_i) \varepsilon_R (r_j) \varepsilon_R (r'_j) \varepsilon_R (r_i) \varepsilon_R (r'_i).
\]
\[
\begin{align*}
&= \sum_{0 \leq i \leq n} s_{m \in R} (s')_n \in R (r')_n \# 1_H + (R \# \xi H)_{m + n - 1} \\
&= \sum_{0 \leq i \leq m} \delta_i, \delta_j, n, (s_i \in R) (s')_m, (r')_m \# 1_H + (R \# \xi H)_{m + n - 1} \\
&= \sum_{0 \leq i \leq m} (s_i \in R) (r_j \in R) (r')_m \# 1_H + (R \# \xi H)_{m + n - 1} \\
&= \sum (s^{(1)} \in R) (s^{(2)} \in R) \# 1_H + (R \# \xi H)_{m + n - 1} \\
&= (sr \# 1_H) + (R \# \xi H)_{m + n - 1} \\
&= \phi ((s + R_m - 1) (r + R - n)) = \phi m_{gr(R)} (\gamma) \\
\end{align*}
\]

Let us check that $\phi$ is unitary. We have

\[
\phi (1_{gr(R)}) = \phi (1_R + R - 1) = \phi (1_R) = 1_R \otimes 1_H = (1_R \otimes 1_H) + (R \# \xi H)_{-1} = 1_B + B - 1 = 1_G.
\]

Summing up we have proved that

\[
gr (Q) \overset{\text{Lem. 5.1}}{=} \overset{\text{Lem. 5.3}}{=} \overset{\text{Prop. 5.4}}{=} D (R \# \xi H) \overset{\text{Prop. 5.3}}{=} D (A)
\]

as bialgebras in $H^Y_D$. Therefore $H^3_Y (D (A), k) = 0$ (the Hochschild cohomology in $H^Y_D$ of the algebra $D (A)$ with values in $k$) if, and only if, $H^3_Y (gr(Q), k) = 0$. In this case, by the foregoing, we get that $Q$ is gauge equivalent to a connected bialgebra in $H^Y_D$.

Now let $E$ be a connected bialgebra in $H^Y_D$ and let $\gamma : E \otimes E \to k$ be a gauge transformation in $H^Y_D$ such that $Q = E^\gamma$. We proved that $A^\xi \cong Q \# H \cong E^\gamma \# H$ as coquasi-bialgebras. By Proposition 2.3, we have that $(E^\# H)^\Gamma = E^\gamma \# H$ as an ordinary coquasi-bialgebra. Recall that two coquasi-bialgebras $A$ and $A'$ are called \textit{gauge equivalent} or \textit{quasi-isomorphic} whenever there is some gauge transformation $\gamma : Q \otimes Q \to k$ in $\text{Vec}_k$ such that $A^\gamma \cong A'$ as coquasi-bialgebras. We point out that, if $A$ and $A'$ are ordinary bialgebras and $A^\gamma \cong A'$, then $\gamma$ comes out to be a unitary cocycle. This is encoded in the triviality of the reassociators of $A$ and $A'$.

\textbf{Theorem 5.6.} Let $A$ be a finite-dimensional Hopf algebra over a field $k$ of characteristic zero such that the coradical $H$ of $A$ is a sub-Hopf algebra (i.e. $A$ has the dual Chevalley Property). If $H^3_Y (D (A), k) = 0$, then $A$ is quasi-isomorphic to the Radford-Majid bosonization $E^\# H$ of some connected bialgebra $E$ in $H^Y_D$ by $H$. Moreover $gr (E) \cong D (A)$ as bialgebras in $H^Y_D$.

\textbf{Proof.} By the foregoing $A^\xi \cong Q \# H \cong E^\gamma \# H = (E^\# H)^\Gamma$ as coquasi-bialgebras. Now $A$ is quasi-isomorphic to $A^\xi$ which is quasi-isomorphic to $E^\# H$ so that $A$ is quasi-isomorphic to $E^\# H$. Moreover

\[
\text{gr} (E) = \text{gr} (E^\gamma) = \text{gr} (Q) \cong D (A)
\]

where the first equality holds by Proposition 2.6.

