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Symmetry-breaking vacua and baryon condensates in AdS/CFT correspondence

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Symmetry-breaking vacua and baryon condensates in AdS/CFT

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Abstract

We study the gravity duals of symmetry-breaking deformations of superconformal field theories, AdS/CFT dual to Type IIB string theory on $\mathrm{AdS}_5 \times Y$ where Y is a Sasaki-Einstein manifold. In these vacua both conformal invariance and baryonic symmetries are spontaneously broken. We present a detailed discussion of the supergravity moduli space, which involves flat form fields on asymptotically conical Calabi-Yau manifolds, and match this to the gauge theory vacuum moduli space. We discuss certain linearised fluctuations of the moduli, identifying the Goldstone bosons associated with spontaneous breaking of non-anomalous baryonic symmetries. The remaining moduli fields are related to spontaneous breaking of anomalous baryonic symmetries. We also elaborate on the proposal that computing condensates of baryon operators is equivalent to computing the partition function of a non-compact Euclidean D3-brane in the background supergravity solution, with fixed boundary conditions at infinity.

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1 Introduction

The AdS/CFT correspondence [1] allows one to understand gauge theory dynamics in terms of string theory on some background spacetime. Properties of strongly coupled gauge theories may then be understood geometrically, leading to non-trivial predictions on both sides of the correspondence. In this paper we will be interested in four-dimensional supersymmetric gauge theories that are dual to Sasaki-Einstein backgrounds in Type IIB string theory. In the infra-red (IR) these are non-trivial interacting superconformal field theories (SCFTs). A number of remarkable predictions of AdS/CFT have been confirmed in this case in recent years. For example, the complete spectrum of mesonic chiral operators may be computed purely geometrically [2, 3, 4, 5] using the results of [6, 7].

The AdS/CFT correspondence describes deformations of CFTs, as well as the conformal fixed points themselves. For instance, one may perturb a CFT in the ultra-violet (UV) by adding a relevant operator to the Lagrangian. Or one may consider different vacua of the theory, leading to spontaneous symmetry-breaking. In either case the field theory then flows under renormalisation group (RG) flow to the IR, where interesting dynamics may arise. In this paper we present a study of symmetry-breaking vacua of $\mathcal{N}=1$ SCFTs which are dual to Type IIB backgrounds of the form $\mathrm{AdS}_5 \times Y$, where Y is a Sasaki-Einstein five-manifold [8, 9, 10, 11]. A particular emphasis will be on the spontaneous breaking of certain global symmetries in these vacua.

The SCFTs that we discuss may be thought of as IR conformal fixed points of certain four-dimensional supersymmetric *quiver gauge theories*. The matter content and interactions of a quiver gauge theory are determined from combinatorial data, namely a directed graph (a quiver) together with a set of closed paths in this graph, encoded in the superpotential. The gauge theories of interest describe the effective worldvolume theory for D3-branes placed at an isolated conical Calabi-Yau singularity. The quiver data is then related to the algebraic geometry of this singularity and its resolutions. There are now many examples in the literature in which both the quiver gauge theory and the Calabi-Yau cone metric that describes the IR SCFT are known rather explicitly [9, 12, 13, 14, 15, 16, 17, 18, 19]. However, in the paper we aim to keep the discussion as general as possible, without resorting to specific examples.

A key point about the quiver gauge theories of interest is that they have rather large classical vacuum moduli spaces (VMSs). The VMS is obtained by minimising the classical potential of the theory. Since the gauge theory may be engineered by placing pointlike D3-branes at a Calabi-Yau singularity, it is expected that the VMS at least contains the corresponding Calabi-Yau. This is simply because the moduli space of a pointlike object on some manifold should at least contain that manifold. Similarly, for N D3-branes one expects to find the Nth symmetric product. In fact the classical VMS of a superconformal quiver gauge theory, referred to as the "master space" in [20], has a rather complicated structure. The centre of the gauge group, which is a torus $U(1)^{\chi}$, dynamically decouples in the IR and becomes a *global symmetry* group of the superconformal theory. Here $\chi \in \mathbb{N}$ denotes the number of nodes in the quiver. A $U(1)^{\chi-1}$ subgroup acts non-trivially on the VMS $\mathscr{M}$. As we shall see, this VMS has a complicated fibration structure. The fibres are themselves fibrations in which the base space is a *mesonic moduli space* and the fibres are generically tori $U(1)^{\chi-1}$. The $U(1)^{\chi-1}$ global symmetry acts in the obvious way on these fibres, and thus a generic point in $\mathscr{M}$ spontaneously breaks this symmetry. For the gauge theory on N D3-branes at an isolated Calabi-Yau singularity, the mesonic moduli spaces are expected to be Nth symmetric products of various Calabi-Yau *resolutions* of the singularity. For example, for a D3-brane at an abelian orbifold singularity it is a rigorous result that the moduli space $\mathcal{M}$ contains *all* possible Calabi-Yau resolutions of the orbifold [21]. The fibration structure of $\mathcal{M}$ contains all of these resolutions, which is why it is so complicated. Gauge-invariant chiral operators may be interpreted as the holomorphic functions on $\mathcal{M}$. The operators carrying non-zero charge under $U(1)^{\chi-1}$ may be written as determinants of bifundamental fields in the quiver gauge theory, and are therefore often referred to as *(di)baryon* operators. The global symmetry group is thus a *baryonic symmetry group.*

One of the main results of the paper will be the identification of the space $\mathcal{M}$ with a moduli space of certain supergravity backgrounds, as well as the identification of the $U(1)^{\chi-1}$ symmetry acting on this space. Generically, each of these backgrounds is the gravity dual of a renormalisation group flow from a SCFT in the UV to another[1] SCFT in the IR. Here the "UV theory" is itself the IR conformal fixed point of the $\mathcal{N}=1$ quiver theory (which is defined up to Seiberg dualities). One may then think of the RG flow as proceeding in two steps: first one flows from the far UV to a SCFT which admits an $\mathrm{AdS}_5 \times Y$ dual description. In a vacuum which spontaneously breaks conformal and baryonic symmetries of this theory, the RG flow proceeds towards a new theory in the deep IR. The supergravity backgrounds we discuss describe the second step, generalising the analysis of the resolved conifold in [23].

In fact the IR theory will be richer than just a SCFT – it will also contain additional low-energy degrees of freedom arising from the spontaneous breaking of the global baryonic symmetries. For each of these symmetries we obtain a corresponding massless Ramond-Ramond (RR) modulus in the supergravity solution, together with its supersymmetric partner. We interpret these as fluctuations along the flat directions in the VMS given by acting with a generator of the broken $U(1)^{\chi-1}$ symmetry group. An important feature of the theories that we describe is that only a *subgroup* of this global symmetry group is an *exact* quantum symmetry. The massless RR modes corresponding to these directions may then be identified as *Goldstone bosons.* This has been studied recently in [24] for the conifold theory, where $\chi = 2$ and the unique baryonic $U(1)$ symmetry is non-anomalous. More generally, the remaining global baryonic symmetries are anomalous, and are therefore only *approximate* symmetries of the theory. Of course, the directions in the classical moduli space that are related to symmetries

[1]Although it is possible to flow to a SCFT with a non-trivial dual Sasaki-Einstein geometry, as discussed in [22] for example, for simplicity we will focus here on the case that the IR theory is $\mathcal{N}=4$ super-Yang-Mills (SYM).

which are broken by anomalies might be lifted by possible quantum corrections. It follows that the associated massless modes discussed above may be lifted in the full quantum theory. On the other hand, the Goldstone bosons are protected since they represent motion in directions associated to true symmetries of the quantum theory; their existence and masslessness is guaranteed by Goldstone's theorem. As we discuss, the supergravity realisation of this statement is that the non-anomalous baryonic symmetries are dual to *gauge symmetries* in the bulk, which thus cannot possibly be anomalous. The anomalous symmetries, on the other hand, are dual to global symmetries of the supergravity backgrounds that do no come from any gauge symmetry.

An interesting aspect of these vacua is that baryon operators acquire non-zero vacuum expectation values (VEVs). As discussed above, classically the baryon operators may be identified with holomorphic functions on the VMS. In AdS/CFT these are conjectured [25, 26, 27] to be dual to D3-branes wrapped on three-submanifolds $\Sigma \subset Y$. One cannot therefore apply the standard AdS/CFT prescription [28, 29] to compute their one-point functions in the dual gravity description. Following a suggestion by E. Witten, it has been proposed [30] that in order to compute baryon VEVs one should perform a path integral over Euclidean D3-branes with fixed boundary condition at infinity. Several features of this idea have been succesfully verified in [22]. In this paper we will investigate in more detail this prescription for computing condensates of a reasonably large class of baryon operators. As we will explain, these are the baryon operators whose string duals are D3-branes wrapped on a smooth[2] Σ together with a flat line bundle. Although this is far from being the complete set of baryon operators, it is the simplest set of baryons which may be described in terms of classical configurations in the bulk. We will evaluate the D3-brane path integral in the semi-classical approximation, which will reduce the computation to a sum over worldvolume gauge instantons. A central issue is that, due to the non-compactness of the D3-brane worldvolume, we will have to pay particular attention to the transformation properties of the action under gauge transformations in the bulk. We will discuss a prescription to obtain a gauge-invariant, and thus physically meaningful, function on the supergravity moduli space. We also discuss the dependence of these baryon condensates on the Goldstone bosons and other RR moduli.

The outline of the paper is as follows. Section 2 reviews relevant aspects of quiver gauge theories arising from D3-branes at Calabi-Yau singularities. In particular, we include a discussion of anomalous $U(1)$s and a description of quiver gauge theory

[2]For toric submanifolds Σ of a toric Sasaki-Einstein manifold, these are Lens spaces.

moduli spaces. In section 3 we describe the supergravity backgrounds and their moduli. Linearised fluctuations of these supergravity moduli are discussed in section 4. In section 5 we discuss the holographic interpretation of the results. Section 6 presents the calculation of baryon condensates using Euclidean D3-branes wrapped on non-compact four-submanifolds. In section 7 we summarise our results and discuss some open questions and directions for future work. Certain technical material is relegated to several appendices.

2 Quiver gauge theories from Calabi-Yau singularities

In this paper we are interested in vacua of field theories that in the UV are described by superconformal field theories with AdS/CFT duals $\mathrm{AdS}_5 \times Y$, where (Y, g_Y) is a Sasaki-Einstein five-manifold. There is by now an extremely large class of examples of such AdS/CFT dualities where both sides of the correspondence are known explicitly. In these examples, the SCFT conjecturally arises as the IR fixed point of RG flow for a quiver gauge theory. The latter is an effective worldvolume theory for the D3-branes at the singular point of the cone $C(Y)$, describing the interactions of the lightest string modes. Here the metric cone

$$g_{C(Y)} = \mathrm{d}r^2 + r^2 g_Y \tag{2.1}$$

is Ricci-flat and Kähler *i.e.* Calabi-Yau.

Throughout the paper we aim for as general a discussion as possible, rather than focusing on specific examples. We begin in this section by discussing relevant background material about Calabi-Yau singularities and D-brane quiver gauge theories. In particular we will be interested in the general structure of the gauge theory vacuum moduli spaces. Our comments on this will also explain some of the recent results of [31] on counting gauge-invariant BPS operators. Although many of the results of this section are known, the arguments we present are somewhat more general than those typically appearing in the literature. We also hope this section will serve as a useful introduction to the subject.

2.1 Resolutions of Calabi-Yau cones

A Sasaki-Einstein five-manifold (Y, g_Y) may be defined by saying that its cone (2.1) is Ricci-flat Kähler. Roughly speaking, there are currently four known constructions of such Calabi-Yau cones:

- Orbifolds $\mathbb{C}^3/\Gamma$, where $\Gamma \subset SU(3)$ is a finite group. The Sasaki-Einstein link is then simply a quotient $Y = S^5/\Gamma$ of the round five-sphere (one may also take orbifolds (quotients) of some of the examples below).

- Complex cones over del Pezzo surfaces dP_k, where $k = 3, \ldots, 8$. Recall that a del Pezzo surface is the blow-up of $\mathbb{CP}^2$ at k points in general position. Provided $3 \leq k \leq 8$ these admit Kähler-Einstein metrics [32, 33]. The complex cone is obtained by taking the canonical line bundle over the del Pezzo and then collapsing the zero section.

- Affine toric singularities. Recently a general existence proof has been presented [34]. These include the explicit Sasaki-Einstein manifolds $Y^{p,q}$ [35, 12] and $L^{a,b,c}$ [36, 16] as links of the singularities.

- Quasi-homogeneous hypersurface singularities. Again these are existence arguments. For a review see [37], or the recent book [38].

Note that some examples fall into more than one of the classes above. In all cases the cone $C(Y)$ is an affine variety. When we wish to emphasize the algebro-geometric nature of the Calabi-Yau cone we will denote it by $C(Y) = Z$ *i.e.* Z is the zero set of a set of polynomials on $\mathbb{C}^D$ for some D. $C(Y)$ has an isolated singularity at the point $p = \{r = 0\}$ unless $C(Y) = \mathbb{C}^3$.

In this paper we will be interested in *resolutions* of such conical singularities. This requires two steps. First, we need to resolve the underlying singularity of Z, as a complex variety. Second, we need to find a Ricci-flat Kähler metric on that resolution that approaches the conical metric at infinity. We now discuss these two steps in more detail.

Technically, a resolution of Z is a smooth variety X together with a proper birational map $\pi : X \to Z$, such that $X \setminus E \cong Z \setminus p$ is a biholomorphism for some *exceptional set* $E \subset X$. Thus, roughly, the singular point p is resolved by replacing it with E. Singularities may always be resolved by a theorem of Hironaka. However, for our purposes we require the resolution X to be Calabi-Yau; that is, to have trivial canonical bundle. For reasons that we will not need to go in to, such a resolution is called *crepant*. In the first three examples of Calabi-Yau cones above, the corresponding singularity Z always admits a crepant resolution. However, the case of quasi-homogeneous hypersurface singularities is quite different. For example, the constructions reviewed in [37] lead to at least 68 different Sasaki-Einstein metrics on S^5. However, as pointed out in [2], none of the corresponding hypersurface singularities admit a crepant (Calabi-Yau) resolution. We suspect that the dual SCFTs do not admit a description in terms of a quiver gauge theory for these examples. Some of these 68 metrics on S^5 come in quite large families, the largest having complex dimension 5. This would mean that the dual SCFT has (at least) a 6-dimensional space of exactly marginal deformations, including the constant string coupling and its RR axion partner.

Assuming $Z = C(Y)$ is such that it admits a crepant resolution X, there are some immediate topological consequences for X. We begin by noting that the singularity Z is *Gorenstein*. By definition this means that it has a holomorphic $(3,0)$-form Ω on the smooth part $Z \setminus p$. In fact existence of a Ricci-flat Kähler cone metric on Z implies that Ω is homogeneous degree 3 under the radial vector $r\partial/\partial r$. Thus in particular Ω is square-integrable

$$\int_U \Omega \wedge \bar{\Omega} \,<\, \infty \tag{2.2}$$

in a small neighbourhood U of the singularity p at $r = 0$, as one sees by writing the integral in polar coordinates. This implies that the singularity Z is *rational*, and hence Z is a canonical singularity, in the sense of Reid. See, for example, [39] for an introduction to these concepts. It follows that for any crepant[3] resolution X the cohomology groups $H^*(X;\mathbb{Z})$ of X depend only on Z; that is, any two crepant resolutions X and X' have the same cohomology groups [40, 41]. For the examples listed above there are often many different choices of crepant resolution. If we denote

$$b_k(X) \,=\, \dim H_k(X;\mathbb{R}) \tag{2.3}$$

the Betti numbers of X, then it was shown in [42] (Theorem 5.2) that $b_1(X) = b_5(X) = b_6(X) = 0$. Moreover, as also proven in [42], $H^2(X;\mathbb{Z})$ is isomorphic to the Picard group of X, which in turn is generated by holomorphic line bundles. So all of the second cohomology of X is represented by closed $(1,1)$-forms; these may be represented by curvature forms on the holomorphic line bundles.

Throughout this paper we will also make the additional *assumption* that

$$b_3(X) \,=\, 0\ . \tag{2.4}$$

[3]This statement fails for non-crepant resolutions. For example, the conifold $Z = \{x^2+y^2+z^2+w^2 = 0\} \subset \mathbb{C}^4$ has a crepant small resolution $X = O(-1) \oplus O(-1) \to \mathbb{CP}^1$, which has $b_4(X) = 0$, but also has a *non-crepant* resolution $X = O(-1) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ with $b_4(X) = 1$.

This is satisfied by crepant resolutions of the first three types of Calabi-Yau cones in the list above. In general we do not know of a proof that this must always hold in the cases of interest in this paper[4]. Later in the paper we will give some further physical justifications for the assumption (2.4).

The assumption (2.4) has the following consequences. Consider the long exact cohomology sequence[5]

$$\begin{aligned} 0 \;\cong\; & H^1(Y;\mathbb{R}) \longrightarrow H^2(X,Y;\mathbb{R}) \longrightarrow H^2(X;\mathbb{R}) \longrightarrow \\ & H^2(Y;\mathbb{R}) \longrightarrow H^3(X,Y;\mathbb{R}) \;\cong\; 0 \; . \end{aligned} \tag{2.5}$$

Here we have used $b_1(Y) = 0$, which follows from Myers' theorem since (Y, g_Y) has positive Ricci curvature. The last isomorphism follows from Poincaré duality and (2.4). The exact sequence (2.5) implies, using Poincaré duality and the universal coefficients theorem, that[6]

$$b_3(Y) \;=\; b_2(X) - b_4(X) \; . \tag{2.6}$$

This relation will be important throughout the paper. We also note that the Euler characteristic of X is given by

$$\chi = \chi(X) \equiv \sum_{i=0}^{6} (-1)^i \dim\, H_i(X;\mathbb{R}) = 1 + b_2(X) + b_4(X) \; . \tag{2.7}$$

Having chosen a Calabi-Yau resolution X, we would like to put a Ricci-flat Kähler metric on it that is asymptotically conical *i.e.* approaches the cone metric (2.1) near infinity. If X were *compact*, Yau's theorem [44] would imply that X has a unique Ricci-flat Kähler metric in each Kähler class in $H^2(X;\mathbb{R})$. Unfortunately, there is currently no general analogue of Yau's theorem for existence and uniqueness of asymptotically conical metrics. However, provided the boundary conditions are right, one generally expects results about compact manifolds to hold also for non-compact manifolds, and we believe that being asymptotically a cone is precisely such a good boundary condition. We state this as a conjecture we shall assume:

---

[4] We thank B. Szendröi for discussions on this issue.

[5] Recall that the relative cohomology groups $H^p(X,Y;\mathbb{R})$, where $Y = \partial X$, are equivalent to compactly supported cohomology groups $H^p_{\mathrm{cpt}}(X;\mathbb{R})$.

[6] For toric X this relation may also be proven using Pick's theorem, by triangulating the toric diagram [43]. In fact (2.6) still holds even when $b_3(X) \neq 0$ [42], although the argument is then much more involved.

Conjecture: If $\pi : X \to Z$ is a crepant resolution of an isolated singularity $Z = C(Y)$, where $C(Y)$ admits a Ricci-flat Kähler cone metric, then X admits a unique Ricci-flat Kähler metric in each Kähler class in $H^2(X;\mathbb{R})$ that is asymptotic to a cone over the Sasaki-Einstein manifold (Y, g_Y).

Despite the lack of a general theorem, there are nonetheless several important results that go some way in this direction[7]. In [46], Joyce proves that any crepant resolution X of an orbifold $\mathbb{C}^3/\Gamma$, with $\Gamma \subset SU(3)$, admits a unique asymptotically conical Ricci-flat Kähler metric that is asymptotic to a cone over S^5/Γ. Thus the result we want is true for the simplest class of Sasaki-Einstein five-manifolds, namely quotients of the round five-sphere. On the other hand, in [47] (see also [48]) it is proven that, under certain mild technical assumptions, $X = \bar{X} \setminus D$ admits a Ricci-flat Kähler metric that is asymptotic to a cone, provided that the divisor $D \subset \bar{X}$ in the compact Kähler manifold (or orbifold) $\bar{X}$ admits a Kähler-Einstein metric of positive Ricci curvature. Note here that one is essentially compactifying X to $\bar{X}$ by adding a divisor D at infinity. In fact, it is a conjecture of Yau that every complete Ricci-flat Kähler manifold may be compactified this way. The metrics of [47, 48] are asymptotic to cones over regular, or quasi-regular, Sasaki-Einstein manifolds. This result has very recently been extended in [49] to the case that is D toric, but does not necessarily admit a Kähler-Einstein metric. Thus these are complete Ricci-flat Kähler metrics that are asymptotic to irregular toric Calabi-Yau cones. There are also a handful of explicit constructions, including: cohomogeneity one ansätze, including the resolved conifold metric on $O(-1) \oplus O(-1) \to \mathbb{CP}^1$ [50]; the Calabi ansatz [51] and its variants, for example studied in [52, 53] and more recently extended by Futaki in [54]; and, finally, constructions using Hamiltonian two-forms [55, 56, 57]. However, all of these latter explicit constructions are rather special, and rely on the presence of certain symmetries. In all cases, the Einstein equations reduce to solving ODEs, which is why it is possible to find explicit solutions.

We regard the above paragraph as rather convincing evidence for the conjecture. By assuming its validity we shall also obtain a consistent picture of the space of supergravity solutions that are dual to the supersymmetric symmetry-breaking vacua of interest.

[7] Since submitting the first draft of this paper to the archive, the paper [45] has appeared giving a proof of this conjecture in the case that the cohomology class of the Kähler form ω is compactly supported, *i.e.* $[\omega] \in H^2_{\mathrm{cpt}}(X;\mathbb{R}) \subset H^2(X;\mathbb{R})$. This substantially generalises the results of Joyce and Tian-Yau mentioned below.

2.2 Quivers and fractional branes

An interesting problem is to determine the effective worldvolume theory for D-branes placed at the singular point $\{r = 0\}$ of the cone $C(Y)$. In general this is very hard. However, if the Ricci-flat Kähler cone is either an orbifold $\mathbb{C}^3/\Gamma$, a complex cone over a del Pezzo surface, or a toric variety, then this worldvolume theory is believed to be a quiver gauge theory. The orbifold case is understood best [58], where the gauge theory may be constructed via standard orbifold techniques and leads to the Mckay quiver. For del Pezzo surfaces the quiver may be constructed from a special type of exceptional collection of sheaves on the del Pezzo surface – see, for example, [59] and in particular [60]. For toric varieties the quiver theory is believed to be described by a certain bipartite tiling of a two-torus called a dimer. For a recent review, see [61], [62]. Note that in all these cases the singularity $Z = C(Y)$ indeed admits a crepant resolution X with $b_3(X) = 0$. In this paper we shall assume our Calabi-Yau cone singularity $C(Y)$ is such that the worldvolume theory of N D3-branes at the singularity is described by a quiver gauge theory. This includes all of the above-mentioned cases.

A quiver is simply a directed graph. If V denotes the set of vertices and A the set of arrows, then we have head and tail maps $h, t : A \to V$. A *representation* of a quiver is an assignment of a $\mathbb{C}$-vector space U_v to each vertex $v \in V$ and a linear map $\phi_a : U_{t(a)} \to U_{h(a)}$ for each arrow $a \in A$. In particular, to specify a representation we must specify a dimension vector $\mathbf{n} \in \mathbb{N}^{|V|}$, so $\dim_{\mathbb{C}} U_v = n_v$. This data also leads to the notion of a quiver gauge theory. This is an $\mathcal{N} = 1$ gauge theory in four dimensions specified as follows:

- The gauge group
$$G = \prod_{v \in V} U(n_v) \tag{2.8}$$
is a product of unitary groups.

- To each arrow $a \in A$ we associate a chiral superfield Φ_a transforming in the fundamental representation of the gauge group $U(n_{h(a)})$ and the anti-fundamental representation of the gauge group $U(n_{t(a)})$. The fields are therefore often called bifundamental fields.

- In addition one must specify a superpotential
$$W = \sum_{l = a_1 \cdots a_k \in L} \lambda_l \,\mathrm{Tr}[\Phi_{a_1} \cdots \Phi_{a_k}] \ . \tag{2.9}$$

Here L is a set of closed oriented paths in the quiver, so $l = a_1 \cdots a_k$ denotes such a loop with $h(a_1) = t(a_k)$. The fact the loop is closed allows one to take a trace to obtain a gauge-invariant object. The complex numbers λ_l are coupling constants.

