Complete set of essential parameters of effective theory

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The present paper continues the series [1] - [5] devoted to the systematic study of effective scattering theories. We consider matrix elements of the effective Lagrangian monomials (in the interaction picture) of arbitrary high dimension D and show that the full set of corresponding coupling constants contains parameters of both kinds: essential and redundant. Since it would be pointless to formulate renormalization prescriptions for redundant parameters, it is necessary to select the full set of the essential ones. This is done in the present paper.

I. INTRODUCTION AND PRELIMINARIES

In this paper we continue constructing the renormalization scheme suitable for the *effective scattering theory*. Let us recall that the theory is called effective if the interaction Lagrangian in the interaction picture contains all the monomials consistent with the given algebraic (linear) symmetry (see [6]). Effective scattering theory (EST) is just the effective theory only designed for perturbative calculations of the S-matrix elements. Green functions may contain divergent contributions: we are only interested in Dyson's perturbation scheme of calculating the S matrix. Though the effective theory is renormalizable by construction (see [4] - [6] and the references therein) it presents no interest until the "problem of couplings" is solved. Thus it is necessary to point out an infinite number of renormalization prescriptions (RP's) that allow one to fix the finite parts of counterterms. If this is done arbitrarily, the theory loses its predictive power. Unfortunately, we do not have an infinite number of corresponding physical principles needed to avoid the problem. Therefore, one must either indicate new (sufficiently powerful) principles or radically reduce the number of free parameters in the theory. Anyway, we need to know the whole list of free parameters which S-matrix depends upon.

In the recent paper [1] the part of the problem of couplings have been analyzed by the example of 2-leg 1-loop graph (conventionally called "self energy"). It was shown that in order to avoid the problem one needs to revise the concept of one-particle reducibility/irreducibility (1PR/1PI) and to redefine the notion of "self energy". In this case one only needs to rely upon two conventional physical conditions (the pole position and the wave function normalization). This turns out quite sufficient for obtaining the well-defined 2-leg function which can be used as the insertion both in external and internal lines of Feynman graphs of the effective scattering theory. The possibility of introducing the alternative definition of reducibility is based on the fact that the analytical expression for arbitrary Feynman graph of the effective scattering theory can be identically rewritten in terms of the *minimal lines* and *minimal vertices* by means of the operation called as *line reduction*.

It seems us pertinent to briefly recall the definitions of the above terms. The effective vertex is called minimal with respect to its line with momentum k if it does not contain the "killing" factor $k^2 - m^2$ (here m stands for the particle mass; it is implied that $m \neq 0$). The line is called minimal if the adjacent vertex (or both vertices, when the line in question is internal) is minimal with respect to it. The graph is called minimal if all its lines are minimal (these definitions along with many examples have been suggested in [4] and [5]). At last, the parameters that appear in the analytic expression for the minimal effective vertex are called the *minimal resultant parameters*.

Thus one concludes that the S-matrix elements only depend on the minimal parameters and thus do not require introducing RP's fixing the non-minimal ones. In turn, this means that the minimal resultant parameters are the only ones needed to fix the physical content of the EST.

Another problem waiting for its solution is the following. To calculate graphs one needs to construct a convenient form of recording of the n-leg effective Lagrangian monomials in the interaction picture. The point is that the form suggested in [1] is not convenient for calculations because it is excessively general: it contains many identical terms. So, we need to suggest the more suitable form (without loss of generality) and, hence, to suggest the complete set of essential coupling constants (see [7]) needed to fix the 4-leg minimal effective vertices[1]. To put it another way: it is

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[1] When writing the term "vertex" we mean the corresponding Lagrangian monomial (or - simpler - monomial) multiplied by a constant.

necessary to point out the complete set of basic 4-leg Lagrangian monomials (basis) with certain highest dimension D (see footnote [2]). The linear span of these basic monomials creates the relevant linear space of corresponding physical dimension. The coefficients at the individual basic monomials present the essential coupling constants.

It is the problem of constructing the basic set of monomials (more correctly, of the corresponding matrix elements in the momentum representation) which we solve in the given paper. This issue is important in the renormalization theory, because it would make no sense to formulate the renormalization prescriptions (RP's) that correspond to redundant parameters.