More generally, given $A$ a (finite-dimensional) Hopf algebra whose coradical $H$ is a sub-Hopf algebra, then if $H$ is also semisimple, we expect that $A$ is quasi-isomorphic to the Radford-Majid bosonization $E^\# H$ of some connected bialgebra $E$ in $H^Y_D$ by $H$. See e.g. [GM, Corollary 3.4 and the proof therein] and [AAGMV, AAG] for a further clue in this direction.

\section{6. Examples}

We notice that the Hochschild cohomology of a finite-dimensional Nichols algebras has been computed in few examples. We consider here those Nichols algebras to compute $H^3_Y (B (V), k)$. 
6.1. Braiding of Cartan type. Let $A = (a_{ij})_{1 \leq i,j \leq \theta}$ be a finite Cartan matrix, $\Delta$ the corresponding root system, $(\alpha_i)_{1 \leq i \leq \theta}$ a set of simple roots and $W$ its Weyl group. Let $w_0 = s_{i_1} \cdots s_{i_M}$ be a reduced expression of the element $w_0 \in W$ of maximal length as a product of simple reflections, $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_j), 1 \leq j \leq M$. Then $\beta_j \neq \beta_k$ if $j \neq k$ and $\Delta^+ = \{ \beta_j | 1 \leq j \leq M \}$, see § page 25 and Proposition 3.6.

Let $\Gamma$ be a finite abelian group, $\hat{\Gamma}$ its group of characters. $\mathcal{D} = (\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A)$ is a datum of finite Cartan type associated to $\Gamma$ and $A$ if $g_i \in \Gamma$, $\chi_j \in \hat{\Gamma}, 1 \leq i,j \leq \theta$, satisfy

$$\chi_i(g_i) \neq 1, \chi_j(g_j)\chi_j(g_i) = \chi_i(g_i)^{\alpha_j} \quad \text{for all } i,j.$$  

Set $q = (q_{ij})_{1 \leq i,j \leq \theta}$, where $q_{ij} = \chi_j(g_i)$.

In what follows $V$ denotes the Yetter-Drinfeld module over $k\hat{\Gamma}$, $\dim V = \theta$, with a fixed basis $x_1, \ldots, x_\theta$, where the action and the coaction over each $x_i$ is given by $\chi_i$ and $g_i$, respectively. Then the associated braiding is $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all $i,j$. Let $B_q = B(V)$. The tensor algebra $T(V)$ is $\mathbb{N}_0^\theta$-graded with grading $a_i$ for each $x_i$. For $\beta = \sum_{i=1}^\theta a_i a_i \in \Delta^+$, set

$$g_\beta = g_1^{a_1} \cdots g_\theta^{a_\theta}, \quad \chi_\beta = \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \quad q_\beta = \chi_\beta(g_\beta).$$

Given $\alpha, \beta \in \Delta^+$, we denote $q_{\alpha \beta} = \chi_\beta(g_\alpha)$.

We assume as in [AS2, MPSW] that the order of $q_{ii}$ is odd for all $i$, and not divisible by 3 for each connected component of the Dynkin diagram of $A$ of type $G_2$. Therefore the order of $q_{ii}$ is the same for all the $i$ in the same connected component $J$. Given $\beta \in J$, we denote by $N_\beta$ the order of the corresponding $q_{ii}$ in $J$, which is also the order of $q_\beta$.