The relation between the singularity Z and the quiver is a large technical subject which is still not very well-understood. It is clearly very difficult to attack the problem of determining the worldvolume theory directly since we cannot quantise strings on $C(Y)$. However, one can circumvent this problem to some extent by replacing the Type IIB string by the topological string. The latter is independent of the Kähler class, and so does not see any difference between $Z = C(Y)$ and the crepant resolution (X, g_X). Moreover, the topological string is sufficient for addressing certain questions which are holomorphic in nature, such as the matter content and superpotential above. Space-filling D-branes on X are described in terms of the topological string as coherent sheaves on X, or more precisely its derived category. This is the mathematical way to understand the problem, which is then defined purely algebro-geometrically. However, we will not go into the details of this here.

For our purposes, all that we need to know is that there conjecturally exists a special set of D-branes, called *fractional branes*, which form a basis for all other D-branes at the singularity. Once we have resolved the singularity to a large smooth (X, g_X) one may think of D-branes as submanifolds of spacetime $\mathbb{R}^{1,3} \times X$ on which open strings may end. The space-filling D-brane charges on X are then determined by their homology[8] class in $H_*(X; \mathbb{Z})$. A complete basis therefore requires $1 + b_2(X) + b_4(X)$ fractional branes, corresponding to the charge of a D3-brane, and wrapped D5-branes and D7-branes, respectively. From (2.7) this is the Euler characteristic $\chi = \chi(X)$ of X. The nodes of the quiver are in 1-1 correspondence with these fractional brane basis elements. Thus

$$|V| = \chi \ . \tag{2.10}$$

The charges of any D-brane on X may then be used to expand the D-brane in terms of this basis. The ranks of the gauge groups $\mathbf{n} \in \mathbb{N}^{|V|}$ in the quiver are the coefficients in this expansion. More precisely, this identifies the unitary group factor $U(n_v)$ in (2.8) as the gauge group on the vth fractional brane. The bifundamental fields Φ_a describe the massless strings stretching between the fractional branes. Thus the quiver gauge theory is essentially a description of a D-brane at the singular point of $Z =$

[8] More precisely they are K-theory classes, but we will ignore this subtlety.

$C(Y)$ as a marginally bound state of the fractional branes, which should be mutually supersymmetric at the singular point.

Since a quiver gauge theory is in general chiral, it will typically suffer from various anomalies. In particular, gauge anomaly cancellation for the gauge group $U(n_v)$, corresponding to a triangle diagram with three gluons for this gauge group, is equivalent to

$$\sum_{a\in A|h(a)=v} n_{t(a)} - \sum_{a\in A|t(a)=v} n_{h(a)} \;=\; 0 \tag{2.11}$$

for all $v \in V$. Note this is $|V|$ equations for $|V|$ variables $n \in \mathbb{N}^{|V|}$. A gauge anomaly would of course lead to an inconsistent quantum theory, so one may wonder where the condition (2.11) comes from in string theory. As originally pointed out for del Pezzo surfaces in [59], this condition should be understood simply as charge conservation on X. In general, a space-filling D-brane wrapped on a $(k-3)$-submanifold $\Sigma_{k-3} \subset X$ has a charge in $H^{9-k}(X, Y; \mathbb{R})$. Here $k = 3, 5, 7$ are the possible Dk-branes. A Dk-brane is a magnetic source for the RR flux G_{8-k}, which thus satisfies

$$\mathrm{d}G_{8-k} \;=\; \frac{2\pi M}{\mu_k}\,\delta(\Sigma_{k-3}) \;. \tag{2.12}$$

Here

$$\mu_k \;=\; \frac{1}{(2\pi)^k \alpha'^{(k+1)/2}} \tag{2.13}$$

is the charge of a Dk-brane, $\delta(\Sigma_{k-3})$ is a delta-function representative of the Poincaré dual to Σ_{k-3} in X, and M is the number of wrapped branes. Thus $[\delta(\Sigma_{k-3})] \in H^{9-k}(X, Y; \mathbb{R})$ represents the charge of a single space-filling D-brane on Σ_{k-3}. The modified Bianchi identity (2.12) implies that the image of this in $H^{9-k}(X; \mathbb{R})$ is zero. There is a long exact cohomology sequence

$$\cdots \longrightarrow H^{8-k}(Y; \mathbb{R}) \xrightarrow{\beta} H^{9-k}(X, Y; \mathbb{R}) \longrightarrow H^{9-k}(X; \mathbb{R}) \longrightarrow \cdots \;. \tag{2.14}$$

Thus the only allowed D-brane charges on X are elements of $H^{9-k}(X, Y; \mathbb{R})$ that are images under β of $H^{8-k}(Y; \mathbb{R})$. The latter group measures the flux of G_{8-k} at infinity.

In the case at hand, we have $k = 3, 5, 7$. It is easy to show that $\beta(H^5(Y; \mathbb{R})) \cong \mathbb{R}$, $\beta(H^3(Y; \mathbb{R})) \cong \mathbb{R}^{b_3(Y)}$, since $b_3(X) = 0$, and $\beta(H^1(Y; \mathbb{R})) \cong 0$. Here the last relation follows since $H^1(Y; \mathbb{R}) = 0$ by Myers' theorem.

Thus there is only a $(b_3(Y) + 1)$-dimensional space of space-filling D-brane charges. The anomaly cancellation condition (2.11) is identified with this charge conservation condition. This implies there is a $(b_3(Y)+1)$-dimensional space of solutions to the $|V| = \chi = 1 + b_2(X) + b_4(X)$ linear equations in (2.11). In other words, the skew part of the adjacency matrix of the quiver has kernel of dimension $(b_3(Y)+1)$. We shall use this later in the paper.

2.3 Anomalous $U(1)$s

Throughout the paper a crucial role is played by the $|V| = \chi$ central $U(1)$ factors in the gauge group $G = \prod_{v \in V} U(n_v)$. In the quiver gauge theory these are dynamical $U(1)$ gauge fields. However, there are anomalies in addition to those already discussed in the previous subsection, namely mixed $\mathrm{Tr}[SU(n_v)^2 U(1)_{v'}]$ triangle anomalies. As is well-known, such anomalies often occur in string theory, and are cancelled via a form of Green-Schwarz mechanism. The anomalous combinations of $U(1)$ gauge fields become massive in the process, and are described by the Stückelberg action. This has recently been discussed in some detail for the case of del Pezzo singularities in [63]. Due to the importance of the anomalous $U(1)$s in our later discussion of AdS/CFT, we shall here review the salient features. At the same time this will allow us to generalise the del Pezzo results of [63] to any asymptotically conical Ricci-flat Kähler (X, g_X). In particular, the Stückelberg scalars are related to certain L^2 harmonic forms on (X, g_X), which we show indeed exist by appealing to a recent mathematical result. More practically, this section will allow us to review various properties of RR fields and their gauge symmetries, and also introduce notation used later in the paper.

We begin by noting that on the spacetime $\mathcal{M} = \mathbb{R}^{1,3} \times X$ one is free to turn on various fields without affecting the background metric. Firstly, there is the constant dilaton field ϕ, which determines the string coupling constant $g_s = \exp\phi$. This is paired under the $SL(2;\mathbb{R})$ symmetry of Type IIB supergravity with a constant axion field C_0. These combine into the axion-dilaton

$$\tau = C_0 + i \exp(-\phi) \ . \tag{2.15}$$

Secondly, we may turn on various flat form fields. In particular, we may turn on a flat B field, and flat RR fields C_2 and C_4 on X. The classification of such fields, up to gauge equivalence, is discussed in appendix C. However, in the presence of a non-trivial B field the gauge transformations of the RR fields are twisted. Recall that the gauge symmetries of string theory require that gauge transformations of RR potentials are also accompanied by transformations of higher rank potentials. Consider, for instance, the $SL(2;\mathbb{Z})$, or equivalently large gauge, transformation

$$C_0 \to C_0 + 1 \ . \tag{2.16}$$

This must be accompanied by the transformations

$$C_2 \to C_2 + B \ , \qquad\qquad C_4 \to C_4 + \frac{1}{2} B \wedge B \ . \tag{2.17}$$

One way to see this is to note that the gauge-invariant RR field strength may be written as

$$\tilde{G} = \mathrm{d}C - H_3 \wedge C = \mathrm{e}^{B}\, \mathrm{d}(\mathrm{e}^{-B} C) \ . \tag{2.18}$$

C is a formal sum of RR form potentials

$$C = \sum_{p \geq 0} C_{2p} \ . \tag{2.19}$$

The combination $C\,\mathrm{e}^{-B}$, which appears in the Chern-Simons action of D-branes to be discussed below, transforms as

$$C\,\mathrm{e}^{-B} \to C\,\mathrm{e}^{-B} + \mathrm{d}\Lambda \tag{2.20}$$

where Λ is a formal sum of odd-degree forms. General gauge transformations may also be written this way, if one allows $\mathrm{d}\Lambda$ to be any closed form with appropriately quantised periods[9].

In the case at hand, $\mathcal{M} = \mathbb{R}^{1,3} \times X$ is contractible to X. Thus we may turn on the following (non-torsion) flat fields

$$C_2 \ = \ \frac{2\pi}{\mu_1} \sum_{M=1}^{b_2(X)} c_2^M \Upsilon^M \ , \qquad\qquad B \ = \ \frac{2\pi}{\mu_1} \sum_{M=1}^{b_2(X)} b^M \Upsilon^M \tag{2.21}$$

$$C_4 \ = \ \frac{2\pi}{\mu_3} \sum_{A=1}^{b_4(X)} c_4^A \, \Xi^A \ . \tag{2.22}$$

The factors of μ_k are related to the normalisation of large gauge transformations, which are in turn determined by the D-brane Wess-Zumino couplings. The Υ^M are closed two-forms with integer periods, generating the lattice $H^2_{\mathrm{free}}(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})/H^2_{\mathrm{tors}}(X;\mathbb{Z})$.

[9] Λ is then roughly instead a formal sum of connection forms on gerbes.

Similarly, the Ξ^A are closed four-forms with integer periods, generating the lattice $H^4_{\rm free}(X;\mathbb{Z})$. Before taking into account large gauge transformations we may view the flat RR fields as a vector

$$(C_0, [C_2], [C_4]) \in H^0(X;\mathbb{R}) \oplus H^2(X;\mathbb{R}) \oplus H^4(X;\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}^{b_2(X)} \oplus \mathbb{R}^{b_4(X)} \cong \mathbb{R}^\chi \tag{2.23}$$

The lattice of large gauge transformations is

$$\begin{aligned}\Lambda^X_B \;&=\; \Bigg\{ \left(n, \frac{2\pi}{\mu_1}\sigma + nB, \frac{2\pi}{\mu_3}\kappa + \frac{2\pi}{\mu_1}\sigma\wedge B + \frac{1}{2}nB\wedge B\right) \mid n \in \mathbb{Z}, \\ &\qquad \sigma \in H^2_{\rm free}(X;\mathbb{Z}), \kappa \in H^4_{\rm free}(X;\mathbb{Z}) \Bigg\} \subset \mathbb{R}^\chi \ .\end{aligned} \tag{2.24}$$

Thus the flat RR fields, parameterised by the χ constants C_0 and c_2^M, c_4^A in (2.21), (2.22), respectively, live in the twisted torus

$$(C_0, [C_2], [C_4]) \in \mathbb{R}^\chi / \Lambda^X_B \ . \tag{2.25}$$

Once we have resolved the singularity to a large smooth (X, g_X), the χ fractional branes may be described as certain space-filling D3-D5-D7 bound states. In the remainder of this subsection we study the dynamics of the $U(1)$ gauge fields on $\mathbb{R}^{1,3}$ using this large-volume description, essentially following [63]. We focus on a single fractional brane, and assume for simplicity that it has a non-zero D7-brane charge. We shall denote the compact four-submanifold in X that the brane wraps by Σ_4, which gives rise to a homology class $[\Sigma_4] \in H_4(X;\mathbb{Z})$. At large volume the worldvolume theory of the fractional brane is described by the Born-Infeld and Chern-Simons actions. The Born-Infeld action is

$$S_{BI} \;=\; -T_7 \int_{\mathbb{R}^{1,3}\times\Sigma_4} \mathrm{d}\sigma^8 \, \mathrm{Tr}\sqrt{-\det(h + 2\pi\alpha' F - B)} \ . \tag{2.26}$$

Here T_k is the Dk-brane tension, related to the charge (2.13) by

$$g_s T_k \;=\; \mu_k \ , \tag{2.27}$$

and σ_α, $\alpha = 0, \ldots, 7$ denote worldvolume coordinates. h denotes the induced metric on the worldvolume from its embedding into spacetime $\mathcal{M} = \mathbb{R}^{1,3} \times X$, B is the pull-back of the NS two-form, and F denotes the curvature of a $U(n)$ gauge field for a worldvolume gauge bundle E of rank n. The induced metric and the B field are understood to be multiplied by a unit $n \times n$ matrix in these formulae. Recall that B is not gauge-invariant, but rather transforms as $B \to B + \mathrm{d}\lambda$ where λ is a one-form. In fact large gauge transformations may also be included if $\mu_1\lambda$ is taken to be a connection one-form on some line bundle over the spacetime $\mathcal{M}$. Thus $[\mu_1 \mathrm{d}\lambda/2\pi] \in H^2(\mathcal{M};\mathbb{Z})$. At the same time the worldvolume gauge field F transforms as

$$F \to F + \mu_1 \iota^* \mathrm{d}\lambda \ , \tag{2.28}$$

where ι denotes the embedding. Again, for non-abelian F a unit $n \times n$ matrix is understood in this formula.

The Chern-Simons terms are given by[10]

$$S_{CS} = \mu_7 \int_{\mathbb{R}^{1,3}\times\Sigma_4} C \,\mathrm{Tr}\, \mathrm{e}^{2\pi\alpha' F - B} \sqrt{\frac{\hat{\mathcal{A}}(4\pi^2\alpha' R_T)}{\hat{\mathcal{A}}(4\pi^2\alpha' R_N)}} \ . \tag{2.29}$$

Here μ_7 is the D7-brane charge (2.13) and the curvature terms will play no role in our discussion, so we shall ignore them. The topology of the gauge bundle E over Σ_4 induces D5-brane and D3-brane charges on the D7-brane via (2.29).

The worldvolume gauge field A, with field strength F, dimensionally reduces to a $U(n)$ gauge field on $\mathbb{R}^{1,3}$. Since we are only interested in the $U(1)$s we study here only the abelian part of this gauge field, which we denote $\mathcal{A}$. Its field strength is $\mathcal{F}$. At low energies this has a standard kinetic term

$$-\frac{1}{4g^2}\int_{\mathbb{R}^{1,3}} \mathcal{F} \wedge *_4 \mathcal{F} \tag{2.30}$$

where the gauge coupling g may be related to the Born-Infeld volume of Σ_4. The flat background fields (2.21), (2.22), together with the topology of the gauge bundle E over Σ_4, also induce an effective θ-angle term

$$\frac{1}{32\pi^2}\int_{\mathbb{R}^{1,3}} \theta\, \mathcal{F} \wedge \mathcal{F} \tag{2.31}$$

where

$$\begin{aligned} \frac{\theta}{8\pi} \;=\; & \int_{\Sigma_4} \Big\{ C_0 \Big[\mathrm{ch}_2(E) - \mathrm{ch}_1(E) \wedge b^M \Upsilon^M + \frac{1}{2}\mathrm{ch}_0(E) b^M b^N \Upsilon^M \wedge \Upsilon^N \Big] \\ & + c_2^M \Upsilon^M \wedge \big[\mathrm{ch}_1(E) - \mathrm{ch}_0(E) b^N \Upsilon^N \big] + c_4^A \mathrm{ch}_0(E) \Xi^A \Big\} \ . \end{aligned} \tag{2.32}$$

Summation is understood over repeated indices. $\mathrm{ch}(E)$ denote the Chern characters of the bundle E, so in particular $\mathrm{ch}_0(E) = n$ is the rank of the gauge bundle, or

[10] See, for example, [64]. The normal bundle couplings are given in [65].

equivalently number of D7-branes. Note that the above, slightly technical, discussion of large gauge transformations in string theory is crucial for seeing that the expression (2.32) is a well-defined angle.

Now consider fluctuations of the background form fields. If one has a k-form field C_k on $\mathbb{R}^{1,3} \times X$ then one will obtain a massless dynamical scalar field φ on $\mathbb{R}^{1,3}$ via an ansätz

$$C_k = \varphi \psi \tag{2.33}$$

provided the k-form ψ is closed, co-closed and L^2 normalisable on (X, g_X). The last condition ensures that the kinetic energy of φ is finite. Thus in particular ψ is an L^2 harmonic k-form on (X, g_X). We denote the space of such forms by $\mathcal{H}^k_{L^2}(X, g_X)$. For (X, g_X) asymptotically conical, the number of such harmonic forms is known [66]. Theorem 1A of the latter reference says that for a complete asymptotically conical manifold (X, g_X) of real dimension m with boundary ∂X the following natural isomorphisms[11] hold:

$$\mathcal{H}^k_{L^2}(X, g_X) \cong \begin{cases} H^k(X, \partial X; \mathbb{R}), & k < m/2 \\ f(H^{m/2}(X, \partial X; \mathbb{R})) \subset H^{m/2}(X; \mathbb{R}), & k = m/2 \\ H^k(X; \mathbb{R}), & k > m/2 \end{cases} . \tag{2.34}$$

Thus the space of L^2 harmonic forms is topological. It follows that the only L^2 harmonic forms on (X, g_X) are $\mathcal{H}^2_{L^2}(X, g_X) \cong H^2(X, \partial X; \mathbb{R}) \cong H_4(X; \mathbb{R})$ and $\mathcal{H}^4_{L^2}(X, g_X) \cong H^4(X; \mathbb{R})$. There are hence $b_4(X)$ L^2 harmonic two-forms and four-forms on (X, g_X), respectively. Since X is complete and asymptotically a cone, these forms are also closed and co-closed.

Thus only $b_4(X)$ of the $b_2(X)$ constants in (2.21) may be interpreted as VEVs of massless dynamical axions in $\mathbb{R}^{1,3}$, whereas all of the constants c_4^A in (2.22) are VEVs of massless dynamical axions. We focus in the following only on the RR fields, and write the dynamical fields

$$C_2 = \frac{2\pi}{\mu_1} c_2^A \Upsilon^A , \qquad C_4 = \frac{2\pi}{\mu_3} c_4^A \, \Xi^A \tag{2.35}$$

where now c_2^A and c_4^A are massless scalar fields on $\mathbb{R}^{1,3}$, and $\Upsilon^A \in \mathcal{H}^2_{L^2}(X, g_X)$, $\Xi^A \in \mathcal{H}^4_{L^2}(X, g_X)$. In fact since the Hodge dual of an L^2 harmonic form is also an L^2 harmonic

[11] Here f is the forgetful map $H^k(X, \partial X; \mathbb{R}) \xrightarrow{f} H^k(X; \mathbb{R})$, that forgets that a class has compact support.

form, we may clearly take

$$\Xi^A = R^{AB} *_6 \Upsilon^B \tag{2.36}$$

for some constant matrix $R = (R^{AB}) \in GL(b_4(X);\mathbb{R})$, where $*_6$ denotes the Hodge operator on (X, g_X). Recall that RR fields are self-dual, so we must also turn on

$$C_6 \;=\; \frac{2\pi}{\mu_5}\tilde{c}_2^A \wedge \Xi^A \tag{2.37}$$

and more correctly write

$$C_4 \;=\; \frac{2\pi}{\mu_3}(c_4^A\, \Xi^A + \tilde{c}_4^A \wedge \Upsilon^A) \ . \tag{2.38}$$

Here $\tilde{c}_2^A$ and $\tilde{c}_4^A$ are dynamical two-form potentials on $\mathbb{R}^{1,3}$. Self-duality then requires

$$\mathrm{d}c_2^A \;=\; \frac{\mu_1}{\mu_5}R^{BA} *_4 \mathrm{d}\tilde{c}_2^B, \qquad\qquad \mathrm{d}\tilde{c}_4^A \;=\; R^{BA} *_4 \mathrm{d}c_4^B \ . \tag{2.39}$$

Note also that C_2 itself describes a two-form on $\mathbb{R}^{1,3}$. Altogether these terms produce a coupling

$$\int_{\mathbb{R}^{1,3}} c \wedge \mathcal{F} \tag{2.40}$$

where

$$\begin{aligned} c \;=\; & \frac{\mu_1}{2\pi}C_2 \int_{\Sigma_4}\Big[\mathrm{ch}_2(E) - \mathrm{ch}_1(E)\wedge b^M\Upsilon^M + \frac{1}{2}\mathrm{ch}_0(E)b^M b^N \Upsilon^M \wedge \Upsilon^N\Big] \\ & + \tilde{c}_4^A \int_{\Sigma_4} \Upsilon^A \wedge \big[\mathrm{ch}_1(E) - \mathrm{ch}_0(E) b^M \wedge \Upsilon^M\big] + \tilde{c}_2^A \int_{\Sigma_4} \mathrm{ch}_0(E)\Xi^A \ . \end{aligned} \tag{2.41}$$

The interesting part of the effective Lagrangian for the $U(1)$ gauge field $\mathcal{A}$ on $\mathbb{R}^{1,3}$ is then

$$\mathcal{L} = -\frac{1}{4g^2}\mathcal{F} \wedge *_4\mathcal{F} + c \wedge \mathcal{F} + \frac{\theta}{32\pi^2}\mathcal{F}\wedge\mathcal{F} \tag{2.42}$$

where the two-form c and scalar θ are given by (2.41) and (2.32), respectively. The precise formulae are not particularly important. The important point to notice is that c is linear in the $1 + b_4(X) + b_4(X) = \chi - b_3(Y)$ variables C_2, $\tilde{c}_2^A$ and $\tilde{c}_4^A$, respectively. The field C_2 is non-dynamical since (X, g_X) has infinite volume. Thus, for fixed B field flux b^M on X, c depends linearly on $2b_4(X)$ dynamical two-forms. There are χ different fractional branes, wrapping different cycles in X or with different gauge bundles E, each with a gauge field $\mathcal{A}_s$, $s = 1, \ldots, \chi$. By taking linear combinations of fractional $U(1)$s with no C_2 term in c we obtain generically $2b_4(X)$ linearly independent gauge fields with an effective Lagrangian of the form (2.42) with $c \neq 0$, and also a finite kinetic term for c. Here the kinetic term for c comes from the bulk IIB supergravity kinetic terms for the RR fields. A standard change of variable then shows that the dual scalar to c in $\mathbb{R}^{1,3}$ is a Stückelberg field, and thus (2.42) describes the action for a *massive gauge field*. Indeed, the equation of motion for c, in the presence of a $c \wedge \mathcal{F}$ coupling, is given by

$$\mathrm{d} *_4 \mathrm{d}c = \mathcal{F} \ . \tag{2.43}$$

A duality transformation to a scalar field ρ involves interchanging equations of motion and Bianchi identities. Thus one defines ρ satisfying

$$\mathrm{d}\rho = *_4 \mathrm{d}c - \mathcal{A} \tag{2.44}$$

so that (2.43) is automatic. The equation of motion for ρ is then

$$\mathrm{d} *_4 (\mathrm{d}\rho + A) = 0 \ . \tag{2.45}$$

The definition (2.44) implies that a gauge transformation $\mathcal{A} \to \mathcal{A} + \mathrm{d}\lambda$ must be accompanied by a transformation $\rho \to \rho - \lambda$. Thus ρ is a Stückelberg scalar, and the gauge field $\mathcal{A}$ is in fact massive. The dual scalar is, from (2.39), a linear combination of the dynamical scalars c_2^A, c_4^A, which in turn enter the expression for the θ-angle (2.32). Thus the $2b_4(X)$ Stückelberg fields that give masses to $2b_4(X)$ of the fractional brane $U(1)$s are linear combinations of θ-angles. It is precisely this fact that allows the additional triangle anomalies to be cancelled. On the other hand, the $\chi - 2b_4(X) = 1 + b_3(Y)$ linear combinations of fractional brane $U(1)$s with no $\tilde{c}_2^A$ and $\tilde{c}_4^A$ fields in c remain massless. Note that the overall $U(1)$, often referred to as the centre of mass $U(1)$, should essentially decouple from everything.

