PRACTICAL NUMBERS IN LUCAS SEQUENCES

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Abstract. A practical number is a positive integer n such that all the positive integers m ≤ n can be written as a sum of distinct divisors of n. Let \((u_n)_{n \geq 0}\) be the Lucas sequence satisfying \(u_0 = 0, u_1 = 1, \) and \(u_{n+2} = au_{n+1} + bu_n\) for all integers \(n \geq 0,\) where \(a\) and \(b\) are fixed nonzero integers. Assume \(a(b+1)\) even and \(a^2 + 4b > 0.\) Also, let \(A\) be the set of all positive integers \(n\) such that \(|u_n|\) is a practical number. Melfi proved that \(A\) is infinite. We improve this result by showing that \(\#A(x) \gg x/\log x\) for all \(x \geq 2,\) where the implied constant depends on \(a\) and \(b.\) We also pose some open questions regarding \(A.\)

1. Introduction

A practical number is a positive integer \(n\) such that all the positive integers \(m \leq n\) can be written as a sum of distinct divisors of \(n.\) The term “practical” was coined by Srinivasan [7]. Let \(P\) be the set of practical numbers. Estimates for the counting function \(#P(x)\) were given by Hausman and Shapiro [1], Tenenbaum [10], Margenstern [2], Saias [5], and, finally, Weingartner [12], who proved that there exists a constant \(C > 0\) such that
\[
\#P(x) = Cx \cdot \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right)
\]
for all \(x \geq 3,\) settling a conjecture of Margenstern [2].

In analogy with well-known conjectures about prime numbers, Melfi [4] proved that every positive even integer is the sum of two practical numbers, and that there are infinitely many triples \((n, n + 2, n + 4)\) of practical numbers. Let \((u_n)_{n \geq 0}\) be a Lucas sequence, that is, a sequence of integers satisfying \(u_0 = 0, u_1 = 1,\) and \(u_{n+2} = au_{n+1} + bu_n\) for all integers \(n \geq 0,\) where \(a\) and \(b\) are two fixed nonzero integers. Also, let \(A\) be the set of all positive integers \(n\) such that \(|u_n|\) is a practical number. From now on, we assume \(a^2 + 4b > 0\) and \(a(b+1)\) even. We remark that, in the study of \(A,\) assuming \(a(b+1)\) even is not a loss of generality. Indeed, if \(a(b+1)\) is odd then \(u_n\) is odd for all \(n \geq 1\) and, since 1 is the only odd practical number, it follows that \(A = \{1\}.\) Melfi [3, Theorem 10] proved the following result.

Theorem 1.1. The set \(A\) is infinite. Precisely, \(2^j \cdot 3 \in A\) for all sufficiently large positive integers \(j,\) how large depending on \(a\) and \(b,\) and hence
\[
\#A(x) \gg \log x,
\]
for all sufficiently large \(x > 1.\)

In this paper, we improve Theorem 1.1 to the following:

Theorem 1.2. For all \(x \geq 2,\) we have
\[
\#A(x) \gg \frac{x}{\log x},
\]
where the implied constant depends on \(a\) and \(b.\)

We leave the following open questions to the interested readers:

(Q1) Does \(A\) have zero natural density?

(Q2) Can a nontrivial upper bound for \(#A(x)\) be proved?

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(Q3) Are there infinitely many nonpractical $n$ such that $|u_n|$ is practical?
(Q4) Are there infinitely many practical $n$ such that $|u_n|$ is nonpractical?
(Q5) What about practical numbers in general integral linear recurrences over the integers?

**Notation.** For any set of positive integers $S$, we put $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for all $x \geq 1$, and $\# \mathcal{S}(x)$ denotes the counting function of $\mathcal{S}$. We employ the Landau–Bachmann “Big Oh” notation $O$, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated. As usual, we write $\mu(n)$, $\phi(n)$, $\sigma(n)$, and $\omega(n)$, for the Möbius function, the Euler’s totient function, the sum of divisors, and the number of prime factors of a positive integer $n$, respectively.