Below we use the same notations as those in the Ref. [1]:

$$\partial^{[n]} \stackrel{\text{def}}{=} \partial^{\mu_1} \ldots \partial^{\mu_n}, \qquad \partial_{[n]} \stackrel{\text{def}}{=} \partial_{\mu_1} \ldots \partial_{\mu_n}$$

and

$$\partial^{[a,b,c]} \stackrel{\text{def}}{=} \partial^{\alpha_1,\ldots,\alpha_a;\beta_1,\ldots,\beta_b;\gamma_1,\ldots,\gamma_c} ; \qquad \partial^{[a,b]}_{[c]} \stackrel{\text{def}}{=} \partial^{\alpha_1,\ldots,\alpha_a;\beta_1,\ldots,\beta_b}_{\gamma_1,\ldots,\gamma_c} .$$

(Latin letters show the number of Lorentz indices which are denoted by Greek letters.)

II. 3-LEG EFFECTIVE VERTEX

In the field theory two Lagrangian vertices (densities) that differ from one another by a surface terms arising from the integration by parts in action are considered equivalent. In certain cases this allows one to simplify the form of the vertex record. Let us illustrate this with the simplest example.

Let the 3-leg vertex has the form

$$V = G : \phi \partial^\mu \phi \partial_\mu \phi : . \tag{1}$$

Here the sign $: \ldots :$ stands for the normal product and G is the dimensional coupling constant (with $D = 1$ in the units of m). Recall that inside the normal product the field operators commute.

Integrating by parts we obtain the equivalent vertex (as usually, the surface term is neglected). Since we consider the interaction picture field then

$$\partial^\mu \partial_\mu \phi = -m^2 \phi . \tag{2}$$

This allows one to simplify (1) as follows:

$$V = G : \phi \partial^\mu \phi \partial_\mu \phi : \simeq \widetilde{V} = -G : \phi \partial^\mu \phi \partial_\mu \phi : + G m^2 : \phi^3 : + \ldots , . \tag{3}$$

Here we use the symbol $A \simeq B$ to show the field-theoretic equivalence of the operators A and B (the ellipsis stands for the neglected term out of integral). From the relation (3) one concludes that the initial vertex V is equivalent to the triple vertex $: \phi^3 :$

$$V = G : \phi \partial^\mu \phi \partial_\mu \phi : \simeq \frac{G m^2}{2} : \phi^3 : \equiv g_3 : \phi^3 :$$

which is more simple than the initial one[3].

Now let us turn to a consideration of the general form of 3-leg vertex in effective theory. The method allowing one to construct the n-leg effective vertices was presented in [1]. It is certainly general. For example, in the framework of this method the 3-leg vertex takes the form ($G^{s_1 s_2 s_3}$ stand for the corresponding coupling constants)

$$V_3 = \sum_{s_1,s_2,s_3=0}^{\infty} G^{s_1 s_2 s_3} : \left(\partial^{[s_1]} \partial_{[s_2]} \phi \right) \left(\partial^{[s_2]} \partial_{[s_3]} \phi \right) \left(\partial^{[s_3]} \partial_{[s_1]} \phi \right) : . \tag{4}$$

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[2] When using the term "dimension" we mean the "physical dimension" D calculated in units of m. Keep in mind, however, that, starting from the next Section, we deviate from this rule and express the physical dimension in units of m^2.

[3] Note that the similar trick could not be performed with the vertices of the form

$$V = G^{00n} : \phi \partial^{[n]} \phi \partial_{[n]} \phi :$$

where $n \geqslant 2$. In the latter case one needs to compare the relevant matrix elements – see below.

The problem is that this form is excessively general because it contains many identical terms. This makes difficult the interpretation of coupling constants and complicates calculations.

