Cohen class of time-frequency representations and operators: boundedness and uncertainty principles

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Paolo Boggiatto*, Evanthia Carypis† and Alessandro Oliaro‡
Department of Mathematics, University of Torino, Italy

Abstract

This paper presents a proof of an uncertainty principle of Donoho-Stark type involving ε-concentration of localization operators. More general operators associated with time-frequency representations in the Cohen class are then considered. For these operators, which include all usual quantizations, we prove a boundedness result in the L^p functional setting and a form of uncertainty principle analogous to that for localization operators.

Keywords: Uncertainty Principles; Time-Frequency Representations; Pseudo-differential Operators.

1 Introduction

Uncertainty principles (UP) appear in harmonic analysis and signal theory in a variety of different forms involving not only the couple $(f, \widehat{f})$ formed by a signal (function or distribution) and its Fourier transform, but essentially every representation of a signal in the time-frequency space. Among the wide literature on this topic we refer for example to [2, 3, 4, 8, 11, 14, 15, 17, 19, 21].

In this paper we consider the case where the couple $(f, \widehat{f})$ is substituted by a couple $(T_1 f, T_2 f)$, where T_1, T_2 are operators by which, in some sense, the concentration of the signal f is "tested". The consequent uncertainty statement is then of the following type: if the tests yield functions which are sufficiently concentrated on some domains of the time-frequency space, then the Lebesgue measure of these domains can not be "too small".

*e-mail: paolo.boggiatto@unito.it
†e-mail: evanthia.carypis@unito.it
‡e-mail: alessandro.oliaro@unito.it

We make now precise the type of operators that are used, in which sense "concentration" is intended, and what is meant by "too small".

The class of operators that we consider is strictly connected with the *Cohen class* of time-frequency representations, which consists of sesquilinear forms of the type

$$Q_\sigma(f,g)(x,\omega) = \sigma * \mathrm{Wig}(f,g)(x,\omega), \tag{1}$$

where

$$\mathrm{Wig}(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot t} f\left(x+t/2\right) \overline{g\left(x-t/2\right)}\, dt \tag{2}$$

is the *Wigner transform* and σ is the *Cohen kernel*. We shall shortly write $Q_\sigma(f)$ for the quadratic form $Q_\sigma(f,f)$. Clearly the signals f,g must be chosen in functional or distributional spaces such that the convolution (1) makes sense.

The Cohen class finds its justification in applied signal analysis as it actually coincides with the class of quadratic *covariant* time-frequency representations. More precisely, let Q be any sesquilinear form (non a priori in the Cohen class); a very natural requirement is that a translation in time $\tau_a f(x) = f(x-a)$ of the signal should reflect into the same translation of its representation along the time-axis, i.e. $Q(\tau_a f)(x,\omega) = Qf(x-a,\omega)$. On the other hand a modulation $\mu_b f(x) = e^{2\pi i b x} f(x)$ should reflect into a translation by the same parameter b along the frequency-axis, i.e. $Q(\mu_b f)(x,\omega) = Qf(x,\omega-b)$. It can be proved that these two requirements, called *covariance property*, actually characterize, under some minor technical hypothesis, the Cohen class among all quadratic representations (see e.g. [18], Thm 4.5.1).

As described in [6], [7], we can associate an operator T^a_σ, depending on a symbol a, with each time-frequency representation Q_σ, by the formula:

$$(T^a_\sigma f, g) = (a, Q_\sigma(g,f)). \tag{3}$$

Formula (3) can be understood, e.g. in the Lebesgue setting, as follows:

$$\begin{aligned} &Q_\sigma : L^q(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \to L^r(\mathbb{R}^{2d}), \\ &T_\sigma : a \in L^{r'}(\mathbb{R}^{2d}) \to B\big(L^p(\mathbb{R}^d), L^{q'}(\mathbb{R}^d)\big), \end{aligned}$$

where $1 < q < \infty$, $1 < r \leq \infty$, $1 \leq p \leq \infty$, and $\frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1$. For simplicity we write $\mathrm{Wig}\,(f)$ and $Q_\sigma(f)$ when $f = g$.
More generally, if $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, formula (3) defines a continuous linear map $T^a_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, and actually it establishes a bijection between operators and sesquilinear forms, we refer to [6] for details and general functional

settings. The operators T^a_σ, obtained by (3) in correspondence with representations Q_σ in the Cohen class, will be called *Cohen operators*. Referring to (3), we actually remark that $(T^a_\sigma f, g) = (a, Q_\sigma(g, f)) = ((a * \overline{\widetilde{\sigma}}, \mathrm{Wig}(g, f)))$, therefore, viewed as operators independently of quantization rules, all Cohen operators are Weyl operators (cfr. equation (9) and Proposition 14 (a)).

Due however to the freedom in the choice of the Cohen kernel σ, we recapture by (3) all types of quantizations used in pseudo-differential calculus (Weyl, Kohn-Nirenberg, localization, etc.). A particular family of operators of this kind is considered in [1], see Remark 13.

When the symbol a is the characteristic function of a measurable set in $\mathbb{R}^{2d}$ it is natural to look at Cohen operators as a generalized way of expressing the concentration of energy. In this spirit we shall consider couples of these operators applied to a signal f as the substitute for the couple $(f, \widehat{f})$ in the formulations of the UP of Donoho-Stark type in Sections 3 and 5. More precisely in Section 3 we shall consider the particular case of *localization operators*, see (8), correcting a flaw in the estimate of a Donoho-Stark type UP appearing in [5], whereas in Section 5 a similar UP in the general case of Cohen operators is presented. Sections 2 and 4 are dedicated to some L^p-boundedness results for Wigner (and Gabor) transforms and for general Cohen class operators respectively, which are preliminary to the results of the corresponding following sections.

Although a vast literature is available on L^p-boundedness, the norm estimates of Section 2 improve existing results as found in [9] and [27], and those in Section 4 furnish extensions of results for Weyl operators to Cohen operators.

Concerning the meaning of "concentration" and "not too small" sets we refer to the classical Donoho-Stark UP which we recall next (see e.g. [18], Thm. 2.3.1).

Definiton 1. Given $\varepsilon \geq 0$, a function $f \in L^2(\mathbb{R}^d)$ is *ε-concentrated* on a measurable set $U \subseteq \mathbb{R}^d$ if

$$\Big(\int_{\mathbb{R}^d \setminus U} |f(x)|^2 dx \Big)^{1/2} \leq \varepsilon \|f\|_2.$$

Theorem 2 (Donoho-Stark)**.** *Suppose that $f \in L^2(\mathbb{R}^d)$, $f \neq 0$, is ε_T-concentrated on $T \subseteq \mathbb{R}^d$, and $\widehat{f}$ is ε_Ω-concentrated on $\Omega \subseteq \mathbb{R}^d$, with T, Ω measurable sets in $\mathbb{R}^d$ and $\varepsilon_T, \varepsilon_\Omega \geq 0$, $\varepsilon_T + \varepsilon_\Omega \leq 1$. Then*

$$|T||\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2. \tag{4}$$

As showed by Remark 13, estimate (4) is actually improved by Propositon 12 of Section 2.