We end this section with a comparison to the quiver gauge theory formula for anomaly cancellation. Let $q \in \mathbb{Z}^{|V|}$ denote a vector of charges specifying a subgroup $U(1)_q \subset U(1)^{|V|} \subset G$. Recall that $|V| = \chi$. Then cancellation of the triangle anomaly $\mathrm{Tr}[U(1)_q SU(n_v)^2]$ is equivalent to

$$\sum_{a \in A | h(a) = v} n_{t(a)} q_{t(a)} - \sum_{a \in A | t(a) = v} n_{h(a)} q_{h(a)} = 0 \ . \tag{2.46}$$

The following neat argument is due to [67]. Recall that anomaly cancellation (2.11) for the gauge group $G = \prod_{v \in V} U(m_v)$ requires

$$\sum_{a \in A | h(a) = v} m_{t(a)} - \sum_{a \in A | t(a) = v} m_{h(a)} = 0 \tag{2.47}$$

for all $v \in V$. As explained in the previous subsection, there exist $b_3(Y) + 1$ linearly independent solutions to (2.47). We may thus solve (2.46) by setting

$$q_v = \frac{m_v}{n_v} \tag{2.48}$$

assuming $n_v \neq 0$ for all $v \in V$ for the D3-brane worldvolume theory of interest. A discussion of why this should be the case may be found in [67]. Note, however, that taking $\mathbf{m} = \mathbf{n}$ leads to $q = (1, 1, \ldots, 1)$. This corresponds to the overall diagonal $U(1)$ under which nothing is charged. Thus one has $b_3(Y)$ non-anomalous $U(1)$s, precisely as we have argued above using a large-volume description of the fractional branes.

2.4 Marginal couplings and superconformal quivers

In the IR all $U(1)$s dynamically decouple, anomalous or otherwise. The overall diagonal $U(1)$ completely decouples from everything as nothing is charged under it. The massless non-anomalous $U(1)$s decouple since their gauge coupling goes to zero in the IR, while the massive $U(1)$s decouple because they are massive. In the IR theory we will therefore encounter only global $U(1)$ symmetries, and these likewise split as non-anomalous and anomalous, as global symmetries. The IR gauge group will thus be

$$SG \equiv \prod_{v \in V} SU(n_v) \ . \tag{2.49}$$

A necessary condition for an IR fixed point of the quiver gauge theory is that the β functions of all coupling constants vanish. For a quiver gauge theory the vanishing of the NSVZ β functions, which are exact in perturbation theory, is given by[12]

$$\begin{aligned} 0 &= \hat{\beta}_{1/g_v^2} \equiv 2n_v + \sum_{a \in A | h(a) = v} (R_a - 1) n_{t(a)} + \sum_{a \in A | t(a) = v} (R_a - 1) n_{h(a)} && (2.50) \\ 0 &= \hat{\beta}_{\lambda_l} \equiv -2 + \sum_{a \in \text{loop } l} R_a \ . && (2.51) \end{aligned}$$

[12]The expressions $\hat{\beta}$ in these formulae are not the actual NSVZ β functions, but are rather proportional to them.

Here g_v, $v \in V$, are the gauge couplings while λ_l, $l \in L$, are the superpotential couplings. Recall that L is a set of oriented loops in the quiver. We have also defined

$$R_a = \frac{2}{3}(1 + \gamma_a) \tag{2.52}$$

where $\gamma_a = \gamma_a(\{g_v\}_{v \in V}, \{\lambda_l\}_{l \in L})$ is the anomalous dimension of the bifundamental field Φ_a. Thus γ_a, and hence R_a, are functions of $|V| + |L|$ couplings.

Setting the β functions (2.50) to zero gives $|V| + |L|$ linear equations in the variables $\{R_a\}_{a \in A}$. However, notice from (2.46) that if $\{R_a^*\}$ is a zero of the β functions, then so is $\{R_a^* + \mu Q_a\}$ for any real number $\mu \in \mathbb{R}$, where

$$Q_a = q_{t(a)} - q_{h(a)} \ . \tag{2.53}$$

If we instead regard the β functions as functions of the *couplings*, then this simple argument shows that, generically, the space of *marginal* couplings will be at least $b_3(Y)$-dimensional. Indeed, the non-anomalous $U(1)$ symmetries are directly related to the number of marginal couplings via the above argument. We will see how this happens in the dual AdS description in section 3.1.

For the theory on N D3-branes, we also expect in general a linear relation

$$N \sum_{l \in L} \hat{\beta}_{\lambda_l} = \sum_{v \in V} \hat{\beta}_{1/g_v^2} \tag{2.54}$$

leading to another marginal direction. In particular, we conjecture that the relation (2.54) should correspond to the constant string coupling in the dual AdS background. One can show that (2.54) is indeed an identity for toric quiver gauge theories as follows (for further discussion, see the recent paper [68]). For a toric quiver gauge theory on N D3-branes at a toric Calabi-Yau singularity one has $n_v = N$ for all $v \in V$. Also, each field Φ_a appears precisely twice in the superpotential W. These statements imply

$$N \sum_{l \in L} \hat{\beta}_{\lambda_l} = 2N \sum_{a \in A} R_a - 2N|L| \tag{2.55}$$

$$\sum_{v \in V} \hat{\beta}_{1/g_v^2} = 2N|V| + 2 \sum_{a \in A} (R_a - 1)N \ . \tag{2.56}$$

Thus

$$\sum_{v \in V} \hat{\beta}_{1/g_v^2} - N \sum_{l \in L} \hat{\beta}_{\lambda_l} = 2N(|V| - |A| + |L|) = 0 \ , \tag{2.57}$$

where the last relation follows from Euler's theorem applied to the brane tiling [69].

Finally, note that coupling constants in $\mathcal{N} = 1$ gauge theories are always complex. In particular, the gauge couplings g_v are paired with θ-angles. Thus one expects *at least* a $(b_3(Y) + 1)$-dimensional space of complex marginal couplings.

2.5 Classical vacuum moduli space

In this section we review the classical vacuum moduli space $\mathscr{M}$ of a quiver gauge theory with gauge group (2.49). This may be referred to[13] as the "master space" [20]. The main purpose of this subsection is to describe a (singular) fibration structure of $\mathscr{M}$ in which the fibres are constructed from mesonic moduli spaces. We shall see in section 3 that much of the structure of this classical VMS is reproduced in the dual supergravity solutions. There exist two complementary descriptions of the VMSs of interest, namely the *Kähler quotient* description and the *GIT quotient* description, and we will discuss both below. The former is perhaps more familiar to physicists and indeed is the most convenient for the purposes of the paper.

As discussed in the previous subsection, in the IR the gauge group is given by (2.49). The classical VMS of such a quiver gauge theory is the space of constant matrix-valued fields Φ_a minimising the potential. This is equivalent to setting the F-terms to zero

$$\frac{\partial W}{\partial \Phi_a} = 0 , \quad a \in A \tag{2.58}$$

and also the D-terms to zero

$$\mu_v = 0 , \quad v \in V . \tag{2.59}$$

Here μ denotes the vector of D-terms

$$\begin{aligned}\mu_v \;=\; & -\sum_{a\in A|\ t(a)=v}\left[\Phi_a^\dagger\Phi_a - \frac{1}{n_v}\mathrm{Tr}(\Phi_a^\dagger\Phi_a)1_{n_v\times n_v}\right] \\ & +\sum_{a\in A|\ h(a)=v}\left[\Phi_a\Phi_a^\dagger - \frac{1}{n_v}\mathrm{Tr}(\Phi_a\Phi_a^\dagger)1_{n_v\times n_v}\right] .\end{aligned} \tag{2.60}$$

Finally, one must identify configurations related by the action of the gauge group SG given by (2.49).

A bifundamental field Φ_a in vacuum is just an $n_{h(a)} \times n_{t(a)}$ matrix. The space of all such fields may then be thought of as

$$\mathbb{C}^D = \bigoplus_{a\in A} \mathbb{C}^{n_{h(a)}\times n_{t(a)}} . \tag{2.61}$$

Since the superpotential W is a polynomial in the Φ_a, the F-term equations (2.58) cut out an affine algebraic set

$$\mathscr{Z} = \{\mathrm{d}W = 0\} \subset \mathbb{C}^D . \tag{2.62}$$

[13] The terminology is apparently due to A. Bertram.

Since W is invariant under G, and thus also $SG \subset G$, it follows that SG acts on $\mathcal{Z}$. The process of setting the D-terms to zero and quotienting by the action of the gauge group is then by definition the *Kähler quotient*

$$\mathcal{M} = \mathcal{Z}//SG \ . \tag{2.63}$$

Indeed, the D-terms (2.59) are, up to a factor of i, the moment map for the action of SG on $\mathbb{C}^D$ equipped with its standand flat Kähler structure[14]. Note the subtraction of the trace terms ensures that μ_v is traceless, as required for an element of the (dual) Lie algebra of $SU(n_v)$. The vacuum moduli space $\mathcal{M}$ inherits a Kähler metric from the flat metric on $\mathbb{C}^D$.

We may alternatively construct $\mathcal{M}$ algebro-geometrically. It is useful to introduce the complexified gauge group

$$SG_{\mathbb{C}} = \prod_{v\in V} SL(n_v;\mathbb{C}) \ . \tag{2.64}$$

In Geometric Invariant Theory (GIT) it is natural to define the quotient of $\mathcal{Z}$ by $SG_{\mathbb{C}}$ in terms of the ring of invariants of $SG_{\mathbb{C}}$

$$\mathcal{M} = \mathcal{Z}//SG_{\mathbb{C}} = \mathrm{Spec}\ \mathbb{C}[\mathcal{Z}]^{SG_{\mathbb{C}}} \ . \tag{2.65}$$

This is also the ring of *semi-invariants* of $G_{\mathbb{C}}$. The construction (2.65) realises $\mathcal{M}$ as an affine set. In more detail, the ring of invariants of $SG_{\mathbb{C}}$ is finitely generated as the group $SG_{\mathbb{C}}$ is reductive (it is the complexification of a compact Lie group). One may thus pick a set of d generators, for some d. This realises $\mathcal{M}$ as an affine set in $\mathbb{C}^d$, the relations among the generators being the defining equations. It then follows from a general theorem that the GIT quotient (2.65) is isomorphic to the Kähler quotient (2.63), as complex manifolds defined as the complement of the singular points.

Such moduli spaces, for certain examples of simple quiver gauge theories on a D3-brane at a Calabi-Yau singularity, have recently been investigated in detail in [20]. Very little is known in general about the detailed structure of these moduli spaces. A notable point is that, in general, $\mathcal{M}$ is reducible. Also, so far in the paper we have ignored the fact that the quiver gauge theory is usually far from unique: the different quiver theories are valid in different regions of Kähler moduli space. However, they are all in the same universality class, flowing in the IR to the same superconformal fixed point. The quivers are then related by a form of Seiberg duality. As first pointed out

[14] This assumes that one takes canonical kinetic terms for the bifundamental fields.

for the complex cone over dP_3 in [70], the moduli space $\mathcal{M}$ is not always invariant under Seiberg duality. However, it was conjectured in [20] that there is a top-dimensional irreducible component of $\mathcal{M}$ that is invariant. The discussion below should probably be applied to this irreducible component of $\mathcal{M}$. Having said this, the structure of $\mathcal{M}$ that we wish to describe is so general that these precise details will not be important for our purposes.

The global symmetry group $U(1)^\chi \subset G$ acts holomorphically on $\mathcal{M}$, preserving its Kähler structure. In fact, recall that no field is charged under the diagonal $U(1)_{\mathrm{diag}} \subset U(1)^\chi$. The effectively acting group is in fact the torus

$$\mathbb{T} \, = \, U(1)^{\chi-1} \, \cong \, U(1)^\chi / U(1)_{\mathrm{diag}} \, . \tag{2.66}$$

We may thus in particular take the Kähler quotient of $\mathcal{M}$ by $\mathbb{T}$. Since the dual Lie algebra of $U(1)^\chi$ is isomorphic to $\mathbb{R}^\chi$, we may also pick a non-zero moment map level, or FI parameter in physics language, $\zeta \in \mathbb{R}^\chi$ satisfying

$$\sum_{v \in V} \zeta_v \, = \, 0 \, . \tag{2.67}$$

This is equivalent to quotienting the space of F-term solutions $\mathcal{Z}$ by the gauge group $G = \prod_{v \in V} U(n_V)$ with moment map

$$\mu_v(\zeta) \;\; \equiv \;\; - \sum_{a \in A | \; t(a)=v} \Phi_a^\dagger \Phi_a + \sum_{a \in A | \; h(a)=v} \Phi_a \Phi_a^\dagger + \frac{1}{n_v} \, \zeta_v \, 1_{n_v \times n_v} \, . \tag{2.68}$$

We shall denote this quotient by

$$\mathcal{M}(\zeta) = \mathcal{Z} //_{\zeta} G \, . \tag{2.69}$$

This is usually called the mesonic moduli space with FI parameters given by the vector ζ.

It is perhaps worth stressing that the above quotient by G (as opposed to that by SG) is not physically relevant; it may be regarded as a mathematical trick that is useful for describing the global structure of the physical moduli space, which is $\mathcal{M}$. Indeed, picking a point $p \in \mathcal{M}$ determines a vector $\zeta \in \mathbb{R}^{\chi-1} \subset \mathbb{R}^\chi$ via setting $\mu_v(\zeta) = 0$ in (2.68). Notice that the sum of quadratic terms in the bifundamental fields is necessarily proportional to the identity matrix $1_{n_v \times n_v}$ since any point in $\mathcal{M}$ satisfies $\mu_v = 0$ in (2.60). This gives a well-defined map

$$\Pi : \mathcal{M} \to \mathbb{R}^{\chi-1} \, . \tag{2.70}$$

The mesonic moduli space $\mathcal{M}(\zeta)$ is then $\Pi^{-1}(\zeta)/U(1)^{\chi-1}$. For a generic (smooth) point, the group $U(1)^{\chi-1}$ acts freely and thus $\Pi^{-1}(\zeta)$ is a $U(1)^{\chi-1}$ fibration over $\mathcal{M}(\zeta)$. Thus $\mathcal{M}$ fibres over $\mathbb{R}^{\chi-1}$ where the fibres are themselves fibrations with base space $\mathcal{M}(\zeta)$ and generic fibre $U(1)^{\chi-1}$.

In fact, not all values of $\zeta \in \mathbb{R}^{\chi-1}$ are realised. The set of ζ for which $\Pi^{-1}(\zeta)$ is non-empty correspond to points in a convex cone in $\mathbb{R}^{\chi-1} \subset \mathbb{R}^{\chi}$ [71]. This cone in $\mathbb{R}^{\chi-1}$ is further subdivided into a set of *chambers* C, which are the open interiors of convex rational polyhedral cones, with boundaries between chambers being known as *walls*. Mesonic moduli spaces with FI parameters inside the same chamber C are all isomorphic to the same complex manifold $\mathcal{M}_C$, although they have distinct Kähler forms. The Kähler class locally varies linearly with the FI parameters. As one crosses a wall from one chamber into another, the mesonic moduli space undergoes a form of small birational transformation called a *flip* [71]. The moduli spaces corresponding to FI parameters on the walls are singular. Thus, strictly speaking, the map Π in (2.70) is not a fibration, since the fibres are only locally isomorphic. Across the walls in $\mathbb{R}^{\chi-1}$ the fibres change topology.

For applications to AdS/CFT, where the quiver gauge theory describes the theory on N coincident D3-branes transverse to X, one *expects*[15] that

$$\mathcal{M}_N(\zeta) \cong \mathrm{Sym}^N X \; = \; X^N/S_N \tag{2.71}$$

is the Nth symmetrised product of X. Here the set of dimension vectors $\mathbf{n}$, which we have suppressed in the notation (2.69), are of course fixed in terms of N. The space $\mathcal{M}_1(\zeta) \equiv X(\zeta) = X$ is naturally the vacuum moduli space of a single pointlike D3-brane on X. Thus the dual geometry is expected to arise as the classical vacuum moduli space[16] for the gauge theory on a D3-brane. The singular cone geometry $C(Y)$ corresponds to the zero moment map level $\zeta = 0$. Thus we identify

$$C(Y) \; = \; \mathcal{M}_1(0) \; . \tag{2.72}$$

From the above discussion regarding convex cones and chambers in FI space, setting $\zeta \neq 0$ in the quiver gauge theory with gauge group G corresponds to (partially) resolving the moduli space (2.72). Indeed, in [73] it was proven that for toric quiver gauge theories

[15]When there is an equivalence of derived categories, there is always an irreducible component of the vacuum moduli space isomorphic to the original cone – see [72].

[16]If the singularity $\{r = 0\}$ is not isolated the situation may be more complicated. In this case the mesonic moduli space contains the dual geometry, but typically also has other branches.

described by dimers the identification (2.72) indeed holds, and moreover $\mathcal{M}_1(\zeta)$ is a toric *crepant* resolution of $C(Y)$ for generic ζ. These results are also known to hold for orbifolds $C(Y) = \mathbb{C}^3/\Gamma$ with $\Gamma \subset SU(3)$ a finite group [74]. Indeed, for orbifold quiver gauge theories it is known that all crepant resolutions $\pi : X \to \mathbb{C}^3/\Gamma$ arise in this way [21]. The results of [31] for certain examples of toric theories strongly suggest this is true in general for toric quivers.

In the preceding discussion we have reviewed the description of $\mathcal{M}$ as a certain fibration over the space of "FI paramaters". This will be sufficient for comparing with the dual gravity VMS that we discuss in the next section. In the remainder of this subsection we will describe in more detail the GIT quotient point of view. Although this is slightly technical, and will not affect most of the rest of the paper, it provides a description of baryon operators as holomorphic functions on $\mathcal{M}$ which sheds light on the recent counting results presented in [31].

Thus, consider the GIT quotient by the *complexification* $\mathbb{T}_{\mathbb{C}} \cong (\mathbb{C}^*)^{\chi-1}$ of the torus (2.66). This also acts on $\mathcal{M}$, which we now consider as an affine set. In order to obtain the analogue of a non-zero moment map level ζ, we need to pick a *character* of $\mathbb{T}$. The character lattice $\Lambda \subset \mathtt{t}^*$ is by definition the set of all one-dimensional representations of $\mathbb{T}$. On picking a basis, this is

$$\mathbb{Z}^{\chi-1} \cong \Lambda \subset \mathtt{t}^* \cong \mathbb{R}^{\chi-1} . \tag{2.73}$$

Picking a character $q \in \Lambda$ specifies an action of $\mathbb{T}_{\mathbb{C}}$ on $\mathcal{M} \times \mathbb{C}$, namely

$$\mathcal{M} \times \mathbb{C} \ni (p, z) \mapsto (\lambda \cdot p, \chi_q(\lambda) z), \qquad \lambda \in \mathbb{T}_{\mathbb{C}} . \tag{2.74}$$

In a basis, we may write $\lambda = (\lambda_1, \ldots, \lambda_{\chi-1}) \in (\mathbb{C}^*)^{\chi-1}$, $q = (q_1, \ldots, q_{\chi-1}) \in \mathbb{Z}^{\chi-1}$ and then

$$\chi_q(\lambda) = \prod_{i=1}^{\chi-1} \lambda_i^{q_i} . \tag{2.75}$$

We may now perform the GIT quotient for the action of $\mathbb{T}_{\mathbb{C}}$ on $\mathcal{M} \times \mathbb{C}$, using the character q (or rather χ_q). This picks out the set of holomorphic functions on $\mathcal{M}$ of charge kq, with $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. To see this, note that the invariant regular functions on $\mathcal{M} \times \mathbb{C}$ are spanned by functions[17] of the form $f_{kq} z^k$, where f_{kq} is a holomorphic function on $\mathcal{M}$ of charge $kq \in \Lambda$. Thus

$$\mathbb{C}[\mathcal{M} \times \mathbb{C}]^{\mathbb{T}_{\mathbb{C}}(q)} = \bigoplus_{k \in \mathbb{Z}_+} \{f_{kq}\} . \tag{2.76}$$

[17] The *function* $f(p, z) = z$ transforms with the opposite weight to the *coordinate* z.

This is a graded ring, graded by k, and we may thus take the projective Proj, rather than the affine Spec, of (2.76). This may be defined concretely as follows. One first takes a finite set of generators w_i, $i = 1, \ldots, d$, of the ring (2.76), which may be taken to be homogeneous under the grading. From this we could construct the corresponding affine variety in $\mathbb{C}^d$. However, instead we do something different. Let w_a, $a = 1, \ldots, m$, be the generators of homogeneous degree zero. Then we define Proj of (2.76) to be the zero set of the relations between the generators in $\mathbb{C}^m \times (\mathbb{C}^{d-m} \setminus 0) \subset \mathbb{C}^d$, quotiented by the $\mathbb{C}^*$ action given by the grading on the generators. This produces the quotient space $\mathcal{M} //_q \mathbb{T}_{\mathbb{C}}$ together with an ample line bundle over it. The holomorphic sections of this line bundle are by definition the charge q holomorphic functions on $\mathcal{M}$, which is the degree one piece $k = 1$ of the ring (2.76). It is then a fairly standard result (see, for example, [71] and references therein) that the Kähler quotient $\mathcal{M}(q)$ with FI parameter $q \in \mathtt{t}^*$ is the same as the GIT quotient using the lattice point q:

$$\mathcal{M}(q) \cong \mathcal{M} //_q \mathbb{T}_{\mathbb{C}} \cong \mathcal{M}_C \ . \tag{2.77}$$

Here $q \in C \subset \mathtt{t}^*$ lies in the chamber C, and recall that the underlying space $\mathcal{M}_C$ depends only on the chamber: the choice of point $q \in \Lambda \cap C$ determines an ample line bundle over $\mathcal{M}_C$ in the GIT quotient, and a (quantised) Kähler form in the Kähler quotient. One also naturally gets a morphism

$$\pi : \mathcal{M}(q) \to \mathcal{M}(0) \tag{2.78}$$

via the inclusion of the invariant functions on $\mathcal{M}$ in (2.76). For the gauge theory on a single D3-brane, $\mathcal{M}_1(0) = C(Y)$. For a toric quiver gauge theory, corresponding to a D3-brane at the singular point of an affine toric singularity, it was proven in [73] that (2.78) is indeed a toric crepant resolution of $C(Y)$.

The description of the mesonic moduli spaces in terms of a *geometric quotient* of $\mathcal{M}$ by the complex torus $\mathbb{T}_{\mathbb{C}} \cong (\mathbb{C}^*)^{\chi-1}$ is standard [75]. The points of $\mathcal{M}$ under the group action are separated into unstable, semi-stable and stable points, where the stable points are a subset of the semi-stable points. The unstable points, which we denote S_q, are thrown away in the quotient. On the other hand for *generic* q one expects all other points to be stable – see [73] for a discussion of this for toric quiver moduli spaces. Then the statement is that $\mathcal{M} \setminus S_q$ is a $\mathbb{T}_{\mathbb{C}} \cong (\mathbb{C}^*)^{\chi-1}$ fibration over $\mathcal{M}(q) \cong \mathcal{M}_C$. When there are semi-stable but not stable points $\mathcal{M} \setminus S_q$ is no longer a fibration over the mesonic moduli space. For example, this is certainly true when $q = 0$.

In the abstract language above, the gauge-invariant BPS operators are classically just the holomorphic functions on $\mathcal{M}$. These form the coordinate ring $\mathbb{C}[\mathcal{M}]$. The BPS meson operators are the subset of these that have zero charge $q = 0$ under the baryonic torus $\mathbb{T}$. Alternatively, these are the ring of invariants $\mathbb{C}[\mathcal{Z}]^{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$, rather than $SG_{\mathbb{C}}$. On the other hand, the baryon operators are by definition the gauge-invariant BPS operators with non-zero change q under $\mathbb{T}$. For $N = 1$, a baryonic operator of charge $q \in \Lambda$ is, by the above GIT construction, the same thing as an ample divisor of the mesonic moduli space $X = \mathcal{M}(q)$. These statements explain the counting of baryon operators presented in [31], since an ample divisor may be identified with a quantised Kähler class on X. Thus counting baryon operators according to their baryonic charge indeed involves summing over mesonic moduli spaces X that are resolutions of the Calabi-Yau cone, and on each X summing over quantised Kähler classes (ample line bundles).

