ON THE EQUIVALENCE OF STOCHASTIC
COMPLETENESS, LIOUVILLE AND KHAS’MINSKII
CONDITION IN LINEAR AND NONLINEAR SETTING

LUCIANO MARI AND DANIELE VALTORTA

Abstract. Set in Riemannian environment, the aim of this paper is to
present and discuss some equivalent characterizations of the Liouville
property relative to special operators, in some sense modeled after the
$p$-Laplacian with potential. In particular, we discuss the equivalence
between the Liouville property and the Khas’minskii condition, i.e. the
existence of an exhaustion function which is also a supersolution for
the operator outside a compact set. This generalizes a previous result
obtained by one of the authors and answers to a question in [25].

Sui quisque laplaciani faber

1. Introduction

In what follows, let $M$ denote a connected Riemannian manifold of di-

dimension $m$, with no boundary. We stress that no completeness assumption

is required. The relationship between the probabilistic notions of stochastic

completeness and parabolicity (respectively the non-explosion and the recur-

rence of the Brownian motion on $M$) and function-theoretic properties of

has been the subject of an active area of research in the last decades. Deep

connections with the heat equation, Liouville type theorems, capacity theory

and spectral theory have been described, for instance, in the beautiful survey

[8]. In [22] and [21], the authors showed that stochastic completeness and

parabolicity are also related to weak maximum principles at infinity. This

characterization reveals to be fruitful in investigating many kinds of geometric

problems (for a detailed account, see [23]). Among the various conditions

equivalent to stochastic completeness, the following two are of prior interest
to us:

Date: May 16, 2013.

2010 Mathematics Subject Classification. Primary 31C12 (potential theory on Rie-

mannian manifolds, Secondary: 35B53(Liouville theorems), 58J65 (stochastic equations

and processes on manifolds), 58J05 (elliptic equations on manifolds).

Key words and phrases. Khas’minskii condition, stochastic completeness, parabolicity.
- \([L^\infty, \text{Liouville}]\) for some (any) \(\lambda > 0\), the sole bounded, non-negative, continuous weak solution of \(\Delta u - \lambda u \geq 0\) is \(u = 0\);
- \([\text{weak maximum principle}]\) for every \(u \in C^2(M)\) with \(u^* = \sup_M u < +\infty\), and for every \(\eta < u^*\),
\[
\inf_{\Omega_\eta} \Delta u \leq 0, \quad \text{where} \quad \Omega_\eta = u^{-1}\{(\eta, +\infty)\}.
\]

R.Z. Khas’minskii [11] has found the following criterion for stochastic completeness. We recall that a \(w \in C_0^0(M)\) is called an exhaustion if it has compact sublevels \(w^{-1}((-\infty, t])\), \(t \in \mathbb{R}\).

**Theorem 1.1** (Khas’minskii test, [11]). *Suppose that there exists a compact set \(K\) and a function \(w \in C^0_0(M) \cap C^2_0(M \setminus K)\) satisfying for some \(\lambda > 0\):

\(i\) \(w\) is an exhaustion;
\(ii\) \(\Delta w - \lambda w \leq 0\) on \(M \setminus K\).

Then \(M\) is stochastically complete.

A very similar characterization holds for the parabolicity of \(M\). Namely, among many others, parabolicity is equivalent to:
- every bounded, non-negative continuous weak solutions of \(\Delta u \geq 0\) on \(M\) is constant;
- for every non-constant \(u \in C^2(M)\) with \(u^* = \sup_M u < +\infty\), and for every \(\eta < u^*\),
\[
\inf_{\Omega_\eta} \Delta u < 0, \quad \text{where} \quad \Omega_\eta = u^{-1}\{(\eta, +\infty)\}.
\]

Note that the first condition is precisely case \(\lambda = 0\) of the Liouville property above. As for Khas’minskii type conditions, it has been proved by M. Nakai [19] and Z. Kuramochi [14] that the parabolicity of \(M\) is indeed equivalent to the existence of a so-called Evans potential, that is, a harmonic function \(w\) defined outside a compact set \(K\) and such that \(w = 0\) on \(\partial K\). To the best of our knowledge, an analogue of such equivalence for stochastic completeness has still to be proved, and this is the starting point of the present work.

With some modifications, it is possible to define the Liouville property, the Khas’minskii test and Evans potentials also for \(p\)-Laplacians or other nonlinear operators, and the aim of this paper is to prove that in this more general setting the Liouville property is equivalent to the Khas’minskii test, answering in the affirmative to a question raised in [25] (question 4.6). After that, a brief discussion on the connection with appropriate definitions of the weak maximum principle is included. The final section will be devoted to the existence of Evans type potentials in the particular setting of radially symmetric manifolds. To fix the ideas, we cite the main theorem in the “easy case” of the \(p\)-Laplacian, and then introduce the more general (and more
technical) operators to which our theorem applies. Recall that for a function \( u \in W^{1,p}(\Omega) \), the \( p \)-laplacian \( \Delta_p \) is defined weakly as:

\[
\int_{\Omega} \phi \Delta_p u = - \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle
\]

where \( \phi \in C^\infty_c(\Omega) \) and integration is with respect to the Riemannian measure.

**Theorem 1.2.** Let \( M \) be a Riemannian manifold and let \( p > 1, \lambda \geq 0 \). Then, the following conditions are equivalent.

\((W)\) The weak maximum principle for \( C^0 \) holds for \( \Delta_p \), that is, for every non-constant \( u \in C^0(M) \cap W^{1,p}_{\text{loc}}(M) \) with \( u^* = \sup_M u < \infty \) and for every \( \eta < u^* \) we have:

\[
\inf_{\Omega_\eta} \Delta_p u \leq 0 \quad (< 0 \text{ if } \lambda = 0)
\]

weakly on \( \Omega_\eta = u^{-1}\{ (\eta, +\infty) \} \).

\((L)\) Every non-negative, \( L^\infty \cap W^{1,p}_{\text{loc}} \) solution \( u \) of \( \Delta_p u - \lambda u^{p-1} \geq 0 \) is constant (hence zero if \( \lambda > 0 \)).

\((K)\) For every compact \( K \) with smooth boundary, there exists an exhaustion \( w \in C^0(M \setminus K) \cap W^{1,p}_{\text{loc}}(M \setminus K) \) such that

\[
w > 0 \quad \text{on } M \setminus K, \quad w = 0 \quad \text{on } \partial K, \quad \Delta_p w - \lambda u^{p-1} \leq 0.
\]

Up to some minor changes, the implications \((W) \iff (L)\) and \((K) \implies (L)\) have been shown in [24], Theorem A, where it is also proved that, in \((W)\) and \((L)\), \( u \) can be equivalently restricted to the class \( C^1(M) \). In this respect, see also [25], Section 2. On the other hand, the second author in [30] has proved that \((L) \implies (K)\) when \( \lambda = 0 \). The proof developed in this article covers both the case \( \lambda = 0 \) and \( \lambda > 0 \), is easier and more straightforward and, above all, does not depend on some features which are typical of the \( p \)-Laplacian.

### 2. Definitions and main theorems

**Notational conventions.** We set \( \mathbb{R}^+ = (0, +\infty), \mathbb{R}^+_0 = [0, +\infty), \) and \( \mathbb{R}^-, \mathbb{R}^-_0 \) accordingly; for a function \( u \) defined on some set \( \Omega \), \( u^* \) and \( u_* \) will denote, respectively, the sup and inf of \( u \) on \( \Omega \), where the sup / inf has to be intended in the sense of Lebesque spaces if \( u \) is not continuous; we will write \( K \subset \Omega \) whenever the set \( K \) has compact closure in \( \Omega \); \( \text{Lip}_{\text{loc}}(M) \) denotes the class of locally Lipschitz functions on \( M \); with \( u \in \text{H}^1_{\text{loc}}(M) \) we mean that, for every \( \Omega \subset M, u \in C^{0,\alpha}(\Omega) \) for some \( \alpha \in (0,1] \) possibly depending on \( \Omega \). We will use the symbol \( Q \overset{\Delta}{=} \ldots \) to define the quantity \( Q \) as . . . .

In order for our techniques to work, we will consider quasilinear operators of the following form. Let \( A : TM \to TM \) be a Caratheodory map, that
is if \( \pi : TM \to M \) is the bundle projection, \( \pi \circ A = A \), moreover every representation \( \tilde{A} \) of \( A \) in local charts satisfies

- \( \tilde{A}(x, \cdot) \) continuous for a.e. \( x \in M \)
- \( \tilde{A}(\cdot, v) \) measurable for every \( v \in \mathbb{R}^m \)

Note that every continuous bundle map satisfies these assumptions. Furthermore, let \( B : M \times \mathbb{R} \to \mathbb{R} \) be of Caratheodory type, that is, \( B(\cdot, t) \) is measurable for every fixed \( t \in \mathbb{R} \), and \( B(x, \cdot) \) is continuous for a.e. \( x \in M \). We shall assume that there exists \( p > 1 \) such that, for each fixed open set \( \Omega \subset M \), the following set of assumptions \( \mathcal{S} \) is met:

\begin{enumerate}
  \item \([A(X)|X] \geq a_1|X|^p \quad \forall X \in TM \)  
  \item \([A(X)] \leq a_2|X|^{p-1} \quad \forall X \in TM \)  
  \item \( A \) is strictly monotone, i.e. \( \langle A(X) - A(Y) | X - Y \rangle_p \geq 0 \) for every \( x \in M, X, Y \in T_xM \), with equality if and only if \( X = Y \)  
  \item \(|B(x,t)| \leq b_1 + b_2|t|^{p-1} \) for \( t \in \mathbb{R} \)  
  \item for a.e. \( x \), \( B(x,\cdot) \) is monotone non-decreasing  
  \item for a.e. \( x \), \( B(x,t) t \geq 0 \), \( (B3) \)
\end{enumerate}

where \( a_1, a_2, b_1, b_2 \) are positive constants possibly depending on \( \Omega \). As explained in remark 4.2 we could state our main theorem relaxing condition \( B1 \) to:

\begin{enumerate}
  \item \([B(x,t)] \leq b(t) \quad \forall t \in \mathbb{R} \) \( (B1^+) \)
\end{enumerate}

for some positive and finite function \( b \), however for the moment we assume \( B1 \) to avoid some complications in the notation, and explain later how to extend our result to this more general case.

We define the operators \( \mathcal{F}, A, B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^* \) by setting

\[
A : \quad u \mapsto \left[ \phi \in W^{1,p}(\Omega) \mapsto \int_\Omega \langle A(\nabla u) \nabla \phi \rangle \right]
\]

\[
B : \quad u \mapsto \left[ \phi \in W^{1,p}(\Omega) \mapsto \int_\Omega B(x, u(x)) \phi \right]
\]

\[
\mathcal{F} \doteq A + B.
\]

With these assumptions, it can be easily verified that both \( A \) and \( B \) maps to continuous linear functionals on \( W^{1,p}(\Omega) \) for each fixed \( \Omega \subset M \). We define the operators \( L_A, L_\mathcal{F} \) accordingly to the distributional equality:

\[
\int_M \phi L_A u \doteq - \langle A(u), \phi \rangle, \quad \int_M \phi L_\mathcal{F} u \doteq - \langle \mathcal{F}(u), \phi \rangle
\]
for every \( u \in W^{1,p}_{\text{loc}}(M) \) and \( \phi \in C_{c}^{\infty}(M) \), where \( <,> \) is the duality. In other words, in the weak sense

\[
L_{\mathcal{F}}u = \text{div}(A(\nabla u)) - B(x,u) \quad \forall \ u \in W^{1,p}_{\text{loc}}(M).
\]

Example 2.1. The \( p \)-Laplacian defined in (3), corresponding to the choices \( A(X) \doteq |X|^{p-2}X \) and \( B(x,t) \doteq 0 \), satisfies all the assumptions in \( \mathcal{S} \) for each \( \Omega \subseteq M \). Another admissible choice of \( B \) is \( B(x,t) \doteq \lambda |t|^{p-2}t \), where \( \lambda \geq 0 \). For such a choice,

\[
(6) \quad L_{\mathcal{F}}u = \Delta_{p}u - \lambda |u|^{p-2}u
\]

is the operator of Theorem 1.2. We stress that, however, in \( \mathcal{S} \) we require no homogeneity condition neither on \( A \) nor on \( B \).