2 L^p-continuity of the Gabor and Wigner distributions

Sharp L^p-boundedness estimates for the Gabor transform (or *short-time Fourier transform*, STFT)

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot t} f(t)\overline{g(t-x)}dt$$

with window g, and applied to a signal f, shall be needed later on in this paper. They are consequence of Young's inequality with optimal constants (Babenko-Beckner constants) which we recall here: if $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ and $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$, then $f*g \in L^r(\mathbb{R}^d)$ and

$$\|f*g\|_r \leq (A_p A_q A_{r'})^d \|f\|_p \|g\|_q,$$

where $A_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$.

The boundedness of Gabor and Wigner transform has been widely studied, in several functional settings, cf. for example [6], [13], [25], [26], [28]. Here we focus on Legesgue spaces, for which an estimate with sharp constant can be found for the one-dimensional case in [23], whereas a more general result but with no sharp estimate is proved in [6] (Proposition 3.1). We improve here the boundedness result of [6] with an estimation of the constant of the type of [23].

For $2 \leq p < +\infty$, $1 \leq q < +\infty$, $p \geq \max\{q, q'\}$, let us set

$$H(p,q) = \left(\frac{q}{p}\right)^{d/p} \frac{(p-q)^{(p-q)d/2pq}(qp-p-q)^{(qp-p-q)d/2pq}}{(q-1)^{(q-1)d/2q}(p-2)^{(p-2)d/2p}}. \tag{5}$$

As it is easily verified, for every $q \in [1,+\infty)$, we have $\lim_{p\to+\infty} H(p,q) = 1$, therefore it is convenient to extend (5) by setting $H(q,+\infty)=1$, and also $H(+\infty,+\infty)=1$. With this agreement we have the following result.

Theorem 3. *Let $2 \leq p \leq \infty$ and $p' \leq q \leq p$ (so that also $p' \leq q' \leq p$). Then the Gabor transform defines a bounded sesquilinear map*

$$V : (f,g) \in L^q(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d) \to V_g f \in L^p(\mathbb{R}^{2d}),$$

and

$$\|V_g f\|_p \leq H(p,q)\|f\|_q \|g\|_{q'}. \tag{6}$$

Proof. The case $p = +\infty$ is a trivial application of Young's inequality which immediately yields the estimate $\|V_g f\|_\infty \leq \|f\|_q \|g\|_{q'}$. Suppose now that $p < +\infty$. We recall that the Gabor transform can be written as Fourier transform of a product (cf. [18], Thm. 3.3.2), namely $V_g f(x,\omega) = (f \cdot \tau_x \overline{g})\widehat{\ }(\omega)$, where the translation operator τ_x is defined as $\tau_x h(t) := h(t-x)$. We have the following estimation:

$$\begin{aligned}
\|V_g f\|_p &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |(f \cdot \tau_x \overline{g})\widehat{\ }(\omega)|^p d\omega \right)^{\frac{1}{p} \cdot p} dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^d} \|(f \cdot \tau_x \overline{g})\widehat{\ }(\omega)\|_p^p dx \right)^{1/p} \\
&\leq \left(\int_{\mathbb{R}^d} A_{p'}^{pd} \|f \cdot \tau_x \overline{g}\|_{p'}^p dx \right)^{1/p} \\
&= A_{p'}^d \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)|^{p'} |\widetilde{\overline{g}}(x-y)|^{p'} dy \right)^{p/p'} dx \right)^{1/p} \\
&= A_{p'}^d \left(\int_{\mathbb{R}^d} \left(|f|^{p'} * |\widetilde{\overline{g}}|^{p'}(x) \right)^{p/p'} dx \right)^{\frac{1}{p}\frac{p'}{p'}} \\
&= A_{p'}^d \| |f|^{p'} * |\widetilde{\overline{g}}|^{p'} \|_{p/p'}^{1/p'},
\end{aligned}$$

where $\widetilde{g}(x) = g(-x)$. Let us now apply the Young's inequality to $|f|^{p'}$ and $|\widetilde{\overline{g}}|^{p'}$ which are in $L^{\frac{q}{p'}}$ and $L^{\frac{q'}{p'}}$ respectively, and denote $s = \frac{q}{p'} = \frac{q(p-1)}{p}$, $t = \frac{q'}{p'} = \frac{q(p-1)}{p(q-1)}$, and $r = \frac{p}{p'} = p-1$. Then

$$\begin{aligned}
A_{p'}^d \| |f|^{p'} * |\widetilde{\overline{g}}|^{p'} \|_{p/p'}^{1/p'} &\leq A_{p'}^d \left(A_{\frac{q(p-1)}{p(q-1)}} A_{\frac{q(p-1)}{p}} A_{\frac{p-1}{p-2}} \right)^{d/p'} \| |f|^{p'} \|_s^{1/p'} \| |\widetilde{\overline{g}}|^{p'} \|_t^{1/p'} \\
&= H(p,q) \|f\|_q \|g\|_{q'},
\end{aligned}$$

where we have

$$\begin{aligned}
H(p,q) &= A_{p'}^d \left(A_{\frac{q(p-1)}{p(q-1)}} A_{\frac{q(p-1)}{p}} A_{\frac{p-1}{p-2}} \right)^{d/p'} \\
&= \left(\frac{q}{p} \right)^{d/p} \frac{(p-q)^{(p-q)d/2pq} (qp-p-q)^{(qp-p-q)d/2pq}}{(q-1)^{(q-1)d/2q} (p-2)^{(p-2)d/2p}}.
\end{aligned}$$

□

Remark 4. For $q = 2$, we have $H(p,2) = \left(\frac{2}{p}\right)^{d/p}$, which is the constant appearing in [18] (Sec. 3.3). In the case $q = p$ the constant becomes $H(p,q) = H(p) = \left(\frac{p^{(p-2)}}{(p-1)^{p-1}}\right)^{d/2p} = \left(\frac{p'^{1/p'}}{p^{1/p}}\right)^{d/2}$, which is the Babenko-Beckner constant $A_{p'}$.

In view of the well-known formula

$$\mathrm{Wig}\,(f,g)(x,\omega) = 2^d e^{4\pi i x\cdot\omega} V_{\widetilde{g}} f(2x, 2\omega) \tag{7}$$

(see for instance [18], Lemma 4.3.1), we have $\|\,\mathrm{Wig}\,(f,g)\|_p = 2^{\frac{p-2}{p}d}\|V_{\widetilde{g}} f\|_p$ and any L^p- boundedness result for the Gabor transform automatically transfers to a corresponding result for the Wigner transform. More explicitly:

Proposition 5. *For $2 \leq p \leq +\infty$ and $p' \leq q \leq p$, the Wigner transform is a bounded map*

$$\mathrm{Wig}\,: L^q(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$$

and

$$\|\,\mathrm{Wig}\,(f,g)\|_p \leq C(p,q)\|f\|_q\|g\|_{q'},$$

where $C(p,q) = 2^{\frac{p-2}{p}d} H(p,q)$, and $H(p,q)$ is defined by (5).

