Mean-variance target-based optimisation for defined contribution pension schemes in a stochastic framework

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Abstract

We solve a mean-variance optimisation problem in the accumulation phase of a defined contribution pension scheme. In a general multi-asset financial market with stochastic investment opportunities and stochastic contributions, we provide the general forms for the efficient frontier, the optimal investment strategy, and the ruin probability. We show that the mean-variance approach is equivalent to a "user-friendly" target-based optimisation problem which minimises a quadratic loss function, and provide implementation guidelines for the selection of the target. We show that the ruin probability can be kept under control through the choice of the target level. We find closed-form solutions for the special case of stochastic interest rate following the Vasiček (1977) dynamics, contributions following a geometric Brownian motion, and market consisting of cash, one bond and one stock. Numerical applications report the behaviour over time of optimal strategies and non-negative constrained strategies.

Keywords: Mean-variance approach; defined contribution pension scheme; stochastic optimal control; martingale method; efficient frontier; ruin probability.

JEL: C61, D81, D90, G11, G22.

1 Introduction and motivation

Defined contribution (DC) pension schemes are becoming more and more important in the pension systems of most industrialised countries and are replacing the defined benefit (DB) schemes that were more frequent in the past. It is well known that the investment risk, which is borne by the sponsor in DB pension schemes, is faced by the member in DC pension schemes and its analysis is therefore of the utmost importance nowadays.

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The optimal investment strategy in the accumulation phase (i.e. prior to retirement) in a DC framework has been derived in the literature with a variety of objective functions (mainly maximisation of expected utility of final wealth) and financial market structures, see, among many others, Boulier et al. (2001), Haberman and Vigna (2002), Deelstra et al. (2003), Devolder et al. (2003), Battocchio and Menoncin (2004), Cairns et al. (2006), Di Giacinto et al. (2011).

The long-term investment planning of pension schemes is less frequently cast in the framework of a mean-variance portfolio selection. Mean-variance problems for DC plans are solved in He and Liang (2013), Yao et al. (2013), Vigna (2014), Guan and Liang (2015) and Wu et al. (2015).

The likely reason for the scarcity of literature is the well-known difficulty in solving the mean-variance optimisation problem in both a discrete multi-period framework and in continuous time. The first solution in continuous time to this kind of problem was found in Richardson (1989), and subsequently by Bajeux-Besnainou and Portait (1998), both through the so-called martingale approach. In the first paper, the financial market consists of a riskless and a risky asset, and there is no derivation of the efficient frontier. In the second paper, the interest rate is stochastic, the efficient frontier is derived, and explicit solutions are found in the special case of the Vasiček (1977) model. Li and Ng (2000) and Zhou and Li (2000) solved, respectively in the multiperiod framework and in continuous time, the mean-variance problem by transforming it into a standard stochastic optimal control problem. Since then, a number of extensions have been following.

Even if the choice of the most appropriate point on the efficient frontier is relevant for matching at best investors' preference and, thus, practically implement the mean-variance approach, the literature devotes little attention to this issue. One of the main contributions of this paper is to enhance the comprehension of how to select the correct subjective level of risk/reward for a member of a DC pension scheme. We interpret the mean-variance problem as a target-based problem and provide a closed-form one-to-one simple relationship between the target (in terms of final wealth to be reached) and the appropriate level of risk/reward. Furthermore, we show how to keep under control the ruin probability through the choice of the target. Finally, we provide a lowest threshold for the target as a function of the initial wealth and the expected present value of future contributions.

We stress the importance of targets in DC pension funds. Let us consider, for instance, the so-called replacement ratio, i.e. the ratio between the pension rate and the final salary. The achievement of a minimum replacement ratio was guaranteed in DB pension schemes, but is not in DC pension schemes. The possibility of selecting a suitable wealth-target at retirement might enable members of a DC plan to get close to a desired replacement ratio, and might help reducing the inequity among pension fund members belonging to different cohorts that is typical of DC plans (see e.g. Knox, 1993).

The equivalence between mean-variance criterion and the target-based approach is one of the characteristics that make the mean-variance preferences appealing with respect to other types of preferences. Due to this equivalence, the identification of the risk profile for the mean-variance investor can be done via the selection of a final target at retirement, while it is done via the selection of an abstract risk aversion coefficient

for other common types of preferences (e.g., the relative risk aversion coefficient for power preferences, the absolute risk aversion coefficient for exponential preferences etc.). For the average pension fund member it is easier to select a wealth target rather than an abstract index. Our selection of mean-variance preferences is also motivated by the evidence that the performance of most investment funds is determined according to mean-variance criteria (see Chiu and Zhou, 2011).

The relationship between targets and points on the mean-variance efficient frontier was introduced by Zhou and Li (2000), and in the context of a DC pension plan was pointed out by Vigna (2014) in a Black and Scholes financial market with constant contribution. In this paper, we extend Vigna (2014) to a more general complete financial market with an arbitrary number of risky assets, risk sources and state variables, and stochastic contribution.

A second contribution of our work is the analytical solution of the mean-variance problem in a DC pension plan in a quite general financial framework. A rather important special case with stochastic interest rate and stochastic salary is solved explicitly and analysed in detail.

As a third contribution, we propose an empirical methodology for implementing the non-negativity constraints on portfolio shares. This issue is usually neglected by the literature on DC pension funds with stochastic interest rate. Furthermore, the comparison between optimal non-constrained strategies and suboptimal constrained strategies is presented. The constrained strategies turn out to be similar to empirical investment strategies actually adopted by DC pension schemes in UK.

The remainder of the paper is organised as follows. In Section 2 we outline the general financial market and derive the wealth dynamics. In Section 3 the mean-variance optimisation problem is solved using the embedding technique introduced by Zhou and Li (2000) and the martingale approach; the optimal portfolio, the ruin probability and the efficient frontier are provided analytically. In Section 4 the equivalence between the mean-variance approach and the target-based approach is shown, and guidelines for the practical implementation of the mean-variance model are provided. Section 5 contains a numerical application and presents a special case with financial market consisting in a riskless asset, one bond and one stock. Two stochastic state variables are considered: the riskless interest rate following the Vasiček (1977) dynamics and the contribution following a geometric Brownian motion. The optimal portfolio and the efficient frontier are analysed with different risk profiles, and suboptimal strategies with non-negative weights are introduced and studied. Section 6 concludes. All proofs are gathered in Appendix.

2 The framework

The financial market is arbitrage free, complete, frictionless, and continuously open at any time $t \in [0, T]$. The risk is described by a set of n independent Brownian motions $W(t)$, defined on the complete filtered probability space $\{\mathbf{\Omega}, \mathcal{F}(t), \mathbb{P}\}$, where $\mathcal{F}(t)$ is the filtration generated by the Brownian motions and $\mathbb{P}$ is the real-world probability measure. The financial market is described by the following variables:

- s state variables $z(t)$ (with $z(0) = z_0 \in \mathbb{R}^s$ known) whose values solve the stochastic differential

equation (SDE)

$$\underset{s\times 1}{dz\left(t\right)}=\underset{s\times 1}{\mu_z\left(t,z\right)}dt+\underset{s\times n}{\Omega\left(t,z\right)}\underset{n\times 1}{dW\left(t\right)}; \tag{1}$$

- one riskless asset whose price $G\left(t\right)$ solves the (ordinary) differential equation

$$dG(t)=G(t)r\left(t,z\right)dt,$$

where $r\left(t,z\right)$ is the spot instantaneously riskless interest rate;

- n risky assets whose prices $P\left(t\right)$ (with $P\left(0\right)=P_0\in\mathbb{R}^n$ known) solve the matrix stochastic differential equation

$$\underset{n\times 1}{dP\left(t\right)}=\underset{n\times n}{I_P}\left[\underset{n\times 1}{\mu\left(t,z\right)}dt+\underset{n\times n}{\Sigma\left(t,z\right)}\underset{n\times 1}{dW\left(t\right)}\right], \tag{2}$$

where I_P is the $n\times n$ square diagonal matrix that reports the prices $P_1,P_2,...,P_n$ on the diagonal and zero elsewhere.

Drift and diffusion terms in (1) and (2) are assumed to satisfy the usual conditions for the existence and uniqueness of a strong solution to the SDEs.

The absence of arbitrage and completeness imply the existence of a unique risk-neutral equivalent martingale measure $\mathbb{Q}$. There exists a unique vector of market prices of risk $\xi\left(t,z\right)$ which solves the linear system $\Sigma\left(t,z\right)\xi\left(t,z\right)=\mu\left(t,z\right)-r\left(t,z\right)\mathbf{1}$, where $\mathbf{1}$ is a vector of 1's (i.e. $\exists\Sigma\left(t,z\right)^{-1}$). Assuming that $\xi(t,z)$ satisfies the Novikov's condition, the Girsanov theorem applies and the Wiener processes $dW\left(t\right)$ can be rewritten under $\mathbb{Q}$ as follows:

$$dW^{\mathbb{Q}}\left(t\right)=\xi\left(t,z\right)dt+dW\left(t\right). \tag{3}$$

The Radon-Nikodym derivative is (the prime denotes transposition):

$$m\left(t_0,t\right)=e^{-\frac{1}{2}\int_{t_0}^{t}\xi(u,z)'\xi(u,z)du-\int_{t_0}^{t}\xi(u,z)'dW(u)}\Longleftrightarrow\begin{cases}dm\left(t_0,t\right)=-m\left(t_0,t\right)\xi\left(t,z\right)'dW\left(t\right),\\ m\left(t_0,t_0\right)=1.\end{cases}$$

Thus, given any $t-$measurable random variable $Z\left(t\right)$, the following relationship holds true

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[Z\left(t\right)\right]=\mathbb{E}_{t_0}\left[Z\left(t\right)\cdot m\left(t_0,t\right)\right], \tag{4}$$

where $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\bullet\right]$ and $\mathbb{E}_{t_0}\left[\bullet\right]$ are the expected values, under the risk neutral and the real world probabilities respectively, conditional to $\mathcal{F}\left(t_0\right)$.