Example: the conifold

Since the discussion above is all rather abstract, we include here a simple example. The gauge theory on a D3-brane at the singular point of the conifold[18] consists of four chiral fields A_i, B_i, $i = 1, 2$. The gauge group is $G = U(1)^2$, with the fields A_i, B_i carrying charges $(1, -1)$ and $(-1, 1)$, respectively. The superpotential is zero, and thus the classical VMS is simply $\mathcal{M} = \mathbb{C}^4$, parameterised by the VEVs of the above bifundamental fields. We introduce standard coordinates z_1, z_2, z_3, z_4 on $\mathbb{C}^4$. The overall $U(1)$ decouples, as always, and the charges under the remaining $U(1)$ are $(1, 1, -1, -1)$. The moment map is then, up to normalisation,

$$\mu = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 \ . \tag{2.79}$$

Picking a point $p \in \mathcal{M} = \mathbb{C}^4$ thus determines a real number $\mu(p) \in \mathbb{R}$. This is the same as the map Π in (2.70). By a slight abuse of terminology, we refer to $\mu(p)$ as the value of the FI parameter. The mesonic moduli spaces are then given by

$$\mathcal{M}(\zeta) = \mathbb{C}^4 //_{\zeta} U(1) = \{p \in \mathbb{C}^4 \mid \mu(p) = \zeta\}/U(1) \ . \tag{2.80}$$

The underlying complex variety X of $\mathcal{M}(\zeta)$ depends on the sign of ζ: for $\zeta > 0$ one obtains the resolved conifold $X_+ = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$; for $\zeta = 0$ one obtains the conifold $X_0 = \{u, v, x, y \in \mathbb{C}^4 \mid u^2 + v^2 + x^2 + y^2 = 0\}$; whereas for $\zeta < 0$ one obtains the

[18] Ordinary double point singularity, in mathematical terminology.

other small resolution of the conifold X_-, obtained by flopping the $\mathbb{CP}^1$ in X_+. Thus the space of FI parameters in $\mathbb{R}$ is fan consisting of two one-dimensional chambers $\mathbb{R}_\pm$, together with a point: $\mathbb{R} = \mathbb{R}_- \cup \{0\} \cup \mathbb{R}_+$. The Kähler class of $\mathcal{M}(\zeta)$, equipped with its induced Kähler metric, depends linearly on ζ. Roughly, one may interpret $|\zeta|$ as the size of the $\mathbb{CP}^1$. Picking a point $p \in \mathcal{M} = \mathbb{C}^4$ hence determines, via (2.80), a point in a $U(1)$ fibre over a point in one of X_0 or $X_\pm$, together with a Kähler class on $X_\pm$. The $U(1)$ fibre is non-degenerate everywhere, except over the singular point of the conifold X_0.

The unstable points, in the GIT quotient description, are $S_+ = \{z_1 = z_2 = 0\} \cong \mathbb{C}^2$ and $S_- = \{z_3 = z_4 = 0\} \cong \mathbb{C}^2$. Thus we may define $\mathcal{M}_\pm = \mathcal{M} \setminus S_\pm$. Then $\mathcal{M}_\pm$ is a $\mathbb{C}^*$ fibration over $X_\pm$.

3 Gravity backgrounds

3.1 AdS$_5$ backgrounds

We begin our discussion by recalling the well-known $\mathrm{AdS}_5 \times Y$ solutions of Type IIB supergravity, where (Y, g_Y) is a Sasaki-Einstein five-manifold. In particular we present a discussion of the various background moduli described by flat form fields. These will play a crucial role in our subsequent discussion of deformations of the conformal backgrounds.

Consider placing N D3-branes at the tip of the Ricci-flat Kähler cone (2.1). The corresponding solution of Type IIB supergravity is given by

$$g_{10} = H^{-1/2} g_{\mathbb{R}^4} + H^{1/2} g_{C(Y)} \tag{3.1}$$

$$G_5 = (1 + *_{10}) \mathrm{d}H^{-1} \wedge \mathrm{vol}_4 \tag{3.2}$$

where the function H is given by

$$H = 1 + \frac{L^4}{r^4} \ . \tag{3.3}$$

Here $g_{\mathbb{R}^4}$ is four-dimensional Euclidean space, with volume form vol_4, and L is a constant given by

$$L^4 = \frac{(2\pi)^4 g_s (\alpha')^2 N}{4\mathrm{vol}(Y)} \ . \tag{3.4}$$

The near-horizon limit of this system of D3-branes may be obtained by simply dropping the additive constant from the function (3.3), which results in the product background $\mathrm{AdS}_5 \times Y$. Here AdS_5, or rather its Euclidean version which is hyperbolic space, is realised in horospherical coordinates. Specifically, the metric (3.1) becomes

$$g_{10} = \frac{L^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{L^2}g_{\mathbb{R}^4} + L^2 g_Y \ . \tag{3.5}$$

The AdS/CFT correspondence conjectures that Type IIB string theory on this background is dual, in the large N limit, to a four-dimensional $\mathcal{N} = 1$ superconformal field theory. The latter may be regarded as living on the conformal boundary of the five-dimensional hyperbolic space in (3.5). The conformal compactification of hyperbolic space may be described topologically as adding an S^4 conformal boundary to a five-dimensional open ball. In the horospherical coordinates above, this S^4 boundary of AdS_5 is the union of $r = \infty$, which is a copy of $\mathbb{R}^4$, with the point $r = 0$.

Actually, more precisely, for fixed spacetime metric (3.5) there will in general be a family of corresponding AdS backgrounds, obtained by turning on various flat background fields. These correspond to exactly marginal directions in a family of $\mathcal{N} = 1$ superconformal field theories. In the remainder of the subsection we give a careful summary of these marginal deformations.

Firstly there is the constant axion-dilaton τ in (2.15). Secondly, we may turn on a flat RR C_2 field and its $SL(2;\mathbb{R})$ partner, a flat NS B field. In the current set-up, with $\mathcal{M} = \mathrm{AdS}_5 \times Y$, the spacetime is contractible to Y. Note that before taking into account large gauge transformations, we may view the non-torsion flat RR fields as a vector $(C_0, [C_2]) \in \mathbb{R} \oplus \mathbb{R}^{b_3(Y)} \cong \mathbb{R}^{b_3(Y)+1}$, where $[C_2] \in H^2(Y;\mathbb{R}) \cong \mathbb{R}^{b_3(Y)}$. The lattice of large gauge transformations is then given by

$$\Lambda_B^Y = \left\{ \left(n, \frac{2\pi}{\mu_1}\sigma + nB \right) \mid n \in \mathbb{Z}, \sigma \in H^2_{\mathrm{free}}(Y;\mathbb{Z}) \right\} \subset \mathbb{R}^{b_3(Y)+1} \ . \tag{3.6}$$

The flat RR fields thus live in the torus

$$([C_0], [C_2]) \in \mathbb{R}^{b_3(Y)+1}/\Lambda_B^Y \cong U(1)^{b_3(Y)+1} \ . \tag{3.7}$$

The C_2 field and B field pair naturally into the complex combination $\tau B - C_2$. Note that when $H^3_{\mathrm{tors}}(Y;\mathbb{Z})$ is non-trivial it is possible to turn on torsion G_3 and H fields. These should correspond to discrete parameters labelling the dual SCFTs.

In principle we might also have been able to turn on a flat RR C_4 field, in addition to the background flux (3.2). However, since $b_1(Y) = 0$ by Poincaré duality we have $H^4(Y;\mathbb{R}) = 0$. There is hence no room for such a flat field. Such fields will play an important role once we deform the AdS background geometry in the next subsection.

The above flat fields, including the axion-dilaton, may be identified with the $(b_3(Y)+1)$-dimensional space of marginal couplings discussed in section 2.4, in the case that the dual SCFT has a quiver gauge theory description. Note, however, that both B and C_2 are periodic variables. Although there is some field theory understanding of this for simple examples, such as the conifold with $(Y, g_Y) = T^{1,1}$, a general account seems to be lacking at present. Marginal deformations also arise if there is a non-trivial moduli space of Sasaki-Einstein metrics on Y, as often occurs in the constructions of Sasaki-Einstein manifolds as links of hypersurface singularities in [37]. There may also be metric deformations that take us outside the class of Sasaki-Einstein backgrounds, notably the β-deformations of [76] for toric Sasaki-Einstein manifolds. We will not consider either of these possibilities in the present paper.

3.2 Symmetry-breaking backgrounds

The quantum field theories dual to the above backgrounds are in vacua in which all scalar operators have zero VEVs. Indeed, a non-zero VEV will break conformal invariance, leading to a renormalisation group flow via an associated Higgs effect. The aim of this paper is to consider more general field theory vacua in which various operators have non-zero VEVs. This was first discussed for $\mathcal{N} = 4$ Yang-Mills theory in [78], and for orbifolds and the conifold theory in [23]. Here we wish to extend the discussion to general Sasaki-Einstein backgrounds with dual field theories described by quiver gauge theories. We discussed the *classical* space of such vacua in section 2.5. In the remainder of the section we would like to construct the corresponding dual supergravity solutions.

At energies well above the highest scale set by the VEVs, one expects the physics to be well-described by the original $\mathcal{N} = 1$ superconformal field theory. The latter is thus the UV theory in this set-up. As usual in AdS/CFT, one may describe field theories that are conformal at high energies by a dual gravitational background that is *asymptotic* to an AdS solution. One should therefore look for supergravity solutions that are asymptotic to $\mathrm{AdS}_5 \times Y$. However, as emphasized in [23], if the dual field theory is defined on S^4 one does not expect to find vacua of the type discussed in section 2.5: the conformal coupling of scalar fields to the positive scalar curvature of S^4 prevents them from acquiring a VEV. Instead, one should regard the "boundary" of AdS_5 to be $\mathbb{R}^4$, given by $r = \infty$ in the horospherical coordinates (3.5), so that the dual field theory is defined on flat $\mathbb{R}^4$. We thus seek supergravity solutions that have an asymptotic region which approaches the large r region of (3.5). The solutions of interest will also have other asymptotic regions, as we shall describe momentarily.

There are two natural ways of deforming the AdS backgrounds in section 3.1 in this manner:

- *Mesonic deformations*: where one moves some or all of the stack of N D3-branes away from the singularity $r = 0$ of $C(Y)$.
- *Baryonic deformations*: where one de-singularizes the Calabi-Yau cone $C(Y)$, replacing it by a (possibly still singular) Ricci-flat Kähler manifold (X, g_X) that is asymptotic to a cone over the Sasaki-Einstein manifold (Y, g_Y).

Actually these names are slightly misleading, since generically meson and baryon operators obtain VEVs in both types of vacua. However, the space of mesonic deformations is naturally isomorphic to the gauge theory mesonic moduli space at zero FI parameter. Also, for certain baryonic deformations no meson operator obtains a VEV. To be more precise, if $\pi : X \to Z$ is a crepant resolution of the singularity $Z = C(Y)$, then in backgrounds where all of the D3-branes are located on the exceptional set (the set of points in X mapping to the singular point $r = 0$ of $C(Y)$) one expects that no meson operator obtains a VEV. Classically this is because the meson operators are the holomorphic functions $\mathbb{C}[Z]$, which, if not constant, vanish at $r = 0$. For example, the backgrounds discussed in [22] are all of this form.

For any such (X, g_X) above we may construct a family of supersymmetric Type IIB backgrounds, asymptotic to $\mathrm{AdS}_5 \times Y$ in the above sense, as follows. The ten-dimensional metric is

$$g_{10} \;=\; H^{-1/2} g_{\mathbb{R}^4} + H^{1/2} g_X \ , \tag{3.8}$$

with G_5-flux still given by (3.2). We pick m points x_i, $i = 1, \ldots, m$, and place N_i D3-branes at the ith point. Thus

$$\sum_{i=1}^{m} N_i \;=\; N \tag{3.9}$$

and the function H, which is sourced by the D3-branes, satisfies

$$\Delta_x H \;=\; -\frac{(2\pi)^4 g_s (\alpha')^2 N}{\sqrt{\det g_X}} \sum_{i=1}^{m} \frac{N_i}{N} \delta^6(x - x_i) \ . \tag{3.10}$$

Here Δ is the Laplacian on (X, g_X). The warp factor H thus satisfies the Laplace equation on $X \setminus \{x_1, \ldots, x_m\}$. The boundary conditions may be described as follows. Since (X, g_X) is asymptotic to a cone over Y we may require the solution for H to approach $H_{\mathrm{AdS}} = L^4/r^4$ for large r. This, together with the D3-brane charge relation (3.9), precisely ensures that the Type IIB background is asymptotic to the large r region of (3.5), with L given by (3.4). Near to the ith stack of D3-branes $x_i \in X$, the function H behaves as

$$H(x) = \frac{L_i^4}{\rho(x, x_i)^4}(1 + o(1)) \ . \tag{3.11}$$

Here $\rho(x, x_i)$ is the geodesic distance from x_i to x, and

$$L_i^4 = \frac{(2\pi)^4 g_s (\alpha')^2 N_i}{4\mathrm{vol}(S^5)} \ , \tag{3.12}$$

provided that x_i is a smooth point of X. As discussed in [22], if (X, g_X) has a conical singularity at x_i, with corresponding Sasaki-Einstein link (Y_i, g_{Y_i}), then $\mathrm{vol}(S^5)$ is replaced by $\mathrm{vol}(Y_i)$ in (3.12). The singular nature of H at x_i implies that the metric (3.8) develops a "throat" near to this point. In fact it approaches the metric in a neighbourhood of $r = 0$ in the AdS background

$$g_{10} = \frac{L_i^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{L_i^2} g_{\mathbb{R}^4} + L_i^2 g_{S^5} \ . \tag{3.13}$$

Again, if the point x_i is a conical singularity, the round S^5 is replaced by (Y_i, g_{Y_i}).

Notice that when all N of the D3-branes are placed at the same point, so $m = 1$ in the above notation, the function H is simply the Green's function on (X, g_X). Provided (X, g_X) is smooth and complete, we argued in [22] that there always exists a unique positive solution to (3.10) with the required boundary behaviour – this follows from standard theorems about Green's functions on manifolds with non-negative Ricci curvature and appropriate volume growth. For $m > 1$ stacks of D3-branes we may then simply take an appropriate linear combination of Green's functions as solution to (3.10). More generally, when (X, g_X) contains singularities (such as a mesonic background with $X = C(Y)$), we do not know of any general theorems that guarantee existence of a unique solution to (3.10). However, it is very reasonable to conjecture this to be true, at least when (X, g_X) contains only conical singularities. Indeed, for the homogeneous case of $(Y, g_Y) = T^{1,1}$ one may construct [30] explicit solutions to (3.10) on the conifold. We will nevertheless focus on the case that (X, g_X) is smooth in the present paper.

The supergravity backgrounds with $m = 1$ may be interpreted as a renormalisation group flow from the initial $\mathcal{N} = 1$ superconformal field theory to $\mathcal{N} = 4$ Yang-Mills in the IR, with gauge group $SU(N)$ [23]. As we explain later in the paper, there may be additional light particles in the IR, namely Goldstone bosons associated to the spontaneous breaking of non-anomalous baryonic symmetries. When the branes are separated, $m > 1$, the interpretation of the background is a little more subtle: there are m regions in which the supergravity solution approaches a neighbourhood of $r = 0$ in the AdS solution (3.13). The natural interpretation is thus that the theory flows, in the extreme IR, to a non-trivial fixed point that is a *product* of m superconformal field theories. When the points x_i are all smooth, the factors in this product are $\mathcal{N} = 4$ Yang-Mills with gauge groups $SU(N_i)$, as suggested in [23]. More generally it is natural to conjecture that the IR theory is a product of the $\mathcal{N} = 1$ superconformal field theories dual to (Y_i, g_{Y_i}). Such theories have been discussed in [77], where the IR theory itself was conjectured to be dual to the union of m AdS_5 spaces, with conformal boundaries identified. Note also that the supergravity approximation is valid only when all N_i are large, or equivalently the AdS radii L_i are large compared to the string scale.

Naively ignoring this last point, the space of supergravity metrics for fixed (X, g_X) is naturally given[19] by the symmetric product $\mathrm{Sym}^N X$, describing the positions of the N D3-branes. Of course, such symmetric products arise in the classical gauge theory as mesonic moduli spaces, as we reviewed in section 2.5. Fixing a non-zero FI parmeter $\zeta \in \mathbb{R}^{\chi-1}$ for the gauge theory on a single D3-brane, the corresponding mesonic moduli space $\mathscr{M}_1(\zeta) = X(\zeta)$ is a resolution $\pi : X(\zeta) \to Z$ of the Calabi-Yau singularity $Z = C(Y)$. Indeed, this is known to be a crepant resolution for orbifold [74] and toric [73] quiver gauge theories. We expect this to be true in general. Note also that the Kähler class in $H^2(X;\mathbb{R})$ of the induced metric on $X(\zeta)$ varies linearly with ζ. Thus, provided $X \cong X(\zeta)$ for some FI parameter ζ, the space of supergravity metrics obtained by varying the positions of the D3-branes is the same as the corresponding mesonic moduli space. Of course, the caveat to this statement is that the supergravity solutions are strictly valid only when the D3-branes are in large "clumps".

The above discussion raises the question of how to characterise those X which are of the form $X \cong X(\zeta)$ for some FI parameter ζ. Certainly not all Calabi-Yau's (X, g_X), asymptotic to a cone over (Y, g_Y), are of this form. Firstly X must be a crepant resolution of Z. For example, the deformed conifold is a de-singularization of the conifold, but clearly this cannot arise as a mesonic moduli space. The deformed conifold is therefore, at least for generic couplings, not relevant for the vacua of interest [23].

[19] In fact an exception to this is when $X = \mathbb{C}^3$. In this case the translational symmetry of $\mathbb{C}^3$ may be used to fix the centre of mass of the D3-brane positions at the origin [78, 23], resulting in $\mathrm{Sym}^{N-1}\mathbb{C}^3$.

Note in this example one has $b_3(X) = 1$. Another, more physical, justification for the assumption (2.4) is that if there are odd-dimensional cycles on X then one may wrap D-branes over these cycles to obtain topologically stable domain walls in $\mathbb{R}^4$ – see, for exampe, [79, 80]. In particular, if $b_3(X) \neq 0$ one may wrap D5-branes to obtain such domain walls. These connect different vacua of the theory. Such backgrounds therefore have qualitatively different physics from those without odd-dimensional cycles. Even for toric quiver gauge theories it is not known whether *all* toric crepant resolutions X of $C(Y)$ are of the form $X(\zeta)$ for some ζ. For abelian orbifolds $C(Y) = \mathbb{C}^3/\Gamma$ this is true [21], and the baryon counting results of [31] certainly suggest that it is true in general for toric theories. Thus, at least for orbifold and toric quiver theories, it seems that all crepant resolutions of the conical singularity should arise as dual descriptions of the supersymmetric vacua of interest.

3.3 Form field moduli

As discussed in section 3.1, for the AdS background $\mathrm{AdS}_5 \times Y$ one is free to turn on various flat background fields, corresponding to a choice of marginal couplings in the dual field theory. The supergravity backgrounds discussed in section 3.2 are asymptotic to $\mathrm{AdS}_5 \times Y$, in the sense that there is an asymptotic region that approaches a neighbourhood of $r = \infty$ in (3.5). Thus $\mathbb{R}^4 \times Y$ is a boundary component of the full spacetime $\mathcal{M}$. One must extend the fields on this boundary over $\mathcal{M}$, and thus in particular over X, to obtain a solution to supergravity. Note that the spacetime $\mathcal{M}$, with metric (3.8), is globally of the form $\mathbb{R}^4 \times (X \setminus \{x_1, \ldots, x_m\})$. Thus $\mathcal{M}$ also has m asymptotic regions that look like a neighbourhood of $r = 0$ in (3.13). Near each such region the set $r = \epsilon$, with $\epsilon > 0$ small, is a copy of $\mathbb{R}^4 \times S^5$. More generally, when the ith set of N_i D3-branes are placed at a conical singularity of X with Sasaki-Einstein link (Y_i, g_i), this boundary is replaced by $\mathbb{R}^4 \times Y_i$. The restriction of form fields on $\mathcal{M}$ to this "internal" boundary thus naturally determines the IR superconformal field theory. We should therefore regard the spacetime $\mathcal{M}$ as having $m + 1$ boundary components: the UV boundary $\mathbb{R}^4 \times Y$, and the m components of the IR boundary, which if X is smooth are all diffeomorphic to $\mathbb{R}^4 \times S^5$.

The dynamical fields of interest in this section are the RR fields and the NS B field. Consider a generic p-form field strength G with $(p-1)$-form potential C on a spacetime $\mathcal{M}$. This means that locally $G = \mathrm{d}C$. We assume that G is a fixed field strength on $(\mathcal{M}, g_{10})$, satisfying the relevant equation of motion. We may then pick a particular potential C°, defined locally in coordinate patches, such that $G = \mathrm{d}C^\circ$. Any other C field giving rise to the same field strength G is then given by

$$C = C^\circ + C^\flat \ , \tag{3.14}$$

where $C^\flat$ is a *flat* C field, *i.e.* it is closed. The small gauge transformations on C are of the form

$$C \to C + \mathrm{d}\lambda \tag{3.15}$$

where λ is a $(p-2)$-form. This immediately leads to the cohomology group $H^{p-1}(\mathcal{M};\mathbb{R})$, classifying the space of C fields modulo gauge equivalence. Of course, inclusion of large gauge transformations typically leads to $U(1)$ coefficients instead, and for RR fields there is a twisting by the B field, as discussed in section 2.3.

In the present situation $H^{p-1}(\mathcal{M};\mathbb{R}) \cong H^{p-1}(X;\mathbb{R})$ for the cases $p=3$, $p=5$ of interest. That is, deleting a finite number of points from a smooth manifold X does not affect the cohomology in these degrees, as one easily proves using a simple Mayer-Vietoris sequence. However, we do not want to think of $H^2(X;\mathbb{R})$ as classifying, say, flat C_2 field moduli of the backgrounds. The reason is that the restriction $H^2(X;\mathbb{R}) \to H^2(Y;\mathbb{R})$ gives the marginal couplings of the UV theory, which should be regarded as fixed boundary data. We would like to instead classify fields on X that induce the same field at infinity. Thus we are interested in the kernel of the map $H^{p-1}(X;\mathbb{R}) \to H^{p-1}(Y;\mathbb{R})$, which is the same as the image of the map $H^{p-1}(X,Y;\mathbb{R}) \to H^{p-1}(X;\mathbb{R})$ by the long exact cohomology sequence for (X,Y).

In fact, as we explain further below, and also in section 6, we would like to interpet the form field moduli as living in $H^{p-1}(X,Y;\mathbb{R})$ itself, rather than its image in $H^{p-1}(X;\mathbb{R})$. The elements of $H^{p-1}(X,Y;\mathbb{R})$ that map to zero in $H^{p-1}(X;\mathbb{R})$ are, again by the long exact sequence for (X,Y), images of $H^{p-2}(Y;\mathbb{R})$. This may be realised concretely as follows. Take an element $\lambda \in H^{p-2}(Y;\mathbb{R})$. By the Hodge theorem we may represent λ by a harmonic form. Let f be a smooth function on X that is equal to 1 on Y and is identically zero outside a tubular neighbouhood of Y in X. Then $\mathrm{d}(f\lambda) = \mathrm{d}f \wedge \lambda$ makes sense as a closed compactly supported $(p-1)$-form on X. In fact, such forms precisely represent the image of $H^{p-2}(Y;\mathbb{R})$ in $H^{p-1}(X,Y;\mathbb{R})$. Although such an expression is exact, and thus a pure gauge mode, the gauge generator λ is *non-zero* on Y. Such gauge transformations are always associated with global symmetries in gauge theory, and indeed later we will identify these with the $b_3(Y)$ non-anomalous global $U(1)$ symmetries associated to the RR four-form (so $p=5$ in the above discussion).

A more refined treatment of the form field moduli thus treats them as compactly supported cohomology classes. We may describe this explicitly by requiring $C^\flat$ in (3.14) to be zero on Y, and λ in (3.15) to also be zero on Y. The gauge for $C\mid_{\partial\mathcal{M}}$ is thus held fixed. This leads to the relative/compactly supported cohomology group

$$H^{p-1}(\mathcal{M},\partial\mathcal{M};\mathbb{R}) \ , \tag{3.16}$$

modulo large gauge transformations, which will be twisted by B for RR fields. Again, in the present situation one may replace $\mathcal{M}$ by X, since deleting a finite number of points from a smooth X will not affect the cohomology in the degrees of interest. Note that the result (3.16) is independent of the choice of C°.