By [AS2] there exist homogeneous elements $x_\beta$ of degree $\beta$, $\beta \in \Delta^+$, such that the Nichols algebra $B_q$ of $V$ is presented by generators $x_1, \ldots, x_\theta$ and relations

$$(\text{ad}_x x_i)^{1-a_{ij}} x_j = 0, \quad 1 \leq i \neq j \theta;$$

$$x_{\beta}^N = 0, \quad \beta \in \Delta^+.$$  

Moreover $\{x_{\beta}^{n_1} \cdots x_{\beta}^{n_M} \mid 0 \leq n_i < N_{\beta_i} \}$ is a basis of $B_q$.

We shall prove that $H^2_{\mathcal{D}}(B_q, k) = 0$. We need first some technical results.

**Lemma 6.1.** Let $\alpha, \beta \in \Delta^+$. Then either $g_\alpha g_\beta^N \neq e$, or else $\chi_\alpha \chi_\beta^{-N} \neq e$.

**Proof.** Suppose on the contrary that $g_\alpha g_\beta^N = e$, $\chi_\alpha \chi_\beta^{-N} = e$. Then

$$q_\alpha = \chi^{-1}_\alpha(g_\alpha^{-1}) = \chi_\beta^{-N}(g_\beta^N) = q_\beta^N = 1,$$

since $q_\beta$ is a root of unity of order $N_\beta$. But this is a contradiction, since $q_\alpha \neq 1$. □

**Lemma 6.2.** Let $\alpha, \beta, \gamma \in \Delta^+$ be pairwise different. Then either $g_\alpha g_\beta g_\gamma \neq e$, or else $\chi_\alpha \chi_\beta \chi_\gamma \neq e$.

**Proof.** Suppose on the contrary that $g_\alpha g_\beta g_\gamma = e$ and $\chi_\alpha \chi_\beta \chi_\gamma = e$. Then

$$(33) \quad q_\alpha = \chi^{-1}_\alpha(g_\alpha^{-1}) = \chi_\beta \chi_\gamma(g_\beta g_\gamma) = q_\beta q_\gamma q_\beta q_\gamma q_\beta q_\gamma q_\beta = q_\alpha g_\beta g_\gamma q_\alpha q_\gamma q_\alpha q_\gamma = q_\gamma = q_\alpha q_\beta q_\alpha q_\beta.$$  

Notice that $\alpha, \beta, \gamma$ belong to the same connected component. Indeed, if $\gamma$ belongs to a different connected component, then $q_\beta q_\gamma = q_\alpha q_\gamma = 1$. Thus $q_\beta = q_\alpha q_\gamma$, so $q_\gamma^2 = 1$, which is a contradiction. Therefore we may assume that the Dynkin diagram is connected.

One can prove that $q_{\alpha, (\alpha)} = q_\alpha$ for every $\alpha \in \Delta$. As we observed that $\Delta^+ = \{ \beta_j | 1 \leq j \leq M \}$, we deduce that for every $\beta \in \Delta^+$ there is some $j$ such that $q_{\beta} = q_j$. One can prove that there is some $q \in k$ such that $q_{\alpha} = q^{(\alpha,\alpha)/2}$ and $q_{\alpha, \gamma} q_{\alpha, \gamma} = q^{(\alpha,\gamma)}$, where $(\cdot, \cdot)$ is the invariant bilinear form on the simple Lie algebra $\mathfrak{g}$ associated with the finite Cartan matrix [Bo, Ch. VI, §1, Proposition 3 and Definition 3] and the basis of the root systems given in [Bo, Ch. VI, §4] should be normalized in such a way that $q = q_\beta, (\delta, \delta) = 2$ for each short root $\delta \in \Delta$. Note that $q_{\alpha} = q^{(\alpha,\alpha)/2} \neq 1$ for all $\alpha$ as $(\alpha, \alpha) \neq 0$. Thus

- $q_{\alpha} = q_{\beta} = q_\gamma = q$ if the Dynkin diagram is simply laced,
- $q_{\alpha, \beta}, q_{\beta, \gamma} \in \{q, q^2\}$ if the Dynkin diagram has a double arrow,
- $q_{\alpha, \beta}, q_\gamma \in \{q, q^3\}$ if the Dynkin diagram is of type $G_2$.  