As suggested in [1] one can make use of the integration by parts (together with the boundary condition at infinity) in order to simplify the form (4). Taking away the derivatives from the first factor and lowering and raising indices one can obtain the expression given in [1]. Further simplification is also possible but it requires comparison of matrix elements in the space of 3-particle states (together with the energy-momentum conservation low). From this comparison one finds that an arbitrary 3-leg vertex in the interaction picture is equivalent to the simplest one:

$$D^{s_1 s_2 s_3} : \left(\partial^{[s_1]} \partial_{[s_2]} \phi \right) \left(\partial^{[s_2]} \partial_{[s_3]} \phi \right) \left(\partial^{[s_3]} \partial_{[s_1]} \phi \right) \simeq g_3 : \phi^3 : \ .$$

So, the final form of the effective 3-leg vertex reads:

$$V_3 = g_3 : \phi^3 : \ . \tag{5}$$

Here g_3 is the true resultant coupling constant that has the dimension of mass: $D(g_3) = D(m) = 1$

In what follows we shall often omit the sign of the normal ordering though it will be implied.

III.   4-LEG EFFECTIVE VERTICES

Let us now analyze the more complicated object – the 4-leg effective vertex. Precisely as above we will use two different principles: a) Zero boundary condition at infinity and b) Two vertices are equivalent if the corresponding matrix elements are the same.

The obviously general form of the 4-leg effective vertex reads

$$V_4 = \sum_{s_1 \ldots s_6 = 0}^{\infty} g^{s_1 \ldots s_6} \left(\partial^{[s_1 s_2 s_3]} \phi \partial_{[s_1 s_4 s_5]} \phi \partial^{[s_4]}_{[s_2 s_6]} \phi \partial^{[s_5 s_6]}_{[s_3]} \phi \right) \ . \tag{6}$$

Let us consider the corresponding Lagrangian monomial with fixed dimensionality

$$L(s_1, \ldots, s_6) \equiv \left(\partial^{[s_1 s_2 s_3]} \phi \right) \left(\partial_{[s_1 s_4 s_5]} \phi \right) \left(\partial^{[s_4]}_{[s_2 s_6]} \phi \right) \left(\partial^{[s_5 s_6]}_{[s_3]} \phi \right) \tag{7}$$

and make use of the integration by parts in order to simplify it. Clearly, integrating by parts $(s_1 + s_2 + s_3)$ times one can remove all the derivatives from the first factor. This means that the expression (7) is just a sum of more simple Lagrangian monomials of the form

$$L^{p,q,r} \simeq L(0,0,0,p,q,r) = \phi \partial^{[p]}_{[q]} \phi \partial^{[q]}_{[r]} \phi \partial^{[r]}_{[p]} \phi \ . \tag{8}$$

So, it is shown that integration by parts allows one to use the 3-index monomials (and, hence, the 3-index coupling constants[4]) in place of the 6-index ones. However, this is not the result that we would like to get. The point is that the form (8) still contains identical contributions. Therefore it is necessary to further simplify the monomials of this kind. Let us do this.

As above, we may reduce the order of derivative operator in the second factor in (8): $p + q \to p + q - 1$. Integration by parts allows one to get the following relation (the corresponding calculation is simple but tedious)

$$L^{p,q,r} = m^2 L^{p-1,q,r} - L^{p-1,q+1,r} - L^{p-1,q,r+1} \ . \tag{9}$$

Thus we can make the first index equal zero and need to consider only the 2-index vertices of the form

$$L^{0qr} = \phi \partial_{[q]} \phi \partial^{[q]}_{[r]} \phi \partial^{[r]} \phi \ . \tag{10}$$

These are the monomials which could seem to be the basic ones. Nevertheless, this is not true. The thing is that the monomials (10) are obviously symmetric in their indices:

$$L^{0qr} = L^{0rq} \ , \tag{11}$$

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[4] This form was used in [1], in that case the problem of identical terms was not important.

therefore only those of them are independent which have $q \leqslant r$. If we want (as we do) to work with the monomials that have the dimensionality $D \leqslant N$, (in the units of m^2) then forms $L^{r,N-r}$ with $r \leqslant [N/2]$ might be considered independent. Here the symbol $[x]$ stands for the integer part of x.

At last, considering the monomials with $q = 1$ we see that

$$L^{1,r} \simeq \frac{m^2}{2} L^{0,r} - \frac{1}{2} L^{0,r+1}.$$

This means that the monomials of the form $L^{1,r}$ should be considered dependent and, therefore, excluded from the set of basic ones.