For the B field and C_2 field this leads to the group $H^2(X,Y;\mathbb{R})/H^2_{\rm free}(X,Y;\mathbb{Z})$, classifying flat C_2 and B fields on X with fixed value on the boundary. The RR four-form C_4 is slightly more involved. This has a non-trivial background flux G_5 given by (3.2). This field strength is not exact since the flux of G_5 over Y is equal to N, the number of D3-branes. There is thus no globally defined potential C_4° on $\mathcal{M}$ with $\mathrm{d}C_4^\circ = G_5$. However, this doesn't change the discussion much: we may instead define C_4° in an open covering of spacetime by coordinate patches, glued by transition forms across overlaps. Such a choice also fixes a gauge choice $C_4\mid_{\partial\mathcal{M}}$ at infinity. Again, this cannot be a globally defined four-form, either on the UV boundary or on any connected component of the IR boundary. More importantly, the choice of background C_4° depends on the positions $x_1,\ldots,x_m$ of the D3-branes and also on the metric g_X on X. Recall that the latter is conjecturally fixed by a choice of Kähler class $[\omega_X]\in H^2(X;\mathbb{R})$. Thus we should more correctly write $C_4^\circ(\{x_i\},[\omega_X])$, and fix a gauge choice $C_4^\circ(\{x_i\},[\omega_X])\mid_{\partial\mathcal{M}}= C_4\mid_{\partial\mathcal{M}}$ at infinity. This shows that the C_4 field is naturally fibred over the mesonic moduli space, whereas the other supergravity gauge fields are not.

The space of RR field moduli may thus be described as follows. By Poincaré duality we have

$$H^2_{\rm free}(X,Y;\mathbb{Z}) \;\cong\; H_4^{\rm free}(X;\mathbb{Z}) \ , \qquad \text{and} \qquad H^4_{\rm free}(X,Y;\mathbb{Z}) \;\cong\; H_2^{\rm free}(X;\mathbb{Z}) \ . \tag{3.17}$$

The ranks of these groups are thus $b_4(X)$ and $b_2(X)$, respectively. Thus, before taking into account large gauge transformations, the different (non-torsion) RR fields may be described by a vector

$$([C_2^\flat],[C_4^\flat]) \in \mathbb{R}^{b_4(X)}\oplus\mathbb{R}^{b_2(X)} \cong \mathbb{R}^{\chi-1} \tag{3.18}$$

where $\chi = \chi(X)$ is the Euler number given by (2.7). The lattice of large gauge transformations is

$$\Lambda_B^{X,Y} = \left\{ \left(\frac{2\pi}{\mu_1}\sigma, \frac{2\pi}{\mu_1}\sigma \wedge B + \frac{2\pi}{\mu_3}\kappa \right) \mid \sigma \in H^2_{\mathrm{free}}(X, Y; \mathbb{Z}), \right.$$
$$\left. \kappa \in H^4_{\mathrm{free}}(X, Y; \mathbb{Z}) \right\} \subset \mathbb{R}^{\chi-1} \ . \tag{3.19}$$

The space of RR field moduli, modulo discrete torsion fields, is then described by the twisted torus

$$([C_2^\flat], [C_4^\flat]) \in \mathbb{R}^{\chi-1}/\Lambda_B^{X,Y} \cong U(1)^{\chi-1} \ . \tag{3.20}$$

One should compare this to (3.7).

3.4 Comparison: gauge theory and gravity vacua

We conclude this section by comparing the supergravity backgrounds to the classical vacuum moduli space structure described in section 2.5. In order to construct a gravity background we must first pick a complex manifold X that resolves $Z = C(Y)$. Since there are N units of G_5 flux through Y at infinity, to preserve Poincaré symmetry we must choose where to put N pointlike D3-branes on X. This naturally leads to the symmetric product $\mathrm{Sym}^N X$ as moduli space, precisely as one expects for a mesonic moduli space in the gauge theory. Although, as we noted, once one includes the backreaction of the D3-branes on the geometry, the supergravity approximation breaks down unless these D3-branes are in large "clumps". Thus this matching is perhaps rather better than one might have expected.

As explained in section 2.5, the gauge theory moduli space $\mathcal{M}$ may be viewed (2.70) as a fibration over $\mathbb{R}^{\chi-1}$. The latter is divided into chambers, and over each chamber $C \subset \mathbb{R}^{\chi-1}$ the fibres are all isomorphic. In particular, each fibre is a $U(1)^{\chi-1}$ bundle over the mesonic moduli space $\mathcal{M}_C$. In the case at hand, one expects $\mathcal{M}_C = \mathrm{Sym}^N X$ for some crepant resolution X of $C(Y)$. A point $\zeta \in C$ in particular determines a classical Kähler class on $\mathcal{M}_C$, with the Kähler class varying linearly with ζ.

It should now be clear how one matches this to the parameters of the supergravity backgrounds. By our conjecture in section 2.1, there is a $b_2(X)$-dimensional space of asymptotically conical Ricci-flat Kähler metrics on X, determined by their Kähler class in the Kähler cone in $H^2(X; \mathbb{R})$. These may be identified with $b_2(X)$ of the coordinates of $\zeta \in C$. We identify the remaining $b_4(X)$ "FI parameters" with the B field periods, which live in $H^2(X,Y;\mathbb{R})/H^2_{\text{free}}(X,Y;\mathbb{Z})$. On the other hand, the periods of B in $H^2(Y;\mathbb{R})/H^2_{\text{free}}(Y;\mathbb{Z})$ partly determine the marginal couplings of the UV SCFT. One puzzle here is that B is periodic in string theory, whereas in the classical gauge theory the FI parameters and marginal gauge couplings are real numbers. However, this is a somewhat standard issue. Indeed, in some cases the periodicity of B is known to be related to Seiberg duality – see, in particular, [81] and [82]. Thus one would not expect to see this periodicity in the classical gauge theory, which in particular involves choosing a fixed Seiberg phase. The RR field moduli in (3.20), which indeed form a torus $U(1)^{\chi-1}$ due to large gauge transformations, are then identified with the $U(1)^{\chi-1}$ fibres over $\mathcal{M}_C$. Supersymmetry pairs the Kähler class with C_4, and the B field with C_2. In the classical VMS, this is reflected by the complexification $(\mathbb{C}^*)^{\chi-1}$ of the global baryonic symmetry group. This appears in the GIT description of obtaining the mesonic moduli spaces $\mathcal{M}_C$ as a quotient of $\mathcal{M}$. We thus obtain a surprisingly good matching between the classical gauge theory moduli space and the space of supergravity backgrounds described in this section.

Notice also that, for fixed choice of smooth Ricci-flat Kähler background (X, g_X), positions of the N D3-branes on X and B field modulus, the space of RR field moduli form a group under addition, and that this group is isomorphic to $U(1)^{\chi-1}$. In this way we obtain an action of $U(1)^{\chi-1}$ on the moduli space of gravity backgrounds. Given that we are identifying the latter with the symmetry-breaking vacua in the dual field theory, it is natural to interpet[20] this $U(1)^{\chi-1}$ with the group of baryonic symmetries in the dual field theory, described in section 2. In fact this group has a natural $U(1)^{b_3(Y)}$ subgroup. Specifically, the C_4 moduli in $H^4(X,Y;\mathbb{R})$ that are images of $H^3(Y;\mathbb{R})$ are, as explained in the previous subsection, naturally related to global symmetries on Y. Since these global symmetries come from gauge symmetries of RR fields, in particular they cannot be anomalous. This identifies the RR gauge symmetries coming from $H^3(Y;\mathbb{R})/H^3_{\text{free}}(Y;\mathbb{Z}) \cong U(1)^{b_3(Y)}$ with the non-anomalous baryonic $U(1)$ symmetries in the field theory. This is a very satisfying check that the picture we have outlined so far is consistent.

It would be interesting to study the global structure of these supergravity moduli spaces in more detail. For example, one could try to relate the Chern classes of the torus bundle $U(1)^{\chi-1}$ over a mesonic moduli space $\mathcal{M}_C$ in the classical VMS to the

[20]One may construct gravity backgrounds in which only part of the global symmetry group is spontaneously broken by taking X to be singular. For simplicity we shall not consider this here.

number	gravity	gauge theory
1	$\tau = C_0 + i\mathrm{e}^{-\phi}$	marginal coupling
$b_3(Y)$	$\tau B - C_2$	marginal coupling
$b_4(X)$	$\tau B - C_2$	anomalous $U(1)$
$b_4(X)$	$\omega_X + iC_4$	anomalous $U(1)$
$b_3(Y)$	$\omega_X + iC_4$	non-anomalous $U(1)$

Table 1: Gravity moduli and their interpretation in the dual quiver gauge theory.

fibration structure of the RR field moduli (3.20) over the corresponding supergravity moduli space of D3-brane positions, which is naturally isomorphic to $\mathcal{M}_C$. As we have already remarked, the construction of the C_4 field certainly depends on position in this moduli space via (3.2). One approach to this would be to investigate the induced Kähler metric on the supergravity moduli space. A similar situation was studied in [83], where a RR modulus field is indeed fibred over a mesonic moduli space, with the curvature of the corresponding line bundle being a Kähler form on the mesonic moduli space. For this to make sense globally, the Kähler class should be quantised (although this point was not addressed in [83]). This is precisely what happens in the classical GIT description of the mesonic moduli space, where $\zeta = q$ is a lattice point and is thus "quantised". It would also be interesting to investigate how different resolutions X_1 and X_2 are glued together across the walls between chambers, and in particular what happens to the RR fibres in this process.

4 Linearised fluctuations

In this section we consider certain linearised fluctuations of the background fields, by allowing them to depend on position in $\mathbb{R}^4$. We shall argue that the relevant modes require the existence of certain L^2 harmonic forms, with respect to appropriate metrics, where the L^2 condition is required in order for the fluctuations to be normalisable (have finite kinetic energy). We then appeal to mathematical results on the existence and asymptotic expansions of such forms. The AdS/CFT interpretation of these modes is postponed to the next section.

The gauge-invariant form fields of Type IIB supergravity may be obtained by expanding the RR multi-form (2.18) in forms of definite degree:

$$\tilde{G}_3 = G_3 - H_3\, C_0 \ , \tag{4.1}$$
$$\tilde{G}_5 = G_5 - H_3 \wedge C_2 \ , \tag{4.2}$$
$$H_3 = \mathrm{d}B \ , \tag{4.3}$$

where

$$G_3 = \mathrm{d}C_2 \ , \tag{4.4}$$
$$G_5 = \mathrm{d}C_4 \ . \tag{4.5}$$

These expressions automatically solve the relevant Bianchi identities. The five-form field strength $\tilde{G}_5$ is required to be self-dual

$$\tilde{G}_5 = *\tilde{G}_5 \ ; \tag{4.6}$$

the equation of motion is then implied by the Bianchi identity. The equations of motion for the remaining fields are

$$\nabla^2\phi = \mathrm{e}^{2\phi}|\mathrm{d}C_0|^2 - \frac{1}{2}\mathrm{e}^{-\phi}|H_3|^2 + \frac{1}{2}\mathrm{e}^{\phi}|\tilde{G}_3|^2 \tag{4.7}$$
$$\mathrm{d}^\dagger(\mathrm{e}^{2\phi}\mathrm{d}C_0) = \mathrm{e}^{\phi}\langle H, \tilde{G}_3\rangle \tag{4.8}$$
$$\mathrm{d}(\mathrm{e}^{-\phi} * H_3) = -\tilde{G}_5 \wedge \tilde{G}_3 + \mathrm{e}^{\phi}\,\mathrm{d}C_0 \wedge *\tilde{G}_3 \tag{4.9}$$
$$\mathrm{d}(\mathrm{e}^{\phi} * \tilde{G}_3) = \tilde{G}_5 \wedge H_3 \tag{4.10}$$
$$\begin{aligned} R_{mn} = {} & \frac{1}{2}\partial_m\phi\partial_n\phi + \frac{1}{2}\mathrm{e}^{2\phi}\partial_m C_0 \partial_n C_0 + \frac{1}{96}\tilde{G}_{5mpqrs}\tilde{G}_{5n}{}^{pqrs} \\ & + \frac{1}{4}\left(\mathrm{e}^{-\phi}H_{3mpq}H_{3n}{}^{pq} + \mathrm{e}^{\phi}\tilde{G}_{3mpq}\tilde{G}_{3n}{}^{pq}\right) - \frac{1}{8}g_{mn}\left(\mathrm{e}^{-\phi}|H_3|^2 + \mathrm{e}^{\phi}|\tilde{G}_3|^2\right) \ . \end{aligned} \tag{4.11}$$

where recall that ϕ is the dilaton and C_0 is the RR axion. The angle brackets and modulus signs denote the natural pointwise inner products and norms for p-forms, respectively. Thus if $a_{m_1\cdots m_p}$, $b_{m_1\cdots m_p}$ denote the components of two p-forms a, b then $\langle a, b\rangle = \frac{1}{p!}a_{m_1\cdots m_p}b^{m_1\cdots m_p}$, and $|a|^2 = \langle a, a\rangle$. The operator $\mathrm{d}^\dagger = - * \mathrm{d}*$ denotes the codifferential, the formal adjoint to the exterior derivative d.

It will turn out that, for the linearised fluctuations of interest, it is consistent to vary C_2, C_4 and B, while keeping the metric, the dilaton and the axion fixed. Noting that our backgrounds have $G_3 = H_3 = 0$, and constant ϕ, C_0, it is straightforward to obtain the linearised equations of motion. The linearisations of the first two equations of motion (4.7), (4.8) are trivially satisfied in our backgrounds. The remaining linearised equations of motion give

$$\begin{aligned}
\delta\tilde{G}_5 &= *\delta\tilde{G}_5 && (4.12)\\
\mathrm{e}^{-\phi}\mathrm{d}*\delta H_3 &= -G_5\wedge(\delta G_3 - C_0\delta H_3) && (4.13)\\
\mathrm{e}^{\phi}\mathrm{d}*(\delta G_3 - C_0\delta H_3) &= G_5\wedge\delta H_3 && (4.14)\\
0 &= G_{5mpqrs}\delta\tilde{G}_{5n}{}^{pqrs} + \delta\tilde{G}_{5mpqrs}G_{5n}{}^{pqrs} && (4.15)
\end{aligned}$$

where

$$\begin{aligned}
\delta G_3 &= \mathrm{d}\delta C_2 && (4.16)\\
\delta\tilde{G}_5 &= \mathrm{d}\delta C_4 - C_2\wedge\mathrm{d}\delta B && (4.17)\\
\delta H_3 &= \mathrm{d}\delta B\ . && (4.18)
\end{aligned}$$

In section 4.4 we examine the linearised equation of motion for the metric separately. The metric modes in warped non-compact backgrounds are considerably more complicated than for Calabi-Yau compactifications [84], and we shall only give a partial treatment. In the next three subsections we shall allow C_2, B and C_4 to fluctuate in turn, imposing a natural ansatz and then solving the resulting linearised equations of motion. Having done this, it will be immediately clear that all of these modes may be turned on simultaneously, and that this leads to the same equations. Thus the modes are completely decoupled from each other.

4.1 C_2 field moduli

We begin with the RR two-form C_2, since this is technically the simplest. Let ψ^A, $A = 1,\ldots,b_4(X)$, be representatives for a basis of $H^2_{\mathrm{free}}(X,Y;\mathbb{Z})$. That is, the ψ^A are closed two-forms on X which have integral periods and vanish when restricted to $Y = \partial X$. Then we may write the C_2 moduli in (3.20) as

$$C_2^\flat = \frac{1}{\mu_1}\varphi^A\psi^A\ . \tag{4.19}$$

As in section 2.3, a sum over repeated indices is understood. The φ^A, which are periodic constants, determine the C_2 moduli.

Consider now a fluctuation of C_2, where φ^A may depend on the coordinates of $\mathbb{R}^4$. Thus we write

$$\delta C_2 = \frac{1}{\mu_1}\delta\varphi^A\psi^A \tag{4.20}$$

where $\delta\varphi^A$ are functions on $\mathbb{R}^4$. One must check whether such a perturbation satisfies the linearised Type IIB supergravity equations (4.13), (4.14). The right hand side of (4.13) is identically zero, as one sees by noting the form of the background G_5-flux given by (3.2). Thus the B field is not sourced by the fluctuation (4.20). The equation of motion (4.14) for G_3, on the other hand, requires

$$\mathrm{d}(*_4 \mathrm{d}\delta\varphi^A \wedge *_6\psi^A) \;=\; 0 \tag{4.21}$$

where $*_6$ denotes the Hodge dual operator on (X, g_X). Notice that the warp factor H has dropped out of the computation. Assuming the $\delta\varphi^A$ are linearly independent this equation implies that

$$\mathrm{d} *_6 \psi^A \;=\; 0 \tag{4.22}$$

for all A, and the resulting equation for $\delta\varphi^A$ is the equation of motion for a massless scalar field on $\mathbb{R}^4$. Since ψ^A is both closed and co-closed on (X, g_X), it is a harmonic two-form $\psi^A \in \mathcal{H}^2(X, g_X)$.

The variation of the ten-dimensional kinetic term is proportional to

$$\frac{1}{2}\int_{\mathcal{M}} \delta G_3 \wedge *_{10}\delta G_3 \quad\propto\quad e_{AB}\int_{\mathbb{R}^4} \mathrm{d}\delta\varphi^A \wedge *_4\mathrm{d}\delta\varphi^B \ , \tag{4.23}$$

where

$$e_{AB} \;=\; \int_X \psi^A \wedge *_6\psi^B \ . \tag{4.24}$$

Note that e_{AB} is a symmetric matrix. It may therefore be diagonalised by an orthogonal change of basis for the ψ^A, accompanied by a corresponding change of basis for the fields $\delta\varphi^A$. In such a basis one obtains canonical kinetic terms on the right hand side of (4.23), with

$$e_{AB} \;=\; \delta_{AB}\int_X \psi^A \wedge *_6\psi^A \qquad \text{(no sum)} \ . \tag{4.25}$$

Notice again that the warp factor H has essentially dropped out of the calculation. The constants e_{AB} are finite precisely when the ψ^A are L^2 normalisable on (X, g_X). Thus $\psi^A \in \mathcal{H}^2_{L^2}(X, g_X)$. Using (2.34) we see that there are indeed precisely $b_4(X)$ L^2 harmonic two-forms ψ^A on (X, g_X), as required by the analysis above.

Finally, let us consider the asymptotics of the forms ψ^A for large r. By construction we require the ψ^A to be closed and co-closed; this of course implies they are harmonic, but in general the converse is not true. However, provided (X, g_X) is complete and one considers L^2 harmonic forms, harmonic is indeed equivalent to closed and co-closed. On any asymptotically conical manifold, a closed and co-closed form ψ has an asymptotic large r expansion[21] [85] of the form

$$\psi \;=\; \psi_0 + o(r^\gamma) \;. \tag{4.26}$$

Here ψ_0 is closed and co-closed on the *cone* $C(Y)$, and has L^2 norm, with respect to the cone metric,

$$\|\psi_0\| \;=\; r^\gamma \;. \tag{4.27}$$

Even more precisely, ψ_0 is one of the homogeneous modes listed in appendix A, and thus γ takes only a countable set of special values. The notation $o(r^\gamma)$ in (4.26) denotes those forms whose norms are $o(r^\gamma)$ in the limit $r \to \infty$.

In the case at hand, we have $p = 2$, $n = 3$, in the notation of appendix A. Table 4 implies that the only modes that are L^2 (denoted L^2_∞ in the appendix) are of type II and III$^-$. Modes of type II require a harmonic one-form on (Y, g_Y), and since $b_1(Y) = 0$ we see that there are no modes of type II. Thus

$$\psi_0 \;=\; r^{-1-\sqrt{1+\mu}}\mathrm{d}\beta^{(1)}_\mu - (1+\sqrt{1+\mu})r^{-2-\sqrt{1+\mu}}\mathrm{d}r \wedge \beta^{(1)}_\mu \tag{4.28}$$

where $\beta^{(1)}_\mu$ is a co-closed one-form on (Y, g_Y) which is an eigenfunction of the Laplacian Δ_Y with eigenvalue $\mu > 0$. In particular, this gives

$$\gamma \;=\; -3 - \sqrt{1+\mu} \;. \tag{4.29}$$

Note also that

$$\psi\mid_Y = \lim_{r\to\infty} \psi\mid_{Y_r} = \lim_{r\to\infty}\Big(r^{-1-\sqrt{1+\mu}}\mathrm{d}\beta^{(1)}_\mu + o(r^\gamma)\Big) \;=\; 0 \;. \tag{4.30}$$

This is consistent with the fact that we require the fluctuations to preserve the boundary conditions at infinity. Of course, this analysis is not sufficient to determine which particular mode $\beta^{(1)}_\mu$ is associated to each ψ^A.

4.2 B field moduli

The fluctuations of the B field are rather similar. One added complication, however, is that $\tilde{G}_5$ is no longer invariant. If we write

$$\delta B \;=\; \frac{1}{\mu_1}\delta\sigma^A\psi^A \tag{4.31}$$

[21] We thank T. Pacini for discussions on the existence of this expansion.

then we may keep $\tilde{G}_5$ invariant, and thus in particular also self-dual, if we also vary

$$\delta C_4 \;=\; \frac{1}{\mu_1^2}\varphi_A \delta\sigma^A \psi^A \wedge \psi^A \ . \tag{4.32}$$

Note that the form $\psi^A \wedge \psi^A$ represents a class in $H^4(X, Y; \mathbb{R})$. Apart from these minor differences, the analysis is identical to that in the previous subsection, with similar conclusions. In fact, supersymmetry pairs the C_2 field with the B field, and thus this is expected.

4.3 C_4 field moduli

In order to satisfy the self-duality condition (4.12) we take the following ansatz, essentially as in [24]

$$\delta G_5 \;=\; \frac{1}{\mu_3}(1 + *_{10})\left(\mathrm{d}\delta\vartheta^M \wedge \Psi^M\right) \ . \tag{4.33}$$

The fluctuation of C_4 that leads to this will be described below. Here the $\delta\vartheta^M$, $M = 1, \ldots, b_2(X)$, are $b_2(X)$ functions on $\mathbb{R}^4$, and the Ψ^M are representatives of $H^4_{\mathrm{free}}(X, Y; \mathbb{Z})$. Thus the Ψ^M are closed four-forms on X with integral periods that vanish on Y. The linearised Bianchi identity implies that the scalars $\delta\vartheta^M$ satisfy the equation of motion

$$\mathrm{d} *_4 \mathrm{d}\delta\vartheta^M \;=\; 0 \ , \tag{4.34}$$

together with the requirement that

$$\mathrm{d}(H^{-1} *_6 \Psi^M) \;=\; 0 \ . \tag{4.35}$$

Recall that $H(x_i)^{-1} = 0$ at the locations x_i, $i = 1, \ldots, m$, of the m stacks of D3-branes. Since the Ψ^M are closed and co-closed on $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$ they define harmonic four-forms $\Psi^M \in \mathcal{H}^4(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$. Equivalently, their duals

$$\Phi^M \;\equiv\; H^{-1} *_6 \Psi^M \tag{4.36}$$

define harmonic two-forms $\Phi^M \in \mathcal{H}^2(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$.

In [24] the equations for such a harmonic form on the warped resolved conifold were written down, and it was argued that there exists a unique solution such that the two-form (denoted W in [24]) is L^2 normalisable in the warped metric Hg_X. Below we show that the results of [24] may be generalised. After dualising the scalar fields $\delta\vartheta^M$ to a corresponding set of two-forms δa^M on $\mathbb{R}^4$,

$$\mathrm{d}\delta a^M \;=\; *_4 \mathrm{d}\delta\vartheta^M \ , \tag{4.37}$$

the fluctuation of C_4 that gives rise to (4.33) is

$$\delta C_4 \;=\; \frac{1}{\mu_3}\left(\delta\vartheta^M \Psi^M + \delta a^M \wedge \Phi^M\right) \ . \tag{4.38}$$

Since $\tilde{G}_5$ is a self-dual five-form its kinetic term vanishes identically. Moreover, since the fluctuation (4.33) is self-dual it automatically solves (4.12) and (4.15). Following [24] we impose a normalisablity condition obtained by inserting the variation in C_4 due to $\delta\vartheta^M$ into the ten-dimensional action. This gives the four-dimensional kinetic term

$$f_{MN}\int_{\mathbb{R}^4} \mathrm{d}\delta\vartheta^M \wedge *_4 \mathrm{d}\delta\vartheta^N \tag{4.39}$$

where the constants f_{MN} are defined by

$$f_{MN} \;=\; \int_X H^{-1}\Psi^M \wedge *_6 \Psi^N \ . \tag{4.40}$$

As before, an orthogonal change of basis leads to

$$f_{MN} \;=\; \delta_{MN}\int_X H^{-1}\Psi^M \wedge *_6 \Psi^M \qquad \text{(no sum)} \ . \tag{4.41}$$

The constants f_{MN} are therefore finite when the Ψ^M are L^2 normalisable on $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$, or equivalently $\Phi^M \in \mathcal{H}^2_{L^2}(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$.