Remark 6. In the particular case $q = p$, Proposition 5 reads $\mathrm{Wig}\,(f,g)\|_p \leq 2^{-d}4^{d/p}A_p^d\|f\|_p\|g\|_{p'}$, a result which appears in Wong [27].

We also recall that the boundedness result of Proposition 5 without estimation of the boundedness constant appears in [6], Prop 3.1. From Prop. 3.2 of the same paper [6] one can further deduce that in the remaining cases for q and p we do not have boundedness. We can therefore summarize the situation as follows

Proposition 7. *Let $p, q \in [1,\infty]$, then*

$$\mathrm{Wig}\,: L^q(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$$

is bounded if and only if $p' \leq q \leq p$. (Remark that this means no boundedness for $p < 2$).

Motivated by Proposition 7, it is natural to investigate the cases

$$\mathrm{Wig}\,: L^r(\mathbb{R}^d) \times L^s(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$$

where not necessarily r, s are conjugate indices and, along these lines, we present next a discussion that will yield to a complete characterization of the cases of boundedness of the Wigner transform on Lebesgue spaces, which, although not strictly needed in the following sections, in our opinion has an interest in itself. The same results, with suitably adapted constants, hold for the Gabor transform.

We start by the case of non-boundedness on the diagonal of the indices space.

Proposition 8. *Let $p \in [1, \infty]$, then the map* $\mathrm{Wig} : L^q(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$ *is not continuous if $q \neq 2$.*

Proof. Consider a function $f \in \mathcal{S}(\mathbb{R}^d)$ and let $f_\lambda(x) = f(\lambda x)$ be the corresponding dilation by $\lambda > 0$. Let $\varphi_\lambda(x) = \frac{f_\lambda(x)}{\|f_\lambda\|_2}$ be the normalization of f_λ, then an easy computation gives

$$\|\varphi_\lambda\|_q = \frac{\lambda^{\left(\frac{1}{2}-\frac{1}{q}\right)d}}{\|f\|_2}\|f\|_q.$$

Given f as above,

$$\mathrm{Wig}\,(\varphi_\lambda)(x, \omega) = \frac{1}{\|f\|_2^2}\,\mathrm{Wig}\,(f)\left(\lambda x, \frac{\omega}{\lambda}\right),$$

and $\mathrm{Wig}\,(\varphi_\lambda) \in \mathcal{S}(\mathbb{R}^{2d}) \subset L^p(\mathbb{R}^{2d})$. However, $\|\varphi_\lambda\|_q \to 0$ for $\lambda \to +\infty$ if $q < 2$, whereas $\|\,\mathrm{Wig}\,(\varphi_\lambda)\|_p$ is constant, as

$$\|\,\mathrm{Wig}\,(\varphi_\lambda)\|_p = \frac{\|\,\mathrm{Wig}\,(f)\|_p}{\|f\|_2^2}.$$

Analogously, $\|\varphi_\lambda\|_q \to 0$ for $\lambda \to 0$ if $q > 2$, but $\|\,\mathrm{Wig}\,(\varphi_\lambda)\|_p$ still remain constant. Hence, we cannot have continuity of $\mathrm{Wig} : L^q \times L^q \to L^p$ for $q \neq 2$. □

We extend now the non-boundedness result to the general case.

Proposition 9. *Let $r, s, p \in [1, \infty]$, with $r \neq s'$, then the map* $\mathrm{Wig} : L^r(\mathbb{R}^d) \times L^s(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$ *is not continuous.*

Proof. Suppose, on the contrary, that there exists p and $r \neq s'$ such that

$$\mathrm{Wig} : L^r(\mathbb{R}^d) \times L^s(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$$

is bounded. From the sesquilinearity of Wig, we would then also have a bounded map

$$\mathrm{Wig}\ : L^s(\mathbb{R}^d) \times L^r(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d}).$$

Well-known interpolation theorems would therefore yield a bounded map

$$\mathrm{Wig}\ : L^{(\frac{\theta}{r}+\frac{1-\theta}{s})^{-1}}(\mathbb{R}^d) \times L^{(\frac{\theta}{s}+\frac{1-\theta}{r})^{-1}}(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$$

for any $\theta \in [0,1]$. For $\theta = 1/2$ we would then obtain boudnedness in the case

$$\mathrm{Wig}\ : L^q(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$$

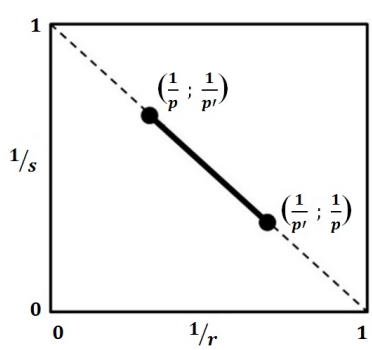
with $\frac{1}{q} = \frac{1}{2}(\frac{1}{r} + \frac{1}{s})$, which contradicts Proposition 8. $\square$

Proposition 7 together with Proposition 9 finally yield the following proposition which provides the complete picture of the situation, and is more conveniently expressed in geometric terms.

Proposition 10. *Let $r, s, p \in [1, \infty]$, then*

$$\mathrm{Wig}\ : L^r(\mathbb{R}^d) \times L^s(\mathbb{R}^d) \to L^p(\mathbb{R}^{2d})$$

is bounded if and only if the point $(\frac{1}{r}; \frac{1}{s})$ lies on the segment $\left[(\frac{1}{p}; \frac{1}{p'}), (\frac{1}{p'}; \frac{1}{p})\right]$, which should be considered empty for $p < 2$, as shown by the following picture.



3 A Donoho-Stark UP for localization operators

We revise in this section the content of Sections 2 and 3 of [5] correcting a flaw in the proof of Theorem 6, which expresses a Donoho-Stark uncertainty principle in terms of localization operators and leads to an improvement of the classical Donoho-Stark estimate, which now we specify correctly.

Let us first recall that a localization operator is a map of the type:

$$f \longrightarrow L^a_{\phi,\psi} f = \int_{\mathbb{R}^{2d}} a(x,\omega) V_\phi f(x,\omega)\, \mu_\omega \tau_x \psi \, dx d\omega \tag{8}$$

acting on $L^2(\mathbb{R}^d)$, with symbol $a \in L^q(\mathbb{R}^{2d})$, for $q \in [1,\infty]$, and "window" functions $\phi, \psi \in L^2(\mathbb{R}^d)$. Here $\mu_\omega \tau_x \psi(t) = e^{2\pi i \omega t}\psi(t-x)$ are *time-frequency shifts* of $\psi(t)$. We refer to the available vast literature (e.g. [9, 10, 15, 16, 18, 29]) for the motivations and the meaning of these operators in time-frequency and harmonic analysis, as well as for extensions to more general functional settings. Their use in the Donoho-Stark UP relies on the following boundedness estimate, which appears in [5] (Lemma 4).