Let $B\left(t,T\right)$ be the price in t of a zero-coupon bond expiring in T, and $\sigma_B\left(t,T\right)$ the (vector) diffusion term of $\frac{dB(t,T)}{B(t,T)}$. It is well known that a so-called "forward probability measure" ($\mathbb{F}_T$) can be defined as

follows

$$dW^{\mathbb{Q}}(t) = \sigma_B(t,T)\,dt + dW^{\mathbb{F}_T}(t), \tag{5}$$

and, given any $T-$measurable random variable $Z(T)$, we can write

$$\mathbb{E}_t^{\mathbb{Q}}\left[Z(T)\,e^{-\int_t^T r(u,z)du}\right] = \mathbb{E}_t^{\mathbb{F}_T}[Z(T)]\,\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u,z)du}\right] = \mathbb{E}_t^{\mathbb{F}_T}[Z(T)]\,B(t,T), \tag{6}$$

where the new *numeraire* of the economy is $B(t,T)$ (Björk, 2009). $\mathbb{F}_T$ will be useful for simplifying the role of contributions in the evolution of the pension fund's wealth.

Remark 1. The forward probability measure is needed to split the expected value of a product into the product of two expected values, as in (6). In this way, also the derivative of the expected value can be written in a much simpler way.

The stochastic contribution paid by the member into the fund's wealth $X(t)$ per time unit is $c(t,z) > 0$. If $w(t) \in \mathbb{R}^n$ contains the monetary amount invested at time t in each risky asset (i.e. a *portfolio*) and satisfies the usual "admissible" properties (Karatzas and Shreve, 1998), the wealth dynamics are given by

$$dX(t) = \left(X(t)\,r(t,z) + c(t,z) + w(t)'\left(\mu(t,z) - r(t,z)\,\mathbf{1}\right)\right)dt + w(t)'\,\Sigma(t,z)\,dW(t). \tag{7}$$

3 Optimisation problem

3.1 Mean-Variance and Target-Based approaches

We assume that the representative member of the pension fund is a mean-variance optimiser who solves the following problem (where $\mathbb{V}_t[\cdot]$ is the variance operator, conditional to $\mathcal{F}(t)$):

$$(P_\alpha) \qquad \min J(w(\cdot)) \equiv \alpha\mathbb{V}_0[X(T)] - \mathbb{E}_0[X(T)], \tag{8}$$

where $\alpha > 0$ is a measure of risk aversion. If $w^*(t)$ solves (P_α) for some $\alpha > 0$ and $X^*(T)$ is the associated wealth level, the set $(\mathbb{V}_0[X^*(T)], \mathbb{E}_0[X^*(T)])$ is called the *efficient frontier*.

Even if $\mathbb{V}_t[\cdot]$ does not satisfy the "smoothing" property of the expected value, Zhou and Li (2000) show that Problem (8) can be approached by solving a corresponding standard linear quadratic control problem:

$$(P_\gamma) \qquad \min J(w(\cdot)) \equiv \mathbb{E}_0\left[\frac{1}{2}\left(X(T) - \gamma\right)^2\right], \tag{9}$$

with a suitable relationship between α and γ. Problem (P_γ) can be interpreted as a target-based approach where γ plays the role of a target. Since Kahneman and Tversky (1979), who underlined the importance of reference points in decision making, the benchmark tracking has been widely used in portfolio selection[1] and

[1] See, among many others, Gaivoronski et al. (2005), He and Zhou (2011) and Jin and Zhou (2013).

pension funds optimisation.[2] With this interpretation in mind, the difficult task of selecting a proper abstract risk aversion coefficient α is transformed into the easier task of selecting a final target γ. Implementation guidelines for the selection of γ are provided in Section 4. We now focus on the solution of Problem (9) and defer to Section 4 the equivalence result demonstrated in Zhou and Li (2000) and the implementation guidelines.

Remark 2. It is important to stress that, differently from the classical investment-consumption problem, in Problems (P_α) and (P_γ) the control variable is *only* the investment strategy. The contribution stream $c(s,z)$ paid into the fund is *not* (and cannot be) a decision variable. Indeed, while in a DB pension scheme the yearly contributions are calculated by the actuary in order to fund future (defined) benefits, in a DC pension scheme the periodic contribution is defined in the rules of the scheme as a fixed percentage of the salary, and cannot be modified. Thus, the contribution is stochastic because the salary (or labour income) is stochastic, but cannot be controlled by the member or the employer.

Remark 3. The solution to (P_α) is not time-consistent. Indeed, Problems (P_γ) and (P_α) are equivalent only at time 0. Addressing the time-inconsistency with the so-called "sophisticated" approach would imply solving the time-consistent version of (P_α) (Basak and Chabakauri, 2010 and Björk and Murgoci, 2010). Instead, here we assume that the investor adopts the so-called "pre-commitment" approach and solves (P_γ).

3.2 Optimal wealth, optimal portfolio, ruin probability

Problem (9) can be equivalently solved through either dynamic programming or martingale approach. We use the second method and, accordingly, the control variable of optimisation problem (P_γ) is now $X(T)$:

$$\begin{aligned} &\inf_{X(T)} \mathbb{E}_0 \left[\tfrac{1}{2} \left(X(T) - \gamma \right)^2 \right] \\ &\text{s.t. } \mathbb{E}_0^{\mathbb{Q}} \left[-\int_0^T c(s,z)\, e^{-\int_0^s r(u,z)du} ds + X(T)\, e^{-\int_0^T r(u,z)du} \right] \leq x_0. \end{aligned} \tag{10}$$

First, we solve for the optimal wealth.

Proposition 1. *The optimal wealth of Problem* (10) *is*

$$X^*(T) = \gamma - (\gamma - \chi_T)\, B(0,T)\, \mathbb{E}_0 \left[e^{2\Phi(0,T)} \right]^{-1} e^{\Phi(0,T)}, \tag{11}$$

where

$$\Phi(t,T) = -\int_t^T r(u,z)\, du - \frac{1}{2} \int_t^T \xi(u,z)' \, \xi(u,z)\, du - \int_t^T \xi(u,z)' \, dW(u), \tag{12}$$

$$\chi_T = \frac{x_0 + \int_0^T \mathbb{E}_0^{\mathbb{F}_s} \left[c(s,z) \right] B(0,s)\, ds}{B(0,T)}. \tag{13}$$

[2] See, among many others, Blake et al. (2013), Josa-Fombellida and Rincón-Zapatero (2008) and Josa-Fombellida and Rincón-Zapatero (2012).

Proof. The proof is in the appendix. □

Remark 4. The quantity χ_T given by (13) is key in the implementation of the target-based approach: it is the wealth achievable at time T without risk. This is due to the fact that its numerator is the sum of the initial wealth and the expected present value of all the future contributions and its denominator is the discount factor from T to 0. Thus, χ_T is the compounded value in T of initial wealth and contributions. In Section 4.1 we will see that χ_T is the lowest threshold for the final target.

Then, we find the optimal portfolio.

Proposition 2. *The optimal portfolio of Problem* (10) *is*

$$\begin{aligned} w^*(t) &= \left(\gamma B(t,T) - \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right] B(t,s)\, ds - X^*(t)\right)(\Sigma')^{-1}\xi \\ &\quad + (\Sigma')^{-1}\Omega' \frac{\partial\left(\gamma B(t,T) - \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right] B(t,s)\, ds\right)}{\partial z(t)} \\ &\quad + \left(X^*(t) + \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right] B(t,s)\, ds - \gamma B(t,T)\right)(\Sigma')^{-1}\Omega' \frac{\frac{\partial \mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{\partial z(t)}}{\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}. \end{aligned} \tag{14}$$

Proof. The proof is in the appendix. □

Remark 5. If $\Phi(t,T)$ is Gaussian, the following simplification holds:

$$\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]^{-1} \frac{\partial}{\partial z(t)} \mathbb{E}_t\left[e^{2\Phi(t,T)}\right] = 2\frac{\partial}{\partial z(t)}\left(\mathbb{E}_t\left[\Phi(t,T)\right] + \mathbb{V}_t\left[\Phi(t,T)\right]\right). \tag{15}$$

We notice in particular what follows:

- the optimal portfolio has three components: the first "standard" Merton's speculative (mean-variance) component, a second component hedging against the changes in both contributions and target w.r.t. the state variables, and a third component hedging against the semi-elasticity in $\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]$ w.r.t. the state variables;
- the speculative component is proportional to the distance between the optimal wealth increased by the discounted contributions and the discounted target, i.e. the investor takes more risk to approach the target. Consequently, *ceteris paribus*, the higher the target, the higher the speculative component; the higher the discounted contributions, the lower the speculative component.

The wealth $X^*(t)$ in (11) is accumulated at the end of the management period through both the return obtained on the financial market and the compounded contributions. Thus, the optimising agent will be able to get contributions back, at least partially, only if the final wealth is strictly positive, while he/she will not in case of ruin.

The ruin probability can be computed in closed form as follows:

$$\mathbb{P}\left\{X^*(T) < 0\right\} = \mathbb{P}\left\{\Phi(0,T) > \ln\left(\frac{\gamma}{\gamma - \chi_T}\right) + \ln\left(\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^{-1}\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]\right)\right\}, \tag{16}$$

and is definitely relevant, because it is (or it should be) one of the drivers in the practical implementation of the model (see Section 4.2). Indeed, due to (16) and expectedly, the higher the target γ (or, equivalently, its distance from the lowest threshold χ_T), the higher the ruin probability.

However, for the member of the fund it is rather important to know also the probability that the final wealth will be higher than the compounded contributions (given by χ_T as in (13)). In our framework, given (11), this probability can be calculated in closed form as follows:

$$\mathbb{P}\left\{X^*(T) \geq \chi_T\right\} = \mathbb{P}\left\{\Phi(0,T) \leq \ln\left(\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^{-1}\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]\right)\right\}. \tag{17}$$

We highlight that if $\Phi(0,T)$ is Gaussian, then the terms in the probabilities above simplify as follows:

$$\ln\left(\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^{-1}\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]\right) = \mathbb{E}_0\left[\Phi(0,T)\right] + \frac{3}{2}\mathbb{V}_0\left[\Phi(0,T)\right].$$

We notice that the probability that the final wealth is higher than the compounded value of all the contributions (given by (17)) depends neither on the fund's target (γ), nor on the value of the compounded contributions (χ_T). The intuition is the following: both χ_T and $X^*(T)$ contain the compounded contributions and, accordingly, the probability in (17) depends only on the stochastic behaviour of both interest rate and market price of risk. Finally, because the target γ does not affect the probability in (17), its choice in the practical implementation of the model (Section 4.2) will be determined only by the ruin probability.