Example 2.2. More generally, as in [24] and in [28], for each function \( \varphi \in C_{c}^{0}(\mathbb{R}^{+}) \) such that \( \varphi > 0 \) on \( \mathbb{R}^{+} \), \( \varphi(0) = 0 \), and for each symmetric, positive definite 2-covariant continuous tensor field \( h \in \Gamma(\text{Sym}_{2}(TM)) \), we can consider differential operators of type

\[
L_{\varphi,h}u = \text{div} \left( \frac{\varphi(|\nabla u|)}{|\nabla u|} h(\nabla u, \cdot) \right),
\]

where \( \sharp \) is the musical isomorphism. Due to the continuity and the strict positivity of \( h \), the conditions (A1) and (A2) in \( \mathcal{S} \) can be rephrased as

\[
(7) \quad a_{1}t^{p-1} \leq \varphi(t) \leq a_{2}t^{p-1}.
\]

Furthermore, if \( \varphi \in C^{1}(\mathbb{R}^{+}) \), a sufficient condition for (M) to hold is given by

\[
(8) \quad \frac{\varphi(t)}{t} h(X,X) + \left( \frac{\varphi'(t)}{t} - \frac{\varphi(t)}{t} \right) \langle Y|X \rangle h(Y,X) > 0
\]

for every \( X,Y \) with \( |X| = |Y| = 1 \). The reason why it implies the strict monotonicity can be briefly justified as follows: for \( L_{\varphi,h} \), (M) is equivalent to requiring

\[
(9) \quad \frac{\varphi(|X|)}{|X|} h(X,X - Y) - \frac{\varphi(|Y|)}{|Y|} h(Y,X - Y) > 0 \quad \text{if} \quad X \neq Y.
\]

In the nontrivial case when \( X \) and \( Y \) are not proportional, the segment \( Z(t) = Y + t(X - Y) \), \( t \in [0,1] \) do not touch the zero vector, so that

\[
F(t) = \frac{\varphi(|Z|)}{|Z|} h(Z,Z')
\]

is \( C^{1} \). Condition (8) implies that \( F'(t) > 0 \). Hence, integrating we get \( F(1) > F(0) \), that is, (9). We observe that, if \( h \) is the metric tensor, the strict monotonicity is satisfied whenever \( \varphi \) is strictly increasing on \( \mathbb{R}^{+} \) even without any differentiability assumption on \( \varphi \).
Example 2.3. Even more generally, if \( A \) is of class \( C^1 \), a sufficient condition for the monotonicity of \( A \) has been considered in [1], Section 5 (see the proof of Theorem 5.3). Indeed, the authors required that, for every \( x \in M \) and every \( X \in T_xM \), the differential of the map \( A_x : T_xM \to T_xM \) at the point \( X \in T_xM \) is positive definite as a linear endomorphism of \( T_X(T_xM) \). This is the analogue, for Riemannian manifolds, of Proposition 2.4.3 in [27].

We recall the concept of subsolutions and supersolutions for \( L_F \).

Definition 2.4. We say that \( u \in W^{1,p}_{\text{loc}}(M) \) solves \( L_Fu \geq 0 \) (resp. \( \leq 0 \), \( = 0 \)) weakly on \( M \) if, for every non-negative \( \phi \in C^\infty_c(M) \), \( <F(u), \phi> \leq 0 \), (resp., \( \geq 0 \), \( = 0 \)). Solutions of \( L_Fu \geq 0 \) (resp., \( \leq 0 \), \( = 0 \)) are called (weak) subsolutions (resp. supersolutions, solutions).

Remark 2.5. Note that, since \( B \) is Caratheodory, \([B3]\) implies that \( B(x, 0) = 0 \) a.e. on \( M \). Therefore, the constant function \( u = 0 \) solves \( L_Fu = 0 \). Again by \([B3]\), positive constants are supersolutions.

Following [24] and [25], we present the analogues of the \( L^\infty \)-Liouville property and the Khas'minskii property for the nonlinear operators constructed above.

Definition 2.6. Let \( M \) be a Riemannian manifold, and let \( A, B, F \) be as above.

- We say that the \( L^\infty \)-Liouville property (\( L \)) for \( L^\infty \) (respectively, \( \text{H" o\l}_{\text{loc}} \)) functions holds for the operator \( L_F \) if every \( u \in L^\infty(M) \cap W^{1,p}_{\text{loc}}(M) \) (respectively, \( \text{H" o\l}_{\text{loc}}(M) \cap W^{1,p}_{\text{loc}}(M) \)) essentially bounded, satisfying \( u \geq 0 \) and \( L_Fu \geq 0 \) is constant.

- We say that the Khas’minskii property (\( K \)) holds for \( L_F \) if, for every pair of open sets \( K \Subset \Omega \Subset M \) with Lipschitz boundary, and every \( \varepsilon > 0 \), there exists an exhaustion function
  \[
  w \in C^0(\overline{M\setminus K}) \cap W^{1,p}_{\text{loc}}(M\setminus K)
  \]
  such that
  \[
  w > 0 \quad \text{on } M\setminus K, \quad w = 0 \quad \text{on } \partial K,
  \]
  \[
  w \leq \varepsilon \quad \text{on } \Omega\setminus K, \quad L_Fw \leq 0 \quad \text{on } M\setminus K.
  \]
  Such a \( w \) will be called a Khas’minskii potential relative to the triple \((K, \Omega, \varepsilon)\).

- A Khas’minskii potential \( w \) relative to some triple \((K, \Omega, \varepsilon)\) is called an Evans potential if \( L_Fw = 0 \) on \( M\setminus K \). The operator \( L_F \) has the Evans property (\( E \)) if there exists an Evans potential for every triple \((K, \Omega, \varepsilon)\).

The main result in this paper is the following
Theorem 2.7. Let $M$ be a Riemannian manifold, and let $A, B$ satisfy the set of assumptions $\mathcal{A}$, with $(B1+)$ instead of $(B1)$. Define $A, B, F$ as in (5), and $L_A, L_F$ accordingly. Then, the conditions $(L)$ for Hölder, $(L)$ for $L^\infty$ and $(K)$ are equivalent.

Remark 2.8. It should be observed that if $L_F$ is homogeneous, as in (6), the Khas’minskii condition considerably simplifies as in $(K)$ of Theorem 1.2. Indeed, the fact that $\delta w$ is still a supersolution for every $\delta > 0$, and the continuity of $w$, allow to get rid of $\Omega$ and $\epsilon$.

Next, in Section 5 we briefly describe in which way $(L)$ and $(K)$ are related to the concepts of weak maximum principle and parabolicity. Such relationship has been deeply investigated in [23], [24], whose ideas and proofs we will closely follow. With the aid of Theorem 2.7, we will be able to prove the next Theorem 2.11. To state it, we shall restrict to a particular class of potentials $B(x,t)$, those of the form $B(x,t) = b(x)f(t)$ with

$$b, b^{-1} \in L^\infty_{\text{loc}}(M), \quad b > 0 \text{ a.e. on } M;$$

$$f \in C^0(\mathbb{R}), \quad f(0) = 0, \quad f \text{ is non-decreasing on } \mathbb{R}.$$  

Clearly, $B$ satisfies $(B1+), (B2)$ and $(B3)$. As for $A$, we require $(A1)$ and $(A2)$, as before.

Definition 2.9. Let $A, B$ be as above, define $A, B, F$ as in (5) and $L_A, L_F$ accordingly.

$(W)$ We say that $b^{-1}L_A$ satisfies the weak maximum principle for $C^0$ functions if, for every $u \in C^0(M) \cap W^{1,p}_{\text{loc}}(M)$ such that $u^* < +\infty$, and for every $\eta < u^*$,

$$\inf_{\Omega_\eta} b^{-1}L_A u \leq 0 \quad \text{weakly on } \Omega_\eta = u^{-1}\{(\eta, +\infty)\}.$$

$(W_{pa})$ We say that $b^{-1}L_A$ is parabolic if, for every non-constant $u \in C^0(M) \cap W^{1,p}_{\text{loc}}(M)$ such that $u^* < +\infty$, and for every $\eta < u^*$,

$$\inf_{\Omega_\eta} b^{-1}L_A u < 0 \quad \text{weakly on } \Omega_\eta = u^{-1}\{(\eta, +\infty)\}.$$

- We say that $F$ is of type 1 if, in the potential $B(x,t)$, the factor $f(t)$ satisfies $f > 0$ on $\mathbb{R}^+$. Otherwise, when $f = 0$ on some interval $[0, T]$, $F$ is called of type 2.

Remark 2.10. $\inf_{\Omega_\eta} b^{-1}L_A u \leq 0$ weakly means that, for every $\epsilon > 0$, there exists $0 \leq \phi \in C^\infty_c(\Omega_\eta)$, $\phi \neq 0$ such that

$$- < A(u), \phi > < \epsilon \int b \phi.$$
Similarly, with \( \inf_{\Omega} b^{-1} L_A u < 0 \) weakly we mean that there exist \( \varepsilon > 0 \) and \( \phi \in C^\infty_c(\Omega) \), \( \phi \not\equiv 0 \) such that \( -\langle A(u), \phi \rangle < -\varepsilon \int b \phi \).

**Theorem 2.11.** In the assumptions \([10]\) for \( B(x,t) = b(x)f(t) \), and \([A1], [A2]\) for \( A \), the following properties are equivalent:

- The operator \( b^{-1} L_A \) satisfies \((W)\);  
- Property \((L)\) holds for some (hence any) operator \( F \) of type 1;  
- Property \((K)\) holds for some (hence any) operator \( F \) of type 1;  

Furthermore, in the same assumptions, the next equivalence holds:

- The operator \( b^{-1} L_A \) is parabolic;  
- Property \((L)\) holds for some (hence any) operator \( F \) of type 2;  
- Property \((K)\) holds for some (hence any) operator \( F \) of type 2;  

In the final Section 6 we address the question whether \((W)\), \((K)\), \((L)\) are equivalent to the Evans property \((E)\). Indeed, it should be observed that, in Theorem 2.7, now growth control on \( B \) as a function of \( t \) is required at all. On the contrary, as we will see, the validity of the Evans property forces some precise upper bound for its growth. To better grasp what we shall expect, we will restrict to the case of radially symmetric manifolds. For the statements of the main results, we refer the reader directly to Section 6.

3. Technical tools

In this section we introduce some technical tools, such as the obstacle problem, that will be crucial to the proof of our main theorems. Throughout this section, we will always assume that the assumptions in \( S \) are satisfied, if not explicitly stated. First, we state some basic results on subsolutions-supersolutions such as the comparison principle, which follows from the monotonicity of \( A \) and \( B \).

**Proposition 3.1.** Assume \( w \) and \( s \) are a super and a subsolution defined on \( \Omega \). If \( \min\{w - s, 0\} \in W^{1,p}_0(\Omega) \), then \( w \geq s \) a.e. in \( \Omega \).

**Proof.** This theorem and its proof, which follows quite easily using the right test function in the definition of supersolution, are standard in potential theory. For a detailed proof see [1], Theorem 4.1. \( \square \)

Next, we observe that \( A, B \) satisfy all the assumptions for the subsolution-supersolution method in [13] to be applicable.

**Theorem 3.2** ([13], Theorems 4.1, 4.4 and 4.7). Let \( \phi_1, \phi_2 \in L^{\infty}_{\text{loc}} \cap W^{1,p}_{\text{loc}} \) be, respectively, a subsolution and a supersolution for \( L_F \) on \( M \), and suppose that \( \phi_1 \leq \phi_2 \) a.e. on \( M \). Then, there is a solution \( u \in L^{\infty}_{\text{loc}} \cap W^{1,p}_{\text{loc}} \) of \( L_F u = 0 \) satisfying \( \phi_1 \leq u \leq \phi_2 \) a.e. on \( M \).
A fundamental property is the strong maximum principle, which follows from the next Harnack inequality.

**Theorem 3.3** ([27], Theorems 7.1.2, 7.2.1 and 7.4.1). Let \( u \in W^{1,p}_{\text{loc}}(M) \) be a non-negative solution of \( L_A u \leq 0 \). Let the assumptions in \( \mathcal{S} \) be satisfied. Fix a relatively compact open set \( \Omega \Subset M \).