Lemma 11. *Let $\phi, \psi \in L^2(\mathbb{R}^d)$, $q \in [1,\infty]$ and consider the quantization (see (8)):*

$$L_{\phi,\psi} : a \in L^q(\mathbb{R}^{2d}) \to L^a_{\phi,\psi} \in B(L^2(\mathbb{R}^d)).$$

Then the following estimation holds

$$\|L^a_{\phi,\psi}\|_{B(L^2)} \leq \left(\frac{1}{q'}\right)^{d/q'} \|\phi\|_2 \|\psi\|_2 \|a\|_q,$$

with $\frac{1}{q} + \frac{1}{q'} = 1$, and setting $(\frac{1}{q'})^{\frac{1}{q'}} = 1$ for $q = 1$.

We shall use the previous result to obtain an uncertainty principle involving localization operators in the special case where the symbol is the characteristic function of a set, expressing therefore *concentration* of energy on this set when applied to signals in $L^2(\mathbb{R}^d)$. In this case they are also known as *concentration operators*. The proof makes use of some tools from the pseudo-differential theory which we now recall in the L^2 functional framework, for more general settings and references see e.g. [7], [20], [30].

Given $f, g \in L^2(\mathbb{R}^d)$ we can associate an operator to the Wigner transform by using relation (3), and we call it *Weyl pseudo-differential operators*:

$$(W^a f, g)_{L^2(\mathbb{R}^d)} = (a, \mathrm{Wig}(g,f))_{L^2(\mathbb{R}^{2d})}, \tag{9}$$

where $a \in L^2(\mathbb{R}^{2d})$. More explicitly this is a map of the type

$$f \in L^2(\mathbb{R}^d) \longrightarrow W^a f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\omega} a\left(\frac{x+y}{2}, \omega\right) f(y)\, dy\, d\omega \in L^2(\mathbb{R}^d).$$

The fundamental connection between Weyl and localization operators is expressed by the formula which yields localization operators in terms of Weyl operators:

$$L^a_{\phi,\psi} = W^b, \qquad \text{with} \;\; b = a * \mathrm{Wig}(\widetilde{\psi}, \widetilde{\phi}), \tag{10}$$

with $\psi, \phi \in L^2(\mathbb{R}^d)$ and where, for a generic function $u(x)$, we use the notation $\widetilde{u}(x) = u(-x)$.
Of particular importance for our purpose will be the fact that Weyl operators with symbols $a(x,\omega)$ depending only on x, or only on ω, are multiplication operators, or Fourier multipliers respectively. More precisely we have

$$\begin{array}{lll} a(x,\omega) = a(x) & \Longrightarrow & W^a f(x) = a(x) f(x) \\ a(x,\omega) = a(\omega) & \Longrightarrow & W^a f(x) = \mathcal{F}^{-1}[a(\omega)\widehat{f}(\omega)](x). \end{array} \tag{11}$$

We fix now some notations.
Let $T \subseteq \mathbb{R}^d_x$, $\Omega \subseteq \mathbb{R}^d_\omega$ be measurable sets, and write for shortness $\chi_T = \chi_{T\times\mathbb{R}^d}$ and $\chi_\Omega = \chi_{\mathbb{R}^d\times\Omega}$, in such a way that $\chi_T = \chi_T(x)$ and $\chi_\Omega = \chi_\Omega(\omega)$.
For $\lambda > 0$, let $h_\lambda(x) = e^{-\pi\lambda x^2}$, $\Phi_\lambda = h_\lambda/\|h_\lambda\|_2$, $\varphi_\lambda = h_\lambda/\|h_\lambda\|_1$.
Moreover, for $\lambda_1, \lambda_2 > 0$, let

$$L_1 f = L^{\chi_T}_{\Phi_{\lambda_1}} f = \int_{\mathbb{R}^{2d}} \chi_T(x) V_{\Phi_{\lambda_1}} f(x,\omega) \mu_\omega \tau_x \Phi_{\lambda_1} dx d\omega \tag{12}$$

and

$$L_2 f = L^{\chi_\Omega}_{\Phi_{\lambda_2}} f = \int_{\mathbb{R}^{2d}} \chi_\Omega(\omega) V_{\Phi_{\lambda_2}} f(x,\omega) \mu_\omega \tau_x \Phi_{\lambda_2} dx d\omega \tag{13}$$

be the two localization operators with symbols χ_T, χ_Ω and windows $\Phi_{\lambda_1}, \Phi_{\lambda_2}$ respectively. We can state now the main result of this section which is an UP involving the ε-concentration of these two localization operators and is the corrected version of [5], Thm. 6.

Theorem 12. *With the previous assumptions on T, Ω, L_1, L_2, suppose that $\varepsilon_T, \varepsilon_\Omega > 0$, $\varepsilon_T + \varepsilon_\Omega \leq 1$, and that $f \in L^2(\mathbb{R}^d)$ is such that*

$$\|L_1 f\|_2^2 \geq (1-\varepsilon_T^2)\|f\|_2^2 \quad \textit{and} \quad \|L_2 f\|_2^2 \geq (1-\varepsilon_\Omega^2)\|f\|_2^2. \tag{14}$$

Then

$$|T||\Omega| \geq \sup_{r\in[1,\infty)} (1-\varepsilon_T-\varepsilon_\Omega)^{2r}(2r)^{-d}\left(\frac{(r+1)^{r+1}}{(r-1)^{r-1}}\right)^{d/2}. \tag{15}$$

Proof. Writing the operators L_j, $j = 1, 2$, defined in (12) and (13) as Weyl operators we have:

$$L_1 f = W^{F_1} f, \quad \text{with} \quad F_1(x,\omega) = (\chi_T(x) \otimes 1_\omega) * \mathrm{Wig}\,(\Phi_{\lambda_1})(x,\omega)$$

$$L_2 f = W^{F_2} f, \quad \text{with} \quad F_2(x,\omega) = (1_x \otimes \chi_\Omega(\omega)) * \mathrm{Wig}\,(\Phi_{\lambda_2})(x,\omega).$$

An explicit calculation yields:

$$\mathrm{Wig}(\Phi_{\lambda_j})(x,\omega) = \varphi_{2\lambda_j}(x)\varphi_{2/\lambda_j}(\omega), \quad (j = 1, 2).$$

therefore we have

$$F_1(x,\omega) = (\chi_T * \varphi_{2\lambda_1})(x),$$
$$F_2(x,\omega) = (\chi_\Omega * \varphi_{\frac{2}{\lambda_2}})(\omega)$$

in particular, F_1 depends only on x, and F_2 only on ω.