If the Dynkin diagram is simply laced, then, by ([10], we have $q_{\beta \gamma} q_{\gamma \beta} = q_{\alpha q_{\gamma \alpha}} = q_{\alpha \beta} q_{\beta \alpha} = q^{-1}$. Then $q^{(\alpha, \gamma)} = q^{-1}$. Now set $n(\alpha, \beta) := 2(\alpha, \beta)/(\beta, \beta) = (\alpha, \beta)$. Then $n(\alpha, \beta)$ is symmetric whence, by [10], Ch. VI, §1, page 148] we have $(\alpha, \gamma) = -1$ as the order of $q$ is odd, so $\alpha + \gamma \in \Delta^+$. By [10], VI, §1, Corollary, page 149] now the same argument we used above shows that also $(\alpha, \beta) = -1 = (\gamma, \beta)$ and hence $(\alpha + \gamma, \beta) = -2$, so $\alpha + \beta + \gamma \in \Delta^+$, since $\alpha + \gamma \neq -\beta$ (as $\alpha + \gamma$ and $\beta$ are both in $\Delta^+$). But $q_{\alpha + \beta + \gamma} = q_{\alpha} q_{\beta} q_{\gamma}$, and $\alpha \beta q_{\beta} q_{\gamma} q_{\alpha} = q_{\beta} q_{\gamma} q_{\alpha} = 1$, which is a contradiction.

If the Dynkin diagram has a double arrow, then $\alpha$, $\beta$, $\gamma \in \{q, q^2\}$. If $\alpha = q = q_\gamma$, then the proof follows as for the simply-laced case because $n(u, v) = n(v, u)$ for $u, v \in \{\alpha, \beta, \gamma\}$. If $\alpha = q = \beta = 1$ and $(\alpha, \gamma) = -2 < 0$ by the conditions on the order of $q$, so again $\alpha + \beta + \gamma \in \Delta^+$; but again we obtain $q_{\alpha + \beta + \gamma} = 1$, which is a contradiction. The proof for $q = q = q^2$ and $q_\gamma = q$ follows analogously.

Finally, if the Dynkin diagram is of type $G_2$, then a similar analysis gives a contradiction. □

For each $1 \leq k \leq M$, set $B_q(k)$ as the subspace of $B_q$ spanned by $\{x_{\beta_1}^{n_1} \cdots x_{\beta_k}^{n_k} | 0 \leq n_i < N_{\beta_i}\}$. By [10], this gives an algebra filtration, and the graded algebra $Gr B_q$ associated to this filtration is presented by generators $x_\beta, \beta \in \Delta^+$, and relations

$$x_\beta x_\gamma = q_{\beta \gamma} x_\gamma x_\beta, \quad x_\beta^{N_{\beta}} = 0, \quad \beta < \gamma \in \Delta^+.$$  

In [MPSW], $Gr B_q$ is viewed as an algebra in $\mathbb{A}^k YD$, (which (as an algebra) is the Nichols algebra of Cartan type $A_1 \times \cdots \times A_1, M$ copies, with action and coaction on $x_\beta$ given by $\chi_\beta, g_\beta$, respectively.

By [MPSW], Theorem 4.1, $H^*(Gr B_q, k)$ is the algebra generated by $\xi_\beta, \eta_\beta, \beta \in \Delta^+$, where $\deg \xi_\beta = 2, \deg \eta_\beta = 1$, and relations

$$\xi_\beta \xi_\gamma = q_{\beta \gamma}^{N_{\beta} N_{\gamma}} \xi_\gamma \xi_\beta, \quad \eta_\beta \xi_\gamma = q_{\beta \gamma}^{N_{\beta} N_{\gamma}} \xi_\gamma \eta_\beta, \quad \eta_\beta \eta_\gamma = -q_{\beta \gamma} \eta_\gamma \eta_\beta, \quad \beta, \gamma \in \Delta^+.$$  

As we assume that all the $q_{\alpha}$ have odd order, we deduce in particular from the last equality that $q_{\beta}^{2} = 0$ for all $\beta \in \Delta^+$. As an algebra in $\mathbb{A}^k YD$, the action and coaction on $\xi_\beta$ is given by $\chi_\beta^{-N_{\beta}}, g_\beta^{-N_{\beta}}$, while the action and coaction on $\eta_\beta$ is given by $\chi_\beta^{-1}, g_\beta^{-1}$.