Does it mean that the most general independent 4-field vertex in the effective Lagrangian of the single-scalar theory can be presented as the linear sum of terms of the form

$$V_4 = \sum_{N=0,2,3,\ldots}^{\infty} \sum_{q=0,2,3,\ldots}^{E(N/2)} g_{q,N-q} L^{q,N-q}(\phi) \tag{12}$$

where $g_{q,N-q}$ stand for the corresponding independent coupling constants[5]? The answer is no. The reason is that among different interaction picture operators $L^{q,N-q}$ acting on the space of four-particle states $|k_1, k_2, k_3, k_4\rangle$ (here $k_i^2 = m^2$ and $k_1 + k_2 + k_3 + k_4 = 0$ due to the conservation law) there are many dependent ones. We discuss this point in the next Section.

IV.   THE SPACE OF MATRIX ELEMENTS.

Let us denote M^{qr} the matrix element of the 4-field Lagrangian monomial L^{qr} between the vacuum and 4-particle state $|k_1, k_2, k_3, k_4\rangle$ under the conditions $k_1 + k_2 + k_3 + k_4 = 0$ and $k_i^2 = m^2$.

It is possible to show (see below) that

$$M^{22} \simeq \frac{1}{2} M^{04} - \frac{4}{3} M^{03} + M^{02} - \frac{1}{18} M^{00}, \tag{13}$$

$$M^{23} \simeq \frac{1}{10} M^{05} + 0 M^{04} - \frac{1}{2} M^{03} + \frac{1}{2} M^{02} - \frac{1}{30} M^{00}. \tag{14}$$

These relations demonstrate that the corresponding monomials are linearly dependent and, therefore, the constants g_{22} and g_{23} are not independent parameters of a theory.

It might seem that the elements M^{0N} create a basis in the linear space M_N of the above-described 4-field monomials $\langle 0| L^{q,r} |k_1, k_2, k_3, k_4\rangle$ with $q + r = N$. However, for arbitrary N this is not true. For example, one can deduce the relation

$$M^{24} = -\frac{2}{3} M^{33} + \frac{1}{6} M^{06} - \frac{2}{5} M^{05} + \frac{1}{2} M^{04} - \frac{2}{3} M^{03} + \frac{1}{2} M^{02} - \frac{1}{30} M^{00} \tag{15}$$

which demonstrates that M^{24} is just a linear combination of another (*linearly independent*!) matrix elements with $N \leqslant 6$. Therefore it must be excluded from the set of basic monomials of the index $N \geqslant 6$.

We find it pertinent to note that the number N indicates the physical dimension $D_N = N$ (in the units of m^2) of the relevant matrix element M^{qr} from the space M_N. This matrix element is a homogeneous polynomial (with the homogeneity degree N) that depends on three Mandelstam variables s, t, u

$$s = (k_1 + k_2)^2, \quad t = (k_1 + k_3)^2, \quad u = (k_2 + k_3)^2,$$

restricted by the conditions $s + t + u = 4m^2$ and $k_i^2 = m^2$:

$$M^{qr}(s,t,u) \equiv \langle 0| \int dx : \phi \partial_q \phi \partial_r^q \phi \partial^r \phi : |k_1, k_2, k_3, k_4\rangle =$$
$$= (2\pi)^4 \cdot 4 \cdot (-1/2)^{q+r} \left[\hat{s}^q \left(\hat{t}^r + \hat{u}^r \right) + \hat{t}^q \left(\hat{t}^r + \hat{u}^r \right) + \hat{u}^q \left(\hat{t}^r + \hat{s}^r \right) \right] . \tag{16}$$

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[5] Only the independent coupling constants present a set of essential parameters of a theory.

Here $\hat{s} = s - 2m^2$, $\hat{t} = t - 2m^2$, $\hat{u} = u - 2m^2$. In fact, we are more interested in these matrix elements than in the monomials L^{qr} in coordinate space. The point is that we are studying the effective scattering theory and thus we need to consider the minimal graphs (see [4] and [5]) while the concept of minimality is well defined only in the momentum representation. Therefore we have to provide the full set of constants that allow to fix the graph as a function of relevant kinematic variables. It is more relevant to do this in momentum space.