Remarkably, it turns out that one may argue that precisely $b_2(X)$ such L^2 harmonic forms exist on $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$. Recall that (X, g_X) is a complete asymptotically conical manifold, asymptotic to a cone over (Y, g_Y). To construct the function H we pick m points $x_i \in X$, $i = 1, \ldots, m$, where near to each point H behaves as in (3.11). Thus near to x_i the metric Hg_X looks like

$$\frac{L_i^4}{\rho_i^4}(\mathrm{d}\rho_i^2 + \rho_i^2 g_{S^5}) \ . \tag{4.42}$$

If x_i is a singular point with link (Y_i, g_{Y_i}) then obviously one replaces g_{S^5} with g_{Y_i}. Defining $R_i = L_i^2/\rho_i$ one sees that the metric (4.42) is flat

$$\mathrm{d}R_i^2 + R_i^2 g_{S^5} \ . \tag{4.43}$$

The point x_i is thus at infinity in $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$. On the other hand, near $r = \infty$ the metric Hg_X approaches

$$\frac{L^4}{r^4}(\mathrm{d}r^2 + r^2 g_Y) \ . \tag{4.44}$$

Setting $\rho = L^2/r$ we similarly obtain

$$\mathrm{d}\rho^2 + \rho^2 g_Y \tag{4.45}$$

where $r = \infty$ is the isolated conical singularity $\rho = 0$.

The manifold $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$ thus has an isolated conical singularity, near which the metric looks like the incomplete cone (4.45), and m asymptotically Euclidean regions of the form (4.43). In particular, $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$ is asymptotically conical near to each x_i, which is a point at infinity in the metric Hg_X. If (Y, g_Y) is the round sphere, $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$ is smooth and asymptotically conical and we may apply the results of [66], summarised in (2.34), to determine the L^2 harmonic forms. The UV conformal field theory is $\mathcal{N} = 4$ Yang-Mills, and in this case we find that there are $b_2(X) = 0$ such harmonic forms. More generally, the space of interest has an isolated conical singularity at $\rho = 0$. The L^2 harmonic forms on a *compact* manifold $(\bar{X}, g_{\bar{X}})$ with isolated conical singularities were studied by Cheeger in [86]. If X denotes the smooth part of $\bar{X}$, *i.e.* $\bar{X}$ with the point $\rho = 0$ in (4.45) deleted, then the result for two-forms in dimension six is [86]

$$\mathcal{H}^2_{L^2}(\bar{X}, g_{\bar{X}}) \ \cong \ H^2(X; \mathbb{R}) \ . \tag{4.46}$$

Of course our manifold is not compact, but instead has m asymptotically Euclidean regions. However, because both types of behaviour – asymptotically Euclidean ends and isolated conical singularities – lead to topological results for the L^2 cohomology, one may put the analytic and topological results of [86] and [66] together to show[22] that the L^2 harmonic two-forms on $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$ are given by

$$\mathcal{H}^2_{L^2}(X \setminus \{x_1, \ldots, x_m\}, Hg_X) \ \cong \ H^2(X \setminus \{x_1, \ldots, x_m\}, \cup_{i=1}^m S^5; \mathbb{R}) \ \cong \ H^2(X; \mathbb{R}) \tag{4.47}$$

Here the copies of S^5 are boundaries around the points x_i. Thus there are indeed $b_2(X)$ L^2 harmonic two-forms Φ^M, $M = 1, \ldots, b_2(X)$, on $(X \setminus \{x_1, \ldots, x_m\}, Hg_X)$.

Finally, we consider the asymptotic behaviour of the forms Φ^M as $r \to \infty$. Replacing $\rho = L^2/r$, this becomes $\rho \to 0$. There is then an asymptotic expansion as $\rho \to 0$

$$\Phi \ = \ \Phi_0 + o(\rho^\gamma) \tag{4.48}$$

[22] We are extremely grateful to E. Hunsicker and T. Hausel for discussions on this point.

where Φ_0 is a homogeneous closed and co-closed form on the cone $C(Y)$, $g_{C(Y)} = \mathrm{d}\rho^2 + \rho^2 g_Y$, with norm

$$\|\Phi_0\| \;=\; \rho^\gamma \;. \tag{4.49}$$

Since we require the Φ^M to be L^2 with respect to the metric Hg_X, we are interested in the case $p=2$, $n=3$ and L^2_0 in Table 4. The two possible modes are thus of type I and type III$^+$. Thus

$$\begin{aligned}
(\mathrm{I}) \quad &: \quad \Phi_0 = \alpha_0^{(2)} && (4.50)\\
& \quad\;\; \gamma = -2 && (4.51)\\
(\mathrm{III})^+ \quad &: \quad \Phi_0 = \rho^{-1+\sqrt{1+\mu}}\mathrm{d}\beta_\mu^{(1)} + (-1+\sqrt{1+\mu})\rho^{-2+\sqrt{1+\mu}}\mathrm{d}\rho\wedge\beta_\mu^{(1)} && (4.52)\\
& \quad\;\; \gamma = -3+\sqrt{1+\mu} && (4.53)
\end{aligned}$$

where $\alpha_0^{(2)}$ is a harmonic two-form on (Y, g_Y), and $\beta_\mu^{(1)}$ again denotes a co-closed one-form on (Y, g_Y) which is an eigenfunction of the Laplacian Δ_Y with eigenvalue $\mu > 0$.

To determine which type of asymptotic behaviour we have we may use a topological argument. Consider the long exact sequence

$$\begin{aligned}
0 \;\cong\; H^1(Y;\mathbb{R}) \longrightarrow H^2(X,Y;\mathbb{R}) \xrightarrow{\;f\;} H^2(X;\mathbb{R}) \longrightarrow \\
\longrightarrow H^2(Y;\mathbb{R}) \longrightarrow H^3(X,Y;\mathbb{R}) \;\cong\; 0 \;.
\end{aligned}\tag{4.54}$$

From (4.47) the $b_2(X)$ L^2 harmonic two-forms Φ^M, $M = 1,\ldots,b_2(X)$, define a basis for $H^2(X;\mathbb{R})$. The sequence (4.54) implies that we may choose this basis such that $b_3(Y)$ restrict to non-trivial classes in $H^2(Y;\mathbb{R})$, while $b_4(X) = b_2(X) - b_3(Y)$ restrict to trivial classes in $H^2(Y;\mathbb{R})$. Let us denote these by Φ^I, $I = 1,\ldots,b_3(Y)$, and $\Phi^{b_3(Y)+A}$, $A = 1,\ldots,b_4(X)$, respectively. We have $Y = \lim_{\rho\to 0} Y_\rho$. Thus

$$\begin{aligned}
(\mathrm{I}) \quad &: \quad \Phi\mid_Y = \lim_{\rho\to 0}\left(\alpha_0^{(2)} + o\left(\rho^{-2}\right)\right) = \alpha_0^{(2)} && (4.55)\\
(\mathrm{III})^+ \quad &: \quad \Phi\mid_Y = \lim_{\rho\to 0}\left(\rho^{-1+\sqrt{1+\mu}}\mathrm{d}\beta_\mu^{(1)} + o\left(\rho^{-3+\sqrt{1+\mu}}\right)\right) = 0\;. && (4.56)
\end{aligned}$$

These statements may look slightly odd, given that $\rho^{-2}\to\infty$ as $\rho\to 0$. However, recall that the notation $o(\rho^\gamma)$ refers to forms which have *norms* of order $o(\rho^\gamma)$ as $\rho \to 0$. In a neighbourhood of $\rho = 0$ such a form may be written

$$\phi(\rho) \;=\; \alpha(\rho) + \mathrm{d}\rho\wedge\beta(\rho) \tag{4.57}$$

where $\alpha(\rho)$, $\beta(\rho)$ are forms on Y_ρ. If $\phi(\rho)$ is a p-form, its square norm, to leading order as $\rho \to 0$, is

$$\|\phi(\rho)\|^2 \;=\; \rho^{-2p}\left(\|\alpha(\rho)\|_Y^2 + \rho^2\|\beta(\rho)\|_Y^2\right) \tag{4.58}$$

where $\|\cdot\|_Y$ denotes the pointwise norm on (Y, g_Y). For us $p = 2$ and thus we see that $\|\alpha(\rho)\|_Y$ is $o(1)$ for modes of type I and $o\left(\rho^{-1+\sqrt{1+\mu}}\right)$ for modes of type III$^+$. In both cases

$$\lim_{\rho\to 0}\|\alpha(\rho)\|_Y \;=\; 0 \Longrightarrow \lim_{\rho\to 0}\alpha(\rho) = 0\ . \tag{4.59}$$

In particular note that the harmonic two-forms Φ^I, $I = 1, \ldots, b_3(Y)$, are asymptotic to the $b_3(Y)$ harmonic two-forms on (Y, g_Y). This generalises the warped resolved conifold result of [24]. Note also that the dual four-forms Ψ^M, in either case, satisfy

$$\Psi^M\mid_Y = \lim_{\rho\to 0}\Psi^M\mid_{Y_\rho} =\; 0\ . \tag{4.60}$$

We summarise the properties of the fluctuations discussed so far in Table 2.

number	fluctuations	harmonic	mode	$\mathcal{H}^2_{L^2}(Hg_X)$	$\mathcal{H}^2_{L^2}(g_X)$
$b_3(Y)$	$\delta C_2,\ \delta B$	–	I	–	no
$b_4(X)$	$\delta C_2,\ \delta B$	ψ^A	III$^-$	–	yes
$b_4(X)$	δC_4	$\Phi^{b_3(Y)+A}$	III$^+$	yes	–
$b_3(Y)$	δC_4	Φ^I	I	yes	–

Table 2: Square-integrability of the moduli fluctuations (*cf.* Table 4). The metric fluctuations, that must pair with δC_4, will be discussed in subsection 4.4.

4.4 Metric moduli

In this section we consider linearised fluctuations of the metric. In principle one should write the full set of linearised equations for both metric perturbations and also the form fields C_2, B and C_4 discussed thus far. As mentioned earlier, although we have fluctuated the form fields separately in the previous subsections, it is straightforward to substitute (4.20), (4.31), (4.32), (4.38) into the linearised equations of motion and verify that these modes are in fact completely decoupled. As we discuss in this section, the metric modes are rather more involved. The first problem is to identify the linearised perturbations of asymptotically conical Ricci-flat Kähler metrics on X *i.e.* the tangent space to the latter space. We give a partial treatment of this that will be sufficient to relate the metric modes to the analysis of form field modes in the previous subsections. For example, this will allow us to determine the asymptotic eigenvalues μ in (4.28) for certain examples. The second problem is to understand how to promote these linearised perturbations of the Ricci-flat Kähler metric on X to an ansatz that allows these modes to depend on position in $\mathbb{R}^4$, as we have done in previous subsections. This is surprisingly complicated for warped Calabi-Yau geometries – see, for example, [84] or the very recent paper [87]. From supersymmetry one naively expects to obtain $b_2(X)$ functions on $\mathbb{R}^4$ satisfying the equation for massless scalar fields, which pair with the modes of C_4 discussed in the previous subsection. However, to show this rigorously would require substantially more work, not least since the Calabi-Yau manifolds here are non-compact. We instead simply summarise some of the issues involved, and refer to the literature for further details.

Before discussing the metric modes, we note that it is *not* possible for the massless fields found in the previous subsections to obtain masses by mixing with additional modes that we may turn on. Indeed, this is rather a general statement. Suppose one has scalar fields φ_i, $i = 0, \ldots, k$, with equation of motion of the general form

$$\nabla^2 \varphi_i \;=\; M_{ij}\, \varphi_j + \text{higher order} \tag{4.61}$$

where the form of the higher order terms is irrelevant. The physical masses are obtained by diagonalising the mass matrix M_{ij}. Indeed, we shall encounter precisely such a phenomenon later in the context of KK theory on $\mathrm{AdS}_5 \times Y$, where the C_4 field mixes with metric modes producing a non-trivial 2×2 mass matrix (see equation (5.31)). However, in the case at hand we have shown that setting $\varphi_i \equiv 0$ for all $i = 1, \ldots, k$, with φ_0 a *massless* scalar in four dimensions, solves the equations of motion. Here the fields φ_i, $i = 1, \ldots, k$, are any modes that we have not fluctuated in the previous subsections. This immediately implies that $M_{j0} = 0$ for all $j = 0, \ldots, k$. Thus the mass matrix necessarily has a zero eigenvalue, although note that in the process of diagonalising the mass matrix this massless field will typically be a mixture of φ_0 with the other fields φ_i, $i = 1, \ldots, k$. However, the important point is that there is necessarily a *massless* combination of the modes.

Our conjecture in section 2.1 implies that there should be a $b_2(X)$-dimensional Kähler moduli space for asymptotically conical Ricci-flat Kähler metrics on a crepant resolution X of a Calabi-Yau cone singularity $Z = C(Y)$. We may define the $b_2(X)$ Kähler classes as

$$\xi_M = \int_{S_M} \omega_X \tag{4.62}$$

where S_M, $M = 1, \dots b_2(X)$, denotes a basis for $H_2^{\rm free}(X;\mathbb{Z})$. Note that the exact sequence

$$\begin{aligned} 0 \cong H_3(X,Y;\mathbb{R}) &\longrightarrow H_2(Y;\mathbb{R}) \longrightarrow H_2(X;\mathbb{R}) \longrightarrow \\ &\longrightarrow H_2(X,Y;\mathbb{R}) \longrightarrow H_1(Y;\mathbb{R}) \cong 0 \end{aligned} \tag{4.63}$$

means we may split the S_M into S_I, $I = 1, \dots, b_3(Y)$, and $S_{b_3(Y)+A}$, $A = 1, \dots, b_4(X)$. The former are images of $H_2(Y;\mathbb{R})$ in $H_2(X;\mathbb{R})$ *i.e.* two-cycles on X that arise from two-cycles on Y.

The tangent space to the space of asymptotically conical Ricci-flat Kähler metrics on (X, g_X) should thus be $b_2(X)$-dimensional. We begin by showing that a $b_4(X)$-dimensional subspace of these linearised perturbations indeed exist, and may be identified with the $b_4(X)$ L^2 harmonic two-forms ψ^A that enter the C_2 field and B field fluctuations of sections 4.1 and 4.2, respectively.

We may phrase the equations for a Calabi-Yau metric in terms of the Kähler form ω_X and the holomorphic volume form Ω_X. These satisfy

$$\frac{1}{3!}\omega_X^3 = \frac{i}{8}\Omega_X \wedge \bar{\Omega}_X \tag{4.64}$$

$$\omega_X \wedge \Omega_X = 0 \tag{4.65}$$

$$\mathrm{d}\omega_X = 0 \tag{4.66}$$

$$\mathrm{d}\Omega_X = 0 \; . \tag{4.67}$$

If we fix Ω_X, the linearised equations for $\delta\omega_X$ are then

$$\omega_X^2 \wedge \delta\omega_X = 0 \tag{4.68}$$

$$\delta\omega_X \wedge \Omega_X = 0 \tag{4.69}$$

$$\mathrm{d}\delta\omega_X = 0 \; . \tag{4.70}$$

We now show that one may solve these equations using the basis of two-forms ψ^A for $\mathcal{H}^2_{L^2}(X, g_X) \cong H^2(X,Y;\mathbb{R}) \cong H_4(X;\mathbb{R})$. That is, we take $\delta\omega_X \in \mathcal{H}^2_{L^2}(X, g_X)$. Note that the L^2 condition on $\delta\omega_X$ is the same as that for a corresponding change in the metric δg_X, with the natural norm

$$\|\delta g_X\|^2_{L^2} = \int_X \mathrm{d}^6 y \sqrt{\det g_X}\, g_X^{ii'} g_X^{jj'} \delta g_{X\,ij} \delta g_{X\,i'j'} \; . \tag{4.71}$$

Since (X, g_X) is complete and the harmonic forms are L^2, they are also both closed and co-closed, and thus satisfy (4.70). Here the co-closed condition is the standard gauge-fixing condition $\nabla^i \delta g_{Xij} = 0$ – see, for example, [88].

Suppose that α is a two-form on X, $\alpha \in \Omega^2(X)$. Recall that the complex structure tensor J acts on a two-form α via

$$(J \circ \alpha)_{ij} \;=\; J_i^m J_j^n \alpha_{mn} \; . \tag{4.72}$$

This action squares to the identity. We may thus introduce the projection maps

$$\pi_\pm : \Omega^2(X) \to \Omega^2_\pm(X) \tag{4.73}$$

defined by

$$\pi_\pm \alpha \;=\; \frac{1}{2}\left(\alpha + J \circ \alpha\right) \;=\; \alpha_\pm \; . \tag{4.74}$$

The splitting of real two-forms into the ± 1 eigenspaces corresponds, over $\mathbb{C}$, to the splitting into forms of type $(1,1)$, and types $(2,0)$, $(0,2)$ respectively. In particular, a two-form α_- with eigenvalue -1 under (4.72) may be written as the real part of a $(2,0)$-form $\alpha^{2,0} \in \Omega^{2,0}(X)$; so

$$\alpha_- = \alpha^{2,0} + \overline{\alpha^{2,0}} \tag{4.75}$$

where $\overline{\alpha^{2,0}} \in \Omega^{0,2}(X)$. By a slight abuse of notation, we will refer to α_- as type $(2,0)$ (or equivalently type $(0,2)$).

On a Kähler manifold, if α is harmonic then it is easy to show that $\alpha_\pm$ are in fact separately harmonic. One way to see this is as follows. We note that for any two-form α we have the Weitzenböck formula

$$(\Delta\alpha)_{ij} = -\nabla^2 \alpha_{ij} - 2R_{ij}{}^{mn}\alpha_{mn} - 2R^m_{\;[i}\alpha_{j]m} \tag{4.76}$$

where $\nabla^2 = \nabla^m \nabla_m$. It follows that

$$(\Delta(J \circ \alpha))_{ij} = -\nabla^m \nabla_m \left(J_i{}^p J_j{}^q \alpha_{pq}\right) - 2R_{ij}{}^{mn} J_m{}^p J_n{}^q \alpha_{pq} - 2R^m_{\;[i} J_{j]}{}^p J_m{}^q \alpha_{pq} \; . \tag{4.77}$$

On a Kähler manifold we have

$$\nabla J \;=\; 0 \tag{4.78}$$

and also the curvature identity

$$R_{ij}{}^{mn} J_m{}^p J_n{}^q \;=\; R_{ij}{}^{pq} \;=\; J_i{}^m J_j{}^n R_{mn}{}^{pq} \; . \tag{4.79}$$

In fact on any Riemannian manifold (X, g_X) where the holonomy group is $G \subset O(n)$, the Riemann tensor may be regarded as an element of $\mathrm{Sym}^2\Omega^2_{\mathfrak{g}}(X)$, where $\mathfrak{g}$ is the Lie algebra of G. Here the symmetric product in $\mathrm{Sym}^2\Omega^2(X)$ reflects the algebraic identity $R_{ijpq} = R_{pqij}$. The equation (4.79) is precisely the statement that the Riemann tensor on a Kähler manifold is in $\mathrm{Sym}^2\Omega^2_+(X)$. It follows from (4.77) that when α is harmonic we have $(\Delta(J \circ \alpha)) = J \circ (\Delta\alpha) = 0$, and thus $\Delta\alpha_\pm = 0$.

If we take $\alpha \in \mathcal{H}^2_{L^2}(X, g_X)$, then $\alpha_\pm$ are also both L^2 since it is straightforward to show that

$$\|\alpha\|^2 \;=\; \|\alpha_+\|^2 + \|\alpha_-\|^2 \; . \tag{4.80}$$

Thus $\alpha_\pm \in \mathcal{H}^2_{L^2}(X, g_X)$. As discussed in section 2.1, all the cohomology of X in degree two is of type $(1,1)$. Since α_- is harmonic and of type $(2,0)$, it represents a cohomology class of type $(2,0)$. But since any such class is trivial, it follows from (2.34) that $\alpha_- = 0$ – in particular, note that (2.34) implies that all non-zero L^2 harmonic forms represent non-trivial cohomology classes. Thus α is necessarily of type $(1,1)$. Finally, consider $\omega_X^2 \wedge \alpha$. This is an L^2 harmonic six-form. However, again by (2.34) we see that any such six-form must be zero, since $\mathcal{H}^6_{L^2}(X, g_X) \cong H^6(X;\mathbb{R}) \cong 0$.

This shows that the $b_4(X)$ L^2 harmonic two-forms ψ^A satisfy all of the equations (4.68), (4.69), (4.70). To conclude our proof that these are indeed tangent directions to the space of asymptotically conical Calabi-Yau metrics on X, we must show that taking $\delta\omega_X \in \mathcal{H}^2(X, g_X)$ preserves the asymptotically conical condition – that is, the L^2 forms do not grow too fast. To do this we may again appeal to the results of appendix A. A closed and co-closed form α has an asymptotic expansion

$$\alpha \;=\; \alpha_0 + o(r^\gamma) \tag{4.81}$$

where α_0 is one of the closed and co-closed homogeneous modes listed in the appendix, and $\|\alpha_0\| = r^\gamma$ where the norm is taken with respect to the cone metric. From Table 4, the only possible modes are of type II (of which there are none since $b_1(Y) = 0$) and type III$^-$. We thus have

$$\alpha_0 \;=\; r^{-1-\sqrt{1+\mu}}\mathrm{d}\beta^{(1)}_\mu - (1 + \sqrt{1+\mu})r^{-2-\sqrt{1+\mu}}\mathrm{d}r \wedge \beta^{(1)}_\mu \tag{4.82}$$

with $\mu > 0$. It follows that these perturbations are indeed subleading to the cone metric near infinity. These are of course the same expansions (4.28) for the harmonic two-forms entering the fluctuations δB and δC_2.

Thus there is certainly a $b_4(X)$-dimensional space of linearised perturbations of an asymptotically conical Ricci-flat Kähler manifold, where the perturbations are L^2 with respect to the natural norm (4.71). In fact, this may be understood from supersymmetry in the case without any background D3-branes. Such fluctuations of the metric are then paired by supersymmetry to fluctuations of the RR C_4 potential on the background $\mathbb{R}^4 \times X$, as in section 2.3. In this case the number of L^2 harmonic four-forms on (X, g_X) is given by (2.34), which indeed gives $\dim H^4(X;\mathbb{R}) = b_4(X)$ L^2 modes.

The remaining $b_2(X) - b_4(X) = b_3(Y)$ linearised perturbations are thus not L^2-normalisable. Assuming these exist, their asymptotics may be understood as follows[23]. Consider integrating the closed two-form $\delta\omega_X$ over a two-cycle S_I that is homologous to a two-cycle on Y. This gives a change in the Kähler class

$$\delta\xi_I = \int_{S_I} \delta\omega_X \ . \tag{4.83}$$

Near infinity we may write

$$\delta\omega_X = \alpha(r) + \mathrm{d}r \wedge \beta(r) \tag{4.84}$$

where $\alpha(r)$, $\beta(r)$ are forms on Y_r. Since $\delta\omega_X$ is closed, and also co-closed in the usual gauge $\nabla^i \delta g_{Xij} = 0$ which fixes diffeomorphism invariance, the right hand side of (4.84) will have an asymptotic expansion with leading term given by a closed and co-closed mode of appendix A. The cycle S_I in (4.83) may be represented by a cycle in Y_r. The only mode for which the integral (4.83) is both finite and non-zero is then mode I *i.e.*

$$\delta\omega_X = \alpha_0^{(2)} + o(r^{-2}) \tag{4.85}$$

where $\alpha_0^{(2)}$ is a harmonic two-form on Y. In fact we have

$$\delta\xi_I = \int_{S_I} \alpha_0^{(2)} \ , \tag{4.86}$$

where S_I is regarded as a cycle in Y. Note that such fluctuations are indeed not L^2 normalisable, as one sees from Table 4. Notice that the Kähler perturbation of the resolved conifold, discussed originally in [23], is precisely of this form.