(i) Suppose that \( 1 < p \leq m \), where \( m = \dim M \). Then, for every ball \( B_{4R} \subset \Omega \) and for every \( s \in (0, (p-1)m/p) \), there exists a constant \( C \) depending on \( R \), on the geometry of \( B_{4R} \), on \( m \) and on the parameters \( a_1, a_2 \) in \( \mathcal{S} \) such that

\[
\|u\|_{L^s(B_{2R})} \leq C \left( \text{essinf}_{B_{2R}} u \right).
\]

(ii) Suppose that \( p > m \). Then, for every ball \( B_{4R} \subset \Omega \), there exists a constant \( C \) depending on \( R \), on the geometry of \( B_{4R} \), on \( m \) and on the parameters \( a_1, a_2 \) in \( \mathcal{S} \) such that

\[
\text{esssup}_{B_{2R}} u \leq C \left( \text{essinf}_{B_{2R}} u \right).
\]

In particular, for every \( p > 1 \), each non-negative solution \( u \) of \( L_A u \leq 0 \) on \( M \) is such that either \( u = 0 \) on \( M \) or \( \text{essinf}_{\Omega} u > 0 \) for every relatively compact set \( \Omega \).

**Remark 3.4.** We spend few words to comment on the Harnack inequalities quoted from [27]. In our assumptions \( \mathcal{S} \), the functions \( \bar{a}_2, \bar{a}, b_1, b_2, b \) in Chapter 7, (7.1.1) and (7.1.2) and the function \( a \) in the monotonicity inequality (6.1.2) can be chosen to be identically zero. Thus, in Theorems 7.1.2 and 7.4.1 the quantity \( k(R) \) is zero. This gives no non-homogeneous term in the Harnack inequality, which is essential for us. For this reason, we cannot weaken (A2) to

\[
|A(X)| \leq a_2 |X|^{p-1} + \bar{a}
\]

locally on \( \Omega \), since the presence of non-zero \( \bar{a} \) implies that \( k(R) > 0 \). It should be observed that Theorem 7.1.2 is only stated for \( 1 < p < m \) but, as observed at the beginning of Section 7.4, the proof can be adapted to cover the case \( p = m \).

**Remark 3.5.** In the rest of the paper, we will only use the fact that either \( u \equiv 0 \) or \( u > 0 \) on \( M \), that is, the strong maximum principle. It is worth observing that, for the operators \( L_A = L_{\varphi,h} \) described in Example 2.2, very general strong maximum principles for \( C^1 \) or \( \text{Lip}_{\text{loc}} \) solutions of \( L_{\varphi,h} u \leq 0 \) on Riemannian manifolds have been obtained in [26] (see Theorem 1.2 when \( h \) is the metric tensor, and Theorems 5.4 and 5.6 for the general case). In particular, if \( h \) is the metric tensor, the sole requirements

(11) \( \varphi \in C^0(\mathbb{R}_+^+) \), \( \varphi(0) = 0 \), \( \varphi > 0 \) on \( \mathbb{R}_+^+ \), \( \varphi \) in strictly increasing on \( \mathbb{R}_+^+ \).
are enough for the strong maximum principle to hold for $C^1$ solutions of $L\varphi u \leq 0$. Hence, for instance for $L\varphi$, the two-sided bound (7) on $\varphi$ can be weakened to any bound ensuring that the comparison and strong maximum principles hold, the subsolution-supersolution method is applicable and the obstacle problem has a solution. For instance, besides (11), the requirement (12) $\varphi(0) = 0$, $a_1 t^{p-1} \leq \varphi(t) \leq a_2 t^{p-1} + a_3$ is enough for Theorems 3.1, 3.2, and it also suffices for the obstacle problem to admit a unique solution, as the reader can infer from the proof of the next Theorem 3.11.

**Remark 3.6.** Regarding the above observation, if $\varphi$ is merely continuous then even solutions of $L\varphi u = 0$ are not expected to be $C^1$, nor even $Lip_{loc}$. Indeed, in our assumptions the optimal regularity for $u$ is (locally) some Hölder class, see the next Theorem 3.7. If $\varphi \in C^1(\mathbb{R}^+)$ is more regular, then we can avail of the regularity result in [29] to go even beyond the $C^1$ class. Indeed, under the assumptions

$$\gamma(k + t)^{p-2} \leq \min \left( \varphi'(t), \frac{\varphi(t)}{t} \right) \leq \max \left( \varphi'(t), \frac{\varphi(t)}{t} \right) \leq \Gamma(k + t)^{p-2},$$

for some $k \geq 0$ and some positive constants $\gamma \leq \Gamma$, then each solution of $L\varphi u = 0$ is in some class $C^{1,\alpha}$ on each relatively compact set $\Omega$, where $\alpha \in (0, 1)$ may depend on $\Omega$. When $h$ is not the metric tensor, the condition on $\varphi$ and $h$ is more complicated, and we refer the reader to [24] (in particular, see (0.1) (v), (vi) p. 803).

Part of the regularity properties that we need are summarized in the following

**Theorem 3.7.** Let the assumptions in $\mathcal{S}$ be satisfied.

(i) [17], Theorem 4.8] If $u$ solves $L_F u \leq 0$ on some open set $\Omega$, then there exists a representative in $W^{1,p}(\Omega)$ which is lower semicontinuous.

(ii) [15], Theorem 1.1 p. 251] If $u \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ is a bounded solution of $L_F u = 0$ on $\Omega$, then there exists $\alpha \in (0, 1)$ depending on the geometry of $\Omega$, on the constants in $\mathcal{S}$ and on $\|u\|_{L^\infty(\Omega)}$ such that $u \in C^{0,\alpha}(\Omega)$. Furthermore, for every $\Omega_0 \Subset \Omega$, there exists $C = C(\gamma, \text{dist}(\Omega_0, \partial \Omega))$ such that

$$\|u\|_{C^{0,\alpha}(\Omega_0)} \leq C.$$

**Remark 3.8.** As for (i), it is worth observing that, in our assumptions, both $b_0$ and $a$ in the statement of [17], Theorem 4.8 are identically zero. Although we will not need the following properties, it is worth noting that
any $u$ solving $L_F u \leq 0$ has a Lebesgue point everywhere and is also $p$-finely continuous (where finite).

Next, this simple elliptic estimate for locally bounded supersolutions is useful:

**Proposition 3.9.** Let $u$ be a bounded solution of $L_F u \leq 0$ on $\Omega$. Then, for every relatively compact, open set $\Omega_0 \Subset \Omega$ there is a constant $C > 0$ depending on $p$, $\Omega$, $\Omega_0$, and on the parameters in $\mathcal{F}$ such that

$$\|\nabla u\|_{L^p(\Omega_0)} \leq C(1 + \|u\|_{L^\infty(\Omega)})$$

**Proof.** Given a supersolution $u$, the monotonicity of $B$ assures that for every positive constant $c$ also $u + c$ is a supersolution, so without loss of generality we may assume that $u_* = 0$. Thus, $u_* = \|u\|_{L^\infty(\Omega)}$.

Shortly, with $\|\cdot\|_p$ we denote the $L^p$ norm on $\Omega$, and with $C$ we denote a positive constant depending on $p$, $\Omega$, and on the parameters in $\mathcal{F}$, that may vary from place to place. Let $\eta \in C^\infty_c(\Omega)$ be such that $0 \leq \eta \leq 1$ on $\Omega$ and $\eta = 1$ on $\Omega_0$. Then, we use the non-negative function $\phi = \eta_p(\nabla u - u)$ in the definition of supersolution to get, after some manipulation and from (A1), (A2) and (B3),

$$a_1 \int_\Omega \eta^p |\nabla u|^p \leq pa_2 \int_\Omega |\nabla u|^{p-1} \eta^{p-1}(u_* - u) |\nabla \eta| + \int_\Omega \eta^p B(x, u) u_*$$

Using (B3), the integral involving $B$ is roughly estimated as follows:

$$\int_\Omega \eta^p B(x, u) u_* \leq |\Omega|(b_1 u_* + b_2(u_*)^p) \leq C(1 + u_*)^p,$$

where the last inequality follows by applying Young inequality on the first addendum. As for the term involving $|\nabla \eta|$, using $(u_* - u) \leq u_*$ and again Young inequality $|ab| \leq |a|^p/(pe^p) + \varepsilon |b|^q/q$ we obtain

$$p a_2 \int_\Omega (|\nabla u|^{p-1} \eta^{p-1}(u_* - u) |\nabla \eta|) \leq pa_2 \int_\Omega (|\nabla u|^{p-1} \eta^{p-1}) (u_* |\nabla \eta|) \leq \frac{a_2}{2} \|\eta \nabla u\|_p^p + \frac{a_2 p \varepsilon^q}{q} \|\nabla \eta\|_p^p (u_*)^p$$

Choosing $\varepsilon$ such that $a_2 \varepsilon^{-p} = a_1/2$, inserting (14) and (15) into (13) and rearranging we obtain

$$\frac{a_1}{2} \|\eta \nabla u\|_p^p \leq C \left[ 1 + (1 + \|\nabla \eta\|_p^p)(u_*)^p \right].$$

Since $\eta = 1$ on $\Omega_0$ and $\|\nabla \eta\|_p \leq C$, taking the $p$-root the desired estimate follows. \hfill $\square$
Remark 3.10. We observe that, when $B \neq 0$ we cannot apply the technique of [9], Lemma 3.27 to get a Caccioppoli-type inequality for bounded, non-negative supersolutions. The reason is that subtracting a positive constant to a supersolution does not yield, for general $B \neq 0$, a supersolution. It should be stressed that, however, when $p \leq m$ a refined Caccioppoli inequality for supersolution has been given in in [17], Theorem 4.4.

Now, we fix our attention on the obstacle problem. There are a lot of references regarding this subject (for example see [17], Chapter 5 or [9], Chapter 3 in the case $B = 0$). As often happens, notation can be quite different from one reference to another. Here we try to adapt the conventions used in [9], and for the reader’s convenience we also sketch some of the proofs.

First of all, some definitions. Given a function $\psi : \Omega \to \mathbb{R} \cup \pm \infty$, and given $\theta \in W^{1,p}(\Omega)$, we define the closed convex set

$$K_{\psi, \theta} = \{ f \in W^{1,p}(\Omega) \mid f \geq \psi \text{ a.e. and } f - \theta \in W^{1,0}(\Omega) \}.$$ 

Loosely speaking, $\theta$ fixes the boundary value of the solution $u$, while $\psi$ is the “obstacle"-function. Most of the times, obstacle and boundary function coincide, and in this case we use the convention $K_{\theta} \equiv K_{\theta, \theta}$. We say that $u \in K_{\psi, \theta}$ solves the obstacle problem if for every $\phi \in K_{\psi, \theta}$:

$$< F(u), \varphi - u > \geq 0.$$ 

(16)

Note that for every $\phi \in C_c(\Omega)$ the function $\varphi = u + \phi$ belongs to $K_{\psi, \theta}$, and this implies that the solution to the obstacle problem is always a supersolution. Note also that if we choose $\psi = -\infty$, we get the standard Dirichlet problem with Sobolev boundary value $\theta$ for the operator $F$, in fact in this case any test function $\phi \in C_c(\Omega)$ verifies $u \pm \phi \in K_{\psi, \theta}$, and so inequality in (16) becomes an equality. Next, we address the solvability of the obstacle problem.

Theorem 3.11. In the assumptions $\mathcal{F}$, if $\Omega$ is relatively compact and $K_{\psi, \theta}$ is nonempty, then there exists a unique solution to the relative obstacle problem.

Proof. The proof is basically the same if we assume $B = 0$, as in [9], Appendix 1; in particular, it is an application of Stampacchia theorem, see for example Corollary III.1.8 in [12]. To apply the theorem, we shall verify that $K_{\psi, \theta}$ is closed and convex, which follows straightforwardly from its very definition, and that $F : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is weakly continuous, monotone and coercive. Monotonicity is immediate by properties (M), (B2). To prove that $F$ is weakly continuous, we take a sequence $u_i \to u$ in $W^{1,p}(\Omega)$. By using (A2) and (B1), we deduce from (5) that

$$| < F(u_i), \phi > | \leq \left( (a_2 + b_2) \| u_i \|_{W^{1,p}(\Omega)}^{p-1} + b_1 |\Omega|^{\frac{p-1}{p}} \right) \| \phi \|_{W^{1,p}(\Omega)}$$
Hence the $W^{1,p}(\Omega)^*$ norm of $\{F(u_i)\}$ is bounded, and so from any subsequence we can extract a weakly convergent sub-subsequence $F(u_k) \to z$ in $W^{1,p}(\Omega)^*$, for some $z$. From $u_k \to u$ in $W^{1,p}(\Omega)$, by Riesz theorem we get (up to a further subsequence) $(u_k, \nabla u_k) \to (u, \nabla u)$ pointwise on $\Omega$, and since the maps

$$X \mapsto A(X), \quad t \mapsto B(x, t)$$

are continuous, then necessarily $z = F(u)$. Since this is true for every weakly convergent subsequence $\{F(u_k)\}$, we deduce that the whole $F(u_i)$ converges weakly to $F(u)$. This proves the weak continuity of $F$.