It follows that

$$L_1 f = W^{F_1} f = F_1 f,$$

i.e. L_1 is the multiplication operator by the function F_1 and

$$L_2 f = W^{F_2} f = \mathcal{F}^{-1} F_2 \mathcal{F} f,$$

i.e. L_2 is the Fourier multiplier with symbol F_2. Now, for $j = 1, 2$, we compute

$$\begin{aligned} \|f\|_2^2 &= \|(f - L_j f) + L_j f\|_2^2 \\ &= ((f - L_j f) + L_j f, (f - L_j f) + L_j f) \\ &= \|f - L_j f\|_2^2 + \|L_j f\|_2^2 + (f - L_j f, L_j f) + (L_j f, f - L_j f). \end{aligned} \tag{16}$$

Next we show that $(f - L_j f, L_j f) \geq 0$. For $j = 1$ we have

$$\begin{aligned} (f - L_1 f, L_1 f) &= (f, L_1 f) - (L_1 f, L_1 f) \\ &= \int f \overline{F_1 f} - \int F_1 f \overline{F_1 f} \\ &= \int (1 - F_1) \overline{F_1} |f|^2 \geq 0, \end{aligned}$$

as $F_1 = \chi_T * \varphi_{2\lambda_1}$ is real, non negative, and $\|F_1\|_\infty \leq \|\chi_T\|_\infty \|\varphi_{2\lambda_1}\|_1 = 1$.

Analogously, if $j = 2$ we have

$$\begin{aligned} (f - L_2 f, L_2 f) &= (f, L_2 f) - (L_2 f, L_2 f) \\ &= \big(f, \mathcal{F}^{-1} F_2 \mathcal{F} f\big) - \big(\mathcal{F}^{-1} F_2 \mathcal{F} f, \mathcal{F}^{-1} F_2 \mathcal{F} f\big) \\ &= (\widehat{f}, F_2 \widehat{f}) - (F_2 \widehat{f}, F_2 \widehat{f}) \\ &= \int \widehat{f} \overline{F_2 \widehat{f}} - \int F_2 \widehat{f} \overline{F_2 \widehat{f}} \\ &= \int (1 - F_2) \overline{F_2} |\widehat{f}|^2 \geq 0, \end{aligned}$$

as $F_2 = (\chi_\Omega * \varphi_{\frac{2}{\lambda_2}})$ is real, non negative, and $\|F_2\|_\infty \leq \|\chi_\Omega\|_\infty \|\varphi_{2/\lambda_2}\|_1 = 1$.

Now, from (16), since $(f - L_j f, L_j f) \geq 0$, it follows

$$\|f\|_2^2 = \|f - L_j f\|_2^2 + \|L_j f\|_2^2 + 2(f - L_j f, L_j f)$$

and hence

$$\|f - L_j f\|_2^2 \leq \|f\|_2^2 - \|L_j f\|_2^2. \tag{17}$$

From the hypothesis and (17) we obtain

$$\begin{cases} \|f - L_1 f\|_2^2 \leq \varepsilon_T^2 \|f\|_2^2 & , \\ \|f - L_2 f\|_2^2 \leq \varepsilon_\Omega^2 \|f\|_2^2 & . \end{cases}$$

Considering the composition of L_1 and L_2 we have

$$\begin{aligned} \|f - L_2 L_1 f\|_2 &\leq \|f - L_2 f\|_2 + \|L_2 f - L_2 L_1 f\|_2 \\ &\leq \varepsilon_\Omega \|f\|_2 + \|L_2\| \|f - L_1 f\|_2 \\ &\leq \varepsilon_\Omega \|f\|_2 + 1 \cdot \varepsilon_T \|f\|_2 \\ &= (\varepsilon_\Omega + \varepsilon_T) \|f\|_2, \end{aligned}$$

where the estimation of the operator norm $\|L_2\|_{B(L^2)} \leq \|\Phi_{\lambda_2}\|_2^2 \|\chi_\Omega\|_\infty = 1$ is obtained applying Lemma 11 with $q = \infty$, or directly. Then

$$\begin{aligned} \|L_2 L_1 f\|_2 &\geq \|f\|_2 - \|f - L_2 L_1 f\|_2 \\ &\geq \|f\|_2 - (\varepsilon_\Omega + \varepsilon_T) \|f\|_2 \\ &= (1 - \varepsilon_T - \varepsilon_\Omega) \|f\|_2, \end{aligned} \tag{18}$$

We look now for an upper estimate of $\|L_2 L_1 f\|$. For $r, k \in [1, +\infty)$ we have:

$$\begin{aligned} \|L_2 L_1 f\|_2 = \|F_2 \cdot \widehat{L_1 f}\|_2 &\leq \|F_2\|_{2r} \|\widehat{L_1 f}\|_{2r'} \\ &\leq A^d_{(2r')'} \|F_2\|_{2r} \|L_1 f\|_{(2r')'} \\ &\leq A^d_{(2r')'} \left\| \left(\chi_\Omega * \varphi_{2/\lambda_2} \right) \right\|_{2r} \|\chi_T * \varphi_{2\lambda_1}\|_{(2r')'k} \|f\|_{(2r')'k'} \\ &\leq A^d_{(2r')'} \|\chi_\Omega\|_{2r} \left\| \varphi_{2/\lambda_2} \right\|_1 \|\chi_T\|_{(2r')'k} \|\varphi_{2\lambda_1}\|_1 \|f\|_{(2r')'k'}, \end{aligned}$$

where A_p is defined at the beginning of Section 2.

But $\|\varphi_{2/\lambda_2}\|_1 = \|\varphi_{2\lambda_1}\|_1 = 1$ and, choosing $k = r + 1$ so that $(2r')'k' = 2$ and $(2r')'k = 2r$, we have

$$\|L_2 L_1 f\|_2 \leq A^d_{(2r')'} \|\chi_\Omega\|_{2r} \|\chi_T\|_{2r} \|f\|_2.$$

A direct computation of the Babenko constant yields

$$A_{(2r')'} = \left(\frac{(2r)^{1/r} (r-1)^{(r-1)/2r}}{(r+1)^{(r+1)/2r}} \right)^{1/2},$$

and therefore we obtain

$$\begin{aligned}\|L_2L_1f\|_2 &\leq \left(\frac{(2r)^{1/r}(r-1)^{(r-1)/2r}}{(r+1)^{(r+1)/2r}}\right)^{d/2}\|f\|_2\|\chi_\Omega\|_{2r}\|\chi_T\|_{2r}\\ &= \left(\frac{(2r)^{1/r}(r-1)^{(r-1)/2r}}{(r+1)^{(r+1)/2r}}\right)^{d/2}\|f\|_2|\Omega|^{1/2r}|T|^{1/2r}.\end{aligned} \tag{19}$$