Given the above (partial) understanding of linearised perturbations of asymptotically conical Calabi-Yau manifolds, one would now like to promote these Kähler moduli to

[23]This is essentially taken from [43], although here we make the argument more rigorous by using the asymptotic expansion together with the results of appendix A.

dynamical four-dimensional fields. Recall that the backgrounds of interest are of the form

$$
\begin{aligned}
g_{10} &= H^{-1/2}\eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + H^{1/2}g_{X ij}\mathrm{d}y^{i}\mathrm{d}y^{j} && (4.87)\\
G_5 &= (1+*_{10})\mathrm{d}H^{-1}\wedge \mathrm{vol}_4\ . && (4.88)
\end{aligned}
$$

These are a particular class of the backgrounds considered in [84] and [87]. The moduli may be parametrised by $b_2(X)$ *constant* parameters u^M. Note here that a change in the background Ricci-flat Kähler metric will induce a corresponding change in the warp factor H, which satisfies (3.10). As emphasised in [84], and in sharp contrast to the familiar Calabi-Yau compactifications, it is not possible to promote straightforwardly the moduli to spacetime-dependent scalar fields $u^M(x)$ in four dimensions. The linearised Type IIB equations of motion cannot be solved by a simple ansatz. Instead one must introduce certain off-diagonal modes, called compensator fields, which are proportional to derivatives of the scalar fields $u^M(x)$. The resulting equations, and gauge invariances, are then rather involved. Note that the non-compactness of our geometries will add to these complications. Very recently the paper [87] has appeared, which claims that the compensator fields may be effectively removed by choosing a certain ten-dimensional gauge condition. However, we will postpone a more detailed investigation of these metric modes for future work. We conclude the section by nevertheless noting that the norm of the metric perturbations induced from the ten-dimensional kinetic terms, as studied in [84], [87], is given by the natural *warped* norm

$$
\|\delta g_X\|^2_{Hg_X} = \int_X \mathrm{d}^6y\,\sqrt{\det g_X}\,H\,g_X^{ii'}g_X^{jj'}\delta g_{X\,ij}\delta g_{X\,i'j'}\ . \qquad (4.89)
$$

Compare this with (4.71). In particular, notice that all of the $b_2(X)$ linearised metric perturbations are L^2 with respect to this warped norm, whereas only a $b_4(X)$-dimensional subspace is L^2 with respect to the unwarped norm (4.71). This implies that all of the metric modes will be normalisable with respect to the physical metric (4.89) coming from the kinetic terms. Again, this is expected from supersymmetry, since all $b_2(X)$ modes of C_4 in section 4.3 are normalisable.

5 Spontaneous breaking of baryonic symmetries

In this section we discuss the dual field theory interpretation of the linearised fluctuations described in section 4. For the modes corresponding to non-anomalous baryonic symmetries our analysis will extend and generalise the resolved conifold result of [30]. The holographic interpretation of the modes corresponding to anomalous baryonic symmetries is less obvious. We will see how the analysis of these leads us to predict the existence of certain particular modes in the KK spectrum of $\mathrm{AdS}_5 \times Y$. We will also speculate on the possibility that some of these modes correspond to anomalous baryonic currents.

5.1 Vacuum fluctuations and Goldstone bosons

As discussed in section 2.5, the classical gauge theory has a large VMS $\mathscr{M}$. The potential of the classical theory is identically zero at any point in this moduli space. One thus expects to find massless scalar fields associated to these flat directions in field space. In section 3, and as summarised in section 3.4, we have explained how this classical vacuum moduli space is realised in the dual gravity description. Roughly, the mesonic directions correspond to moving the N pointlike D3-braneson X. The "FI parameters", which are the image of the map Π in (2.70), may be identified with the $b_2(X)$ Kähler moduli and the $b_4(X)$ B field moduli, whereas the $U(1)^{\chi-1}$ fibres over the mesonic spaces may be identified with the RR torus (3.20). In section 4 we have shown that there do indeed exist massless scalar fields on $\mathbb{R}^4$ associated to linearised deformations of these moduli, at least for the B field and RR field moduli – as discussed in section 4.4, the metric moduli would require more work to make this rigorous. Nonetheless, this is clearly a very satisfying result. Notice that we have not attempted to describe massless fields associated with moving the positions of the pointlike D3-branes on X. In principle one could study such deformations, but our main interest in this paper is with the baryonic symmetries and associated directions in moduli space. As we have alluded to earlier, the IR theory is then not simply a SCFT, or even a product of SCFTs, but rather will also include massless particles corresponding to motion along flat directions in the field theory. Notice that the description of the fluctuations in terms of massless fields on $\mathbb{R}^4$ is essentially an application of KK reduction on the *warped* Calabi-Yau X to $\mathbb{R}^4$. However, one may also understand the fluctuations by applying more standard holographic arguments, as we show later in the section. Thus different aspects of the IR theory may be understood using both holographic and KK techniques. This is a very interesting aspect of these gravity backgrounds.

Since the global symmetry group $U(1)^{\chi-1}$ acts on the space of supergravity backgrounds without fixed points, any choice of vacuum spontaneously breaks this symmetry. This precisely happens in the classical field theory also, for a generic point in the VMS $\mathscr{M}$. A spontaneously broken global symmetry generally leads to Goldstone bosons. These are the same as the massless fields referred to above of course – they are flat directions given by acting with a broken symmetry generator. Since the vacua are supersymmetric, these Goldstone fields will have $\mathcal{N}=1$ superpartners. The Goldstones are fluctuations of RR fields, and are hence pseudo-scalars. Their scalar partners come from metric and B-field fluctuations This is precisely the pairing of the RR fields with the Kähler moduli and B field moduli. Thus the linearised fluctuations we have found may be tentatively associated with these Goldstone bosons and their supersymmetric partners.

However, the above, essentially classical, discussion overlooks an important subtlety: in the quantum theory only a $U(1)^{b_3(Y)}$ subgroup of the baryonic symmetry group is non-anomalous, the remaining symmetries being anomalous and thus broken in the field theory by instantons. Their corresponding classically conserved currents are thus not conserved in the quantum theory. By Goldstone's theorem, the massless fields associated to motion in the non-anomalous directions should be *exactly massless* in the quantum theory. Thus the $b_3(Y)$ fluctuations corresponding to modes of C_4 of type I in section 4.3, and the non-normalisable (with respect to (4.71)) Kähler moduli (4.86), should also be exactly massless. Notice that in both cases these modes are constructed from forms that are asymptotic to the $b_3(Y)$ harmonic two-forms on (Y, g_Y) – see equations (4.50) and (4.85), respectively. That $U(1)^{b_3(Y)}$ is an *exact* symmetry of the quantum theory is simple to understand in our gravity dual, as we alluded to earlier: these symmetries come from gauge transformations of the RR four-form. A gauge symmetry is always non-anomalous, otherwise the theory is inconsistent. The relevant gauge transformations are of the form

$$C_4 \rightarrow C_4 + \mathrm{d}K \tag{5.1}$$

where $\mathrm{d}K \mid_{\partial\mathcal{M}}= 0$ but $K \mid_{\partial\mathcal{M}} \neq 0$. Thus $K \mid_{\partial\mathcal{M}}$ defines a class in $H^3(\partial\mathcal{M};\mathbb{R})$. For smooth X, this is isomorphic to $H^3(Y;\mathbb{R})$. The gauge transformation (5.1) then changes the compactly supported cohomology class of C_4, which recall we are identifying as part of the background moduli. In fact the group of global symmetries generated by such gauge transformations is $H^3(Y;U(1))$. The group of components $\overline{H}^3(Y;U(1))$ is, from (C.5), isomorphic to $H^4(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z}) \cong H_{3,\mathrm{tor}}(Y;\mathbb{Z})$. These are discrete non-anomalous baryonic symmetries, and arise only if Y has a non-trivial fundamental group.

Another way to see that the symmetry group $H^3(Y;\mathbb{R})/H^3_{\text{free}}(Y;\mathbb{Z}) \cong U(1)^{b_3(Y)}$ is completely broken by a background with smooth X is to note that (2.4) implies $H^3(X;\mathbb{R})/H^3_{\text{free}}(X;\mathbb{Z}) = 0$. Indeed, baryons are interpreted as D3-branes wrapped on three-submanifolds in Y, which are thus charged under the group $H^3(Y;U(1))$. Since there are no three-cycles on X, such D3-branes may presumably annihilate in the interior of X, as discussed in [23]. Thus all non-anomalous baryonic symmetries are broken by a choice of X. So again we expect to find $b_3(Y)$ massless Goldstone bosons, given by linearised fluctuations of C_4, together with their supersymmetric partners, given by fluctuations of the metric.

The anomalous baryonic symmetries are different, however. The classically conserved currents are not conserved at quantum level, because of the presence of anomalies, as we reviewed in section 2.3. Thus Goldstone's theorem does not apply, and there is *a priori* nothing to prevent quantum corrections lifting the classical massless fields. Correspondingly, in the gravity dual the anomalous baryonic symmetries are not associated to gauge transformations. The corresponding massless modes may in particular be lifted by corrections to the supergravity backgrounds we have been discussing. For example, there may well be corrections to the gravity backgrounds of section 3 coming from D-brane instantons wrapped on compact even-dimensional cycles in X. These presumably couple to the RR moduli in general, but not to the $b_3(Y)$ modes associated to gauge transformations of C_4. We will not pursue this line of thought further here, but instead postpone some speculative comments on this topic to the discussion section 7.2.

5.2 AdS/CFT interpretation: non-anomalous $U(1)$s

In this subsection and the next we present a holographic analysis of the fluctuations of section 4. This requires expanding the fluctuations at large r, which recall involves an asymptotic expansion of closed and co-closed forms on an asymptotically conical manifold. We first discuss the interpretation of the $b_3(Y)$ C_4 field fluctuations of type I. The argument generalises the discussion in [24] for the non-anomalous[24] baryonic $U(1)$ of the conifold model.

Equation (4.50) implies that the four-forms Ψ^I Hodge dual under Hg_X to the L^2

[24]Recall that the resolved conifold has $b_2(X) = b_3(Y) = 1$.

harmonic two-forms $\Phi^I \in \mathcal{H}^2_{L^2}(X \setminus \{x_1, \ldots, x_m\}, H g_X)$ satisfy

$$\Psi^I \sim r^{-3} \mathrm{d}r \wedge \alpha_0^{(3)I} \tag{5.2}$$

to leading order as $r \to \infty$. Here $\alpha_0^{(3)I} \equiv *_Y \alpha_0^{(2)I}$ are the $b_3(Y)$ harmonic three-forms on (Y, g_Y). Making a gauge transformation

$$\delta C_4 \to \delta C_4 + \frac{1}{2\mu_3} \mathrm{d}K \tag{5.3}$$

where K is a three-form with

$$K \sim r^{-2} \delta\vartheta^I \alpha_0^{(3)I} \ , \tag{5.4}$$

the first term in the fluctuation (4.38) may be rewritten

$$\delta C_4 \sim \frac{1}{\mu_3} r^{-2} \mathrm{d}\delta\vartheta^I \wedge \alpha_0^{(3)I} \ . \tag{5.5}$$

Note that the generator of the gauge transformation in (5.3) vanishes at infinity.

The holographic interpretation of this follows from comparing to the situation in an $\mathrm{AdS}_5 \times Y$ background. Here the harmonic three-forms $\alpha_0^{(3)I}$ lead to $b_3(Y)$ massless gauge fields A^I in AdS_5, via the ansatz

$$\delta C_4 = \frac{1}{\mu_3} A^I \wedge \alpha_0^{(3)I} \qquad \text{(AdS background)} \ . \tag{5.6}$$

These are dual to $b_3(Y)$ non-anomalous baryonic currents J^I. The linearised perturbations of the non-conformal background we have found therefore induce, near the UV AdS boundary $r \to \infty$, a perturbation of the gauge fields A^I which behaves as

$$\delta A^I \sim r^{-2} \mathrm{d}\delta\vartheta^I \ . \tag{5.7}$$

According to the AdS/CFT dictionary, the leading order terms of a perturbation in an (approximately) AdS_{d+1} space with metric

$$\mathrm{d}s^2_{\mathrm{EAdS}_{d+1}} = \frac{1}{r^2} \mathrm{d}r^2 + r^2 g_{\mathbb{R}^d} \tag{5.8}$$

admit different interpretations in the dual field theory [89, 23]. In general, for a massive p-form field $\mathcal{A}$, obeying

$$\mathrm{d} *_{d+1} \mathrm{d}\mathcal{A} - m^2 *_{d+1} \mathcal{A} = 0 \ , \tag{5.9}$$

we have the asymptotic expansion [90, 91]

$$\mathcal{A}_{i_1\dots i_p} \sim B_{i_1\dots i_p}\, r^{p+\Delta-d} + C_{i_1\dots i_p}\, r^{p-\Delta} \tag{5.10}$$

where

$$\Delta \;=\; \frac{d}{2} + \sqrt{\left(\frac{d}{2}-p\right)^2 + m^2} \tag{5.11}$$

is the conformal dimension of the dual operator – a formula which is perhaps more familiar for scalar fields, $p = 0$. The first term in (5.10) is non-normalisable, and therefore corresponds to changing the Lagrangian of the CFT. If this term vanishes and only the second normalisable term appears, then we are in a vacuum of the theory where the dual operator has a non-zero expectation value.

Notice that (5.7) is only computing the leading perturbation of A^I as $r \to \infty$ under the linearised perturbations of the background. In particular we are not[25] computing VEVs in the background itself – this would require a treatment as in [92, 93]. However, we may nonetheless naively read off conformal dimensions using these results. Setting $d = 4$, $p = 1$ and $m^2 = 0$ in (5.11) we see that the currents J^I dual to the vector fields A^I have conformal dimension $\Delta = 3$, which is of course correct for a conserved current. Equation (5.10) then implies that

$$\langle \delta J^I \rangle \;=\; \langle \mathrm{d}\, \delta\vartheta^I \rangle \ . \tag{5.12}$$

Standard field theory arguments then allow one to interpret the fields $\delta\vartheta^I$ as Goldstone bosons of the spontaneously broken $U(1)^{b_3(Y)}$ symmetry, as in [30]. Indeed, the classical Noether current for a complex scalar field ϕ is

$$J \;=\; \frac{i}{2}\left(\phi \mathrm{d}\bar{\phi} - \bar{\phi}\mathrm{d}\phi\right) \ . \tag{5.13}$$

If ϕ has a classical VEV ϕ_0, then we may write the Goldstone fluctuation as $\phi = \phi_0 \mathrm{e}^{i\delta\vartheta}$, and then

$$\delta J \;=\; |\phi_0|^2 \mathrm{d}\delta\vartheta \ . \tag{5.14}$$

[25]Note that computing VEVs in the gravity backgrounds, as opposed to their linearised variations under a change of vacuum, would presumably involve finding the general solution to certain (non-linear) ten-dimensional equations with prescribed behaviour at the UV boundary, substituting this into an appropriately holographically renormalised action, and then differentiating this once with respect to the boundary data (sources). Such a computation is clearly well beyond the scope of this paper. For a discussion in the case of $\mathrm{AdS}_5 \times S^5$, we refer the reader to [92, 93].

As mentioned in the previous subsection, supersymmetry should pair these Goldstone pseudo-scalars with scalar superpartners. These are clearly the $b_3(Y)$ non-normalisable Kähler deformations (4.86), although as stressed in section 4.4 we have not shown rigorously that these lead to modes satisfying massless scalar field equations in $\mathbb{R}^4$.

Note that the currents J^I are necessarily components of conserved current multiplets. The lowest components of these superfields may be identified as follows. If we follow the arguments in reference [23], a term of order $r^{-\Delta}$ in the metric, relative to the cone metric, indicates a VEV for a scalar operator of conformal dimension Δ. Since (4.85) is order r^{-2} relative to the cone metric, we see that these metric deformations should be dual to operators of conformal dimension $\Delta = 2$ in the dual field theory. This is precisely as expected for the scalar component of a massless vector multiplet in AdS_5. This leads one, as in the conifold model [23], to identify the $\Delta = 2$ scalar operators with the lowest component of the superfield that contains the non-anomalous baryonic currents J^I. Note these are necessarily axial currents, and thus the associated Goldstone bosons should be pseudo-scalars, precisely as we have found in the supergravity dual. The expression for the $b_3(Y)$ scalar operators follows from our discussion of quiver gauge theories in section 2:

$$\mathcal{U}^I \equiv \mathrm{Tr}\left[\sum_{a\in A} Q^I_a \Phi^\dagger_a \Phi_a\right] . \tag{5.15}$$

Here Q^I_a are the baryonic charges (2.53) with $q = q^I \in \mathbb{Z}^\chi$ being the $b_3(Y)$ linearly independent solutions to the anomaly cancellation condition (2.46). Recall that Φ_a are the bifundamental fields. The superfield version of (5.15) contains the conserved currents J^I as the $\theta\bar{\theta}$ components.

5.3 AdS/CFT interpretation: anomalous $U(1)$s

Our aim now is to discuss a possible holographic interpretation of the moduli fields associated to the remaining $U(1)^{2b_4(X)}$ flat directions, which correspond to anomalous baryonic symmetries. A key ingredient in the arguments of the previous subsection was the comparison of the asymptotic expansion of the fluctuating modes to a background $\mathrm{AdS}_5 \times Y$ analysis. In particular, it is well-known that the KK spectrum contains massless vector multiplets (so-called “Betti” multiplets) for each three-cycle in Y, which are dual to conserved currents. On the other hand, there is no general understanding of anomalous baryonic currents in the context of $\mathrm{AdS}_5 \times Y$ backgrounds.

In order to proceed in analogy with the previous subsection, we will first use the aymptotic expansions of harmonic forms to describe a set of KK modes in AdS_5 to compare with. As we discuss below, these modes must correspond to the set of lowest dimension operators in the dual SCFT acquiring non-zero VEVs. Based on supersymmetry considerations, and in analogy with the non-anomalous $U(1)$s, we will make some comments on the specific form of these operators.

Recall that in section 4.1 we have shown the existence of $b_4(X)$ harmonic two-forms $\psi^A \in \mathcal{H}^2_{L^2}(X, g_X)$ with the following asymptotic expansion (of type III^-, in the notation of appendix A) near infinity

$$\psi^A \ \sim \ \frac{1}{r^{1+\nu^{(1)A}}}\mathrm{d}\beta^{(1)A} - (1+\nu^{(1)A})\frac{1}{r^{2+\nu^{(1)A}}}\mathrm{d}r \wedge \beta^{(1)A} \ , \tag{5.16}$$

where we have defined $\nu^{(1)A} = \sqrt{1+\mu^{(1)A}}$, and $\beta^{(1)A}$ are co-closed one-forms on Y obeying

$$\Delta_Y \beta^{(1)A} \ = \ \mu^{(1)A}\,\beta^{(1)A} \qquad \text{(no sum)} \ . \tag{5.17}$$

The L^2 harmonic two-forms ψ^A are invariant under isometries of (Y, g_Y) that extend to isometries of the resolution (X, g_X). One can prove this using Theorem 3 of [94]. The latter states that Killing vector fields of linear growth (see [94] for the definition) leave the L^2 cohomology classes of (X, g_X) fixed. Killing vector fields on (X, g_X) that are Killing on (Y, g_Y) indeed have linear growth (their norms are $O(r)$), so the theorem applies. In fact the proof of the theorem shows that the Lie derivative of an L^2 harmonic form ψ is L^2 harmonic with L^2 cohomology class zero. However, again using the results of [66] in (2.34), this implies the Lie derivative is zero, and thus ψ is invariant under such isometries. Since r is also invariant, we see that $\beta^{(1)}$ is invariant under the isometries of (Y, g_Y) that extend to isometries of (X, g_X).

We must be slightly careful when writing expressions such as (5.16). Here we have picked an arbitrary basis for $\mathcal{H}^2_{L^2}(X, g_X) \cong H^2(X, Y; \mathbb{R})$. However, we are clearly free to choose a different basis. The issue is then that the set of asymptotic modes $\{\beta^{(1)A}\}$ in one basis is clearly not necessarily the same as in another basis, since the modes correspond only to the *leading order* behaviour of the harmonic forms at infinity. For example, by adding some multiple of the harmonic form with smallest $\mu^{(1)}$ to all the other forms, the leading asymptotic behaviour of all forms in the new basis will be the same. On the other hand, one can also clearly take linear combinations in such a way that the leading terms $\beta^{(1)A}$ are all different (although the set of eigenvalues

$\{\mu^{(1)A}\}$ may of course be degenerate). For, if any two harmonic forms have the same asymptotic $\beta^{(1)}$, one may simply pick a new basis which uses the difference of these forms as one of the basis elements; the latter will then have subleading behaviour. In this way, there exists a set of $b_4(X)$ distinct eigenfunctions $\{\beta^{(1)A}\}$ on (Y, g_Y).

Next, we will show that massive co-closed one-forms on (Y, g_Y) are dual to massive co-closed three-forms, in the sense that given any co-closed one-form $\beta^{(1)}$, one can construct a co-closed three-form $\beta^{(3)}$ with the *same* eigenvalue. In particular, we will prove the following:

$$*_Y\,\beta^{(3)} \;=\; \mathrm{d}\beta^{(1)} \qquad (5.18)$$

$$*_Y\beta^{(1)} \;=\; \frac{1}{\mu}\mathrm{d}\beta^{(3)}\ . \qquad (5.19)$$

The argument is rather simple. Consider a co-closed three-form obeying

$$\Delta_Y\beta^{(3)} \;=\; \mu\,\beta^{(3)}\ . \qquad (5.20)$$

We have for any two-cycle $S\subset Y$

$$\int_S *_Y\beta^{(3)} \;=\; \frac{1}{\mu}\int_S \mathrm{d}\,*_Y\,\mathrm{d}\beta^{(3)} \;=\; 0\ , \qquad (5.21)$$

where in the first equality we used (5.20) and the second is Stokes' theorem. Thus the closed form $*_Y\beta^{(3)}$ is exact, and we may write (5.18) for some $\beta^{(1)}$. Note that by the Hodge decomposition $\beta^{(1)}$ may be taken to be co-closed. Now define the one-form

$$\sigma \;=\; \Delta_Y\beta^{(1)} - \mu\beta^{(1)}\ . \qquad (5.22)$$

It is straightforward to verify that this is closed and co-closed, and so must be harmonic. However, since $b_1(Y)=0$, we have that $\sigma=0$. This shows that

$$\Delta_Y\beta^{(1)} = \mu\beta^{(1)} \qquad (5.23)$$

and also that the relation (5.19) holds. This proves that there exists two sets of $b_4(X)$ one-forms and three-forms $\{\beta^{(1)A},\beta^{(3)A}\}$ on (Y,g_Y), with pairwise equal eigenvalues $\{\mu^A\}$.

Given these forms, we can perform a KK reduction and obtain a corresponding set of modes in AdS$_5$. Let us describe roughly the types of modes that are obtained from these massive forms. Consider, for instance, Kaluza-Klein reduction via the ansätze

$$\begin{aligned}\delta C_4 &= \tfrac{1}{\mu_3}\mathcal{A}^A\wedge\beta^{(3)A}\\ \delta C_2 &= \tfrac{1}{\mu_1}\mathcal{C}^A\wedge\beta^{(1)A}\end{aligned}\qquad \text{(AdS background)}\ . \qquad (5.24)$$

Then, for example, if $\beta^{(3)}$ is a co-closed massive eigenfunction of Δ_Y with eigenvalue μ, satisfying (5.20), we have

$$\delta G_5 = \mathrm{d}\mathcal{A} \wedge \beta^{(3)} - \mathcal{A} \wedge \mathrm{d}\beta^{(3)} \ . \tag{5.25}$$

Then the linearised equation of motion implies

$$\begin{aligned} \mathrm{d} *_5 \mathrm{d}\mathcal{A} - \mu *_5 \mathcal{A} &= 0 & (5.26) \\ \mathrm{d} *_5 \mathcal{A} &= 0 \ , & (5.27) \end{aligned}$$

where $*_5$ is the Hodge operator on AdS$_5$. These are precisely the Proca equations for a massive vector field. A similar consideration applies for reduction of $\mathrm{d}\delta C_2$ on a massive one-form; the $\mathcal{A}^A$ and $\mathcal{C}^A$ in (5.24) thus obey these equations. Note that the Proca equations are gauge-fixed. More generally we should write

$$\delta C_4 = \mathcal{A} \wedge \beta^{(3)} + \varrho\, \mathrm{d}\beta^{(3)} \tag{5.28}$$

so that a gauge transformation

$$\delta C_4 \rightarrow \delta C_4 + \mathrm{d}(f\beta^{(3)}) \tag{5.29}$$

leads to the transformations

$$\mathcal{A} \rightarrow \mathcal{A} + \mathrm{d}f, \qquad \varrho \rightarrow \varrho + f \ . \tag{5.30}$$

The ansatz (5.28) then leads to the Stückelberg action in AdS$_5$, where the scalar ϱ is the Stückelberg field. Of course, by a gauge transformation this scalar may be set to zero, and one recovers the Proca equations.