**Theorem 6.3.** $H^3_{\mathbb{A}^k YD}(B_q, k) = 0$.

*Proof.* First we will prove that $H^3(Gr B_q, k)^D = 0$ for $D := D(\Gamma)$. Now, the invariants are with respect to the $D$-bimodule structure that $H^3(Gr B_q, k)$ inherits from $\text{Hom}(Gr B_q \otimes \cdots \otimes k, k)$ (this is a $D$-bimodule as its arguments are left $D$-modules). Since the left $D$-module structure is induced by the one of $k$, it is trivial. Thus the invariants of $H^3(Gr B_q, k)$ as a $D$-bimodule reduce to the its invariants as a right $D$-module. Since right $D$-modules are equivalent to left $D$-modules, via the antipode of $D$ which is invertible as $D$ is finite-dimensional, the right $D$-module structure of $H^3(Gr B_q, k)$ becomes the structure of object in $\mathbb{A}^k YD$ described above. Thus, in order to prove that $H^3(Gr B_q, k)^D = 0$ we just have to check that the invariants of $H^3(Gr B_q, k)$ as a left-left Yetter-Drinfeld modules are zero.

Now, by the definition relations of $H^*(Gr B_q, k)$, a basis $B$ of $H^3(Gr B_q, k)$ is given by $\{\xi_\alpha \eta_\beta \} \cup \{\eta_\alpha \eta_\beta \eta_\gamma | \alpha < \beta < \gamma\}$. If $v \in H^3(Gr B_q, k)$ is invariant, then $v$ is written as a linear combination of elements in the trivial component. Indeed, write $v = \sum_{b \in B} c_b b$ for some $c_b \in k$, and let $g_b, \chi_b$ be the elements describing the component of $b$. Then

$$v = g \cdot v = \sum_{b \in B} c_b g \cdot b = \sum_{b \in B} c_b \chi_b(g) b, \quad 1 \otimes v = \rho(v) = \sum_{b \in B} c_b \rho \cdot b = \sum_{b \in B} c_b g_b \otimes b.$$  

If $c_b \neq 0$, then $\chi_b(g) = 1$ for all $g \in \Gamma$ so $\chi_b = \epsilon$, and $g_b = 1$. Thus $b$ is invariant. We have so proved that the existence of $v \neq 0$ invariant implies the existence of $b \in B$ invariant. Hence, if $B$ has no invariant element then there is no invariant element at all. Note that, for all $h \in H$, we have $b \cdot (\xi_\alpha \eta_\beta) = (\chi_\alpha \eta_\beta^{-1})(h) \xi_\alpha \eta_\beta$ and $\rho(\xi_\alpha \eta_\beta) = g_\alpha \eta_\beta \otimes \xi_\alpha \eta_\beta$ so that, by Lemma [L1], the element
\(\xi_{\alpha}\eta_{\beta}\) is not \(D\)-invariant. A similar argument, using Lemma 5.2, shows that also \(n_{\alpha}\eta_{\beta}\eta_{\gamma}\) is not \(D\)-invariant. Thus the elements in \(B\) are not \(D\)-invariant, so \(H^{3}(\text{Gr}B_{4},k)^{D} = 0\). Since the elements in \(\{x_{n_{1}}^{\beta_{1}} \cdots x_{n_{k}}^{\beta_{k}} | 0 \leq n_{i} < N_{\beta_{i}}\}\) are eigenvectors for \(D\), we can mimic the argument in [MPSW, Section 5] by taking into account the spectral sequence associated to the filtration of algebras therein; see for example [MPSW, Corollary 5.5] for a similar argument. Thus \(H^{3}_{\text{Gr}}(B_{4},k) \cong H^{3}(B_{4},k)^{D} = 0\).