3.2.1 The efficient frontier

Given the optimal wealth (11), its expected value and variance are:

$$\mathbb{E}_0\left[X^*(T)\right] = \gamma - (\gamma - \chi_T)\,\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^2\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]^{-1}, \tag{18}$$

$$\mathbb{V}_0\left[X^*(T)\right] = (\gamma - \chi_T)^2\,\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^2\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]^{-2}\mathbb{V}_0\left[e^{\Phi(0,T)}\right],$$

and if γ is taken from one equation and plugged into the other, we can obtain the efficient frontier in the mean-standard deviation plan, that is

$$\mathbb{E}_0\left[X^*(T)\right] = \chi_T + \sqrt{\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^{-2}\mathbb{E}_0\left[e^{2\Phi(0,T)}\right] - 1}\sqrt{\mathbb{V}_0\left[X^*(T)\right]}, \tag{19}$$

where the intercept χ_T is the wealth achievable with zero variance (Remark 4).

Remark 6. If $\Phi(t,T)$ is Gaussian, then $\mathbb{E}_0\left[e^{\Phi(0,T)}\right] = e^{\mathbb{E}_0[\Phi(0,T)]+\frac{1}{2}\mathbb{V}_0[\Phi(0,T)]}$ and (19) can be written as $\mathbb{E}_0\left[X^*(T)\right] = \chi_T + \sqrt{e^{\mathbb{V}_0[\Phi(0,T)]}-1}\sqrt{\mathbb{V}_0\left[X^*(T)\right]}$. Furthermore, if r, ξ and c are constant, $\Phi(0,T) = -\left(r+\frac{1}{2}\xi^2\right)T - \xi W(T)$, the intercept is $\chi_T = x_0 e^{rT} + \frac{c}{r}\left(e^{rT}-1\right)$, and the slope is $\sqrt{e^{\xi^2 T}-1}$.

3.3 The case of an incomplete market

In our approach we have taken into account a complete financial market, where the diffusion matrix Σ in (2) is invertible. In other words, in this market it is possible to hedge any risk source by replicating it with a suitable portfolio.

The so-called martingale approach relies precisely on this replicating procedure. The differential of the optimal wealth $X^*(t)$ is given by

$$dX^*(t) = (...)\,dt + \Sigma^*(t,z)\,dW(t),$$

where

$$\begin{aligned}\Sigma^*(t,z) = &\left(\gamma B(t,T) - \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right]B(t,s)\,ds - X^*(t)\right)\xi(t,z)' \\ &+ \frac{\partial}{\partial z}\left(\gamma B(t,T) - \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right]B(t,s)\,ds\right)\Omega(t,z) \\ &+ \left(X^*(t) + \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right]B(t,s)\,ds - \gamma B(t,T)\right)\frac{\frac{\partial}{\partial z}\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}\Omega(t,z),\end{aligned}$$

and we have omitted the drift term since it is immaterial to our aim. The diffusion term of the fund's wealth in (7), i.e. $w(t)'\Sigma(t,z)$, can be set equal to the diffusion term $\Sigma^*(t,z)$ only if $\Sigma(t,z)$ is invertible. In other words, this means that we are looking for a portfolio $w(t)$ that replicates the optimal wealth $X^*(t)$.

If the market is incomplete, and $\left(\Sigma(t,z)'\right)^{-1}$ does not exist, then the replicating portfolio does not exist. This means that a perfect hedging cannot be found. Nevertheless, it is still possible either to over replicate the optimal wealth or to find the portfolio that minimizes the square of the distance between the diffusion terms of the optimal wealth $(\Sigma^*(t,z))$ and of the wealth in (7). This problem can be algebraically written as follows

$$\min_{w(t)}\left(w(t)'\Sigma(t,z) - \Sigma^*(t,z)\right)\left(\Sigma(t,z)'w(t) - \Sigma^*(t,z)'\right),$$

whose solution is

$$w^{**}(t) = \left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}\Sigma(t,z)\Sigma^*(t,z)'.$$

Of course, if $\Sigma(t,z)$ is invertible (i.e. the market is complete) then $\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}\Sigma(t,z) = \left(\Sigma(t,z)'\right)^{-1}$, and the solution presented in the previous sections is retrieved.

Any approach in the case of an incomplete market allows to find a portfolio which does not provide a

perfect hedging, and is, accordingly, more expensive. Furthermore, in case of incompleteness, there exist infinitely many market prices of risk ξ that eliminate any arbitrage opportunity from the market, and the choice of the price is mainly subjective. This means that any solution found in an incomplete market cannot be taken as a benchmark for measuring the cost and the portfolio riskiness with respect to a perfect hedging strategy.

In this paper there is mainly one source of risk that could create incompleteness: the presence of stochastic contributions that cannot be replicated. Thus, if the contributions are driven by an additional idiosyncratic source of risk that cannot be replicated, then the procedure described above should be applied. In the context of pension funds, this is done, for instance, in De Jong (2008), in the presence of pension liabilities driven by an idiosyncratic wage risk with power preferences.

Our assumption of market completeness is common in the literature of DC pension plans, because the financial assets that are correlated with the stochastic contributions are quite liquid on the market. In fact, the contributions are a percentage of the workers wages and there are many theoretical and empirical works dealing with the correlation between wages and both financial markets and interest rate. For instance, Michelacci and Quadrini (2009) presents a framework where the correlation between firms (and financial markets) and wages is justified from a theoretical point of view and also empirically checked. Accordingly, the hypothesis of a complete financial market with respect to the risk of contributions (i.e., wages) does not seem to be too strong.

4 Mean-variance versus target-based

In this section, we summarize Zhou and Li (2000) to show the equivalence between Problems (8) and (9). Then, we interpret the equivalence in our framework, and we finally provide guidelines for the practical implementation of the model.

4.1 Equivalence between (P_α) and (P_γ)

Zhou and Li (2000) show that Problem (8) can be approached through a corresponding standard linear quadratic problem:

$$\min J\left(w\left(\cdot\right)\right) \equiv \mathbb{E}_0\left[\alpha X\left(T\right)^2 - \beta X\left(T\right)\right]. \tag{20}$$

Actually, they show that if $w^*\left(\cdot\right)$ is a solution to (8) with $\alpha = \overline{\alpha}$, then it is also a solution to (20) with $\alpha = \overline{\alpha}$ and $\beta = \overline{\beta}$ satisfying:[3]

$$\overline{\beta} = 1 + 2\,\overline{\alpha}\,\mathbb{E}_0\left[X^*\left(T\right); \overline{\alpha}, \overline{\beta}\right]. \tag{21}$$

[3]In practice, the solution to the *particular* problem $(P_{\overline{\alpha}})$ (with exogenous risk aversion $\overline{\alpha}$), is found by: (i) first, solving the general problem (20), finding the optimal wealth in terms of α and β, and calculating $\mathbb{E}_0\left[X^*(T); \alpha, \beta\right]$; (ii) then, in the optimal solution, replacing α with $\overline{\alpha}$ and β with $\overline{\beta}$ found via (21). The optimal solution to (20) with $\alpha = \overline{\alpha}$ and $\beta = \overline{\beta}$ is the optimal solution to $(P_{\overline{\alpha}})$.

Problem (20) can be conveniently transformed into the equivalent Problem (9), by setting

$$\gamma = \frac{\beta}{2\alpha}. \tag{22}$$

Therefore, Zhou and Li (2000) show that solving (9) is equivalent to solving (8).

In our framework, when the expected value (18) is plugged into (21), we find that β satisfies (hereafter the bar over β is omitted for simplicity):

$$\beta = 1 + 2\alpha\mathbb{E}_0\left[X^*(T);\alpha,\beta\right] = 1 + 2\alpha\left[\frac{\beta}{2\alpha}\left(1 - \frac{\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^2}{\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]}\right) + \chi_T \cdot \frac{\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^2}{\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]}\right],$$

therefore

$$\beta = \mathbb{E}_0\left[e^{\Phi(0,T)}\right]^{-2}\mathbb{E}_0\left[e^{2\Phi(0,T)}\right] + 2\alpha \cdot \chi_T. \tag{23}$$

It is now possible to interpret the equivalence between the mean-variance problem (P_α) and the target-based problem (P_γ) by providing the relationship between α and γ. By plugging (23) into (22), we find:

$$\alpha = \frac{1}{2(\gamma - \chi_T)}\mathbb{E}_0\left[e^{\Phi(0,T)}\right]^{-2}\mathbb{E}_0\left[e^{2\Phi(0,T)}\right], \tag{24}$$

i.e. the higher α the lower γ and vice versa. Indeed, α is a measure of risk aversion in (P_α), while the target γ in (P_γ) is a measure of the risk tolerance. Moreover:

$$\alpha > 0 \iff \gamma > \chi_T, \tag{25}$$

or, in other words, the target must exceed the lowest threshold χ_T. From the economic point of view, it is easy to understand why $\gamma > \chi_T$: χ_T is the final wealth obtainable by investing initial wealth and future contributions in the riskless asset (see (13) and Remark 4), therefore the target must exceed it. If the target were lower than χ_T, the problem would not be interesting, as the investor could invest the portfolio entirely in the riskless asset and obtain even more than the target. According to the quadratic loss function this excess of final wealth would produce a penalty, and the only way to reach the goal would be throwing away money, that is clearly problematic for a rational investor.

Remark 7. Given (11), the strict positivity of $\gamma - \chi_T$ implies that the optimal wealth will never reach the target γ, which thus behaves as an upper threshold for the desired final wealth.

We can now establish the link between mean-variance and target-based problems.

Proposition 3. *Given the financial market and wealth dynamics as in Section 2, there is a one-to-one correspondence between portfolios on the efficient frontier* (19)*, identified by* $\alpha > 0$*, and optimal solutions to target-based problems* (9)*, identified by* $\gamma > \chi_T$*. The relationship between the corresponding* α *and* γ *is given by* (24)*.*

Proof. The proof is in the appendix. □

In other words, the investor's preferences can be described in two different ways: mean-variance or target-based. If the investor has mean-variance preferences described by exogenous $\overline{\alpha}$, he/she has target-based preferences described by a suitable $\overline{\gamma}$. If the investor has target-based preferences described by exogenous $\overline{\gamma}$, he/she has mean-variance preferences described by a suitable $\overline{\alpha}$. The difference is that the set of choice for α is $(0, +\infty)$, independent of the model's parameters, while the set of choice for γ is $(\chi_T, +\infty)$ where χ_T depends on the model's parameters (see (13)).