In place of three dependent dimensional variables $(\hat{s}, \hat{t}, \hat{u})$ it is more convenient to work in terms of three dimensionless ones:

$$\sigma \equiv \frac{\hat{s}}{2m^2}, \quad \tau \equiv \frac{\hat{t}}{2m^2}, \quad \xi \equiv \frac{\hat{u}}{2m^2}; \quad \sigma + \tau + \xi = -1. \tag{17}$$

This gives:

$$M^{qr}(\sigma, \tau, \xi) \simeq m^{2N}(-1)^N \left[\sigma^q \left(\tau^r + \xi^r \right) + \tau^q \left(\xi^r + \sigma^r \right) + \xi^q \left(\sigma^r + \tau^r \right) \right] . \tag{18}$$

Here the inessential numerical factor $4 \cdot (2\pi)^4$ has been omitted. The dimensional factor $D_N = N$ turns out separated and the dependence on the kinematical variables is just a dimensionless 3-variable symmetric polynomial with the leading degree $N = q + r$.

In this Section we will consider not the linear space of Lagrangian monomials itself but the closely related space M_N of matrix elements of these monomials with the dimension $D = N$. In accordance with written above we need to reveal the basic elements of M_N; the relevant coefficients will be precisely the essential parameters that completely fix the vertex in question. No redundant parameters can appear in this case.

Let us – for example – show how can be obtained the relation (13) in this technique. For this we need to write down the explicit formulae for matrix elements of the monomials with, say, $N \leqslant 8$ (see (18)):

$$\begin{aligned}
M^{00} &= D_0(+6); \\
M^{02} &= D_2(-4a_2 + 2); \\
M^{03} &= D_3(-6a_3 - 6a_2 + 2); \\
M^{04} &= D_4(+4a_2^2 - 8a_3 - 8a_2 + 2); \\
M^{05} &= D_5(+10a_2a_3 + 10a_2^2 - 10a_3 - 10a_2 + 2); \\
M^{06} &= D_6(+6a_3^2 - 4a_2^3 + 24a_2a_3 + 18a_2^2 - 12a_3 - 12a_2 + 2); \\
M^{07} &= D_7(-14a_2^2a_3 + 14a_3^2 - 14a_2^3 + 42a_2a_3 + 28a_2^2 - 14a_3 - 14a_2 + 2); \\
M^{08} &= D_8(+4a_2^4 - 16a_2a_3^2 - 48a_2^2a_3 + 24a_3^2 - 32a_2^3 + 64a_2a_3 + 40a_2^2 - 16a_3 - 16a_2 + 2), \\
M^{22} &= D_4(+2a_2^2 + 4a_3); \\
M^{23} &= D_5(+a_2a_3 + a_2^2 + 2a_3); \\
M^{33} &= D_6(+6a_3^2 + 2a_2^3 + 6a_2a_3); \\
M^{24} &= D_6(-3a_3^2 - 2a_2^3 - 4a_2a_3 + a_2^2 + 2a_3); \\
M^{25} &= D_7(-3a_2^2a_3 - 7a_3^2 - 3a_2^3 - 6a_2a_3 + a_2^2 + 2a_3); \\
M^{34} &= D_7(+a_2^2a_3 + 5a_3^2 + a_2^3 + 3a_2a_3); \\
M^{26} &= D_8(+2a_2a_3^2 + 2a_2^4 - 9a_3^2 - 4a_2^3 - 8a_2a_3 + a_2^2 + 2a_3); \\
M^{35} &= D_8(-7a_2a_3^2 - 2a_2^4 - 6a_2^2a_3 + 3a_3^2 + a_2^3 + 3a_2a_3); \\
M^{44} &= D_8(+8a_2a_3^2 + 2a_2^4 + 8a_2^2a_3 + 4a_3^3),
\end{aligned} \tag{19}$$

where two important combinations

$$a_2 \equiv \sigma\tau + \tau\xi + \xi\sigma; \quad a_3 \equiv \sigma\tau\xi \tag{20}$$

have been introduced. Note that, in accordance with Waring formula (see Appendix), the leading terms in all matrix elements are different from zero.

These matrix elements are not the only ones that correspond to all interaction picture monomials L^{qr} with $q + r \leqslant 8$, $q \leqslant r$. The clearly dependent ones L^{1r} (or, better, their matrix elements) have been excluded from our list: as shown in the Section(III) they can be written as linear combinations of the above-presented monomials.