2. Preliminaries on Lucas sequences

In this section we collect some basic facts about Lucas sequences. Let $\alpha$ and $\beta$ be the two roots of the characteristic polynomial $X^2 - aX - b$. Since $a^2 + 4b > 0$ and $b \neq 0$, we have that $\alpha$ and $\beta$ are real, nonzero, and distinct. It is well known that the generalized Binet’s formula

\begin{equation}
\tag{1}
 u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}
\end{equation}

holds for all integers $n \geq 0$. Define

\[ \Phi_n := \prod_{1 \leq k \leq n} \left( \alpha - e^{2\pi i k/n} \beta \right), \]

for each positive integer $n$. It can be proved that $\Phi_n \in \mathbb{Z}$ for all integers $n > 1$ (see, e.g., [9, p. 428]). Furthermore, we have

\begin{equation}
\tag{2}
 u_n = \prod_{d \mid n \atop d > 1} \Phi_d
\end{equation}

and, by the Möbius inversion formula,

\begin{equation}
\tag{3}
 \Phi_n = \prod_{d \mid n} u_{n/d}^\mu(d)
\end{equation}

for all integers $n > 1$. Changing the sign of $a$ changes the signs of $\alpha, \beta$ and turns $u_n$ into $(-1)^{n+1}u_n$, which is not a problem, since for the study of $\mathcal{A}$ we are interested only in $|u_n|$. Hence, without loss of generality, we can assume $a > 0$ and $\alpha > |\beta|$, which in turn implies that $u_n, \Phi_n > 0$ for all integers $n > 0$. We conclude this section with an easy lemma regarding the growth of $u_n$ and $\Phi_n$.

**Lemma 2.1.** For all integers $n > 0$, we have

(i) $u_n \geq u_{n-1}$;
(ii) $u_n = \alpha^{n+O(1)}$;
(iii) $\Phi_n = \alpha^{\phi(n)+O(1)}$;

where the implied constants depend on $a$ and $b$.

**Proof.** If $b > 0$, then (i) is clear from the recursion for $u_n$. Hence, suppose $b < 0$, so that $\beta > 0$. After a bit of manipulations, (i) is equivalent to $\alpha^{n-1}(\alpha - 1) \geq \beta^{n-1}(\beta - 1)$, which in turn follows easily since $\alpha > \beta > 0$. Claim (ii) is a consequence of (1). Setting $\gamma := \beta/\alpha$, by (1) and (3), we get

\[ \Phi_n = \alpha^{\phi(n)} \prod_{d \mid n} \left( \frac{1 - \gamma^{n/d}}{\alpha - \beta} \right)^\mu(d) = \alpha^{\phi(n)} \prod_{d \mid n} \left( 1 - \gamma^{n/d} \right)^\mu(d), \]
for all integers \( n > 1 \), where we used the well-known formulas \( \sum_{d \mid n} \mu(d) \frac{n}{d} = \varphi(n) \) and 
\( \sum_{d \mid n} \mu(d) = 0 \). Therefore, since \(|\gamma| < 1\), we have

\[
|\log(\Phi_n/\alpha^{\varphi(n)})| \leq \sum_{d \mid n} |\log(1 - \gamma^d)| \ll \sum_{d=1}^{\infty} |\gamma|^d \ll 1,
\]

and also (iii) is proved. \( \square \)

3. Preliminaries on practical numbers and close relatives

The following lemma on practical numbers will be fundamental later.

**Lemma 3.1.** If \( n \) is a practical number and \( m \leq 2n \) is a positive integer, then \( mn \) is a practical number.

**Proof.** See [4, Lemma 1]. \( \square \)

Close relatives of practical numbers are \( \varphi \)-practical numbers. A \( \varphi \)-practical number is a positive integer \( n \) such that all the positive integers \( m \leq n \) can be written as

\[
m = \sum_{d \in D} \varphi(d),
\]

where \( D \) is a subset of the divisors of \( n \). This notion was introduced by Thompson [11] while studying the degrees of the divisors of the polynomial \( X^n - 1 \). Indeed, \( \varphi \)-practical numbers are exactly the positive integers \( n \) such that \( X^n - 1 \) has a divisor of every degree up to \( n \).

We need a couple of results regarding \( \varphi \)-practical numbers.

**Lemma 3.2.** Let \( n \) be a \( \varphi \)-practical number and \( p \) be a prime number not dividing \( n \). Then \( pn \) is \( \varphi \)-practical if and only if \( p \leq n + 2 \). Moreover, \( p^n \) is \( \varphi \)-practical if and only if \( p \leq n + 1 \), for every integer \( j \geq 2 \).