4.2 Implementation: choice of the target

The equivalence between (P_α) and (P_γ) has relevant implications. In fact, it may not be easy for an investor to define his/her risk aversion in terms of a pure number $\alpha > 0$. Instead, defining a target γ in terms of monetary units appears to be much more "user-friendly".

Since γ cannot be less than the riskfree level χ_T, we parametrise γ as a multiple of χ_T:

$$\gamma = \kappa \cdot \chi_T, \tag{26}$$

with $\kappa > 1$. The extreme cases for the selection of the target are: (i) $\gamma \to \chi_T$ (or, equivalently, $\kappa \to 1$ or $\alpha \to +\infty$) with infinite risk aversion, and (ii) $\gamma \to +\infty$ (or, equivalently, $\kappa \to +\infty$ or $\alpha \to 0$) with null risk aversion.

Proposition 3 allows the member to identify his/her own risk aversion α (and therefore the corresponding point on the frontier) just by selecting a final target γ, or a multiple $\kappa > 1$ of the lowest threshold χ_T.

Another tool for the practical implementation is the computation of the ruin probability (16), that, due to (26), is

$$\mathbb{P}\left\{X^*(T) < 0\right\} = \mathbb{P}\left\{\Phi(0,T) > \ln\left(1 + \frac{1}{\kappa - 1}\right) + \ln\frac{\mathbb{E}_0\left[e^{2\Phi(0,T)}\right]}{\mathbb{E}_0\left[e^{\Phi(0,T)}\right]}\right\}. \tag{27}$$

In the case of an infinite risk aversion ($\kappa \to 1$) the ruin probability (27) is null, while for null risk aversion ($\kappa \to +\infty$) it is

$$\mathbb{P}\left\{X^*(T) < 0\right\} = \mathbb{P}\left\{\Phi(0,T) > \ln\mathbb{E}_0\left[e^{2\Phi(0,T)}\right] - \ln\mathbb{E}_0\left[e^{\Phi(0,T)}\right]\right\}. \tag{28}$$

Remark 8. When $\Phi(0,T)$ is normally distributed, (28) becomes

$$\mathbb{P}\left\{X^*(T) < 0\right\} = \mathbb{P}\left\{-\frac{\Phi(0,T) - \mathbb{E}_0\left[\Phi(0,T)\right]}{\sqrt{\mathbb{V}_0\left[\Phi(0,T)\right]}} < -\frac{3}{2}\sqrt{\mathbb{V}_0\left[\Phi(0,T)\right]}\right\} = \mathcal{N}\left(-\frac{3}{2}\sqrt{\mathbb{V}_0\left[\Phi(0,T)\right]}\right),$$

where $\mathcal{N}(\bullet)$ is the CDF of a normal standard variable.

Thus, the range of ruin probability associated to Problem (8) (or (9)) is

$$\left(0, \mathbb{P}\left\{\Phi\left(0,T\right) > \ln \mathbb{E}_0\left[e^{2\Phi(0,T)}\right] - \ln \mathbb{E}_0\left[e^{\Phi(0,T)}\right]\right\}\right). \tag{29}$$

Accordingly, with a target strictly greater than χ_T, the non zero ruin probability is the price to be paid in order to be mean-variance efficient. On the other hand, the ruin probability is controllable and can/should indeed affect the choice of the target. Through (27), $\kappa > 1$ can be chosen to set the ruin probability at any desired level in the range (29). We recall that a common rule in investment management is the so-called "safety-first-portfolio", where the expected wealth is maximised under a constraint on the ruin probability (e.g. lower than 10^{-5}, 10^{-3} etc.).

In implementing the model, the members of a DC pension fund should be aware of how the distribution of their final wealth changes with respect to the chosen target, and which is the associated ruin probability. This could be done via disclosure of tables with percentiles of final wealth, as Section 5 illustrates.

It should also be stressed that the "target" is an upper (unreachable) bound for the wealth (see Remark 7).

The importance of full information to the pension fund members is noted also by Looney and Hardin (2009). They find that retirement portfolio of pension funds are overly conservative, and that conservatism diminishes when the investors are provided with prospective probabilities and payoffs over long time horizons.

5 Application to a model with stochastic interest rate and stochastic contribution

In this section we show a market where the state variables are the stochastic riskless interest rate and the contribution (i.e. $z\left(t\right) = \begin{bmatrix} r\left(t\right) & c\left(t\right) \end{bmatrix}'$). Two risky assets are listed on the financial market: a bond and a stock. We define a base scenario by calibrating the model parameters to historical data. Then, we investigate the dynamics of the optimal portfolio and its dependence on the risk aversion. Finally, we discuss the limits of applicability of the optimal strategies, and propose suboptimal strategies that satisfy non-negativity constraints.

5.1 Interest rate and contributions

Let $r\left(t\right)$ follows a Vasiček (1977)'s model and $c\left(t\right)$ a geometric Brownian motion under the real-world probability $\mathbb{P}$. The state variables are assumed to be driven by two independent risk sources:

$$\underbrace{\begin{bmatrix} dr\left(t\right) \\ dc\left(t\right) \end{bmatrix}}_{dz(t)} = \underbrace{\begin{bmatrix} a\left(b - r\left(t\right)\right) \\ c\left(t\right)\mu_c \end{bmatrix}}_{\mu_z(t,z)} dt + \underbrace{\begin{bmatrix} \sigma_r & 0 \\ c\left(t\right)\sigma_{cr} & c\left(t\right)\sigma_{cs} \end{bmatrix}}_{\Omega(t,z)} \underbrace{\begin{bmatrix} dW_r\left(t\right) \\ dW_s\left(t\right) \end{bmatrix}}_{dW(t)}, \tag{30}$$

where all the parameters are constant and both $r(0) = r_0$ and $c(0) = c_0$ are known. We assume that both the market prices of risk (ξ_r for W_r and ξ_s for W_s) are constant, so that the statistical properties of (30) do not change under $\mathbb{P}$ or $\mathbb{Q}$.

The dynamics (30) for the state variables has a double advantage: its analytic tractability and the existence of optimal investment strategy in closed-form. Given (30), the price of the zero coupon bond is (see Vasiček, 1977)

$$B(t,T) = e^{f(t,T)-g(t,T)r(t)}, \tag{31}$$

where

$$f(t,T) = \left(\frac{1-e^{-a(T-t)}}{a} - (T-t)\right)\left(b - \frac{\sigma_r \xi_r}{a} - \frac{1}{2}\frac{\sigma_r^2}{a^2}\right) - \frac{\sigma_r^2\left(1-e^{-a(T-t)}\right)^2}{4a^3},$$
$$g(t,T) = \frac{1-e^{-a(T-t)}}{a}.$$

5.2 The financial market

In order to avoid the distortion of the time dependent bond's duration, we take a constant time-to-maturity (K) zero-coupon bond (like in Boulier et al., 2001; Battocchio and Menoncin, 2004) whose price is $B_K(t) = B(t, t+K)$.

Remark 9. The zero-coupon bond $B_K(t)$ represents the class of the whole bonds whose maturity is K. The maturity of a bond is a measure of its risk and bonds with different maturities belong to different risk classes. If the zero-coupon bond volatility $g(t,T)$ is allowed to vary over time, then also the bond switches from one risk class to another over time. In this case, the portfolio composition would not be homogeneous over time in terms of risk classes. Since we want the bond to belong to the same risk class over the whole management period, then we take into account the bond $B_K(t)$, which can of course be replicated by a suitable portfolio of bonds since we are dealing with a complete market.

Furthermore, we assume that the stock price is driven by both risk sources (W_r and W_s). The dynamics of these two assets are described by the following matrix stochastic equation (with σ_{sr} and σ_s constant):

$$\underbrace{\begin{bmatrix} \frac{dB_K(t)}{B_K(t)} \\ \frac{dS(t)}{S(t)} \end{bmatrix}}_{I_P^{-1}dP(t)} = \underbrace{\begin{bmatrix} r(t) - g(0,K)\sigma_r\xi_r \\ r(t) + \xi_r\sigma_{sr} + \xi_s\sigma_s \end{bmatrix}}_{\mu(t,z)} dt + \underbrace{\begin{bmatrix} -g(0,K)\sigma_r & 0 \\ \sigma_{sr} & \sigma_s \end{bmatrix}}_{\Sigma(t,z)} \underbrace{\begin{bmatrix} dW_r(t) \\ dW_s(t) \end{bmatrix}}_{dW(t)}.$$

Using (3) and (5) the process $c(t)$ and its expected value can be written under the forward probability

measure as follows (recall $\sigma_B(t,T) = \begin{bmatrix} -g(0,K)\sigma_r & 0 \end{bmatrix}'$):

$$\begin{aligned}\frac{dc(t)}{c(t)} &= (\mu_c - \sigma_{cr}\xi_r - \sigma_{cs}\xi_s)\,dt + \sigma_{cr}dW_r^{\mathbb{Q}}(t) + \sigma_{cs}dW_s^{\mathbb{Q}}(t) \\ &= (\mu_c - \sigma_{cr}\xi_r - g(0,K)\sigma_r\sigma_{cr} - \sigma_{cs}\xi_s)\,dt + \sigma_{cr}dW_r^{\mathbb{F}_T}(t) + \sigma_{cs}dW_s^{\mathbb{F}_T}(t),\end{aligned}$$

$$\mathbb{E}_t^{\mathbb{F}_T}[c(T)] = c(t)\,e^{(\mu_c - \sigma_{cr}\xi_r - g(0,K)\sigma_r\sigma_{cr} - \sigma_{cs}\xi_s)(T-t)}. \tag{32}$$

Here, we have assumed that the contribution is correlated with both the bond and the stock. In fact, the previous differential equations imply the following covariances:

$$\begin{aligned}\mathbb{C}\left[\frac{dB_K(t)}{B_K(t)}, \frac{dc(t)}{c(t)}\right] &= -g(0,K)\sigma_r\sigma_{cr}dt, \\ \mathbb{C}\left[\frac{dS(t)}{S(t)}, \frac{dc(t)}{c(t)}\right] &= (\sigma_{sr}\sigma_{cr} + \sigma_s\sigma_{cs})\,dt, \\ \mathbb{C}\left[\frac{dS(t)}{S(t)}, \frac{dB_K(t)}{B_K(t)}\right] &= -g(0,K)\sigma_r\sigma_{sr}dt,\end{aligned}$$

where the values of the parameters can be calibrated on the market data as we are about to show in the following subsections.