To prove the relation (13) one needs to write down the general linear dependence/independens condition in the space M_N of matrix elements of the vertices with dimension $D = N = q + r \leqslant 4$:

$$\alpha_{22}M^{22} + \alpha_{04}M^{04} + \alpha_{03}M^{03} + \alpha_{02}M^{02} + \alpha_{00}M^{00} = 0. \tag{21}$$

Here α_{qr} stand for the arbitrary numerical factors[6]. Now we need to check if this equality can be fulfilled with some factors α_{qr} that are non-zero. If such a solution exists, we conclude that the elements $M^{22} \ldots M^{00}$ are linearly dependent. Otherwise they should be considered linearly independent. In the latter case we can consider them as a basis elements in the space M_4.

As known (see [10]), an arbitrary *symmetric* (in variables σ, τ, ξ) polynomial can be uniquely (identically!) rewritten in the form of an *ordinary* polynomial depending on three elementary symmetric combinations:

$$a_1 = (\sigma + \tau + \xi), \quad a_2 = (\sigma\tau + \tau\xi + \xi\sigma), \quad a_3 = \sigma\tau\xi. \tag{22}$$

In the case under consideration $a_1 = -1$, so, in fact, we deal with the ordinary polynomials M^{ij} that depend on two combinations a_2 and a_3 only.

The left side of the equation (21) is nothing but the sum of several ordinary polynomials depending on two variables a_2 and a_3. To equate this sum to zero one has to equate zero each coefficient of this sum. This gives one the following homogeneous system of equations[7]:

$$2\alpha_{22} + 4\alpha_{04} = 0;$$
$$4\alpha_{22} - 8\alpha_{04} - 6\alpha_{03} = 0;$$
$$0\alpha_{22} - 8\alpha_{04} - 6\alpha_{03} - 4\alpha_{02} = 0;$$
$$0\alpha_{22} + 2\alpha_{04} + 2\alpha_{03} + 2\alpha_{02} + 6\alpha_{00} = 0.$$

This system has a stepped form. The length of the top step equals the number of terms (in the case in question it is two) with the maximal dimension $D = 4$ in the expressions for M^{04} and M^{22} (see (19)). The thickness of this top step equals to one line.

Let us now move one of the two columns starting with the top line (for example, that $\sim \alpha_{22}$) from the left to the right side. This gives:

$$+ 4\alpha_{04} = -2\alpha_{22};$$
$$- 8\alpha_{04} - 6\alpha_{03} = -4\alpha_{22};$$
$$- 8\alpha_{04} - 6\alpha_{03} - 4\alpha_{02} = 0\alpha_{22};$$
$$+ 2\alpha_{04} + 2\alpha_{03} + 2\alpha_{02} + 6\alpha_{00} = 0\alpha_{22}.$$

The obtained system is solvable, the solution being nonzero. Therefore the set $\{M^{22}, M^{04}, M^{03}, M^{02}, M^{00}\}$ certainly contains extra member that should not be included in the list of basic elements. At the same time, at $\alpha_{22} = 0$ the system has the unique solution (equal zero). Therefore the basis may be constructed from the four linearly independent elements: $M^{04}, M^{03}, M^{02}, M^{00}$ (or, equally, from $M^{22}, M^{03}, M^{02}, M^{00}$). The relation (14) can be proved in precisely the same way.

The above-considered example, in fact, shows the most important steps, that should be taken in order to construct a basis in the space M_N of matrix elements of the dimension $D \leqslant N$. It only remains to illustrate the purely technical complications arising when one passages to the case $N \geqslant 6$. For this it is sufficient to analyze the case $N = 6$. Let us do this.