Coercivity on $K_{\psi, \theta}$ follows if we fix any $\varphi \in K_{\psi, \theta}$ and consider a diverging sequence $\{u_i\} \subset K_{\psi, \theta}$ and calculate:

$$\frac{\langle F(u_i) - F(\varphi) \rangle_{L^p(\Omega)}}{\|u_i - \varphi\|_{W^{1,p}(\Omega)}} \geq \frac{\langle A(u_i) - A(\varphi) \rangle_{L^p(\Omega)}}{\|u_i - \varphi\|_{W^{1,p}(\Omega)}} \overset{A1}/\overset{A2}{} \geq a_1 \left( \|\nabla u_i\|_p^p + \|\nabla \varphi\|_p^p \right) - a_2 \left( \|\nabla u_i\|_p^{p-1} \|\nabla \varphi\|_p + \|\nabla u_i\|_p \|\nabla \varphi\|_p^{p-1} \right)$$

This last quantity tends to infinity as $i$ goes to infinity thanks to the Poincarè inequality on $\Omega$:

$$\|u_i - \varphi\|_{L^p(\Omega)} \leq C \|\nabla u_i - \nabla \varphi\|_{L^p(\Omega)}$$

which leads to $\|\nabla u_i\|_{L^p(\Omega)} \geq C_1 + C_2 \|u_i\|_{W^{1,p}(\Omega)}$ for some constants $C_1, C_2$, where $C_1$ depends on $\|\varphi\|_{W^{1,p}(\Omega)}$.

A very important characterization of the solution of the obstacle problem is a corollary to the following comparison, whose proof closely follows that of the comparison Proposition 3.1.

**Proposition 3.12.** If $u$ is a solution to the obstacle problem $K_{\psi, \theta}$, and if $w$ is a supersolution such that $\min\{u, w\} \in K_{\psi, \theta}$, then $u \leq w$ a.e.

**Proof.** Define $U = \{x \mid u(x) > w(x)\}$. Suppose by contradiction that $U$ has positive measure. Since $u$ solves the obstacle problem, using (16) with function $\varphi = \min\{u, w\} \in K_{\psi, \theta}$ we get

$$0 \leq <F(u), \varphi - u> = \int_U \langle A(\nabla u) | \nabla w - \nabla u \rangle + \int_U B(x, u)(w - u).$$

On the other hand, applying the definition of supersolution $w$ with test function $0 \leq \phi = u - \min\{u, w\} \in W^{1,p}_0(\Omega)$ we get

$$0 \leq <F(w), \phi> = \int_U \langle A(\nabla w) | \nabla u - \nabla w \rangle + \int_U B(x, w)(u - w)$$
summing up the two inequalities we get, by (M) and (B2),

\[ 0 \leq \int_U (A(\nabla u) - A(\nabla w))(\nabla w - \nabla u) + \int_U [B(x,u) - B(x,w)](w-u) \leq 0. \]

Since \( A \) is strictly monotone, \( \nabla u = \nabla w \) a.e. on \( U \), so that \( \nabla ((u-w)_+) = 0 \) a.e. on \( \Omega \). Consequently, since \( U \) has positive measure, \( u - w = c \) a.e. on \( \Omega \), where \( c \) is a positive constant. Since \( \min\{u,w\} \in \mathcal{K}_{\psi,\theta} \), we get \( c = u - w = u - \min\{u,w\} \in W^1_p(\Omega) \), contradiction.

**Corollary 3.13.** The solution \( u \) to the obstacle problem in \( \mathcal{K}_{\psi,\theta} \) is the smallest supersolution in \( \mathcal{K}_{\psi,\theta} \).

**Proposition 3.14.** Let \( w_1, w_2 \in W^{1,p}(\Omega) \) be supersolutions. Then, \( w = \min\{w_1, w_2\} \) is a supersolution. Analogously, if \( u_1, u_2 \in W^{1,p}(\Omega) \) are subsolutions, then so is \( u = \max\{u_1, u_2\} \).

**Proof.** Consider the obstacle problem \( \mathcal{K}_w \). By Corollary 3.13 its solution is necessarily \( w \), and so \( w \) is a supersolution being the solution of an obstacle problem. As for the second part of the statement, define \( \tilde{A}(X) = -A(-X) \) and \( \tilde{B}(x,t) = -B(x,-t) \). Then, \( \tilde{A}, \tilde{B} \) satisfy the set of assumptions \( \mathcal{F} \).

Denote with \( \tilde{F} \) the operator associated to \( \tilde{A}, \tilde{B} \). Then, it is easy to see that \( L_{\tilde{F}} u_i \geq 0 \) if and only if \( L_{\tilde{F}}(-u_i) \leq 0 \), and to conclude it is enough to apply the first part with operator \( L_{\tilde{F}} \).

The following version of the pasting lemma generalizes the previous proposition:

**Lemma 3.15.** Let \( w_1 \) be a supersolution on \( \Omega \), and \( w_2 \) be a supersolution on \( \Omega_2 \subset \Omega \). If \( \min\{w_2 - w_1, 0\} \in W^{1,p}_0(\Omega_2) \), then

\[ m = \begin{cases} 
\min\{w_1, w_2\} & \text{on } \Omega_2 \\
w_1 & \text{on } \Omega \setminus \Omega_2
\end{cases} \]

is a supersolution on \( \Omega \). A similar statement is valid for subsolutions.

**Proof.** Since being a supersolution is a local property, we can suppose without loss of generality that \( w_1 \in W^{1,p}(\Omega) \), otherwise we just substitute \( \Omega \) with any relatively compact set contained in it. First of all, we claim that \( m \in W^{1,p}(\Omega) \). In fact, consider a sequence of smooth functions \( \psi_n \) converging in the \( W^{1,p}(\Omega) \) norm to \( w_1 \), and a sequence \( \{\varphi_n\} \subset C^\infty_c(\Omega_2) \) converging to \( \min\{w_2 - w_1, 0\} \) in the \( W^{1,p}(\Omega_2) \) norm. Define \( u = \min\{w_2 - w_1, 0\}\chi_{\Omega_2} \), where \( \chi_{\Omega_2} \) is the indicatrix function of \( \Omega_2 \). In our assumptions, \( \varphi_n \to u \) in \( W^{1,p}(\Omega) \), thus \( u \in W^{1,p}(\Omega) \). Then, it is evident that \( \psi_n + \varphi_n \) is a sequence of smooth functions in \( \Omega \) converging to \( m = w_1 + u \).

To prove the statement we use a technique similar to Proposition 3.12.

Let \( s \) be the solution to the obstacle problem \( \mathcal{K}_m \) on \( \Omega \), then we immediately
have by comparison $s \leq w_1$ a.e. in $\Omega$ and so $s = w_1 = m$ on $\Omega \setminus \Omega_2$. Since $s$ solves the obstacle problem, using $\varphi = m$ in equation (16) we have:

\begin{equation}
0 \leq \langle F(s), m - s \rangle = \int_{\Omega_2} \langle A(\nabla s) | \nabla m - \nabla s \rangle + \int_{\Omega_2} B(x, s)(m - s)\tag{19}
\end{equation}

on the other hand $m$ is a supersolution in $\Omega_2$, and $s - m$ is evidently a non-negative function in $W_0^{1,p}(\Omega_2)$ so:

\begin{equation}
0 \leq \langle F(m), s - m \rangle = \int_{\Omega_2} \langle A(\nabla m) | \nabla s - \nabla m \rangle + \int_{\Omega_2} B(x, m)(s - m)\tag{19}
\end{equation}

summing the two inequalities, we can conclude as in Proposition 3.12 that $\nabla (s - m) = 0$ in $\Omega_2$ with $s - m \in W_0^{1,p}(\Omega_2)$, and so the two functions are equal there. Since on $\Omega \setminus \Omega_2$, $s = w = m$, the thesis is proved.

Note that the statement remains true if we exchange supersolution with subsolutions and min with max thanks to the same trick as in the previous proposition.

□

As for the regularity of solutions of the obstacle problem, we have

**Theorem 3.16** ([17], Theorem 5.4 and Corollary 5.6). If the obstacle $\psi$ is continuous in $\Omega$, then the solution $u$ to $K_{\psi, \theta}$ has a continuous representative in the Sobolev sense. Furthermore, if $\psi \in C^{0, \alpha}(\Omega)$ for some $\alpha \in (0, 1)$, then there exist $C, \beta > 0$ depending only on $p, \alpha, \Omega, \|u\|_{L^\infty(\Omega)}$ and on the parameters in $\mathcal{F}$ such that

\[ \|u\|_{C^{0, \beta}(\Omega)} \leq C(1 + \|\psi\|_{C^{0, \alpha}(\Omega)})\]

**Remark 3.17.** The interested reader should be advised that, in the notation of [17], $b_0$ and $a$ are both zero with our assumptions. Stronger results, for instance $C^{1, \alpha}$ regularity, can be obtained from stronger requirements on $\psi$, $A$ and $B$ which are stated for instance in [17], Theorem 5.14.

In the proof of our main theorem, and to get some boundary regularity results, it will be important to see what happens on the set where the solution of the obstacle problem is strictly above the obstacle.

**Proposition 3.18.** Let $u$ be the solution of the obstacle problem $K_{\psi, \theta}$ with continuous obstacle $\psi$. If $u > \psi$ on the (open) set $D$, then $u$ is a solution of $\mathcal{F}(u) = 0$ on $D$.

**Proof.** Consider any test function $\phi \in C_c^{\infty}(D)$. Since $u > \psi$ on $D$, and since $\phi$ is bounded, by continuity there exists $\delta > 0$ such that $u \pm \delta \phi \in K_{\psi, \theta}$. From the definition of solution to the obstacle problem we have that:

\[ \pm < \mathcal{F}(u), \phi > = \frac{1}{\delta} < \mathcal{F}(u), \pm \delta \phi > = \frac{1}{\delta} < \mathcal{F}(u), (u \pm \delta \phi) - u > \geq 0, \]
hence $\langle \mathcal{F}(u), \phi \rangle = 0$ for every $\phi \in C^\infty_c(D)$, as required.

As for boundary regularity, to the best of our knowledge there is no result for solutions of the kind of obstacle problems we are studying. However, if we restrict ourselves to Dirichlet problems (i.e. obstacle problems with $\psi = -\infty$), some results are available. We briefly recall that a point $x_0 \in \partial \Omega$ is called “regular” if for every function $\theta \in W^{1,p}(\Omega)$ continuous in a neighborhood of $x_0$, the unique solution to the relative Dirichlet problem is continuous in $x_0$, and that a necessary and sufficient condition for $x_0$ to be regular is the famous Wiener criterion (which has a local nature). For our purposes, it is enough to use some simpler sufficient conditions for regularity, so we just cite the following corollary of the Wiener criterion:

**Theorem 3.19** ([6], Theorem 2.5). Let $\Omega$ be a domain, and suppose that $x_0 \in \partial \Omega$ has a neighborhood where $\partial \Omega$ is Lipschitz, then $x_0$ is regular for the Dirichlet problem.

For a more specific discussion of the subject, we refer the reader to [6]. We mention that Dirichlet and obstacle problems have been studied also in metric space setting, and boundary regularity theorems with the Wiener criterion have been obtained for example in [2], Theorem 7.2.

**Remark 3.20.** Note that [6] deals only with the case $1 < p \leq m$, but the other cases follows from standard Sobolev embeddings.

Using the comparison principle and Proposition 3.18 it is possible to obtain a corollary to this theorem which deals with boundary regularity of some particular obstacle problems.

**Corollary 3.21.** Consider the obstacle problem $\mathcal{K}_{\theta, \psi}$ on $\Omega$, and suppose that $\Omega$ has Lipschitz boundary and both $\theta$ and $\psi$ are continuous up to the boundary. Then the solution $w$ to $\mathcal{K}_{\theta, \psi}$ is continuous up to the boundary (for convenience we denote $w$ the continuous representative of the solution).

**Proof.** If we want $\mathcal{K}_{\theta, \psi}$ to be nonempty, it is necessary to assume $\psi(x_0) \leq \theta(x_0)$ for all $x_0 \in \partial \Omega$.

Let $\tilde{\theta}$ be the unique solution to the Dirichlet problem relative to $\theta$ on $\Omega$. Then Theorem 3.19 guarantees that $\tilde{\theta} \in C^0(\Omega)$ and the comparison principle allow us to conclude that $w(x) \geq \tilde{\theta}(x)$ everywhere in $\Omega$.