5.3 The optimal portfolio

In this special case, $\Phi(t,T)$ in (12) is normally distributed:

$$\begin{aligned}\Phi(t,T) &= -\left(b + \frac{1}{2}\left(\xi_s^2 + \xi_r^2\right)\right)(T-t) - (r(t) - b)\frac{1 - e^{-a(T-t)}}{a} \\ &\quad - \int_t^T \left(\frac{1 - e^{-a(T-u)}}{a}\sigma_r + \xi_r\right) dW_r(u) - \int_t^T \xi_s dW_s(u).\end{aligned} \tag{33}$$

and its variance is

$$\begin{aligned}\mathbb{V}_t[\Phi(t,T)] &= \left(\frac{\sigma_r^2}{a^2} + 2\frac{\sigma_r\xi_r}{a} + \xi_r^2 + \xi_s^2\right)(T-t) \\ &\quad - 2\sigma_r\left(\frac{\sigma_r}{a} + \xi_r\right)\frac{1 - e^{-a(T-t)}}{a^2} + \frac{\sigma_r^2}{2}\frac{1 - e^{-2a(T-t)}}{a^3}.\end{aligned} \tag{34}$$

The calculation of the optimal portfolio follows easily.

Proposition 4. *The optimal portfolio is given by*

$$\begin{aligned} w_r^*(t) &= -\frac{\sigma_s\xi_r - \sigma_{sr}\xi_s + 2g(t,T)\sigma_s\sigma_r}{g(0,K)\sigma_r\sigma_s}\left(\gamma B(t,T) - \int_t^T \mathbb{E}_t^{\mathbb{F}_s}[c(s,z)]B(t,s)\,ds - X^*(t)\right) \\ &\quad + \frac{\gamma g(t,T)B(t,T) - \int_t^T \mathbb{E}_t^{\mathbb{F}_s}[c(s)]g(t,s)B(t,s)\,ds}{g(0,K)} \\ &\quad + c(t)\frac{\sigma_s\sigma_{cr} - \sigma_{sr}\sigma_{cs}}{\sigma_r\sigma_s}\frac{\int_t^T e^{(\mu_c - \sigma_{cr}\xi_r - g(0,K)\sigma_r\sigma_{cr} - \sigma_{cs}\xi_s)(s-t)}B(t,s)\,ds}{g(0,K)}, \\ w_s^*(t) &= \frac{\xi_s}{\sigma_s}\left(\gamma B(t,T) - \int_t^T \mathbb{E}_t^{\mathbb{F}_s}[c(s,z)]B(t,s)\,ds - X^*(t)\right) \\ &\quad - c(t)\frac{\sigma_{cs}}{\sigma_s}\int_t^T e^{(\mu_c - \sigma_{cr}\xi_r - g(0,K)\sigma_r\sigma_{cr} - \sigma_{cs}\xi_s)(s-t)}B(t,s)\,ds. \end{aligned} \tag{35}$$

Proof. The proof is in the appendix. □

Observing that $(\gamma - \chi_T)B(0,T) = \gamma B(0,T) - x_0 - \int_0^T \mathbb{E}_0^{\mathbb{F}_s}[c(s,z)]B(0,s)\,ds$, we can prove a sufficient condition for the positivity of the amount invested in stock.

Proposition 5. *If $\sigma_{cs} \leq 0$, then the optimal amount invested in the stock for the mean-variance problem defined by* (8) *is strictly positive at any time $0 \leq t \leq T$.*

Proof. The proof is in the appendix. □

Since the contribution and the stock are assumed to be correlated, the investment in stock satisfies both a speculation purpose and a hedging purpose against the stochastic changes in the contribution. If the coefficient σ_{cs} is negative, then (given $\sigma_s > 0$ without any loss of generality) a positive shock on dW_s positively affects $S(t)$ and negatively affects $c(t)$. Since the contribution is a positive component of the fund's wealth, then the portfolio is hedged against such a risk if the stock is held in portfolio with a positive weight. In fact, in this case, a negative shock in $c(t)$ is compensated by a corresponding increase in $S(t)$, and vice versa.

Remark 10. Bajeux-Besnainou and Portait (1998) solve a similar problem but without contributions. In fact, our results coincide with theirs by setting $c_0 = \mu_c = \sigma_{cr} = \sigma_{cs} = 0$.

5.4 Base scenario

The calibration is performed on four time series (January 1st 1962 - January 1st 2007 – without the subprime crisis): (i) the 3-month US T-Bill interest rate (on secondary market) for $r(t)$, (ii) the 10-year US Bond interest rate (on secondary market) for $B_K(t)$ (with $K = 10$), (iii) S&P 500 for $S(t)$, and (iv) compensation of employees (wages and salaries) for US workers (`research.stlouisfed.org/fred2` series A576RC1A027NBEA) for $c(t)$.

The parameters of the stochastic processes defined in this section are estimated through a least square optimization based on the discretisation of the processes (Florens-Zmirou, 1989).

All the time series (but the last one) are daily and, accordingly, $dt = 1/250$. From (30), the variance of $dr(t)$ is $\sigma_r^2 dt$ and its empirical value is $\mathbb{V}_t\left[dr\left(t\right)\right]$ from which

$$\sigma_r = \sqrt{\mathbb{V}_t\left[dr\left(t\right)\right]/dt} = 0.0158.$$

The values of a and b are obtained from an OLS estimation of the discretised $r\left(t\right)$ in (30) ($\varepsilon\left(t_i\right)$ is a white noise):

$$r\left(t_{i+1}\right) = \underbrace{ab \cdot dt}_{4.2245\times 10^{-5}} + \underbrace{(1 - a \cdot dt)}_{0.99929} r\left(t_i\right) + \varepsilon\left(t_i\right).$$

The initial value of interest rate is set to its long term equilibrium value (i.e. $r_0 = b$). The annual average return on 10-year bonds is about 7.1%, thus

$$\mathbb{E}_t\left[d\ln B_K\left(t\right)\right]/dt = 0.071, \implies r\left(t\right) - g\left(0,10\right)\sigma_r\xi_r - g\left(0,10\right)^2\sigma_r^2/2 = 0.071.$$

If we replace $r\left(t\right)$ with the long term equilibrium value b, we obtain $\xi_r = -0.1912$.

Remark 11. The value of ξ_r is negative because of the negative correlation between the interest rate $r\left(t\right)$ and the value of the bond $B_K\left(t\right)$. Since the expected return on the bond must be higher than the riskless interest rate, then

$$\frac{1}{dt}\mathbb{E}_t\left[\frac{dB_K\left(t\right)}{B_K\left(t\right)}\right] = r\left(t\right) - g\left(0,K\right)\sigma_r\xi_r > r\left(t\right),$$

and since $g\left(0,K\right) > 0$, we have that $\xi_r < 0$.

The empirical and theoretical variances of the S&P log-return, its means, and its covariances with the 10-year bonds return are (b is used instead of $r\left(t\right)$)

$$\begin{cases} \mathbb{V}\left[d\ln S\left(t\right)\right]\cdot\frac{1}{dt} = \sigma_{sr}^2 + \sigma_s^2 = 0.0223, \\ \mathbb{E}_t\left[d\ln S\left(t\right)\right]\cdot\frac{1}{dt} = \underbrace{r\left(t\right)}_{b} + \xi_r\sigma_{sr} + \xi_s\sigma_s - \frac{1}{2}\sigma_{sr}^2 - \frac{1}{2}\sigma_s^2 = 0.067, \\ \mathbb{C}\left[d\ln S\left(t\right), d\ln B_K\left(t\right)\right]\cdot\frac{1}{dt} = -g\left(0,K\right)\sigma_r\sigma_{sr} = -0.0004552, \end{cases}$$

where two solutions for σ_s are found; we take the positive one.

We assume that the contributions are proportional to wages and salaries. The theoretical and empirical mean of the log-difference in the US data, their variances and their covariances with the S&P log-return are[4]

[4]Since wages and asset prices have different frequency (yearly versus daily), in the following system $dt = 1$ and the yearly S&P log-returns have been considered.

Table 1: Parameters for the base scenario, calibrated on the S&P 500, 3-month US T-Bills, 10-year US Bonds time series, and US employees compensation (January 1st 1962 - January 1st 2007). Efficient frontier: intercept $\chi_T = 8.43$, slope 0.99.

Interest rate	Stock	Bond	Wealth	Contributions
$a = 0.1775$	$\sigma_s = 0.1492$	$K = 10$	$x_0 = 1$	$c_0 = 0.0548978$
$b = 0.0595$	$\sigma_{sr} = 0.006162$		$T = 20$	$\mu_c = 0.0683467$
$\sigma_r = 0.0158$	$\xi_s = 0.1322$			$\sigma_{cr} = 0.0244273$
$\xi_r = -0.1913$				$\sigma_{cs} = -0.001343$

$$\begin{cases} \mathbb{E}_t\left[d\ln c\left(t\right)\right]\cdot\frac{1}{dt} = \mu_c = 0.0683467, \\ \mathbb{V}\left[d\ln c\left(t\right)\right]\cdot\frac{1}{dt} = \sigma_{cr}^2 + \sigma_{cs}^2 = 0.0244642^2, \\ \mathbb{C}\left[d\ln S\left(t\right), d\ln c\left(t\right)\right]\cdot\frac{1}{dt} = \sigma_{cr}\sigma_{sr} + \sigma_{cs}\sigma_s = -0.000049855, \end{cases}$$

which has two solutions: $(\sigma_{cr} > 0, \sigma_{cs} < 0)$ and $(\sigma_{cr} < 0, \sigma_{cs} > 0)$.