Consider the linear dependence/independence test:

$$\alpha_{24}M^{24} + \alpha_{33}M^{33} + \alpha_{06}M^{06} + \alpha_{05}M^{05} + \alpha_{04}M^{04} + \alpha_{03}M^{03} + \alpha_{02}M^{02} + \alpha_{00}M^{00} = 0. \tag{23}$$

With account taken of equations (19) this equality takes a form of the following system of equations[8]:

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\begin{aligned}
(D_6) \quad [l_1]: \quad & -2\alpha_{24} + 2\alpha_{33} - 4\alpha_{06} = 0, \\
(D_6) \quad [l_2]: \quad & -3\alpha_{24} + 6\alpha_{33} + 6\alpha_{06} = 0, \\
(D_5) \quad [l_3]: \quad & -4\alpha_{24} + 6\alpha_{33} + 24\alpha_{06} + 10\alpha_{05} = 0, \\
(D_4) \quad [l_4]: \quad & +1\alpha_{24} + 0\alpha_{33} + 18\alpha_{06} + 10\alpha_{05} + 4\alpha_{04} = 0, \\
(D_3) \quad [l_5]: \quad & +2\alpha_{24} + 0\alpha_{33} - 12\alpha_{06} - 10\alpha_{05} - 8\alpha_{04} - 6\alpha_{03} = 0, \\
(D_2) \quad [l_6]: \quad & +0\alpha_{24} + 0\alpha_{33} - 12\alpha_{06} - 10\alpha_{05} - 8\alpha_{04} - 6\alpha_{03} - 4\alpha_{02} = 0, \\
(D_0) \quad [l_7]: \quad & +0\alpha_{24} + 0\alpha_{33} + 2\alpha_{06} + 2\alpha_{05} + 2\alpha_{04} + 2\alpha_{03} + 2\alpha_{02} + 6\alpha_{00} = 0.
\end{aligned}
\tag{24}
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[6] The element M^{13} is absent in accordance with the note above.
[7] Each equation contains terms corresponding to the fixed dimension $D \leqslant N$; the dimensional factors are omitted.
[8] As above, each equation contains terms corresponding to the fixed dimension $D \leqslant 6$

Here the indices $[l_1] - [l_7]$ stand for the line numbers while (D_N) – for the relevant dimensional factors.

The system (24) is stepped. The only problem is that the thickness of the upper step equals two lines: the step is too thick. However, this is just an apparent difficulty that can be easily removed. Indeed, considering the sum of l_1 and l_2 (with corresponding coefficients), one obtains the two new top lines corresponding to D_6:

$$- 2\alpha_{24} + 3\alpha_{33} = 0$$
$$- \alpha_{24} + 2\alpha_{33} + 2\alpha_{06} = 0.$$

This is precisely what we need.

Now, moving the column with α_{24} to the right side, we obtain the quite solvable (the main determinant is nonzero!) system of equations for α_{ij} $(i + j \leqslant 6)$ which shows that M^{24} is just a linear combination of $M^{33}, M^{06}, \ldots, M^{02}, M^{00}$. It can be easily shown that this latter set of matrix elements presents a basis in the space M_6.

It is clear that the order of the arrangement of columns containing α_{24}, α_{33} and α_{06} is unimportant. It is possible to bring the system to a stepped form in any case. For this reason one can consider dependent any one of three elements: M^{24}, M^{33} or M^{06}. This is just a matter of taste. By our agreement, we always consider M^{0r} as independent.

The above-considered example can be easily generalized for the case of arbitrary large N. Let us formulate the general algorithm how to construct the basis in subspace of the dimension $D = N$ under the condition that this has been done already in the subspace with $D \leqslant (N - 1)$:

1. Construct the linear combination of *all* $[N/2]$ different matrix elements M^{qr} with $q + r = N$, $q \leqslant [N/2]$ and all the *basic* matrix elements with $q + r \leqslant N - 1$.

2. Equate this combination to zero. This will give a test of linear dependence/independence.

3. Present the obtained test in the form of a system of linear equations for the coefficients of ordinary two-variable polynomial (with the leading degree N in the aggregate variable σ, τ, ξ) depending on two combinations $a_2(\sigma, \tau, \xi)$ and $a_3(\sigma, \tau, \xi)$ (see (20)).

4. The constructed system of equations will take the stepped form. Every step that corresponds to the dimension factor $D < N$ will have the thickness equal to one line (the basic elements with $D < N$ are considered known!). In contrast, the thickness of the top step may happen to be equal t lines. The lengths of these top lines will be equal to $[N/2]$ elements. Since $[N/2] \geqslant t$ (the total number of all matrix elements $M^{q,N-q}$ with the index N cannot be less then the number of linearly independent ones), it is always possible to rewrite (identically!) the top (thick) step in the completely stepped form. The length of the finally obtained top step may happen to be equal, say, $l + 1$.