Suppose first that $\psi(x_0) < \theta(x_0)$, then in neighborhood $U$ of $x_0$ ($U \subset \Omega$) $w(x) \geq \tilde{\theta}(x) > \psi(x)$. By Proposition 3.18 $L_F w = 0$ on $U$, and so by Theorem 3.19 $w$ is continuous in $x_0$.

If $\psi(x_0) = \theta(x_0)$, consider $w_\epsilon$ the solutions to the obstacle problem $\mathcal{K}_{\tilde{\theta} + \epsilon, \psi}$. By the same argument as above we have that $w_\epsilon$ are all continuous at $x_0$, and
and by the comparison principle $w(x) \leq w_{\epsilon}(x)$ for every $x \in \Omega$ (recall that both functions are continuous in $\Omega$). So we have on one hand:

$$\lim_{x \to x_0} \inf w(x) \geq \lim_{x \to x_0} \inf \psi(x) = \psi(x_0) = \theta(x_0)$$

and on the other:

$$\lim_{x \to x_0} \sup w(x) \leq \lim_{x \to x_0} \sup w_{\epsilon}(x) = \theta(x_0) + \epsilon$$

this proves that $w$ is continuous in $x_0$ with value $\theta(x_0)$. □

Finally, we present some results on convergence of supersolutions and their approximation with regular ones.

**Proposition 3.22.** Let $w_j$ be a sequence of supersolutions on some open set $\Omega$. Suppose that either $w_j \uparrow w$ or $w_j \downarrow w$ pointwise monotonically, for some locally bounded $w$. Then, $w$ is a supersolution. Furthermore, if $\{u_j\}$ is a sequence of solutions of $L_F u_j = 0$ which are locally uniformly bounded in $L^\infty$ and pointwise convergent to some $u$, then $u$ solves $L_F u = 0$.

**Proof.** Suppose that $w_j \uparrow w$. Up to changing the representative in the Sobolev class, by Theorem 3.7 we can assume that $w_j$ is lower semicontinuous. Hence, it has minimum on compact subsets of $\Omega$. Since $w$ is locally bounded and the convergence is monotone up to a set of zero measure, the sequence $\{w_j\}$ turns out to be locally bounded in the $L^\infty$-norm. The elliptic estimate in Proposition (3.9) ensures that $\{w_j\}$ is locally bounded in $W^{1,p}(\Omega)$. Fix a smooth exhaustion $\{\Omega_n\}$ of $\Omega$. For each $j$, up to passing to a subsequence, $w_j \rightharpoonup z_n$ weakly in $W^{1,p}(\Omega_n)$ and strongly in $L^p(\Omega_n)$. By Riesz theorem, $z_j = w$ for every $j$, hence $w \in W^{1,p}_{\text{loc}}(\Omega)$. With a Cantor argument, we can select a sequence, still called $w_j$, such that $w_j$ converges to $w$ both weakly in $W^{1,p}(\Omega_n)$ and strongly in $L^p(\Omega_n)$ for every fixed $n$. To prove that $w$ is a supersolution, fix $0 \leq \eta \in C^\infty_c(\Omega)$, and choose a smooth relatively compact open set $\Omega_0 \Subset \Omega$ that contains the support of $\eta$. Define $M = \max_{\Omega_0} \|w_j\|_{W^{1,p}(\Omega_0)} < +\infty$. Since $w_j$ is a supersolution and $w \geq w_j$ for every $j$,

$$< F(w_j), \eta(w - w_j) > \geq 0.$$

Using (A1) we can rewrite the above inequality as follows:

$$\int \langle A(\nabla w_j)|\eta(\nabla w - \nabla w_j) \rangle \geq -\int \left[ B(x,w_j) + \langle A(\nabla w_j)|\nabla \eta \rangle \right](w - w_j).$$

Using (A1)
Using (B1), (A2) and suitable Hölder inequalities, the RHS can be bounded from below with the following quantity

\[
\begin{align*}
-b_1 \|\eta\|_{L^\infty(\Omega)} & \int_{\Omega_0} (w - w_j) - b_2 \|\eta\|_{L^\infty(\Omega)} \int_{\Omega_0} |w_j|^{p-1} |w - w_j| \\
-a_2 \|\nabla \eta\|_{L^\infty(\Omega)} & \int_{\Omega_0} |\nabla w_j|^{p-1} |w - w_j| \\
\geq -\|\eta\|_{C^1(\Omega)} \left[ b_1 |\Omega_0|^{\frac{p-1}{p}} - b_2 M^{p-1} - a_2 M^{p-1} \right] \|w - w_j\|_{L^p(\Omega_0)} \to 0
\end{align*}
\]

as \( j \to +\infty \). Combining with (20) and the fact that \( w_j \rightharpoonup w \) weakly on \( W^{1,p}(\Omega_0) \), by assumption (M) the following inequality holds true:

\[
(22) \quad 0 \leq \int \eta \langle A(\nabla w) - A(\nabla w_j) |\nabla w - \nabla w_j) \rangle \leq o(1) \quad \text{as} \quad j \to +\infty.
\]

By a lemma due to F. Browder (see [3], p.13 Lemma 3), the combination of assumptions \( w_j \rightharpoonup w \) both locally weakly in \( W^{1,p} \), and (22) for every \( 0 \leq \eta \in C^\infty_c(\Omega) \), implies that \( w_j \to w \) locally strongly in \( W^{1,p} \). Since the operator \( F \) is weakly continuous, as shown in the proof of Theorem [4], this implies that

\[
0 \leq \langle F(w_j), \eta \rangle \longrightarrow \langle F(w), \eta \rangle,
\]

hence \( L_F w \leq 0 \), as required.

The case \( w_j \downarrow w \) is simpler. By the elliptic estimate, \( w \in W^{1,p}_{\text{loc}}(\Omega) \), being locally bounded by assumption. Let \( \{\Omega_n\} \) be a smooth exhaustion of \( \Omega \), and let \( u_n \) be a solution of the obstacle problem relative to \( \Omega_n \) with obstacle and boundary value \( w \). Then, by (3.13) \( w \leq u_n \leq w_j|_{\Omega_n} \), and letting \( j \to +\infty \) we deduce that \( w = u_n \) is a supersolution on \( \Omega_n \), being a solution of an obstacle problem.

The proof of the last part of the Proposition follows exactly the same lines as the case \( w_j \uparrow w \) done before. Indeed, by the uniform local boundedness, the elliptic estimate gives \( \{u_j\} \subset W^{1,p}_{\text{loc}}(\Omega) \). Furthermore, in definition \( \langle F(u_j), \phi \rangle = 0 \) we can still use as test function \( \phi = \eta(u - u_j) \), since no sign of \( \phi \) is required.

\[ \square \]

A couple of corollaries follow from this theorem. It is in fact easy to see that we can relax the assumption of local boundedness on \( w \) if we assume a priori \( w \in W^{1,p}_{\text{loc}}(\Omega) \), and moreover with a simple trick we can prove that also local uniform convergence preserves the supersolution property, as in [9], Theorem 3.78.

**Corollary 3.23.** Let \( w_j \) be a sequence of supersolutions locally uniformly converging to \( w \), then \( w \) is a supersolution.
Proof. The trick is to transform local uniform convergence in monotone convergence. Fix any relatively compact $\Omega_0 \Subset \Omega$ and a subsequence of $w_j$ (denoted for convenience by the same symbol) with $\|w_j - w\|_{L^\infty(\Omega_0)} \leq 2^{-j}$. The modified sequence of supersolutions $\tilde{w}_j \equiv w_j + \frac{3}{2} \sum_{k=j}^{\infty} 2^{-k} = w_j + 3 \times 2^{-j}$ is easily seen to be a monotonically decreasing sequence on $\Omega_0$, and to its limit $w$ is a supersolution on any $\Omega_0$ by the previous proposition. The conclusion follows from the arbitrariness of $\Omega_0$. \hfill \Box

Now we prove that with continuous supersolutions we can approximate every supersolution.

**Proposition 3.24.** For every supersolution $w \in W^{1,p}_{\text{loc}}(\Omega)$, there exists a sequence $w_n$ of continuous supersolutions converging monotonically from below and in $W^{1,p}_{\text{loc}}(\Omega)$ to $w$. The same statement is true for subsolutions with monotone convergence from above.

Proof. Since every $w$ has a lower-semicontinuous representative, it can be assumed to be locally bounded from below, and since $w^{(m)} = \min\{w, m\}$ is a supersolution (for $m \geq 0$) and converges monotonically to $w$ as $m$ goes to infinity, we can assume without loss of generality that $w$ is also bounded above.

Let $\Omega_n$ be a locally finite relatively compact open covering on $\Omega$. Since $w$ is lower semicontinuous it is possible to find a sequence $\phi_m$ of smooth functions converging monotonically from below to $w$ (see [9], Section 3.71 p. 75). Let $w_m^{(n)}$ be the solution to the obstacle problem $K_{w, \phi_m}$ on $\Omega_n$, and define $\bar{w}_m = \min_n \{w_m^{(1)}, w_m^{(2)}\}$. Thanks to the local finiteness of the covering $\Omega_n$, we can prove that $\bar{w}_m$ is a continuous supersolution by showing that this property holds only for the function $\min\{w_m^{(1)}, w_m^{(2)}\}$. Since $w_m^{(i)}$ are continuous functions defined on open sets, their minimum is an upper-semicontinuous function, and by the pasting Lemma 3.15 we can prove that it is also a supersolution, so lower-semicontinuous and so continuous.

Monotonicity of the convergence is an easy consequence of the comparison principle for obstacle problems, i.e. Proposition 3.12. To prove convergence in the local $W^{1,p}$ sense, the steps are pretty much the same as for Proposition 3.22 and the statement for subsolutions follows from the usual trick. \hfill \Box

**Remark 3.25.** With similar arguments and up to some minor technical difficulties, one could strengthen the previous proposition and prove that every supersolution can be approximated by locally Hölder continuous supersolutions.
4. Proof of Theorem 4.1

Theorem 4.1. Let $M$ be a Riemannian manifold, and let $A, B$ satisfy the set of assumptions $\mathcal{S}$. Define $A, B, F$ as in (5), and $L_A, L_F$ accordingly. Then, the following properties are equivalent:

1. (L) for $\text{Hö} \text{ö} \text{l}_{\text{loc}}$ functions,
2. (L) for $L^\infty$ functions,
3. (K).

Proof. (2) $\Rightarrow$ (1) is obvious. To prove that (1) $\Rightarrow$ (2), we follow the arguments in [24], Lemma 1.5. Assume by contradiction that there exists $0 \leq u \in L^\infty(M) \cap W^{1,p}_{\text{loc}}(M)$, $u \neq 0$ such that $L_F u \geq 0$. We distinguish two cases.

- Suppose first that $B(x,u)u$ is not identically zero in the Sobolev sense. Let $u_2 > u^*$ be a constant. By Lemma 1.5, $L_F u_2 \leq 0$. By the subsolution-supersolution method and the regularity Theorem 3.7 there exists $w \in \text{Hö} \text{ö} \text{l}_{\text{loc}}(M)$ such that $u \leq w \leq u_2$ and $L_F w = 0$. Since, by (B2), (B3) and $w$ is non-constant, contradicting property (1).

- Suppose that $B(x,u)u = 0$ a.e. on $M$. Since $u$ is non-constant, we can choose a positive constant $c$ such that both $\{u - c > 0\}$ and $\{u - c < 0\}$ have positive measure. By Proposition 3.14 the function $v = (u - c)_+ = \max\{u - c, 0\}$ is a non-zero subsolution. Denoting with $\chi_{\{u < c\}}$ the indicatrix of $\{u < c\}$, we can say that $L_F v \geq 0 = \chi_{\{u < c\}}v^{p-1}$. Choose any constant $u_2 > v^*$. Then, clearly $L_F u_2 \leq \chi_{\{u < c\}}u_2^{p-1}$. Since the potential

$$\tilde{B}(x,t) = B(x,t) + \chi_{\{u < c\}}(x)|t|^{p-2}t$$

is still a Caratheodory function satisfying the assumptions in $\mathcal{S}$, by Theorem 3.2 there exists a function $w$ such that $v \leq w \leq u_2$ and $L_F w = \chi_{\{u < c\}}w^{p-1}$. By Theorem 3.7 (ii) $w$ is locally Hölder continuous and, since $\{u < c\}$ has positive measure, $w$ is non-constant, contradicting (1).

To prove the implication (3) $\Rightarrow$ (1), we follow a standard argument in potential theory, see for example [24], Proposition 1.6. Let $u \in \text{Hö} \text{ö} \text{l}_{\text{loc}}(M) \cap W^{1,p}_{\text{loc}}(M)$ be a non-constant, non-negative, bounded solution of $L_F u \geq 0$.