Here, we have a degree of freedom since we can chose the solution that is more likely to describe the true financial market. We recall that σ_{cr} (σ_{cs}) measures how the contribution react to the shocks on the interest rate(risky asset). Contributions and interest rate have a macroeconomic variable in common: inflation. We know from Fisher equation that the nominal interest rate and the inflation rate are positively correlated. Furthermore, also wages and inflation are positively correlated since workers tend to ask for higher wages when inflation grows. Thus, it is more reasonable to assume that $\sigma_{cr} > 0$. This choice implies that we must assume $\sigma_{cs} < 0$, but this is again in line with the empirical evidence since Fama and Schwert (1977) and Geske and Roll (1983) show that inflation and stock returns are negatively correlated.

The initial value of the contributions c_0 is set in order to have $\int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c\left(s,z\right)\right]B\left(t,s\right)ds$ equal to the value that would be obtained with a constant contribution $\overline{c} = 0.1$, i.e.

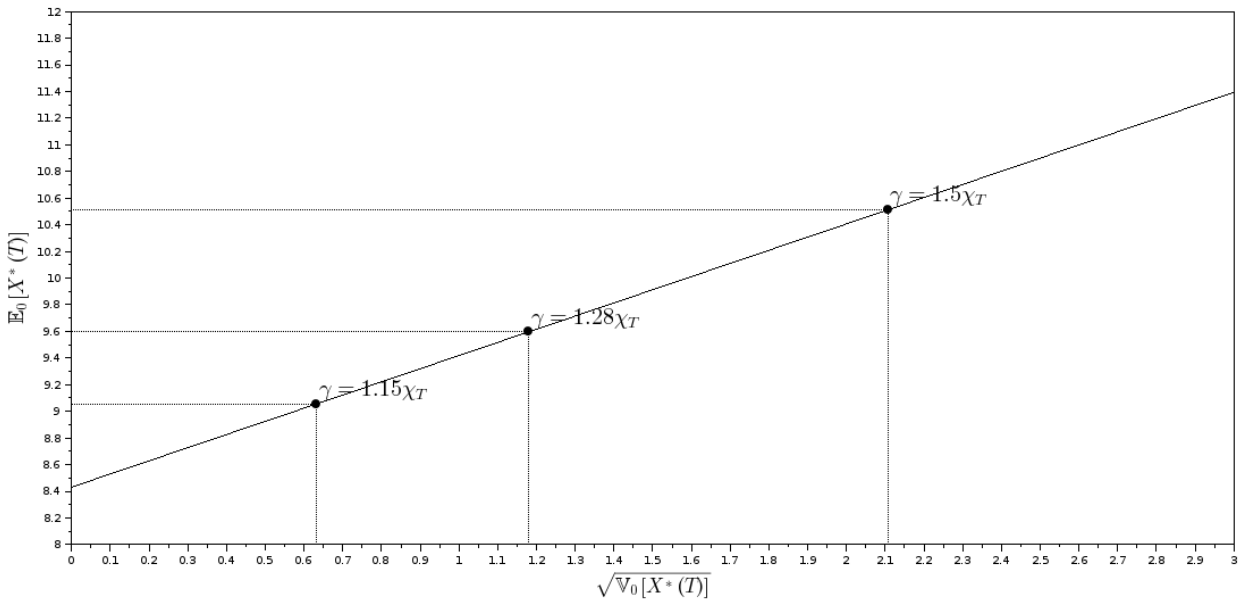
$$\overline{c}\int_0^T B\left(0,s\right)ds = c_0\int_0^T e^{(\mu_c - \sigma_{cr}\xi_r - g(0,K)\sigma_r\sigma_{cr} - \sigma_{cs}\xi_s)s}B\left(0,s\right)ds.$$

Finally, the initial wealth is set to $x_0 = 1$ and the time-horizon to $T = 20$. The values of all the parameters are summarised in Tab. 1. With this set of values, the efficient frontier has intercept $\chi_T = 8.43$ and slope 0.99.

5.5 Optimal portfolio over time with different risk profiles

This section investigates the behaviour of the optimal portfolio over time by means of Monte Carlo simulations. Given the base scenario as in Section 5.4, we find the efficient frontier ($\chi_T = 8.43$ and slope 0.99). The target-based approach requires the member to choose a target $\gamma = \kappa \cdot \chi_T$. Applying (29), in the base scenario the ruin probability lies in the range $(0, 10.8\%)$, depending on the value of $\kappa > 1$. With constant

Figure 1: The efficient frontier in the base scenario ($\chi_T = 8.43$ and slope 0.99) with three optimal portfolios corresponding to three risk aversions (i.e. three ruin probabilities). With $\gamma = 1.15\chi_T$ the ruin probability is 0.01%, with $\gamma = 1.28\chi_T$ the ruin probability is 0.1% and with $\gamma = 1.5\chi_T$ the ruin probability is 0.5%.



interest rate, $c = 0.1$ and $\xi_s = 0.33$ (like in Vigna, 2014), the ruin probability would belong to $(0, 1.34\%)$. The interest rate stochasticity has inflated the maximum ruin probability by a factor 8.

Starting from the base scenario in Section 5.4, three values for κ are chosen, to test three levels of risk aversion:

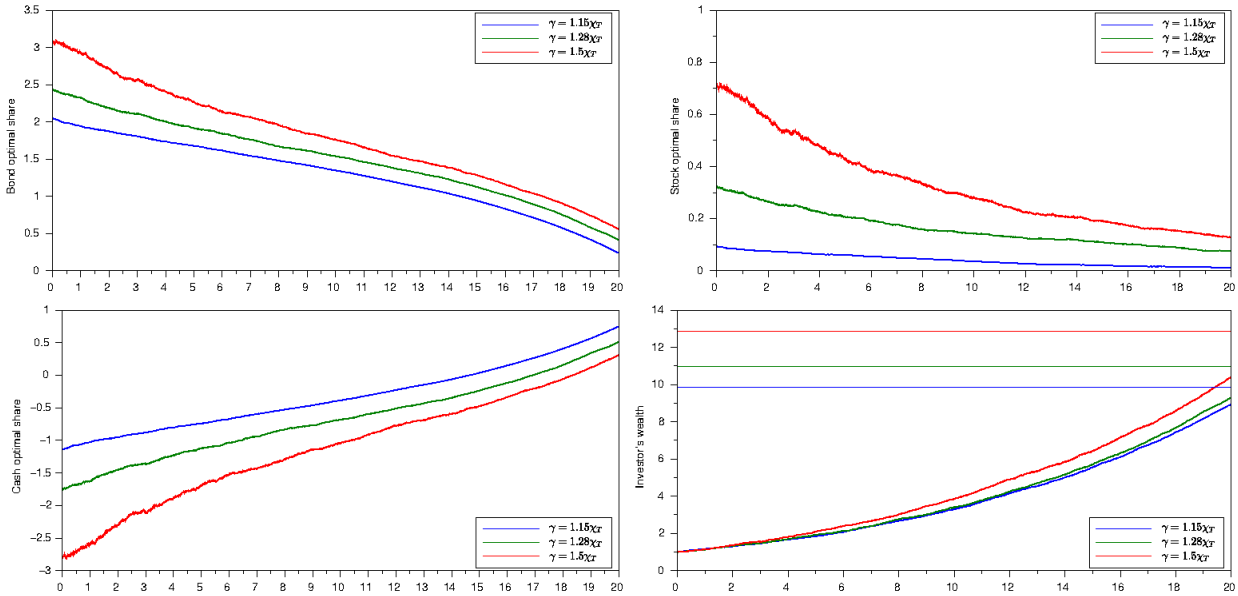
1. high risk aversion with ruin probability $0.01\% \Rightarrow \kappa = 1.15$;
2. medium risk aversion with ruin probability $0.1\% \Rightarrow \kappa = 1.28$;
3. low risk aversion with $\kappa = 1.5 \Rightarrow$ ruin probability 0.5%.

Fig. 1 shows the efficient frontier with the three efficient portfolios associated to the aforementioned risk profiles.

For each risk profile (i.e. for each value of κ) the optimal portfolio share has been derived in $10,000$ Monte Carlo scenarios, see Fig. 2 (time on the abscissa). In particular, for each risk profile the graphs report: the average proportion of wealth invested in bond (top-left), the average proportion of wealth invested in cash (bottom-left), the average proportion of wealth invested in stock (top-right) and the average behaviour of optimal wealth, as compared to the target γ (bottom-right).

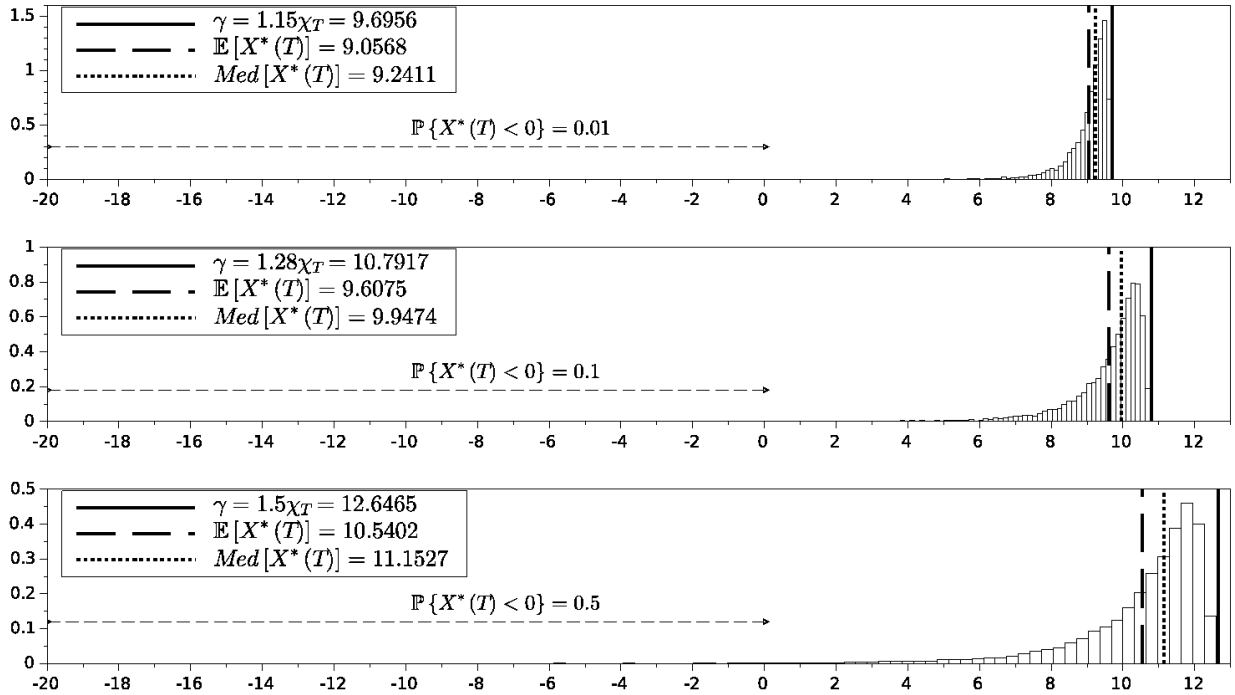
We observe what follows.