5. Move l columns from the left side of the system to the right side. As a result, one obtains ingomogeneous system with nonzero main determinant because the matrix in the left side is triangular (surely, one has to retain non-zero top step!)

6. The columns corresponding to the coefficients α_{qr} in the left side will present the basic matrix elements, while those in the right side – the linearly dependent ones.

From the above analysis it is clear that the number of independent matrix elements with index N is equal to the number of pairs $[i, j]$ such that $2i + 3j = N$ minus one. A specific choice of a set of these elements is arbitrary.

When the basis in the space of matrix elements M^{qr} of the interaction picture monomials L^{qr} with arbitrary fixed maximal index N is constructed, one can point out the complete set of the corresponding essential parameters. Those parameters are nothing but the set of numbers fixing every basic matrix element. As shown above, the basis is just a set of arbitrary two-variable polynomials that depend on two combinations of kinematical variables: a_2 and a_3 (see (20)). This means that we need to point out the numbers fixing those polynomials. Since an arbitrary polynomial is completely fixed by the values of its coefficients, *we can consider those coefficients as the only essential parameters of a theory*. So, we arrive at the conclusion that the complete set of essential parameters of the effective single scalar theory is exhausted by the numbers fixing the coefficients of ordinary polynomials depending on two variables.

V.   CONCLUSION

The effective theory, by definition, deals with the Lagrangian density that contains all terms consistent with a given linear symmetry. When studying such theories, we are faced with a problem that was not previously encountered: the complete list of free parameters that determine the finite S-matrix is unknown. According to the common belief, this

problem does not occur in the theories with a finite number of terms in the interaction Lagrangian. In this latter case, all coupling constants, as a rule, are considered as the independent parameters. One of important results obtained in this paper shows that this is not always true. For example, when the dimensions of the interaction monomials are $6 \leqslant D < \infty$ (in units of mass squared), one has to take into account only a certain part of them in order to avoid doubling of some couplings.

The main result have been formulated in the previous Section. Though it strongly restricts the number of the essential parameters of the single-scalar scattering theory only containing the vertices with *finite* maximal dimension $D_{\max} = N \geqslant 6$, the total number of the essential parameters of *effective* theory (that corresponds to $N \to \infty$) is actually infinite. This supports the common belief that the effective theory itself has no predictive power (see [6] - [8]): it requires infinitely many physical conditions to formulate an infinite number of renormalization prescriptions needed to remove the divergencies from the amplitudes of physical processes.

In subsequent publication we will show that this pessimistic conclusion is wrong. The desired principle does exist and allows one to fix the most part of the essential parameters but several ones.

Acknowledgements

Appendix

Here we recall the reader some facts from the theory of three-variable symmetric polynomials (see, e.g., [10]:

- The 3-variable polynomial $P(x, y, z)$ is called symmetric if

$$P(x, y, z) = P(y, x, z) = P(z, y, x) = P(x, z, y).$$

- An arbitrary symmetric polynomial $P(x, y, z)$ can be presented as just the conventional polynomial depending upon three symmetric combinations

$$\sigma_1 = x + y + z; \quad \sigma_2 = xy + yz + zx; \quad \sigma_3 = xyz.$$

In the theory of 3-variable symmetric polynomials this is known as the main theorem.

- Every power sum

$$S_k \equiv x^k + y^k + z^k$$

can be calculated step by step with the Newton formula

$$S_k = \sigma_1 S_{k-1} - \sigma_2 S_{k-2} + \sigma_3 S_{k-3}$$

and presented as the polynomial in $\sigma_1, \sigma_2, \sigma_3$ with the help of so-called Waring formula (see, e.g. [11], [12]) :

$$\frac{1}{k} S_k = \sum \frac{(-1)^{k - i_1 - i_2 - i - 3}(i_1 + i_2 + i_3 - 1)!}{i_1 i_2 i_3} \sigma_1{}^{i_1} \sigma_2{}^{i_2} \sigma_3{}^{i_3}.$$

Here the summing runs over all sets of non-negative numbers (i_1, i_2, i_3) such that

$$i_1 + 2i_2 + 3i_3 = k.$$

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