We claim that, by the strong maximum principle, $u < u^*$ on $M$. Indeed, let $\overline{A}$ be the operator associated with the choice $A(X) = -A(-X)$. Then, since $\overline{A}$ satisfies all the assumptions in $\mathcal{S}$, it is easy to show that $L_{\overline{A}}(u^* - u) \leq 0$ on $M$. Hence, by the Harnack inequality $u^* - u > 0$ on $M$, as desired.

Let $K \subseteq M$ be a compact set. Consider an $\eta$ such that $0 < \eta < u^*$ and define
the open set \( \Omega_\eta = u^{-1}(\{\eta, +\infty\}) \). From \( u < u^* \) on \( M \), we can choose \( \eta \) close enough to \( u^* \) so that \( K \cap \Omega_\eta = \emptyset \). Let \( x_0 \) be a point such that \( u(x_0) > \frac{u^* + \eta}{2} \), and choose a Khas’minskii potential relative to the triple \((K, \Omega, (u^* - \eta)/2)\). Now, consider the open set \( V \) defined as the connected component containing \( x_0 \) of the open set

\[
\tilde{V} = \{ x \in \Omega_\eta \mid u(x) > \eta + w(x) \}
\]

Since \( u \) is bounded and \( w \) is an exhaustion, \( V \) is relatively compact in \( M \) and \( u(x) = \eta + w(x) \) on \( \partial V \). Since, by (B2), \( L_F(\eta + w) \leq 0 \), and \( L_F u \geq 0 \), this contradicts the comparison Theorem \( 3.1 \).

We are left to the implication (2) \( \Rightarrow \) (3). Fix a triple \((K, \Omega, \varepsilon)\), and a smooth exhaustion \( \{\Omega_j\} \) of \( M \) with \( \Omega \Subset \Omega_1 \). By the existence Theorem \( 3.11 \) with obstacle \( \psi \equiv -\infty \), there exists a unique solution \( h_j \) of

\[
\begin{cases}
L_F h_j = 0 & \text{on } \Omega_j \setminus K \\
h_j = 0 & \text{on } \partial K, \quad h_j = 1 & \text{on } \partial \Omega_j,
\end{cases}
\]

and \( 0 \leq h_j \leq 1 \) by the comparison Theorem \( 3.1 \) with \( h \) continuous up to \( \partial (\Omega_j \setminus K) \) thanks to theorem \( 3.19 \). Extend \( h_j \) by setting \( h_j = 0 \) on \( K \) and \( h_j = 1 \) on \( M \setminus \Omega_j \). Again by comparison, \( \{h_j\} \) is a decreasing sequence which, by Proposition \( 3.22 \), converges locally uniformly in \( M \) to a solution

\[
h \in C^0(M) \cap W^{1,p}(M \setminus K) \quad \text{of} \quad L_F h = 0 \quad \text{on} \quad M \setminus K, \quad h = 0 \quad \text{on} \quad K.
\]

We claim that \( h = 0 \). Indeed, by Lemma \( 3.15 \) \( u = \max\{h, 0\} \) is a non-negative, bounded solution of \( L_F u \geq 0 \) on \( M \). By (1), \( u \) has to be constant, hence the only possibility is \( h = 0 \).

Now we are going to build by induction an increasing sequence of continuous functions \( \{w_n\} \), \( w_0 \equiv 0 \), such that:

(a) \( w_n|_K = 0 \), \( w_n \) are continuous on \( M \) and \( L_F w_n \leq 0 \) on \( M \setminus K \),
(b) for every \( n \), \( w_n \leq n \) on all of \( M \) and \( w_n = n \) in a large enough neighborhood of infinity denoted by \( M \setminus C_n \),
(c) \( \|w_n\|_{L^\infty(\Omega_n)} \leq \|w_{n-1}\|_{L^\infty(\Omega_n)} + \frac{\varepsilon}{2^n} \).

Once this is done, by (c) the increasing sequence \( \{w_n\} \) is locally uniformly convergent to a continuous exhaustion which, by Proposition \( 3.22 \), solves \( L_F w \leq 0 \). Furthermore,

\[
\|w\|_{L^\infty(\Omega)} \leq \sum_{n=1}^{+\infty} \frac{\varepsilon}{2^n} \leq \varepsilon.
\]

Hence, \( w \) is the desired Khas’minskii potential relative to \((K, \Omega, \varepsilon)\).

We start the induction by setting \( w_1 = h_j \), for \( j \) large enough in order for property (c) to hold. Define \( C_1 \) in order to fix property (b). Suppose now that we have constructed \( w_n \). For notational convenience, write \( \bar{w} = w_n \).
Consider the sequence of obstacle problems $\mathcal{K}_{\bar{w}+h_j}$ defined on $\Omega_{j+1}$ and let $s_j$ be their continuous solution by Corollary 3.16 continuous up to the boundary by Corollary 3.21. Take for convenience $j$ large enough such that $C_1 \subset \Omega_j$. Note that $s_j|_{\partial K} = 0$ and since the constant function $n + 1$ is a supersolution, by comparison $s_j \leq n + 1$ and $s_j|_{\Omega_{j+1}\setminus\Omega_j} = n + 1$. So we can extend $s_j$ to a function defined on all of $M$ by setting it equal to 0 on $K$ and equal to $n + 1$ on $M \setminus \Omega_{j+1}$, and in this fashion, by Lemma 3.15 $L_{\mathcal{F}}s_j \leq 0$ on $M \setminus K$. By Corollary 3.13 $\{s_j\}$ is decreasing, and so it has a limit $\bar{s}$ which is still a supersolution on $M \setminus K$ by Proposition 3.22. We take a lower semicontinuous representative of $\bar{s}$, which is granted via Theorem 3.7 (i). We are going to prove that $\bar{s} = \bar{w}$. First, we show that $\bar{s} \leq n$ everywhere. Suppose by contradiction that this is false. Then, since $h_j$ converges locally uniformly to zero, on the open set $A \equiv \{n, \infty\}$ the inequality $s_j > \bar{w} + h_j$ is locally eventually true, so that $s_j$ is locally eventually a solution of $L_{\mathcal{F}}s_j = 0$ by Proposition 3.18 and so $L_{\mathcal{F}}\bar{s} = 0$ on $A$ by Proposition 3.22. By Lemma 3.15 and assumptions $\mathcal{F}$, the function

$$f \doteq \max\{\bar{s} - n, 0\}$$

is a non-negative, non-zero bounded solution of $L_{\mathcal{F}}f \geq 0$. By (2), $f$ is constant, hence zero; therefore $\bar{s} \leq n$. This proves that $\bar{s} = \bar{w} = n$ on $M \setminus C_n$. As for the remaining set, a similar argument than the one just used shows that $\bar{s}$ is a solution of $L_{\mathcal{F}}\bar{s} = 0$ on $V \doteq \{\bar{s} > \bar{w}\}$. Now since $V$ is relatively compact and $\bar{s} = \bar{w}$ on $\partial V$, the comparison principle guarantees that $\bar{s} \leq \bar{w}$ everywhere, which is what we needed to prove. Now, since $s_j \downarrow w$, by Dini’s theorem the convergence is locally uniform and so we can choose $j$ large enough in such a way that $s_j - \bar{w} < \frac{1}{2^n}$ on $\Omega_{n+1}$. Define $w_{n+1} \doteq s_j$, and $C_{n+1}$ in order for $(b)$ to hold, and the construction is completed. \qed

**Remark 4.2.** As anticipated in Section 2, the results of our main theorem are the same if we substitute condition $\text{[B1]}$ with condition $\text{[B1-]}$

$$|B(x,t)| \leq b(t) \quad \text{instead of} \quad |B(x,t)| \leq b_1 + b_2 |t|^{p-1} \quad \text{for} \quad t \in \mathbb{R}$$

Although it is not even possible to define the operator $\mathcal{B}$ if we take $W^{1,p}(\Omega)$ as its domain, this difficulty is easily overcome if we restrict the domain to (essentially) bounded functions, i.e. if we define

$$\mathcal{B} : W^{1,p}(\Omega) \cap L^\infty(\Omega) \to W^{1,p}(\Omega)^*$$

Now consider that each function used in the proof of the main theorem is either bounded or essentially bounded, so it is quite immediate to see all the existence and comparison theorems proved in section 3 along with all the reasoning and tools used in the proof, still work. Consider for example an obstacle problem $\mathcal{K}_{\theta, \psi}$ such that $|\theta| \leq C \geq |\psi|$, and define the operator $\overline{\mathcal{B}}$
relative to the function:

\[ \tilde{B}(x,t) = \begin{cases} 
B(x,t) & \text{for } |t| \leq C + 1 \\
b_1(x, C + 1) & \text{for } t \geq C + 1 \\
b_1(x, -(C + 1)) & \text{for } t \leq -(C + 1) 
\end{cases} \]

\(\tilde{B}\) satisfies evidently condition [B1] so it admits a solution to the obstacle problem, which by comparison Theorem 3.12 is bounded in modulus by \(C\), and now it is evident that this function solves also the obstacle problem relative to the original bad-behaved \(B\).

**Remark 4.3.** As mentioned in Remark 3.25 one could prove that Locally Hölder supersolutions are monotonically and dense in the local \(W^{1,p}\) sense in the set of supersolutions, and this would make implication (2) \(\Rightarrow\) (1) in the main theorem trivial. However, since this result is a little technical and since the subsolution-supersolution technique is of interest in itself, we decided to use this latter method.

5. **On the links with the weak maximum principle and parabolicity: proof of Theorem 2.11**

As already explained in the introduction, throughout this section we will restrict ourselves to potentials \(B(x,t)\) of the form \(B(x,t) = b(x)f(t)\), where

\[ b, b^{-1} \in L^\infty_{\text{loc}}(M), \quad b > 0 \text{ a.e. on } M; \]

\[ f \in C^0(\mathbb{R}), \quad f(0) = 0, \quad f \text{ is non-decreasing on } \mathbb{R}, \]

while we require (A1), (A2) on \(A\).

**Remark 5.1.** As in Remark 3.5 in the case of the operator \(L_\phi\) in Example 2.2 with \(h\) being the metric tensor, (A1) and (A2) can be weakened to (11) and (12).

We begin with the following lemma characterizing (W), whose proof follows the lines of [23].

**Lemma 5.2.** Property (W) for \(b^{-1}L_A\) is equivalent to the following property, which we call (P):

For every \(g \in C^0(\mathbb{R})\), and for every \(u \in C^0(M) \cap W^{1,p}_{\text{loc}}(M)\) bounded above and satisfying \(L_Au \geq b(x)g(u)\) on \(M\), it holds \(g(u^*) \leq 0\).

**Proof.** \((W) \Rightarrow (P)\). From (W) and \(L_Au \geq b(x)g(u)\), for every \(\eta < u^*\) and \(\varepsilon > 0\) we can find \(0 \leq \phi \in C^\infty_c(\Omega_n)\) such that

\[ \varepsilon \int b\phi > -< A(u), \phi > \geq \int g(u)b\phi \geq \inf_{\Omega_n} g(u) \int b\phi \]
Suppose that \( L \). Since 

Furthermore, \( g \) is a non-negative, bounded non-constant solution of \( A \), where

Theorem 2.11 is an immediate corollary of the main Theorem 2.7 and of the following two propositions.

**Proposition 5.3.** If \( b^{-1}L_A \) satisfies \( (W) \), then \( (L) \) holds for every operator \( L_F \) of type 1. Conversely, if \( (L) \) holds for some operator \( F \) of type 1, then \( b^{-1}L_A \) satisfies \( (W) \).

**Proof.** Suppose that \( (W) \) is met, and let \( u \in \text{Hölo}_{\text{loc}} \cap W^{1,p}_{\text{loc}} \) be a bounded, non-negative solution of \( L_F u \geq 0 \). By Lemma 5.2, \( f(u^*) \leq 0 \). Since \( F \) is of type 1, \( u^* \leq 0 \), that is, \( u = 0 \), as desired. Conversely, let \( F \) be an operator of type 1 for which the Liouville property holds. Suppose by contradiction that \( (W) \) is not satisfied, so that there exists \( u \in C^0 \cap W^{1,p}_{\text{loc}} \) such that \( b^{-1}L_A u \geq \varepsilon \) on some \( \Omega_{\eta_0} \). Clearly, \( u \) is non-constant. Since \( f(0) = 0 \), we can choose \( \eta \in (\eta_0, u^*) \) such that \( f(u^* - \eta) < \varepsilon \). Hence, by the monotonicity of \( f \), the function \( u - \eta \) solves

\[
L_A(u - \eta) \geq b(x)\varepsilon \geq b(x)f(u - \eta) \quad \text{on } \Omega_{\eta}.
\]

Thanks to the pasting Lemma 3.15, \( w = \max\{u - \eta, 0\} \) is a non-constant, non-negative bounded solution of \( L_A w \geq b(x)f(w) \), that is, \( L_F w \geq 0 \), contradicting the Liouville property.

