Appendix A: Random generation of $T_{\alpha,\theta}$

Let $Y$ be a standard Exp(1) random variable. Note that

$$P\left(\frac{Y}{T_{\alpha}} > x\right) = \int_0^\infty P(Y > x^{1/\alpha} t) f_{\alpha}(t) dt = \int_0^\infty \exp[-x^{1/\alpha} t] f_{\alpha}(t) dt$$

$$= E\left[ e^{-x^{1/\alpha} T_{\alpha}} \right] = e^{-x} = P(Y > x)$$

so we have $Y = \frac{d}{\alpha}(Y/T_{\alpha})^\alpha$. For $r < \alpha$, $E(Y^{-r/\alpha}) < \infty$, so we find that $E(Y^{-r/\alpha}) = E(T_{\alpha}^r) E(Y^{-r})$ and

$$E(T_{\alpha}^r) = \frac{E(Y^{-r/\alpha})}{E(Y^{-r})} = \frac{\Gamma(1-r/\alpha)}{\Gamma(1-r)}.$$  \hspace{1cm} (1)

The normalizing constant in $f_{\alpha,\theta}(t)$ is $\int_0^\infty t^{-\theta} f_{\alpha}(t) dt = E(T_{\alpha}^{-\theta})$, so set $r = -\theta$ and note that $-\theta < \alpha$. Let $G_a$ be a gamma random variable with shape $a > 0$ and unit rate. Simple moment comparisons using (1) yield the distributional equality $G_{1+\theta/\alpha} \overset{d}{=} \frac{G_{1+\theta/\alpha}}{(1-\alpha)}(G_{1+\theta/\alpha})^\alpha$, which, however, does not provide a way to generate from $T_{\alpha,\theta}$. For this we resort to Devroye (2009). First we recall how to generate a Zolotarev random variable $Z_{\alpha,b}$ for $\alpha \in (0,1)$ and $b = \theta/\alpha > -1$. Let

$$C = \frac{\Gamma(1+b\alpha) \Gamma(1+b(1-\alpha))}{\pi \Gamma(1+b)}$$

and

$$B(u) = A(u)^{-(1-\alpha)} = \frac{\sin(u)}{\sin(\alpha u)^\alpha \sin((1-\alpha)u)^{1-\alpha}}.$$ 

A simple asymptotic argument yields the value $B(0) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$. Then $f(x) = C B(x)^b$, $0 \leq x \leq \pi$. The following bound holds

$$f(x) \leq C B(0)^b e^{-\frac{x^2}{2\sigma^2}}, \text{ with } \sigma^2 = \frac{1}{b\alpha(1-\alpha)}.$$  

This Gaussian upper bound suggests a simple rejection sampler for sampling Zolotarev random variates. Following Devroye (2009), it is most efficient to adapt the sampler to
the value of $\sigma$. If $\sigma \geq \sqrt{2\pi}$, rejection from a uniform random variate is best. Otherwise, use a normal dominating curve as suggested in the bound above. The details are given below.

Algorithm 3 (Sampler of $T_{\alpha, \theta}$)

1. set $b = \theta / \alpha$ and $\sigma = \sqrt{b \alpha (1 - \alpha)}$
2. if $\sigma \geq \sqrt{2\pi}$:
   then repeat: generate $U \sim \text{Unif}(0, \pi)$ and $V \sim \text{Unif}(0, 1)$.
           set $X \leftarrow U$, $W \leftarrow B(X)$.
           until $V \leq (W/B(0))^b$
   else repeat: generate $N \sim \text{N}(0, 1)$ and $V \sim \text{Unif}(0, 1)$.
           set $X \leftarrow \sigma |N|$, $W \leftarrow B(X)$.
           until $X \leq \pi$ and $Ve^{-N^2/2} \leq (W/B(0))^b$
3. generate $G \define G_{1+b(1-\alpha)/\alpha}$
4. set $T \leftarrow 1/(WG^{1-\alpha})^{1/\alpha}$
5. return $T$

References