Finally from (18) and (19) we obtain that, for every $r \in [1, +\infty)$

$$\begin{aligned}1-\varepsilon_\Omega-\varepsilon_T &\leq \frac{\|L_1L_2f\|_2}{\|f\|_2}\\ &\leq \left(\frac{(2r)^{1/r}(r-1)^{(r-1)/2r}}{(r+1)^{(r+1)/2r}}\right)^{d/2}|\Omega|^{1/2r}|T|^{1/2r},\end{aligned}$$

which yields

$$|T||\Omega| \geq \sup_{r\in[1,+\infty)}(1-\varepsilon_T-\varepsilon_\Omega)^{2r}(2r)^{-d}\left(\frac{(r+1)^{r+1}}{(r-1)^{r-1}}\right)^{d/2}.$$

□

Remark 13. (1) The result involves the couple (L_1f, L_2f) and the rectangle $T \times \Omega$ analogously to the Donoho-Stark UP which involves the couple $(f, \widehat{f})$ and the same rectangle.
(2) Similarly to Lieb UP, but unlike Donoho-Stark UP, the estimate is dependent on the dimension d (and improves by increasing d).
(3) The estimate $|T||\Omega| \geq \sup_{r\in[1,\infty)}(1-\varepsilon_T-\varepsilon_\Omega)^{2r}(2r)^{-d}\left(\frac{(r+1)^{r+1}}{(r-1)^{r-1}}\right)^{d/2}$ is stronger then the classical Donoho-Stark estimate. Indeed, for any choice of ε_T, ε_Ω, the inequality $(1-\varepsilon_T-\varepsilon_\Omega)^{2r}\frac{1}{(2r)^d}\left(\frac{(r+1)^{r+1}}{(r-1)^{r-1}}\right)^{d/2} > (1-\varepsilon_T-\varepsilon_\Omega)^2$ can be rewritten as $(1-\varepsilon_T-\varepsilon_\Omega) > (2r)^{\frac{d}{2(r-1)}}\left(\frac{(r-1)}{(r+1)^{\frac{r+1}{r-1}}}\right)^{\frac{d}{4}}$, whose right-hand side vanishes as $r \to 1^+$. For example if $\varepsilon_T + \varepsilon_\Omega = 0.1$, from (4) we get $|\Omega||T| \geq 0.81$ independently from the dimension d, but from (15) we have $|\Omega||T| \geq 0.9138$ for $r = 1.34$, $d = 1$, and $|\Omega||T| \geq 1.1358$ for $r = 1.6$, $d = 2$, etc.
However from Theorem 12 we can not directly affirm that we have an improvement of the Donoho-Stark estimate because our hypotheses are different. The fact that we actually have an improvement is shown in [5], Prop. 10.

(4) Fixing $r = 1$ we have $(1-\varepsilon_1-\varepsilon_2)^{2r}\frac{1}{(2r)^d}\left(\frac{(r+1)^{r+1}}{(r-1)^{r-1}}\right)^{d/2} = (1-\varepsilon_1-\varepsilon_2)^2$ i.e. the lower bound given by Theorem 12 and that of Donoho-Stark UP coincide.
(5) The hypotheses of Theorem 12 depend on the existence of two parameters $\lambda_1, \lambda_2 > 0$, which however do not appear in the conclusion. This is due to the fact that the window functions $\Phi_{\lambda_1}, \Phi_{\lambda_2}$ are L^2- normalized so that the norm of the composition $L_1 L_2$ does not depend on these parameters.
(6) An open question: In the Donoho-Stark UP the case $\varepsilon_T = \varepsilon_\Omega = 0$ is equivalent to $\operatorname{supp} f \subseteq T$, $\operatorname{supp} \widehat{f} \subset \Omega$ and yields $|T||\Omega| \geq 1$, which is trivial because actually from the Benedicks UP we have $|T||\Omega| = +\infty$.
The corresponding case $\varepsilon_T = \varepsilon_\Omega = 0$, for Theorem 12, yields $|T||\Omega| \geq \sup_{r\in[1,\infty)} \frac{1}{(2r)^d}\left(\frac{(r+1)^{r+1}}{(r-1)^{r-1}}\right)^{d/2} = (e/2)^d \approx (1.36)^d$. Is this a meaningful estimate or also in this case actually $|T||\Omega| = +\infty$?

4 Cohen operators: quantizations and boundedness

As mentioned in the Introduction, for any given time-frequency representation $Q_\sigma = \sigma * \mathrm{Wig}$ we can consider, by formula (3), the operator T_σ^a depending on the symbol a. The class of operators that we obtain contains classical pseudodifferential operators, localization, Weyl and τ-Weyl operators, see [6], as well as many other kind of operators of pseudodifferential type, such as the ones associated with the Born-Jordan representation, see [12], and the pseudo-differential operators considered in [1].

The aim of this section is to introduce some basic facts about the correspondence $a \to T_\sigma^a$ and establish a Lebesgue functional setting where these operators act continuously.

The following proposition summarizes the relations of the operators T_σ^a with Weyl operators, Schwartz kernels and adjoints. For simplicity we suppose signals, symbols and Cohen kernels in the Schwartz spaces, letting to the reader the standard extensions to more general settings.

Proposition 14. *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$, $a \in \mathcal{S}(\mathbb{R}^{2d})$. Then*

a) $T^a_\sigma = W^{a*\overline{\widetilde{\sigma}}}$, *where $W^{a*\overline{\widetilde{\sigma}}}$ is the Weyl operator with symbol $a * \overline{\widetilde{\sigma}}$ (and $\widetilde{\sigma}(z) = \sigma(-z)$, $z \in \mathbb{R}^{2d}$).*

b) $(T^a_\sigma)^* = T^{\overline{a}}_{\overline{\sigma}}$, *i.e. the adjoint of a Cohen operator is the Cohen operator corresponding to conjugated sesquilinear form and symbol.*

c) $(T^a_\sigma f, g) = (k, f \otimes \overline{g})$, *i.e.* $k = A\mathcal{F}_2^{-1}[a * \overline{\widetilde{\sigma}}]$ *is the Schwartz kernel of the operator T^a_σ, where $A : \phi(x,t) \mapsto \phi\left(\frac{x+t}{2}, x-t\right)$, and $\mathcal{F}_2$ is the Fourier transform with respect to the second half of variables.*

We omit the proof which is a straightforward computation.

The previous proposition leads to an interesting general view on the behavior of the different rules of association symbol-operator, improperly called "quantizations". By (a) all quantizations can be seen as "deformation" of the Weyl quantization where the symbol at first undergoes a filtering by the convolution with a fixed kernel σ. In absence of filtering, i.e. when $\sigma = \delta$, we have the "pure" Weyl quantization.