**Proposition 5.4.** If \( b^{-1}L_A \) is parabolic, then \( (L) \) holds for every operator \( L_F \) of type 2. Conversely, if \( (L) \) holds for some operator \( F \) of type 2, then \( b^{-1}L_A \) satisfies \( (W_{pa}) \).

**Proof.** Suppose that \( (W_{pa}) \) is met. Since each bounded, non-negative \( u \in \text{Hölo}_{\text{loc}} \cap W^{1,p}_{\text{loc}} \) solving \( L_F u \geq 0 \) automatically solves \( L_A u \geq 0 \), then \( u \) is constant by \( (W_{pa}) \), which proves \( (L) \). Conversely, let \( F \) be an operator of type 2 for which the Liouville property holds, and let \( [0, T] \) be the maximal interval in \( \mathbb{R}_0^+ \) where \( f = 0 \). Suppose by contradiction that \( (W_{pa}) \) is not satisfied, so that there exists a nonconstant \( u \in C^0 \cap W^{1,p}_{\text{loc}} \) with \( b^{-1}L_A u \geq 0 \) on \( M \). For \( \eta \) close enough to \( u^* \), \( u - \eta \leq T \) on \( M \), hence \( w = \max\{u - \eta, 0\} \) is a non-negative, bounded non-constant solution of \( L_A w \geq 0 \) on \( M \), contradicting the Liouville property for \( F \).
6. The Evans property

We conclude this paper with some comments on the existence of Evans potentials on model manifolds. We remark that, for the main Theorems 2.7 and 2.11 to hold, no growth control on $B(x, t)$ in the variable $t$ is required. As we will see, for the Evans property to hold for $L_F$ we shall necessarily assume a precise maximal growth of $B$, otherwise there is no hope to find any Evans potential. This growth is described by the so-called Keller-Osserman condition.

To begin with, we recall that a model manifold $M_o$ is $\mathbb{R}^m$ endowed with a metric $ds^2$ which, in polar coordinates centered at some origin $o$, has the expression $ds^2 = dr^2 + g(r)^2d\theta^2$, where $d\theta^2$ is the standard metric on the unit sphere $S^{m-1}$ and $g(r)$ satisfies the following assumptions:

$$g \in C^\infty(\mathbb{R}^+_0), \quad g > 0 \text{ on } \mathbb{R}^+, \quad g'(0) = 1, \quad g^{(2k)}(0) = 0$$

for every $k = 0, 1, 2, \ldots$, where $g^{(2k)}$ means the $(2k)$-derivative of $g$. The last condition ensures that the metric is smooth at the origin $o$. Note that

$$\Delta r(x) = (m-1)\frac{g'(r(x))}{g(r(x))}, \quad \text{vol}(\partial B_r) = g(r)^{m-1}, \quad \text{vol}(B_r) = \int_0^r g(t)^{m-1}dt.$$ 

Consider the operator $L_\varphi$ of Example 2.2 with $h$ being the metric tensor. If $u(x) = z(r(x))$ is a radial function, a straightforward computation gives

$$\varphi(t) \to +\infty \quad \text{as} \quad t \to +\infty. \quad \text{Let} \quad B(x, t) = B(t) \quad \text{be such that}$$

$$B \in C^0(\mathbb{R}^+_0), \quad B \geq 0 \text{ on } \mathbb{R}^+, \quad B(0) = 0, \quad B \text{ is non-decreasing on } \mathbb{R},$$

and set $B = 0$ on $\mathbb{R}^-$. For $c > 0$, define the functions

$$V_{pa}(r) = \varphi^{-1}\left[cg(r)^{1-m}\right], \quad V_{st}(r) = \varphi^{-1}\left[cg(r)^{1-m}\int_R^r g(t)^{m-1}dt\right]$$

$$z_{pa}(r) = \int_R^r V_{pa}(t)dt, \quad z_{st}(r) = \int_R^r V_{st}(t)dt.$$ 

Note that both $z_{pa}$ and $z_{st}$ are increasing on $[R, +\infty)$. By (24), the functions $u_{pa} = z_{pa} \circ r$, $u_{st} = z_{st} \circ r$ are solutions of

$$L_\varphi u_{pa} = 0, \quad L_\varphi u_{st} = c.$$ 

Therefore, the following property can be easily verified:

**Proposition 6.1.** For the operator $L_F$ defined by $L_Fu = L_\varphi u - B(u)$, properties $(K)$ and $(L)$ are equivalent to either

$$V_{st} \notin L^1(+) \quad \text{for every } c > 0 \text{ small enough, if } B > 0 \text{ on } \mathbb{R}^+,$$
or

\[(27) \quad V_{pa} \notin L^1(+\infty) \quad \text{for every } c > 0 \text{ small enough, otherwise.} \]

Proof. We sketch the proof when \( B > 0 \) on \( \mathbb{R}^+ \), the other case being analogous. If \( V_{st} \in L^1(+\infty) \), then \( u_{st} \) is a bounded, non-negative solution of \( L \varphi u \geq c \) on \( M \setminus B_R \). Choose \( \eta \in (0, u^*) \) in such a way that \( B(u^* - \eta) \leq c \), and proceed as in the second part of the proof of Proposition 5.3 to contradict the Liouville property of \( L_F \). Conversely, if \( V_{st} \notin L^1(+\infty) \), then \( u_{st} \) is an exhaustion. For every \( \delta > 0 \), choose \( c > 0 \) small enough that \( c \leq B(\delta) \).

Since \( \varphi(0) = 0 \), for every \( \rho > R \) and \( \epsilon > 0 \) we can reduce \( c \) in such a way that \( w_{\epsilon,\rho} = \delta \) on \( \partial B_R \), \( w_{\epsilon,\rho} \leq \delta + \epsilon \) on \( B_{\rho} \setminus B_R \), \( L \varphi w_{\epsilon,\rho} = c \leq B(\delta) \leq B(w_{\epsilon,\rho}) \).

As the reader can check by slightly modifying the argument in the proof of (3) \( \Rightarrow \) (1) of Theorem 2.7, the existence of these modified Khas’minskii potentials for every choice of \( \delta, \epsilon, \rho \) is enough to conclude the validity of \( (L) \), hence of \( (K) \). □

Remark 6.2. In the case \( \varphi(t) = t^{p-1} \) of the \( p \)-Laplacian, making the conditions on \( V_{st} \) and \( V_{pa} \) more explicit and using Theorem 2.11 we deduce that, on model manifolds, \( \Delta_p \) satisfies \( (W) \) if and only if

\[
\left( \frac{\text{vol}(B_r)}{\text{vol}(\partial B_r)} \right)^{\frac{1}{p-1}} \notin L^1(+\infty),
\]

and \( \Delta_p \) is parabolic if and only if

\[
\left( \frac{1}{\text{vol}(\partial B_r)} \right)^{\frac{1}{p-1}} \notin L^1(+\infty).
\]

This has been observed, for instance, in [24], see also the end of [25] and the references therein for a thorough discussion on \( \Delta_p \) on model manifolds.

We now study the existence of an Evans potential on \( M_g \). First, we need to produce radial solutions of \( L \varphi u = B(u) \) which are zero on some fixed sphere \( \partial B_R \). To do so, the first step is to solve locally the related Cauchy problem. The next result is a modification of Proposition A.1 of [4].

Lemma 6.3. In our assumptions, for every fixed \( R > 0 \) and \( c \in (0, 1] \) the problem

\[
\begin{align*}
(28) \quad \left\{ \begin{array}{ll}
g^{m-1}(c|z'|) \text{sgn}(z')' = g^{m-1}B(cz) & \text{on } [R, +\infty) \\
z(R) = \vartheta \geq 0, \quad z'(R) = \mu > 0
\end{array} \right.
\end{align*}
\]

has a positive, increasing \( C^1 \) solution \( z_c \) defined on a maximal interval \([R, \rho)\), where \( \rho \) may depend on \( c \). Moreover, if \( \rho < +\infty \), then \( z_c(\rho^-) = +\infty \).
Proof. We sketch the main steps. First, we prove local existence. For every chosen \( r \in (R, R+1) \), denote with \( A_\varepsilon \) the \( \varepsilon \)-ball centered at the constant function \( \vartheta \) in \( C^0([R,r], \| \cdot \|_{L^\infty}) \). We look for a fixed point of the Volterra operator \( T_\varepsilon \) defined by
\[
T_\varepsilon(u)(t) = \vartheta + \frac{1}{c} \int_{R}^{t} \varphi^{-1} \left( \frac{g^{\tau-1}(R)\varphi(c\mu)}{g^{\tau-1}(s)} + \int_{R}^{s} \frac{g^{\tau-1}(\tau)}{g^{\tau-1}(s)} B(cu(\tau))d\tau \right) ds
\]
It is simple matter to check the following properties:

(i) If \(|r - R|\) is sufficiently small, \( T_\varepsilon(A_\varepsilon) \subseteq A_\varepsilon \);

(ii) There exists a constant \( C > 0 \), independent of \( r \in (R, R+1) \), such that \(|T_\varepsilon u(t) - T_\varepsilon u(s)| \leq C|t - s|\) for every \( u \in A_\varepsilon \). By Ascoli-Arzelà theorem, \( T_\varepsilon \) is a compact operator.

(iii) \( T_\varepsilon \) is continuous. To prove this, let \( \{u_j\} \subseteq A_\varepsilon \) be such that \( \|u_j - u\|_{L^\infty} \to 0 \), and use Lebesgue convergence theorem in the definition of \( T_\varepsilon \) to show that \( T_\varepsilon u_j \to T_\varepsilon u \) pointwise. The convergence is indeed uniform by (ii).

By Schauder theorem (\([7]\), Theorem 11.1), \( T_\varepsilon \) has a fixed point \( z_\varepsilon \). Differentiating \( z_\varepsilon = T_\varepsilon z_\varepsilon \) we deduce that \( z_\varepsilon' > 0 \) on \([R,r]\), hence \( z_\varepsilon \) is positive and increasing. Therefore, \( z_\varepsilon \) is also a solution of (28). This solution can be extended up to a maximal interval \([R, \rho]\). If by contradiction the (increasing) solution \( z_\varepsilon \) satisfies \( z_\varepsilon(\rho^-) = z_\varepsilon^* < +\infty \), differentiating \( z_\varepsilon = T_\varepsilon z_\varepsilon \) we would argue that \( z_\varepsilon'(\rho^-) \) exists and is finite. Hence, by local existence \( z_\varepsilon \) could be extended past \( \rho \), a contradiction. \( \square \)

We are going to prove that, if \( B(t) \) does not grow too fast and under a reasonable structure condition on \( M_\mu \), the solution \( z_\varepsilon \) of (28) is defined on \([R, +\infty)\). To do this, we first need some definitions. We consider the initial condition \( \vartheta = 0 \). For convenience, we further require the following assumptions:
\[
\varphi \in C^1(\mathbb{R}^+), \quad a_2^{-1}t^{p-1} \leq t \varphi'(t) \leq a_1 + a_2 t^{p-1} \quad \text{on } \mathbb{R}^+,
\]
for some positive constants \( a_1, a_2 \). Define
\[
K_\mu(t) = \int_{\mu}^{t} s \varphi'(s)ds, \quad \beta(t) = \int_{0}^{t} B(s)ds.
\]
Note that \( \beta(t) \) is non-decreasing on \( \mathbb{R}^+ \) and that, for every \( \mu \geq 0 \), \( K_\mu \) is strictly increasing. By (30), \( K_\mu(+\infty) = +\infty \). We focus our attention on the condition
\[
(\triangleright KO) \quad \frac{1}{K_{\mu}^{-1}(\beta(s))} \notin L^1(+\infty).
\]
This (or, better, it opposite) is called the Keller-Osserman condition. Originating, in the quasilinear setting, from works of J.B. Keller [10] and R. Osserman [20], it has been the subject of an increasing interest in the last years. The interested reader can consult, for instance, [5], [16], [18]. Note that the validity of \( (\text{KO}) \) is independent of the choice of \( \mu \in [0, 1) \), and we can thus refer \( (\text{KO}) \) to \( K_0 = K \). This follows since, by (30), \( K_\mu(t) \asymp t^\rho \) as \( t \to +\infty \), where the constant is independent of \( \mu \), and thus \( K_\mu^{-1}(s) \asymp s^{1/p} \) as \( s \to +\infty \), for some constants which are uniform when \( \mu \in [0, 1) \). Therefore, \( (\text{KO}) \) is also equivalent to

\[
\frac{1}{\beta(s)^{1/p}} \notin L^1(+\infty)
\]

**Lemma 6.4.** In the assumptions of the previous proposition and subsequent discussion, suppose that \( g' \geq 0 \) on \( \mathbb{R}^+ \). If

\[
\frac{1}{K^{-1}((s))} \notin L^1(+\infty),
\]

then, for every choice of \( c \in (0, 1] \), the solution \( z_c \) of (28) is defined on \( [R, +\infty) \).