Figure 2: Optimal portfolios and wealth for three risk profiles: top-left the optimal bond share, top-right the optimal stock share, bottom-left the optimal cash share, bottom-right both the optimal wealth and the target.



- At $t = 0$ the wealth is always heavily invested in bonds and significantly invested in equities, by short selling heavy quantities of the riskless asset (i.e. borrowing).
- The percentage invested in both bonds and equities declines over time, while borrowing decreases (i.e. the negative share invested in cash increases).
- The equity share is between 0 and 1, but this is not the case for bonds and cash; however, with higher κ also the equity share could exceed 1. Strategies with non-negative weights are considered in Section 5.6.
- The average optimal wealth is increasing and approaches the target (the horizontal lines in the bottom-right graph) and never reaches it (as dictated by the model).
- The comparison between the three risk profiles is intuitive. The bond and equity shares are highest for the low risk aversion (red lines), intermediate for the medium risk aversion (green line) and lowest for the high risk aversion (blue line). In particular, the equity share decreases from 70% to 20% for low risk aversion, from 30% to 10% for medium risk aversion, and from 10% to 2% for high risk aversion.
- The bond plays the role of a milder risky asset: despite being less risky than the stock, and therefore being an intermediate asset between cash and stock, the dynamics of the optimal bond share is similar to that of the stock and different from that of the cash.

Figure 3: Distribution of the final wealth ($X^*(T)$) in three scenarios: with high risk aversion ($\gamma = 1.15\chi_T$ – top graph), with medium risk aversion ($\gamma = 1.28\chi_T$ – middle graph), and with low risk aversion ($\gamma = 1.5\chi_T$ – lower graph).



Finally, for the three risk profiles we have analysed the distribution of the final wealth and compared it with the target. Fig. 3 reports the three histograms of the final wealth distribution; the mean and the median are also reported, together with the target γ.

In this framework, where $\Phi(0,T)$ has the variance already computed in (34), and with the parameters gathered in Table 1, the probability that the optimal final wealth is higher than the compounded contributions ($\chi_T = 8.43$) is given by

$$\begin{aligned}\mathbb{P}\{X^*(T) \geq \chi_T\} &= \mathbb{P}\left\{\frac{\Phi(0,T) - \mathbb{E}_0[\Phi(0,T)]}{\sqrt{\mathbb{V}_0[\Phi(0,T)]}} \leq \frac{3}{2}\sqrt{\mathbb{V}_0[\Phi(0,T)]}\right\} \\ &= \mathcal{N}\left(\frac{3}{2}\sqrt{\mathbb{V}_0[\Phi(0,T)]}\right) = 0.8920669,\end{aligned}$$

which is valid in any scenario of risk aversion.

Final wealth follows a shifted log-normal distribution (as the model implies) concentrated on the left of the target γ. With higher risk aversion the distribution is more concentrated near γ, while it is more spread out when risk aversion decreases (indeed, the investment strategy is riskier with a lower risk aversion).

Table 2: Statistics of final wealth for 10,000 Monte Carlo simulations

Risk aversion	Mean	25 percentile	Max	Min	Ruin freq.
High	9.06	8.89	9.69	-2.52	1
Medium	9.61	9.28	10.79	-2.92	5
Low	10.54	10	12.65	-19.82	38

Tab. 2 reports, for each risk profile, some statistics of the final wealth out of 10,000 simulations. In general, a lower risk aversion leads to a higher final wealth: the mean of the final wealth is highest (lowest) for low (high) risk aversion and this happens also for the majority of the scenarios: in 75% of the cases the final wealth lies between 10 and 12.65 for low risk aversion, between 8.89 and 9.69 for high risk aversion, the medium risk profile giving intermediate results.

According to these statistics, one could be tempted to choose *a priori* the low risk aversion profile. However, the price to be paid in order to be richer "on average" is the higher ruin probability: 0.01% for the high risk averse (i.e. 1 case of ruin out of $10,000$ simulations) versus 0.5% for the low risk averse (38 cases of ruin). Furthermore, a higher ruin probability is also associated to a longer left tail of wealth distribution, i.e. to worse results in the bad scenarios. Indeed, the minimum final wealth is -2.52 for the high risk averse, -2.92 for the medium risk averse and -19.82 for the low risk averse. Even if this is indeed a quite rare event, investors should be aware of the possibility of starting with initial wealth equal to 1 and ending up with a final wealth of about -20. Therefore, in a practical application of the model, empirical data on the distribution of final wealth should be clearly disclosed to the pension fund member, to help him/her to decide about the subjective level of trade-off between expected wealth and risk that better describes his/her preferences.

5.6 Implementation issues and cut-shares

The optimal investment strategy and the optimal controlled fund analysed in the previous sections are subject to two implementation limits. The first one is the fact that the optimal investment strategy is unconstrained. As a result, the optimal share invested in bond and cash computed in the previous section often fall outside the range $[0, 1]$. Therefore, in many cases the optimal strategy requires borrowing considerable amounts of money and invest it in the bond. Actually, there may exist regulatory limits on the portfolio shares, and short-selling may be forbidden. In practice, practitioners typically set non-negativity weights in investment portfolios (Jagannathan and Ma, 2003).

The second important issue is ruin. Although controllable, the probability of ruin is strictly positive (apart from the degenerate case). A positive ruin probability is likely to be forbidden or unacceptable.

From the mathematical point of view, ruin can be avoided by solving an optimization problem with constraints on the state variable. This, however, results into a significantly higher degree of complexity and does not guarantee a consequent constrained optimal strategy.

A possible way to overcome both practical issues of ruin and a reasonable investment strategy would be to add constraints on the investment strategy itself. Nevertheless, the mathematical difficulty of an optimization problem with constraints on the control variables is enormous too, and closed-form solutions exist only in very special cases which are, accordingly, strongly model dependent. Therefore, the solution to the constrained problem is beyond the scope of the present paper. To the best of our knowledge, the only paper where there are constraints on the investment strategy in the accumulation phase of a DC pension scheme is Di Giacinto et al. (2011), by means of viscosity solutions.

An alternative tractable way to deal with the above-mentioned applicability issues is to adopt suboptimal investment strategies that are *ex-post* constrained to fall in the range $[0, 1]$. The procedure is the following:

- at any time t, if the optimal shares of cash, bond and stock are in $[0, 1]$ no correction is needed on the optimal shares;
- if some of the shares do not belong to $[0, 1]$, then at least one of the share is negative (because they sum up to 1); there are two sub-cases:
  1. two negative shares and one greater than 1: the negative shares are set to 0, while the remaining share is set to 1;
  2. one negative share and two positive shares (this is the most common situation): we set the negative share to 0 and modify the other two by imposing that: (i) they sum up to 1 and (ii) their ratio is equal to the ratio of the optimal shares.

We call "cut-shares" the shares resulting from this procedure. By construction, the cut-shares belong to the range $[0, 1]$ and they sum up to 1. Furthermore, the cut-shares prevent ruin.

Clearly, the cut-shares are suboptimal, and practitioners should be aware of the reduction in mean-variance efficiency when constraints are introduced (Alexander and Baptista, 2006). However, they are good approximations of the optimal shares. Cut-shares of the same type were applied e.g. by Gerrard et al. (2006) and Vigna (2014) in the context of DC pension schemes with a constant interest rate, and they proved to be satisfactory: with respect to the unrestricted case the effect on the final results turned out to be negligible and the controls resulted to be more stable over time.

In Figure 4 the portfolio obtained adopting the cut-shares is shown. In this framework the main risk that must be hedged is the interest rate risk. Accordingly, most of the wealth is invested in the bond (which has the highest correlation with the interest rate), while the remaining wealth is mainly invested in the stock. Of course, the higher the risk aversion, the lower the amount of wealth invested in the stock (for speculative purposes), and the higher the amount invested in the bond (for hedging purposes). When retirement approaches, the stock cut-share is almost unaffected, while there is a big swap between bond and cash. In particular, between 6 and 2 years before retirement (according to the risk aversion), the cut-share of bond starts reducing, and the cut-share of cash starts increasing. Such a swap is less relevant for a lower

risk averse agent (whose need for hedging is lower), while it is definitely relevant for a high risk averse agent whose bond cut-share goes from about 95% to about 35%.

The suboptimality of the cut-shares is underlined by the fact that the average wealth reached with the cut-shares is always lower than the average optimal wealth reached with the optimal shares computed in the paper (see the bottom-right panel of Figure 4). However, the lower mean of final wealth is in general partially compensated by a lower standard deviation of final wealth. In detail, results not displayed here show that with medium risk aversion, the reduction in mean (of final wealth) is about 10%, while the reduction in standard deviation (of final wealth) is about 13%; with low risk aversion, the reduction in mean is about 16%, while the reduction in standard deviation is about 39%. Further simulations show that in the mean-standard deviation plan the suboptimal portfolios lie on a straight line (starting from χ_T) whose slope is lower than that of the efficient frontier.