**Proof.** See [11, Lemma 4.1]. \( \square \)

**Lemma 3.3.** If \( n \) is an even \( \varphi \)-practical number, and if \( d_1, \ldots, d_s \) are all the divisors of \( n \) ordered so that \( \varphi(d_1) \leq \cdots \leq \varphi(d_s) \), then

\[
\varphi(d_{j+1}) \leq \sum_{i=1}^{j} \varphi(d_i),
\]

for all positive integers \( j < s \).

**Proof.** It is not difficult to see that \( n \) is \( \varphi \)-practical if and only if

\[
\varphi(d_{j+1}) \leq 1 + \sum_{i=1}^{j} \varphi(d_i),
\]

for all positive integers \( j < s \) (see [11, p. 1041]). Hence, we have only to prove that \( n \) even ensures that in (5) the equality cannot happen. If \( j = 1 \) then (4) is obvious since \( \{d_1, d_2\} = \{1, 2\} \), so we can assume \( 1 < j < s \). At this point \( \varphi(d_{j+1}) \) is even, while

\[
1 + \sum_{i=1}^{j} \varphi(d_i)
\]

is odd, because \( \varphi(m) \) is even for all integers \( m > 2 \). Thus, in (5) the equality is not satisfied. \( \square \)

Let \( \theta \) be a real-valued arithmetic function, and define \( B_\theta \) as the set containing \( n = 1 \) and all those \( n = p_1^{a_1} \cdots p_k^{a_k} \), where \( p_1 < \cdots < p_k \) are prime numbers and \( a_1, \ldots, a_k \) are positive integers, which satisfy

\[
p_j \leq \theta \left( \prod_{i=1}^{j-1} p_i^{a_i} \right),
\]

for \( j = 1, \ldots, k \), where the empty product is equal to 1. If \( \theta(n) := \sigma(n) + 1 \), then \( B_\theta \) is the set of practical numbers. This is a characterization given by Stewart [8] and Sierpiński [6].
Weingartner proved a general and strong estimate for $\#\mathcal{B}_\theta(x)$. The following is a simplified version adapted just for our purposes.

**Theorem 3.4.** Suppose $\theta(1) \geq 2$ and $n \leq \theta(n) \leq An$ for all positive integers $n$, where $A \geq 1$ is a constant. Then, we have

$$\#\mathcal{B}_\theta(x) \sim \frac{c_\theta x}{\log x},$$

as $x \to +\infty$, where $c_\theta > 0$ is a constant.

*Proof.* See [12, Theorems 1.2 and 5.1]. \qed

4. **Proof of Theorem 1.2**

The key tool of the proof is the following technical lemma.

**Lemma 4.1.** Suppose that $n$ is a sufficiently large positive integer, how large depending on $a$ and $b$. Let $p$ be a prime number and write $n = p^v m$ for some nonnegative integer $v$ and some positive integer $m$ not divisible by $p$. If $m$ is an even $\varphi$-practical number, $n \in \mathcal{A}$, and $p < m$, then $p^v n \in \mathcal{A}$ for all positive integers $k$.

*Proof.* Clearly, it is enough to prove the claim for $k = 1$. Let $d_1 = 1, d_2 = 2, \ldots, d_s$ be all the divisors of $m$, ordered to that $\varphi(d_1) \leq \cdots \leq \varphi(d_s)$. Furthermore, define

$$N_j := u_n \prod_{i=1}^j \Phi_{p^{v+1} d_i},$$

for $j = 1, \ldots, s$. We shall prove that each $N_j$ is practical. This implies the thesis, since $N_s = u_{pm}$ by (2).

We proceed by induction on $j$. First, by (2) and Lemma 2.1(i), we have

$$\Phi_{p^{v+1} d_1} = \Phi_{p^{v+1}} \leq u_{p^{v+1}} \leq u_{p^v m} = u_n,$$

since $p < m$. Hence, applying Lemma 3.1 and the fact that $u_n$ is practical, we get that $N_1 = u_n \Phi_{p^{v+1} d_1}$ is practical.