**Proof.** From \( [g^{m-1} \varphi(cz')]' = g^{m-1}B(cz) \) and \( g' \geq 0 \) we deduce that

\[
\varphi'(cz')cz'' \leq B(cz), \quad \text{so that} \quad cz' \varphi'(cz')cz'' \leq B(cz)cz' = (\beta(cz))'.
\]

Hence integrating and changing variables we obtain

\[
K_\mu(cz') = \int_0^{cz'} s \varphi'(s)ds \leq \int_0^{cz} B(s)ds = \beta(cz).
\]

Applying \( K_\mu^{-1}(cz') = K_\mu^{-1}(\beta(cz)) \). Since \( z' > 0 \), we can divide the last equality by \( K_\mu^{-1}(\beta(cz)) \) and integrate on \( [R, t] \) to get, after changing variables,

\[
\int_0^{cz(t)} \frac{ds}{K_\mu^{-1}(\beta(s))} \leq t - R.
\]

By \( (\text{KO}) \), we deduce that \( \rho \) cannot be finite for any fixed choice of \( c \). \( \square \)

For every \( R > 0 \), we have produced a radial function \( u_c = (cz_c) \circ r \) which solves \( L\varphi u_c = B(u_c) \) on \( M \setminus B_R \) and \( u_c = 0 \) on \( B_R \). The next step is to guarantee that, up to choosing \( \mu, c \) appropriately, \( u_c \) can be arbitrarily small on some bigger ball \( B_{R_1} \). The basic step is a uniform control of the norm of \( z_c \) on \( [R, R_1] \) with respect to the variable \( c \), up to choosing \( \mu = \mu(c) \) appropriately small. This requires a further control on \( B(t) \), this time on the whole \( \mathbb{R}^+ \) and not only in a neighbourhood of \( +\infty \).
Lemma 6.5. In the assumptions of the previous proposition, suppose further that

\[ B(t) \leq b_1 t^{p-1} \quad \text{on } \mathbb{R}^+. \]

Then, for every \( R_1 > R \) and every \( c \in (0, 1] \), there exists \( \mu > 0 \) depending on \( c \) such that the solution \( z_c \) of (28) with \( \psi = 0 \) satisfies

\[ \|z_c\|_{L^\infty([R,R_1])} \leq K, \]

for some \( K > 0 \) depending on \( R, R_1 \) and on \( a_2 \) in (30) and on \( b_1 \) in (32) but not on \( c \).

Proof. Note that, by (32), (KO) (equivalently, (31)) is satisfied. Hence, \( z_c \) is defined on \( [R, +\infty) \) for every choice of \( \mu, c \). Fix \( R_1 > R \). Setting \( \psi = 0 \) in the expression (29) of the operator \( T_c \), and using the monotonicity of \( g \) and \( z_c \), we deduce that

\[
\begin{align*}
  u_c(t) &\leq \frac{1}{c} \int_R^t \varphi^{-1}(\varphi(c\mu) + \int_R^s B(cu(\tau))d\tau) ds \\
  &\leq \frac{1}{c} \int_R^t \varphi^{-1}(\varphi(c\mu) + (R_1 - R)B(cu_c(s))) ds.
\end{align*}
\]

Differentiating, this gives

\[ \varphi(cu_c'(t)) \leq \varphi(c\mu) + (R_1 - R)B(cu_c(t)). \]

Now, from (30) and (32) we get

\[ c^{p-1}(u_c')^{p-1} \leq a_2 \varphi(c\mu) + a_2(R_1 - R)b_2 c^{p-1} u_c^{p-1}. \]

Choose \( \mu \) in such a way that

\[ \varphi(c\mu) \leq c^{p-1}, \quad \text{that is, } \quad \mu \leq \frac{1}{c} \varphi^{-1}(c^{p-1}) \]

Then, dividing (34) by \( c^{p-1} \) and applying the elementary inequality \( (x+y)^a \leq 2^a (x^a + y^a) \) we obtain the existence of a constant \( K = K(R_1, R, a_2, b_2) \) such that

\[ u_c'(t) \leq K(1 + u_c(t)). \]

Estimate (33) follows by applying Gronwall inequality. \( \Box \)

Corollary 6.6. Let the assumptions of the last proposition be satisfied. Then, for each triple \((B_R, B_{R_1}, \varepsilon)\), there exists a positive, radially increasing solution of \( L_{\varphi}u = B(u) \) on \( M_g \setminus B_R \) such that \( u = 0 \) on \( \partial B_R \) and \( u < \varepsilon \) on \( B_{R_1} \setminus B_R \).
Proof. By the previous lemma, for every $c \in (0, 1]$ we can choose $\mu = \mu(c) > 0$ such that the resulting solution $z_c$ of \((28)\) is uniformly bounded on $[R, R_1]$ by some $K$ independent of $c$. Since, by \((24)\), $u_c = (cz_c) \circ r$ solves $L_\varphi u_c = B(u_c)$, it is enough to choose $c < \varepsilon/K$ to get a desired $u = u_c$ for the triple $(B_R, B_{R_1}, \varepsilon)$.

To conclude, we shall show that Evans potentials exist for any triple $(K, \Omega, \varepsilon)$, not necessarily given by concentric balls centered at the origin. In order to do so, we use a comparison argument with suitable radial Evans potentials. To conclude, we shall show that Evans potentials exist for any triple $(K, \Omega, \varepsilon)$, not necessarily given by concentric balls centered at the origin.

Lemma 6.7. In the assumptions of Lemma 6.4, Let $0 < R$ be chosen, and let $w$ be a positive, increasing $C^1$ solution of

\[
\begin{cases}
[g^{m-1}\varphi(w')]' = g^{m-1}B(w) & \text{on } [R, +\infty) \\
w(R) = 0, & w'(R) = w'_R > 0
\end{cases}
\]

Fix $\hat{R} > R$. Then, for every $c > 0$, there exists $\mu = \mu(c, \hat{R}, \hat{R})$ small enough that the solution $z_c$ of \((28)\), with $R$ replaced by $\hat{R}$, satisfies $cz_c < w$ on $[\hat{R}, +\infty)$.

Proof. Let $\mu$ satisfy $g^{m-1}(R)\varphi(w'_R) > g^{m-1}(\hat{R})\varphi(c\mu)$. Suppose by contradiction that $\{cz_c \geq w\}$ is a closed, non-empty set. Let $r > \hat{R}$ be the first point where $cz_c = w$. Then, $cz_c \leq w$ on $[\hat{R}, r]$, thus $cz'_c(r) \geq w'(r)$. However, from the chain of inequalities

\[
\varphi(w'(r)) = \frac{g^{m-1}(R)\varphi(w'_R)}{g^{m-1}(r)} + \int_R^r B(w(\tau))d\tau > \frac{g^{m-1}(\hat{R})\varphi(c\mu)}{g^{m-1}(r)} + \int_{\hat{R}}^r B(cz'_c(\tau))d\tau = \varphi(cz'_c(r)),
\]

and from the strict monotonicity of $\varphi$ we deduce $w'(r) > cz'_c(r)$, a contradiction.

Corollary 6.8. For each $u$ constructed in Corollary 6.6, and for every $R_2 > R$, there exists a positive, radially increasing solution $w$ of $L_\varphi w = 0$ on $M_g \setminus B_{R_2}$ such that $w = 0$ on $\partial B_{R_2}$ and $w \leq u$ on $M \setminus B_{R_2}$.

Proof. It is a straightforward application of the last Lemma.

We are now ready to state the main result of this section

Theorem 6.9. Let $M_g$ be a model with origin $o$ and non-decreasing defining function $g$. Let $\varphi$ satisfies \((30)\) with $a_1 = 0$, and suppose that $B(t)$ satisfies \((32)\). Define $L_\varphi$ according to $L_\varphi u = L_\varphi u - B(u)$. Then, properties $(K)$, 

\[

\text{Proof.}
\]
(L) (for Hölder or $L^\infty$) and (E) restricted to triples $(K, \Omega, \varepsilon)$ with $o \in K$ are equivalent, and also equivalent to either

\[(36)\quad \left( \frac{\text{vol}(B_r)}{\text{vol}(\partial B_r)} \right)^{-\frac{1}{p-1}} \notin L^1(+\infty) \quad \text{if } B > 0 \text{ on } \mathbb{R}^+,
\]
or

\[(37)\quad \left( \frac{1}{\text{vol}(\partial B_r)} \right)^{-\frac{1}{p-1}} \notin L^1(+\infty) \quad \text{otherwise}.
\]

Proof. From (30), assumptions (36) and (37) are equivalent, respectively, to (26) and (27). Therefore, by Proposition 6.1 and Theorem 2.7, the result will be proved once we show that (L) implies (E) restricted to the triples $(K, \Omega, \varepsilon)$ such that $o \in K$. Fix such a triple $(K, \Omega, \varepsilon)$. Since $o \in K$ and $K$ is open, let $R < \rho$ be such that $B_R \Subset K \Subset \Omega \Subset B_\rho$. By making use of Corollary 6.6 we can construct a radially increasing solution $w_2$ of $L_Fw_2 = 0$ associated to the triple $(B_R, B_\rho, \varepsilon)$. By (L), $u$ must tend to $+\infty$ as $x$ diverges, for otherwise by the pasting Lemma 3.15 the function $s$ obtained extending $w_2$ with zero on $B_R$ would be a bounded, non-negative, non-constant solution of $L_Fs \geq 0$, contradiction. From Corollary 6.8 and the same reasoning, we can produce another exhaustion $w_1$ solving $L_Fw_1 = 0$ on $M \setminus B_\rho$, $w_1 = 0$ on $\partial B_\rho$ and $w_1 \leq w_2$ on $M \setminus B_\rho$. Setting $w_1$ equal to zero on $B_\rho$, by the pasting lemma 3.15 the function $w$ obtained extending $w_2$ with zero on $B_R$ is a global subsolution on $M$ below $w_2$. By the subsolution-supersolution method on $M \setminus K$, there exists a solution $w$ such that $w_1 \leq w \leq w_2$. By construction, $w$ is an exhaustion and $w \leq \varepsilon$ on $\Omega \setminus K$. Note that, by Remark 3.6 from (30) with $a_1 = 0$ we deduce that $w \in C^1(M \setminus K)$. We claim that $w > 0$ on $M \setminus K$. To prove the claim we can avail of the strong maximum principle in the form given in [26], Theorem 1.2. Indeed, again from (30) with $a_1 = 0$ we have (in their notation)

\[pa_2s^p \leq K(s) \leq pa_2s^p \quad \text{on } \mathbb{R}^+, \quad 0 \leq F(s) \leq \frac{b_1}{p}s^p \quad \text{on } \mathbb{R}^+,
\]

hence

\[\frac{1}{K^{-1}(F(s))} \notin L^1(0^+).
\]

The last expression is a necessary and sufficient condition for the validity of the strong maximum principle for $C^1$ solutions $u$ of $L_Fu \leq 0$. Therefore, $w > 0$ on $M \setminus K$ follows since $w$ is not identically zero by construction. In conclusion, $w$ is an Evans potential relative to $(K, \Omega, \varepsilon)$, as desired.

Acknowledgements: The authors would like to thank prof. Anders Bjorn for a helpful e-mail discussion, and in particular for having suggested them the reference to Theorem 3.19.
References


Luciano Mari, Dipartimento di Matematica, Università degli studi di Milano, via Saldini 50, 20133 Milano, Italy, EU
E-mail address: luciano.mari@unimi.it

Daniele Valtorta, Dipartimento di Matematica, Università degli studi di Milano, via Saldini 50, 20133 Milano, Italy, EU
E-mail address: danielevaltorta@gmail.com