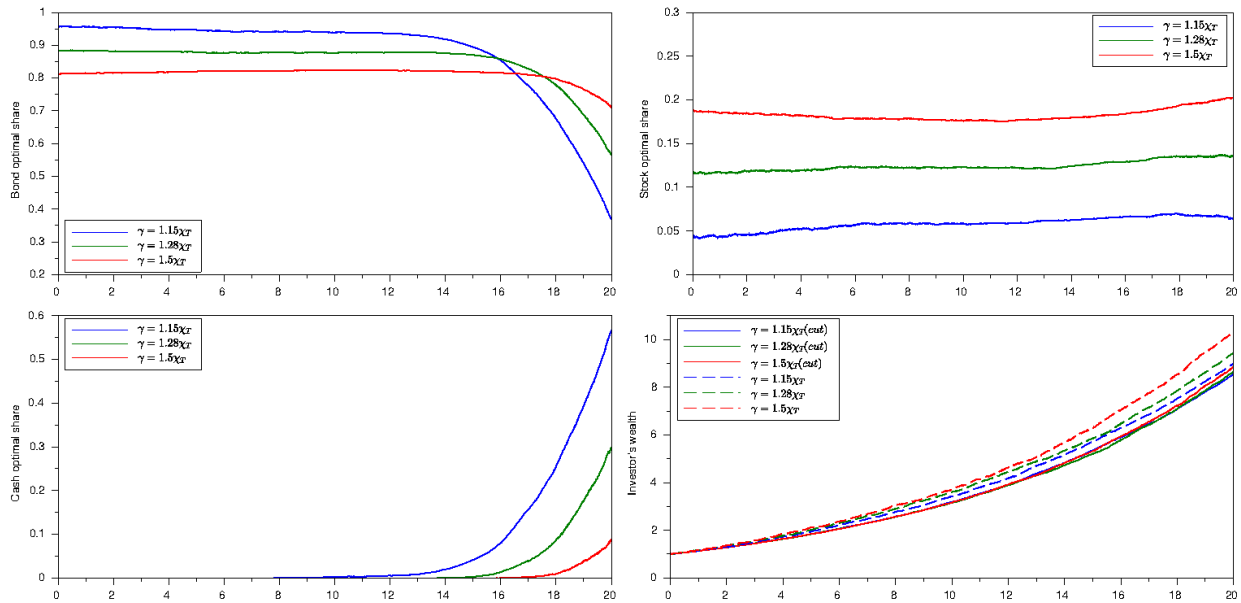
The strategy of the cut-shares is easily implementable, it approximates the optimal mean-variance efficient strategy and reaches the double-goal of no-ruin and feasible strategies. Therefore, the cut-shares strategy can be considered as a reasonable trade-off between the optimal mean-variance efficient strategies of the theoretical model and a realistic implementation of it. Last but not least, the strategies displayed in Figure 4 are in line with the so-called lifestyle strategy, which is widely adopted by DC pension funds in UK (Cairns et al., 2006): the large initial investment in risky assets is progressively switched into cash in the decade prior to retirement. In our framework, since the stock is mainly used for speculative purposes, the remaining of the portfolio must be allocated between a riskier asset (the bond) and the riskless asset. Thus, the amount of the riskier asset is high at the beginning of the period and, during the last period, the portfolio strategy is switched towards the riskless asset.

6 Final remarks

In this paper we have first solved in closed form a mean-variance portfolio problem for a DC pension scheme in a multi-asset complete market with stochastic investment opportunities and stochastic contributions. In such a framework, we have provided a link between the class of mean-variance problems and the class of target-based problems characterised by minimisation of a quadratic loss driven by a target. We have shown a one-to-one correspondence between risk aversion coefficients and targets, providing a suitable financial interpretation. Also the ruin probability is computed in closed form and we show the link between the target and the ruin probability: a higher target can be achieved at the price of a higher ruin probability. Finally, we have investigated a special case with: (i) interest rate following the Vasiček (1977)'s dynamics, (ii) contributions following geometric Brownian motion, and (iii) a financial market consisting of a riskless asset, one bond and one stock.

The interpretation of the mean-variance problem as a target-based problem should make it easier to apply the model to real investment management in DC pension funds. The selection of the correct trade-off between the desired value of the final wealth and the risk that the member is wiling to accept should be

Figure 4: Suboptimal portfolios and wealth for three risk profiles: top-left the bond cut-share, top-right the stock cut-share, bottom-left the cash cut-share, bottom-right the wealth behaviour when strategies are cut (continuous line) or not cut (dashed line)



facilitated by the clear disclosure of tables reporting the distribution of final wealth and the probability of ruin.

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Appendix

Proof of Proposition 1

Recalling (4) and (6), the Lagrangian function of (10) is

$$\mathcal{L} = \mathbb{E}_0 \left[\frac{1}{2} \left(X(T) - \gamma \right)^2 + \lambda X(T) e^{-\int_0^T r(u,z)du} m(0,T) \right] - \lambda x_0 - \lambda \int_0^T \mathbb{E}_0^{\mathbb{F}_s} \left[c(s,z) \right] B(0,s)\, ds,$$

where λ is the Lagrangian multiplier. The derivative of $\mathcal{L}$ with respect to $X(T)$ must be set to zero for each state of the world, i.e.

$$X^*(T) = \gamma - \lambda e^{-\int_0^T r(u,z)du} m(0,T). \tag{36}$$

Now, λ is computed from the constraint in (10) where $X^*(T)$ is substituted, and the inequality is replaced by the equality (since we want the solution to be compatible with the minimum amount of initial wealth):

$$\lambda = \frac{\gamma B(0,T) - \int_0^T \mathbb{E}_0^{\mathbb{F}_s}[c(s,z)] B(0,s)\, ds - x_0}{\mathbb{E}_0\left[e^{-2\int_0^T r(u)du} m^2(0,T)\right]}.$$

By defining the stochastic process $\Phi(t,T)$ as in (12), and noting that $\mathbb{E}_t\left[e^{\Phi(t,T)}\right] = B(t,T)$, then the optimal wealth can be written as in (11).

Proof of Proposition 2

In the optimal solution, the constraint (10) must hold at any instant in time:

$$X^*(t) = -\int_t^T \mathbb{E}_t^{\mathbb{F}_s}[c(s,z)] B(t,s)\, ds + \mathbb{E}_t\left[X^*(T) e^{-\int_t^T r(u,z)du} m(t,T)\right].$$

If the optimal final wealth (36) is plugged into this equation we have:

$$X^*(t) = -\int_t^T \mathbb{E}_t^{\mathbb{F}_s}[c(s,z)] B(t,s)\, ds + \gamma B(t,T) - \lambda m(0,t) e^{-\int_0^t r(u,z)du} \mathbb{E}_t\left[e^{2\Phi(t,T)}\right]. \tag{37}$$

Now, the passages are as follows: (i) $dX^*(t)$ is found through Itō's lemma on (37) (differentiating w.r.t. $m(0,t)$ and $z(t)$), (ii) $\lambda m(0,t) e^{-\int_0^t r(u,z)du} \mathbb{E}_t\left[e^{2\Phi(t,T)}\right]$ is substituted into the diffusion term of $dX^*(t)$ from (37), and (iii) this diffusion term is set equal to the diffusion term of investor's equation in (7) in order to find the portfolio which replicates the optimal wealth. Such a portfolio is given by (14).

Proof of Proposition 3

Consider a target-based problem (P_γ) -(9) with $\overline{\gamma} > \chi_T$. Due to Zhou and Li (2000), the corresponding optimal investment strategy, given by (14), is also the optimal investment strategy of the mean-variance problem (8) with $\overline{\alpha}$ satisfying (24) where $\gamma = \overline{\gamma}$. *Vice versa*, given a mean-variance problem (P_α)-(8) with $\overline{\alpha} > 0$, Zhou and Li (2000) demonstrate that its solution coincides with the solution of the associated (P_γ) problem, with $\overline{\gamma}$ satisfying (24) where $\alpha = \overline{\alpha}$.

Proof of Proposition 4

Given (33), the coefficient of the optimal portfolio third component is (recall (15)):

$$\begin{bmatrix} 2\frac{\partial}{\partial r(t)}\left(\mathbb{E}_t\left[\Phi\left(t,T\right)\right]+\mathbb{V}_t\left[\Phi\left(t,T\right)\right]\right) \\ 2\frac{\partial}{\partial c(t)}\left(\mathbb{E}_t\left[\Phi\left(t,T\right)\right]+\mathbb{V}_t\left[\Phi\left(t,T\right)\right]\right) \end{bmatrix} = \begin{bmatrix} -2\frac{1-e^{-a(T-t)}}{a} \\ 0 \end{bmatrix} = \begin{bmatrix} -2g\left(t,T\right) \\ 0 \end{bmatrix}.$$

The other components of the optimal portfolio are (recall (31) and (32)):

$$\begin{bmatrix} \frac{\partial\left(\gamma B(t,T)-\int_t^T \mathbb{E}_t^{\mathbb{F}_s}[c(s)]B(t,s)ds\right)}{\partial r(t)} \\ \frac{\partial\left(\gamma B(t,T)-\int_t^T \mathbb{E}_t^{\mathbb{F}_s}[c(s)]B(t,s)ds\right)}{\partial c(t)} \end{bmatrix} = \begin{bmatrix} -\gamma g\left(t,T\right)B\left(t,T\right)+\int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c\left(s\right)\right]g\left(t,s\right)B\left(t,s\right)ds \\ -\int_t^T e^{(\mu_c-\sigma_{cr}\xi_r-g(0,K)\sigma_r\sigma_{cr}-\sigma_{cs}\xi_s)(s-t)}B\left(t,s\right)ds \end{bmatrix}.$$

Thus, we can compute

$$\left(\Sigma'\right)^{-1}\xi = \frac{1}{-g\left(0,K\right)\sigma_r\sigma_s}\begin{bmatrix} \sigma_s\xi_r-\sigma_{sr}\xi_s \\ -g\left(0,K\right)\sigma_r\xi_s \end{bmatrix}, \quad \left(\Sigma'\right)^{-1}\Omega' = \frac{1}{-g\left(0,K\right)\sigma_r\sigma_s}\begin{bmatrix} \sigma_s\sigma_r & c\left(t\right)\left(\sigma_s\sigma_{cr}-\sigma_{sr}\sigma_{cs}\right) \\ 0 & -g\left(0,K\right)c\left(t\right)\sigma_r\sigma_{cs} \end{bmatrix},$$

and plug them into (14) for obtaining the desired optimal portfolio.

Proof of Proposition 5

In (35), it is easy to show that $\gamma B\left(t,T\right)-\int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c\left(s\right)\right]B\left(t,s\right)ds-X^*\left(t\right)$ follows a geometric Brownian motion whose initial value $\gamma B\left(0,T\right)-\int_0^T \mathbb{E}_0^{\mathbb{F}_s}\left[c\left(s\right)\right]B\left(0,s\right)ds-x_0$ is positive due to (25).

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