Now assuming that $N_j$ is practical we shall prove that $N_{j+1}$ is practical. Again, since $N_{j+1} = \Phi_{p^{v+1} d_{j+1}} N_j$, thanks to Lemma 3.1 it is enough to show that the inequality

$$\Phi_{p^{v+1} d_{j+1}} \leq u_n \prod_{i=1}^j \Phi_{p^{v+1} d_i} \tag{6}$$

holds. In turn, by Lemma 2.1(ii) and (iii), we have that (6) is implied by

$$n + \varphi(p^{v+1}) \left[ -\varphi(d_{j+1}) + \sum_{i=1}^j \varphi(d_i) \right] \geq C(j + 1), \tag{7}$$

where $C > 0$ is a constant depending only on $a$ and $b$.

On the one hand, since $m$ is an even $\varphi$-practical number, by Lemma 3.3 we have that the term of (7) in square brackets is nonnegative. On the other hand, for sufficiently large $n$, we have

$$n \geq C (\log n / \log 2 + 1) \geq C (\omega(n) + 1) \geq C(j + 1).$$

Therefore, (7) holds and the proof is complete. \qed

We are ready to prove Theorem 1.2. Pick a sufficiently large positive integer $h$, depending on $a$ and $b$, such that the claim of Lemma 4.1 holds for all integers $n \geq 2^h \cdot 3$. Moreover, by Theorem 1.1, we can assume that $2^j \cdot 3 \in \mathcal{A}$ for all integers $j \geq h$. Put $\mathcal{B} := \mathcal{B}_\theta \setminus \{1\}$, where $\theta(n) := \max\{2, n\}$. Note that, as a consequence of Lemma 3.2, all the elements of $\mathcal{B}$ are even $\varphi$-practical numbers. We shall prove that for all $n \in \mathcal{B}$ we have $2^h \cdot 3 n \in \mathcal{A}$. In this way, thanks to Theorem 3.4, we get

$$\#\mathcal{A}(x) \geq \#\mathcal{B} \left( \frac{x}{2^h \cdot 3} \right) \gg \frac{x}{\log x},$$
for all sufficiently large $x$. Hence, since $1 \in \mathcal{A}$, Theorem 1.2 follows.

We proceed by induction on the number of prime factors of $n \in \mathcal{B}$. If $n \in \mathcal{B}$ has exactly one prime factor, then it follows easily that $n = 2^j$ for some positive integer $j$. Hence, we have $2^h \cdot 3n = 2^{h+j} \cdot 3 \in \mathcal{A}$, as claimed.

Now, assuming that the claim is true for all $n \in \mathcal{B}$ with exactly $k \geq 1$ prime factors, we shall prove it for all $n \in \mathcal{B}$ having $k+1$ prime factors. Write $n = p_1^{a_1} \cdots p_{k+1}^{a_{k+1}}$, where $p_1 < \cdots < p_{k+1}$ are prime numbers and $a_1, \ldots, a_{k+1}$ are positive integers. Put also $m := p_1^{a_1} \cdots p_k^{a_k}$. Since $n \in \mathcal{B}$, we have $m \in \mathcal{B}$ and $p_{k+1} < m$. On the one hand, by the induction hypothesis, $2^h \cdot 3m \in \mathcal{A}$. On the other hand, it is easy to see that $m \in \mathcal{B}$ implies $2^h m \in \mathcal{B}$ and $2^h \cdot 3m \in \mathcal{B}$.

First, suppose $p_{k+1} > 3$. Since $2^h \cdot 3m$ is an even $\varphi$-practical number, $2^h \cdot 3m \in \mathcal{A}$, and $p_{k+1} < 2^h \cdot 3m$, by Lemma 4.1 we get that $2^h \cdot 3n = 2^h \cdot 3mp_{k+1}^{a_{k+1}} \in \mathcal{A}$, as claimed.

On the other hand, if $p_{k+1} = 3$ the situation is similar. Since $2^h m$ is an even $\varphi$-practical number, $2^h \cdot 3m \in \mathcal{A}$, and $p_{k+1} < 2^h m$, by Lemma 4.1 we get that $2^h \cdot 3n = 2^h \cdot 3mp_{k+1}^{a_{k+1}} \in \mathcal{A}$, as claimed. The proof is complete.

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