Many characteristics of the quantization can be then read from the Cohen kernel σ. For example it is natural to ask whether the correspondence $a \to T^a_\sigma$ is a quantization in the "classical" sense, i.e. it assigns self-adjoint operators to real symbols. We see from (b) that this happens if and only if the kernel σ is itself real. The fact that the Weyl correspondence is a quantization in the classical sense corresponds to the fact that the Dirac δ is real (a distribution $u \in \mathcal{S}'(\mathbb{R}^{2d})$ is "real" if its value on real-valued test functions is real).

Furthermore, we see from (c) that the correspondence between Schwartz kernels and operators is nothing else than formula (3) where, as sesquilinear form, is taken the skew-tensor product $f \otimes \overline{g}$, which is not in the Cohen class.

An explicit computation shows that the expression of the operator T^a_σ acting on a function f is

$$T^a_\sigma f(t) = \iint_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} e^{2\pi i\xi(t-u)} a(x,\omega)\overline{\sigma\left(x - \frac{t+u}{2}, \omega - \xi\right)} f(u)\, dx\, d\omega\, du\, d\xi, \tag{20}$$

in particular we remark that it is a classical pseudo-differential operator with amplitude

$$a_Q(u,t,\xi) := \iint_{\mathbb{R}^{2d}} a(x,\omega)\overline{\sigma\left(x - \frac{t+u}{2}, \omega - \xi\right)} d\omega\, dx,$$

(see [24]), however we shall not make use of this formula in this context.

With the following property we furnish a Lebesgue functional setting for the action of the Cohen operators.

Theorem 15. *Let $a \in L^r(\mathbb{R}^{2d})$ and $\sigma \in L^s(\mathbb{R}^{2d})$. Moreover, let $q \in [1,2]$ be such that $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{q}$, and $p \in [q, q']$. Then $T : (a, \sigma) \in L^r(\mathbb{R}^{2d}) \times L^s(\mathbb{R}^{2d}) \to T^a_\sigma \in B(L^p(\mathbb{R}^d))$ is a continuous map, and*

$$\|T^a_\sigma\|_{B(L^p)} \leq (A_r A_s A_{q'})^d C(q', p) \|a\|_r \|\sigma\|_s, \tag{21}$$

where $C(q', p) = 2^{\frac{q'-2}{q'}d} H(q', p)$.

Proof. Let $f \in L^p(\mathbb{R}^d)$, and $g \in L^{p'}(\mathbb{R}^d)$. By (3) we obtain the following estimations, where we use Hölder's inequality and Proposition 5 first, Young's inequality with the Babenko-Beckner's constants after:

$$\begin{aligned}
|(T^a_\sigma f, g)| &= |(a * \overline{\widetilde{\sigma}}, \mathrm{Wig}(g, f))| \\
&\leq \|a * \overline{\widetilde{\sigma}}\|_q \|\mathrm{Wig}(g, f)\|_{q'} \\
&\leq \|a * \overline{\widetilde{\sigma}}\|_q C(q', p) \|f\|_p \|g\|_{p'} \\
&\leq (A_r A_s A_{q'})^d C(q', p) \|a\|_r \|\sigma\|_s \|f\|_p \|g\|_{p'}.
\end{aligned}$$

Then the continuity of the operator T^a_σ easily follows from standard arguments. □

Remark 16. If we take $f, g \in L^2(\mathbb{R}^d)$, we can reformulate the previous result. More precisely, let $q \in [1,2]$, $a \in L^r(\mathbb{R}^d)$ and $\sigma \in L^s(\mathbb{R}^d)$. Then $T : (a, \sigma) \in L^r(\mathbb{R}^{2d}) \times L^s(\mathbb{R}^{2d}) \to T^a_\sigma \in B(L^2(\mathbb{R}^d))$, with $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{q}$, is a continuous map, and

$$\|T^a_\sigma\|_{B(L^2)} \leq (A_r A_s A_{q'})^d \left(\frac{2^{q'-1}}{q'} \right)^{d/q'} \|a\|_r \|\sigma\|_s. \tag{22}$$

We also remark that similar types of operators are considered in [1] where boundedness and Schatten-Von Neuman results are obtained. However the estimates in [1] are not given in terms of the Cohen kernel and the type of operators considered there can not cover the totality of the Cohen class operators. Actually the sesquilinear forms associated with the operators in [1] enjoy the following uncertainty principle: if their support has finite measure then at least one of the two entries is null, ([1], Thm. 1.4.3) and this is not shared by all the sesquilinear forms in the Cohen class, see Example 5 in [8].

5 Donoho-Stark UP within the Cohen Class

In this section we give a new formulation of the classical Donoho-Stark UP in the context of the Cohen class, by using the boundedness results founded previously.

Consider the operator T_σ^a associated with a generic time-frequency representation in the Cohen class, cf. (3). We have the following result, that is the analogous for Cohen operators of Theorem 12.

Theorem 17. *Let T, Ω be measurable sets in $\mathbb{R}^d$. Let χ_T be the function $\chi_{T\times\mathbb{R}^d}(x,\omega) = \chi_T(x)$ and χ_Ω be the function $\chi_{\mathbb{R}^d\times\Omega}(x,\omega) = \chi_\Omega(\omega)$, with corresponding operators $T_\sigma^{\chi_T}$ and $T_\sigma^{\chi_\Omega}$ associated with a generic representation Q_σ. We suppose that the kernel σ is such that $F_1(t) = \int_{T\times\mathbb{R}^d} \overline{\sigma}(x-t,\omega)dxd\omega$ and $F_2(\omega) = \int_{\mathbb{R}^d\times\Omega} \overline{\sigma}(x,\eta-\omega)\,dx\,d\eta$ are (real) non negative function satisfying $\|F_j\|_\infty \le 1$, $j = 1,2$. Assume that $T_\sigma^{\chi_T}$ and $T_\sigma^{\chi_\Omega}$ satisfy the ε-concentration condition, i.e.,*

$$\|T_\sigma^{\chi_T} f\|_2^2 \ge (1-\varepsilon_T^2)\|f\|_2^2 \qquad \textit{and} \qquad \|T_\sigma^{\chi_\Omega} f\|_2^2 \ge (1-\varepsilon_\Omega^2)\|f\|_2^2;$$

moreover, suppose that

$$\min\{\|T_\sigma^{\chi_T}\|_{B(L^2)}, \|T_\sigma^{\chi_\Omega}\|_{B(L^2)}\} \le 1. \tag{23}$$