**Remark 6.4.** Notice that \(H^{3}_{\text{Gr}}(B_{4},k) \cong H^{3}(B_{4},k)^{D(\text{Gr})} = 0\) although \(H^{3}(B_{4}\#k\Gamma,k) \cong H^{3}(B_{4},k)^{\Gamma}\) can be non-trivial, see for example [MPSW, Example 5.8].

### 6.2. Braiding of non-diagonal type.

For \(n \geq 3\), \(\mathcal{FK}_{n}\) denotes the quadratic algebra \(\mathcal{FK}\) with a presentation by generators \(x_{ij}\), \(1 \leq i < j \leq n\), and relations

\[
x_{(ij)}x_{(jk)} = x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)},
\]

\[
x_{(jk)}x_{(ij)} = x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)},
\]

\[
x_{(ij)}x_{(kl)} = x_{(kl)}x_{(ij)}, \quad \#\{i,j,k,l\} = 4.
\]

According to [MIS] each \(\mathcal{FK}_{n}\) is a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group \(S_{n}\), generated as an algebra by the vector space \(V_{n}\), with basis \(\{x_{ij} | 1 \leq i < j \leq n\}\). The action is described by identifying \((ij)\) with the corresponding transposition in \(S_{n}\) and then consider the conjugation twisted by the sign, while the coaction is given by declaring \(x_{\sigma}\) a homogeneous element of degree \(\sigma\). Then the braiding on \(V_{n}\) becomes

\[
c(x_{\sigma} \otimes x_{\tau}) = \chi(\sigma,\tau)x_{\sigma\tau\sigma^{-1}} \otimes x_{\sigma}, \quad \chi(\sigma,\tau) = \begin{cases} 1 & \sigma(i) < \sigma(j), \tau = (ij), i < j, \\ -1 & \text{otherwise}, \end{cases}
\]

where \(\sigma\) and \(\tau\) are transpositions. Moreover \(\mathcal{FK}_{n}\) projects onto the Nichols algebra \(\mathcal{B}(V_{n})\). For \(n = 3, 4, 5\), it is known that \(\mathcal{FK}_{n} = \mathcal{B}(V_{n})\) and has dimension, respectively, 12, 576 and 8294400.

The Hochschild cohomology of \(\mathcal{FK}_{3}\) is a consequence of the results in [SV] as follows.

**Theorem 6.5.** \(H^{n}_{\text{Gr}S_{3}\text{-Mod}}(\mathcal{FK}_{3},k)\) is isomorphic to the graded algebra \(k[X,U,V]/(U^{2}V - UV^{2})\), where \(\deg U = \deg V = 2\), \(\deg X = 4\).

**Proof.** By [SV, Theorem 4.19], we have that \(E(B\#kS_{3})\) is isomorphic to the algebra in the claim, where \(B = \mathcal{FK}_{3}\). By [SV, Theorem 2.17], we know that \(E(B\#kS_{3}) \cong E(B)^{kS_{3}}\) as graded algebras. As observed in Remark 4.2, we have that \(E(B) \cong H^{*}(B,k)\). By Remark 4.6 we have \(H^{*}(B,k)^{kS_{3}} \cong H^{*}_{\text{Gr}(\mathcal{FK}_{3},k)}\).

From this result we get \(H^{3}_{\text{Gr}}(\mathcal{FK}_{3},k) = 0\) so that, by Proposition 4.3 we conclude that

**Corollary 6.6.** \(H^{3}_{\text{Gr}}(\mathcal{FK}_{3},k) = 0\).

### References