Define $G_1(t) = \int_{\mathbb{R}^d} \overline{\sigma}(-t,\omega)\,d\omega$ and $G_2(\xi) = \int_{\mathbb{R}^d} \overline{\sigma}(x,-\xi)\,dx$. Then for every $s_1, s_2, p_1, p_2, r \in [1,\infty]$ satisfying $1/s_j + 1/p_j = 1 + 1/2r$, $j = 1,2$, we have

$$|T|^{1/s_1}|\Omega|^{1/s_2} \ge (1-\varepsilon_T-\varepsilon_\Omega)\frac{1}{\left(A_{s_1}A_{s_2}A_{p_1}A_{p_2}A_{\frac{2r}{r+1}}A^2_{\frac{2r}{2r-1}}\right)^d \|G_1\|_{p_1}\|G_2\|_{p_2}}.$$

Remark 18. Taking $s_1 = s_2 := s$ and $p_1 = p_2 := p$, the conclusion of Theorem 17 becomes

$$|T||\Omega| \ge \sup_{\substack{r\in[1,\infty)\\ 1/s+1/p=1+1/2r}} (1-\varepsilon_T-\varepsilon_\Omega)^s \frac{1}{\left(A_s^2A_p^2A_{\frac{2r}{r+1}}A_{\frac{2r}{2r-1}}\right)^{sd}\|G_1\|_p^s\|G_2\|_p^s}.$$

Proof of Theorem 17. Observe at first that from Prop. 14 (a), and (11), for every $f \in L^2$ we have

$$T_\sigma^{\chi_T} f(t) = F_1(t)f(t), \quad \text{and} \quad T_\sigma^{\chi_\Omega} f(t) = \mathcal{F}^{-1}_{\omega\to t}\left[F_2(\omega)\widehat{f}(\omega)\right](t). \tag{24}$$

We can write

$$\begin{aligned}\|f\|_2^2 &= \|(f - T_\sigma^{\chi_T} f) + T_\sigma^{\chi_T} f\|_2^2 \\ &= ((f - T_\sigma^{\chi_T} f) + T_\sigma^{\chi_T} f, (f - T_\sigma^{\chi_T} f) + T_\sigma^{\chi_T} f) \\ &= \|f - T_\sigma^{\chi_T} f\|_2^2 + \|T_\sigma^{\chi_T} f\|_2^2 + (f - T_\sigma^{\chi_T} f, T_\sigma^{\chi_T} f) + (T_\sigma^{\chi_T} f, f - T_\sigma^{\chi_T} f).\end{aligned} \tag{25}$$

We now show that $(f - T_\sigma^{\chi_T} f, T_\sigma^{\chi_T} f) \geq 0$:

$$\begin{aligned}(f - T_\sigma^{\chi_T} f, T_\sigma^{\chi_T} f) &= (f, T_\sigma^{\chi_T} f) - (T_\sigma^{\chi_T} f, T_\sigma^{\chi_T} f) \\ &= \int f\overline{F_1 f} - \int F_1 f\overline{F_1 f} \\ &= \int (1 - F_1)\overline{F_1}|f|^2 \geq 0,\end{aligned}$$

since F_1 is real, non negative, and $\|F_1\|_\infty \leq 1$ by hypothesis. Then, it follows

$$\|f\|_2^2 = \|f - T_\sigma^{\chi_T} f\|_2^2 + \|T_\sigma^{\chi_T} f\|_2^2 + 2(f - T_\sigma^{\chi_T} f, T_\sigma^{\chi_T} f),$$

and hence

$$\|f - T_\sigma^{\chi_T} f\|_2^2 \leq \|f\|_2^2 - \|T_\sigma^{\chi_T} f\|_2^2. \tag{26}$$

From the hypothesis and (26) we obtain

$$\|f - T_\sigma^{\chi_T} f\|_2^2 \leq \varepsilon_T^2 \|f\|_2^2.$$

Reasoning in the same way for $T_\sigma^{\chi_\Omega}$, we get

$$\|f - T_\sigma^{\chi_\Omega} f\|_2^2 \leq \varepsilon_\Omega^2 \|f\|_2^2.$$

We assume for simplicity that $\|T_\sigma^{\chi_\Omega}\|_{B(L^2)} \leq 1$, cf. (23) (in the other case the proof is similar). We then have

$$\begin{aligned}\|f - T_\sigma^{\chi_\Omega} T_\sigma^{\chi_T} f\|_2 &\leq \|f - T_\sigma^{\chi_\Omega} f\|_2 + \|T_\sigma^{\chi_\Omega} f - T_\sigma^{\chi_\Omega} T_\sigma^{\chi_T} f\|_2 \\ &\leq \varepsilon_\Omega \|f\|_2 + \|T_\sigma^{\chi_\Omega}\|_{B(L^2)} \|f - T_\sigma^{\chi_T} f\|_2 \\ &\leq \varepsilon_\Omega \|f\|_2 + 1 \cdot \varepsilon_T \|f\|_2 \\ &= (\varepsilon_\Omega + \varepsilon_T)\|f\|_2.\end{aligned}$$

Then

$$\begin{aligned}\|T_\sigma^{\chi_\Omega} T_\sigma^{\chi_T} f\|_2 &\geq \|f\|_2 - \|f - T_\sigma^{\chi_\Omega} T_\sigma^{\chi_T} f\|_2 \\ &\geq \|f\|_2 - (\varepsilon_\Omega + \varepsilon_T)\|f\|_2 \\ &= (1 - \varepsilon_T - \varepsilon_\Omega)\|f\|_2,\end{aligned}$$

and, from this, it follows that

$$1 - \varepsilon_\Omega - \varepsilon_T \leq \|T_\sigma^{\chi_\Omega} T_\sigma^{\chi_T}\|_{B(L^2)}. \tag{27}$$

From (24), and using Hölder inequality we have, for every $r, k \in [1, \infty]$

$$\begin{aligned}
\|T_\sigma^{\chi_\Omega} T_\sigma^{\chi_T} f\|_2 &= \|F_2 \widehat{F_1 f}\|_2 \\
&\leq A^d_{(2r')'} \|F_2\|_{2r} \|F_1 f\|_{(2r')'} \\
&\leq A^d_{(2r')'} \|F_2\|_{2r} \|F_1\|_{(2r')'k} \|f\|_{(2r')'k'}.
\end{aligned}$$

We choose k such that $(2r')'k' = 2$, that implies $(2r')'k = 2r$. Observe now that $F_1 = \chi_T * G_1$ and $F_2 = \chi_\Omega * G_2$. Then, for every s_1, s_2, p_1, p_2 such that $1/s_j + 1/p_j = 1 + 1/2r$, $j = 1, 2$, by Young inequality we get

$$\|T_\sigma^{\chi_\Omega} T_\sigma^{\chi_T} f\|_2 \leq \left(A_{(2r')'} A_{s_1} A_{s_2} A_{p_1} A_{p_2} A^2_{(2r)'} \right)^d \|\chi_T\|_{s_1} \|\chi_\Omega\|_{s_2} \|G_1\|_{p_1} \|G_2\|_{p_2} \|f\|_2,$$

that, together with (27), implies the thesis. □

Acknowledgments. The authors are thankful to the anonymous referee for the comments and valuable suggestions.

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