However, the above analysis is certainly too naive. The reason for this is that massive modes in AdS$_5$ mix, leading to a non-trivial mass matrix. The physical masses are then the eigenvalues of this mass matrix. This occurs even for scalars fields, for example as discussed in the appendix of [23]. In the case at hand, the RR four-form mixes with metric modes, which recall we have not analysed in any detail. The relevant mass matrix has been worked out for the S^5 case in [95], although their results are easily generalised to a general Sasaki-Einstein manifold (Y, g_Y). The mixing of metric and RR C_4 modes due to massive co-closed three-forms $\beta^{(3)}$ on Y of eigenvalue μ gives rise to a mass matrix (see equation (2.26) of [95])

$$m^2 = \begin{pmatrix} \mu + 8 & 16\mu \\ 1 & \mu \end{pmatrix} \ . \tag{5.31}$$

Here the bottom right hand corner corresponds to the mass in the Proca equation (5.26), as one sees explicitly from the analysis in [95]. The eigenvalues of this matrix are given by

$$m_{\pm}^2 \;=\; \mu + 4 \pm 4\sqrt{\mu+1}\ . \tag{5.32}$$

On the other hand, there is no such mixing for the C_2 and B field modes. In fact these combine straightforwardly into complex modes.

The AdS/CFT dictionary maps these $2b_4(X)$ massive vector fields to some vectorial operators of conformal dimension given by (5.11) in the dual CFT. For the vector fields $\mathcal{A}^A$, picking the positive branch m_+^2 gives rise to conformal dimensions

$$\Delta(\mathcal{A}^A) \;=\; 4 + \sqrt{1+\mu^A}\ . \tag{5.33}$$

Notice that, rather remarkably, the square root for the positive branch factorises, giving the simple surd in (5.33).

On general grounds, these vector fields must of course fit into some supermultiplets in AdS_5. However, to our knowledge, there is no general understanding of the structure of the KK spectrum for a Sasaki-Einstein five-manifold, other than S^5 and $T^{1,1}$. We thus proceed slightly indirectly to gain some intuition on the structure of these multiplets. As we discussed earlier, supersymmetry naturally pairs C_4 with the metric, and C_2 with the B field. Using the argument of [23] one can then show that there are scalar modes s^A associated to the metric with conformal dimensions

$$\Delta(s^A) \;=\; 3 + \sqrt{1+\mu^A}\ . \tag{5.34}$$

These may be read off from the asymptotic expansion (4.82). It is satisfying to see that these conformal dimensions differ precisely by 1 from (5.33). We expect these scalar metric modes to arise from symmetric tensor harmonics on Y.

Moreover, as we show in appendix B, for each of the one-forms $\beta^{1(A)}$ one can also construct a scalar eigenfunction of the Laplacian on (Y, g_Y), defined as

$$f^A \;=\; \beta^{1(A)} \lrcorner\, \eta\ . \tag{5.35}$$

Here η is the canonically defined Killing one-form on a Sasaki-Einstein manifold, metric dual to the Reeb vector field – see, for example, [7]. In particular, we recall that the cone metric on $C(Y)$ has Kähler form

$$\omega_{C(Y)} \;=\; \frac{1}{2}\mathrm{d}(r^2\eta)\ . \tag{5.36}$$

As shown in appendix B, the functions f^A obey

$$\Delta_Y f^A \;=\; E^A f^A \qquad \text{(no sum)} \tag{5.37}$$

where the eigenvalues are

$$E^A \;=\; \mu^A - 2 - 2\sqrt{1+\mu^A}\;. \tag{5.38}$$

These may be used in the KK reduction on Y. In particular, they give rise to a set of $b_4(X)$ scalar fields π^A, by expanding the trace of the metric on Y. These modes mix with scalar modes of C_4, and the eigenvalues of the 2×2 mass matrix are [95]

$$m^2_\pm \;=\; E + 16 \pm 8\sqrt{E+4}\;. \tag{5.39}$$

The AdS/CFT mass-dimension formula then gives

$$\Delta(\pi^A) \;=\; 5 + \sqrt{1+\mu^A}\;, \tag{5.40}$$

where, again, picking the positive branch in (5.39), the surds have simplified rather remarkably. Notice that this value differs precisely by 1 and 2 with respect to $\Delta(\mathcal{A}^A)$ and $\Delta(s^A)$.

This structure, and the comments we shall make below, are suggestive that the modes $(\mathcal{A}^A, s^A, \pi^A)$ may be part of *massive vector multiplets* in AdS_5. On the other hand, the reduction of C_2 and B naturally leads to *complex* massive vector fields in AdS_5, with conformal dimension

$$\Delta(\mathcal{C}^A) \;=\; 2 + \sqrt{1+\mu^A}\;, \tag{5.41}$$

as there is no mass matrix to diagonalise in this case. These should also be part of AdS_5 multiplets, but presently it is not clear to us of which kind.

We do not know if the eigenvalues μ^A are computable in practice in general. *A priori*, the set of eigenvalues also depends on the resolving Calabi-Yau manifold (X, g_X), and thus on the Kähler class, although we have not shown this dependence in the notation[26]. Nevertheless, there is a class of Sasaki-Einstein manifolds where one can determine the eigenvalues explicitly. Note that on any Sasaki-Einstein manifold (Y, g_Y) the contact

[26]Notice, however, that these eigenvalues are simply related to eigenvalues of the scalar Laplacian via (5.38). In particular, $\mu^A = E^A + 4 + 2\sqrt{E^A+4}$.

one-form η is a massive one-form with eigenvalue[27] $\mu = 8$, and gives rise to a L^2-normalisable harmonic two-form on any Calabi-Yau cone

$$\psi_0 \;=\; \mathrm{d}(r^{-4}\eta) \; . \tag{5.42}$$

There is a certain class of models for which there is a Kähler perturbation of a Ricci-flat Kähler metric on X which is exactly the $\mu = 8$ mode (5.42). The corresponding Calabi-Yau singularities are complex cones over Kähler-Einstein surfaces. For example, the Fano surfaces $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$, $dP_0 = \mathbb{CP}^2$ and del Pezzo surfaces with between 3 and 8 blow-ups are of this form. The Calabi-Yau cone singularity may be resolved using the Calabi ansatz [51]. This is an asymptotically conical Ricci-flat Kähler metric on the canonical bundle over the Fano surface, and is completely explicit, up to the Kähler-Einstein metric on the Fano. Thus $b_4(X) = 1$ in these models. The subleading behaviour to the Kähler form on the cone is given by (5.42), and relative to the cone metric this is precisely order r^{-6}, which indicates a dual scalar operator of conformal dimension $\Delta(s) = 6$.

These backgrounds were discussed in an AdS/CFT context in [43]. In fact, in the latter reference it was shown that in such backgrounds there exist two universal KK scalar modes, coming from reduction of the metric, which (in our notation) have masses $m^2(s) = 12$, $m^2(\pi) = 32$, thus giving $\Delta(s) = 6$, $\Delta(\pi) = 8$, respectively. In this case there is one massive mode $\mathcal{A}$ and one massive mode $\mathcal{C}$. These are dual to vector operators with conformal dimensions given by (5.33) and (5.41), which give $\Delta(\mathcal{A}) = 7$ and $\Delta(\mathcal{C}) = 5$, respectively.

We may now proceed, by analogy with the discussion of the non-anomalous $U(1)$ symmetries, to give a holographic interpretation of the asymptotic expansions of the C_4 field modes of type III$^+$ and the normalisable Kähler perturbations, and the C_2 and B field modes of type III$^-$. In terms of the coordinate r, the asymptotic expansions of the harmonic forms used to construct the form field modes take a similar form, namely

$$\delta C_4 : \qquad \Psi^{b_3(Y)+A} \;\sim \mathrm{d}(r^{-1-\sqrt{1+\tilde{\mu}^A}}\tilde{\beta}^{(3)A}) \tag{5.43}$$

$$\delta C_2 : \qquad \psi^{A} \;\sim \mathrm{d}(r^{-1-\sqrt{1+\mu^A}}\beta^{(1)A}) \; . \tag{5.44}$$

[27] In fact it is straightforward to show that all Killing one-forms are co-closed massive one-forms with eigenvalue $\mu = 8$, and that this is a strict lower bound on the spectrum of such massive one-forms. That is, $\mu \geq 8$ with equality if and only if the eigenvalue is associated to a Killing one-form. This is similar to the Lichnerowicz bound used in [2], and is proven in [96]. We note in passing that one can also obtain a strict lower bound on the *second* smallest eigenvalue μ_* by using the Lichnerowicz bound on the smallest non-zero eigenvalue of the scalar Laplacian. In particular, since $E^A \geq 5$, with equality only for $(Y, g_Y) = S^5$, we have that $\mu_* > 15$. This gives corresponding bounds on the conformal dimensions of dual operators. For example, $\Delta(s_*) > 7$.

Here the three-forms $\tilde{\beta}^{(3)A}$ are *a priori* unrelated to the one-forms $\beta^{(1)A}$. However, it would certainly be rather natural if $\tilde{\beta}^{(3)A} = \beta^{(3)A}$, where $\beta^{(3)A}$ are the dual set of forms, in the sense of (5.18), (5.19): if indeed the metric modes s^A introduced above are in the same multiplets as the vectors coming from the asymptotic C_4 modes, then this is necessary by supersymmetry. We do not have a direct geometric proof, but will formally drop the tildes in any case. Again, like many of the issues raised in this section, we will leave further study for future work.

We may perform gauge transformations analogous to (5.3) to obtain the following[28] leading behaviour at large r:

$$\delta C_4 \sim -\frac{1}{\mu_3} r^{-1-\sqrt{1+\mu^A}} \mathrm{d}\delta\vartheta^A \wedge \beta^{(3)A} \tag{5.45}$$

and a similar expression for C_2 and B. Following the logic of the previous subsection, by comparison with the AdS_5 background one concludes that the massive vector modes have leading behaviour

$$\delta \mathcal{A}^A \sim -r^{-1-\sqrt{1+\mu^A}} \mathrm{d}\delta\vartheta^A \ , \tag{5.46}$$

thus indicating VEVs for dual vector operators of conformal dimensions $2 + \sqrt{1+\mu^A}$. However, this is too naive. One may correctly read off the conformal dimension from the expansion of C_2 and B, whereas one obtains the incorrect answer this way from (5.46). As we have explained, this is because the C_4 modes mix with metric modes that we have not fluctuated in the supergravity solution.

Given the discussion of non-anomalous currents in the previous subsection, it seems rather natural to speculate that the massive vector fields $\mathcal{A}^A$ and $\mathcal{C}^A$ should be dual to the $2b_4(X)$ anomalous baryonic currents. At least for the C_4 modes, this may be further motivated by the fact that these modes appear to be part of vector multiplets in AdS_5. These, as we discuss below, are natural candidates to be the gravity dual of anomalous currents [97]. However, the eigenvalues μ^A are just the *leading* terms in the asymptotic expansions. Therefore they correspond to the operators with lowest anomalous dimensions, in an infinite tower of operators getting VEVs (see *e.g.* [23] and [30]). It is then possible that the anomalous baryonic currents are part of vector multiplets but might have larger anomalous dimensions, and thus correspond to subleading terms in the expansions (*cf.* discussion around equations (A.8), (A.9) in appendix A).

[28] In the following we ignore the other term in C_4, which involves the two-forms δa^A. In fact self-duality of the RR fields requires a similar fluctuation in C_6, but this is not important for our analysis.

In the quiver gauge theory, the anomalous baryonic currents may be defined in exactly the same was as the non-anomalous currents, by taking linear combinations of bifundamental bilinears[29] $\Phi_a^\dagger \Phi_a$. Recall that there are $2b_4(X)$ such linear combinations that are anomalous. In particular, the lowest component scalar fields are given by

$$\mathcal{U}_q \equiv \mathrm{Tr}\left[\sum_{a\in A} Q_a \Phi_a^\dagger \Phi_a\right] \tag{5.47}$$

where $Q_a = q_{t(a)} - q_{h(a)}$ is the baryonic charge of Φ_a under the baryonic $U(1)_q$ given by the charge vector $q \in \mathbb{Z}^{\chi-1}$. Then, classically, we have the relation

$$\mathcal{U}_q = \sum_{v\in V} q_v \zeta_v \ . \tag{5.48}$$

This follows from multiplying (2.68) by q_v, taking the trace, and summing over the nodes in the quiver $v \in V$. Thus we see that, classically, the VEV of $\mathcal{U}_q$ is simply an "FI parameter". Of course, this is a completely natural extension of the situation for the non-anomalous currents. However, the operators (5.15) are protected, while *a priori* (5.47) are not known to be protected. Let us denote the corresponding Noether current for $U(1)_q$ by J_q. Classically J_q is conserved for all q, but in the quantum theory we have

$$\partial_\mu J_q^\mu \propto \sum_{v\in V} c_q^v \chi_v \equiv \chi_q \tag{5.49}$$

where the anomaly coefficient is (as in (2.46))

$$c_q^v = \sum_{a\in A|h(a)=v} n_{t(a)} q_{t(a)} - \sum_{a\in A|t(a)=v} n_{h(a)} q_{h(a)} \ . \tag{5.50}$$

Here

$$\chi_v = *_4 \mathrm{Tr}\, F_v \wedge F_v \tag{5.51}$$

is an operator constructed from the curvature of the $SU(n_v)$ gauge field F_v corresponding to the node $v \in V$. As we argued in section 2.3, there is a $b_3(Y)$-dimensional space of charges q for which $c_q^v = 0$ for all v.

The currents J_q sit in (non-conserved) current superfields $\mathcal{J}_q$ in an $\mathcal{N}=1$ supersymmetric theory, while the operators χ_v are part of chiral superfields $\mathcal{O}_v = \mathrm{Tr}\, W_{v\alpha} W_v^\alpha$,

[29]We drop the terms e^V from the expressions.

where $W_{v\alpha}$ is a gauge superfield for the node $v \in V$. In particular, the lowest component of $\mathcal{J}_q$ is given by (5.47). Then the anomaly equation (5.49) becomes the superfield equation

$$\bar{D}^2 \mathcal{J}_q \propto \sum_{v\in V} c_q^v \, \mathcal{O}_v \equiv \mathcal{O}_q \ . \tag{5.52}$$

In the supergravity dual, this equation may be related to a Higgs mechanism in the bulk[30] [97]. The current J_q is dual to a massive vector field, whose transverse mode precisely eats the scalar (the Stückelberg ϱ in the previous paragraph) dual to the anomaly term χ_q. There are then four independent bosonic scalar operators [98], namely the lowest component of $\mathcal{J}_q$ and three independent components in $\mathcal{O}_q$ (the complex gaugino bilinears and the $\mathrm{Tr}F_v^2$ terms). Together with the massive vector, these are the correct number to fill out a massive vector multiplet in AdS_5. Notice that this discussion clarifies that the axions for the anomalous $U(1)$s are not physical degrees of freedom, mirroring the familiar situation reviewed in section 2.3. In particular, it also clarifies that the RR moduli fields have a very different origin in the gravity set-up and in the large volume worldvolume setting [63] of section 2.3.

One can also heuristically understand current non-conservation from a holographic point of view. As discussed in [97], and also in [99] in a different context, in the dual gravity description one introduces an AdS_5 gauge field $\mathcal{A}$ and a scalar ϱ which couple to J and χ on the holographic boundary, respectively, via a coupling

$$\int_{\mathbb{R}^4} (\mathcal{A}_\mu J^\mu + \varrho\, \chi)\, \mathrm{vol}_4 \ . \tag{5.53}$$

The AdS_5 gauge transformation

$$\mathcal{A} \to \mathcal{A} + \mathrm{d}f, \qquad \varrho \to \varrho + f \tag{5.54}$$

then immediately leads to the anomaly equation (5.49) as the associated anomalous Ward identity for the symmetry (5.54). Of course, the gauge field $\mathcal{A}$ here should be identified with a massive gauge field in (5.24), and the gauge transformation (5.54) is the same as the Stückelberg transformation (5.30) which results from RR gauge transformations.

Finally, we return to the interpretation of the fluctuations $\delta\theta^A$ and $\delta\varphi^A$. Assuming that the asympotic expansions contain modes $\mathcal{A}^A$ and $\mathcal{C}^A$ which are dual to the

[30] We are grateful to Y. Tachikawa and F. Yagi for clarifying comments.

baryonic currents, we may tentatively interpret the $2b_4(X)$ massless (in $\mathbb{R}^4$) modes as "pseudo-Goldstone" bosons. Indeed, classically we precisely expect to find these modes in the spectrum. However, since the corresponding symmetries are anomalous, Goldstone's theorem does not guarantee that these modes exist and are massless in the quantum theory, which is why we refer to them as "pseudo-Goldstone" bosons. As we discuss in the concluding section 7.2, there might well be corrections to the supergravity backgrounds of section 3, namely D-brane instanton corrections, which lift these massless fields.

6 Baryon condensates

In this section we describe a Euclidean D3-brane calculation that conjecturally determines the holographic condensates of baryon operators in AdS/CFT. Some basic parts of this calculation were carried out in [22], extending and generalising[31] the warped resolved conifold example in [30]. Our aim here is to abstract this to a fairly general prescription for computing baryon condensates in AdS/CFT, and demonstrate that the result has the features one expects.

Recall that, in the quiver gauge theories of section 2, the gauge-invariant scalar BPS operators may be divided into two sets: the meson operators and the baryon operators. Classically these may be identified with the holomorphic functions on the VMS $\mathcal{M}$. The meson operators are distinguished by being invariant under the baryonic group $U(1)^{\chi-1}$ which acts on $\mathcal{M}$, whereas by definition a baryon operator is charged under this group. In the more mathematical language of section 2.5, the baryon operators are the regular functions on the space of F-term solutions $\mathcal{Z}$ that are semi-invariants (but not invariants) under $G_{\mathbb{C}}$, whereas the meson operators are the invariants under $G_{\mathbb{C}}$. In fact there are very general theorems that state that the meson operators in quiver gauge theories may be written in terms of traces of bifundamental fields [102], whereas the baryon operators may be written in terms of generalised determinants [103]. The classical VEV of an operator $\mathcal{O}$ at the point $p \in \mathcal{M}$ is simply $\mathcal{O}(p) \in \mathbb{C}$.

Of course, the main interest is in the quantum theory at strong coupling. Using AdS/CFT we may identify the space of vacua of the strongly coupled theory with the gravity backgrounds of section 3. In principle, one should be able to compute the condensate of any operator $\langle \mathcal{O} \rangle_p$ in a given vacuum p. It is not clear (to the authors, at least) how one would compute such a condensate directly in quantum field theory.

[31] For related work on the conifold, see [100] and [101].

However, the AdS/CFT correspondence implies that such one-point functions may be computed in the gravity dual of section 3, essentially as a geometric computation in the appropriate large N limit.

The method for computing meson condensates in these gravity backgrounds is more straightforward [23], at least in principle. This is essentially because meson operators are dual to supergravity modes, for which the AdS/CFT correspondence is very well developed. See [92, 93] for a discussion of state-of-the-art techniques for computing holographic VEVs of meson operators in asymptotically AdS_5 ten-dimensional geometries. In contrast, baryon operators are dual to D-brane states, and the method for computing correlation functions of such operators is both conceptually and technically much harder. In this section we elaborate on a method for computing baryonic condensates in the backgrounds described thus far. However, before proceeding to this proposal, we first recall the AdS/CFT description of baryons as wrapped D3-branes.

6.1 Baryon operators in AdS/CFT

Various properties of SCFTs with Sasaki-Einstein duals may be studied in terms of the geometry of the dual background. Particularly well-understood are the operators dual to supergravity modes, where the precise map from geometry to field theory was outlined in the original papers [28, 29]. In the remainder of the paper we are interested in baryon operators, which are dual to D-brane states. Consider a compact three-submanifold $\Sigma \subset Y$. By wrapping a D3-brane on this submanifold we effectively obtain a particle in AdS. This particle will be BPS when the wrapped D3-brane is supersymmetric. In particular, an argument similar to that in section 6.3 implies that a necessary condition is that the cone $C(\Sigma) \subset C(Y)$ is a complex submanifold, or divisor. The D3-brane also carries a worldvolume gauge field with two-form field strength $M = 2\pi\alpha' F - B$, as described in section 2.3. For a D3-brane wrapping $\mathbb{R}_t \times \Sigma$, supersymmetry requires this gauge field to be *flat*, so $M = 0$. Again, this essentially follows from the more general discussion in section 6.3. Flat $U(1)$ gauge fields on Σ are classified, up to gauge equivalence, by the group $H^1(\Sigma; U(1))$ – see the discussion in appendix C. Since $b_1(\Sigma) = 0$ for the three-submanifolds of interest, the long exact coefficient sequence implies that $H^1(\Sigma; U(1)) \cong H^2_{\mathrm{tor}}(\Sigma; \mathbb{Z}) \cong H_1(\Sigma; \mathbb{Z})$. Thus, as originally pointed out in [26], if Σ has non-trivial first homology group, one can turn on distinct flat connections on the worldvolume of a wrapped D3-brane. These flat connections are defined on torsion line bundles over Σ, which we generically denote by

L. Thus $c_1(L) \in H^2_{\mathrm{tor}}(\Sigma;\mathbb{Z})$.

In [26, 25, 27] such wrapped D3-branes were interpreted as *baryonic particles*. The dual operator that creates such a baryonic particle will be referred to as a *baryon operator*. We then have a correspondence

$$(\Sigma, L) \quad \longleftrightarrow \quad \mathcal{B}(\Sigma, L)\ , \tag{6.1}$$

where $\mathcal{B}(\Sigma, L)$ denotes the baryon operator associated to the pair (Σ, L). This also leads one to identify the non-anomalous $U(1)$ baryonic symmetries in the field theory as arising from the topology of Y. As we recalled in section 5.2, massless fluctuations of the RR four-form potential C_4 in the background $\mathrm{AdS}_5 \times Y$ may be expanded in a basis of harmonic three-forms on (Y, g_Y) via the ansatz (5.6). This gives rise to $b_3(Y)$ massless gauge fields A^I in AdS_5 which are dual to the non-anomalous baryonic currents J^I. The baryonic charge of a baryonic particle, arising from a three-submanifold Σ, with respect to the Ith baryonic $U(1)$ is then given by

$$Q^I[\mathcal{B}(\Sigma, L)] \;=\; \int_\Sigma \alpha_0^{(3)I}\ . \tag{6.2}$$

For fixed Σ the operators $\mathcal{B}(\Sigma, L)$, where L is a torsion line bundle on Σ, thus all have equal *non-anomalous* baryonic charge (6.2). They also have equal R-charge, where the latter is determined by the volume of Σ via

$$R[\mathcal{B}(\Sigma, L)] = \frac{N\pi \mathrm{vol}(\Sigma)}{3\mathrm{vol}(Y)}\ . \tag{6.3}$$

Several comments are now in order. Firstly, note that we have two geometric definitions, or at least identifications, of baryon operators: firstly, as holomorphic functions on $\mathcal{M}$; and secondly, as dual objects to a pair (Σ, L). In particular, we have asserted in this subsection that to every (Σ, L) there is a baryon operator $\mathcal{B}(\Sigma, L)$ which we may realise classically as a holomorphic function on the classical gauge theory moduli space $\mathcal{M}$. Although both identifications are geometric, the general relation between them is completely unobvious. Having said that, the level zero mesonic moduli space $\mathcal{M}(0) = \mathrm{Sym}^N C(Y)$ is an affine GIT quotient of $\mathcal{M}$ by the complexified baryonic group $(\mathbb{C}^*)^{\chi-1}$. Thus the complex geometry of $Z = C(Y)$ is certainly contained in $\mathcal{M}$. In fact, for $N = 1$, a baryon operator of definite charge $q \in \mathbb{Z}^{\chi-1}$ under $(\mathbb{C}^*)^{\chi-1}$ defines an ample divisor in the mesonic moduli space $\mathcal{M}(q)$. This follows from the discussion in section 2.5. We then know from [21] and [73] that for orbifold gauge theories and toric quiver gauge theories described by dimers $\pi : \mathcal{M}(q) \to Z$ is a crepant resolution of the cone $Z = C(Y)$. Thus we may take $X = \mathcal{M}(q)$ as an underlying complex manifold for the gravity backgrounds of section 3. Despite recent papers [104, 105, 31] that count baryonic operators in simple examples, this correspondence is still very poorly understood geometrically. The main motivations for the identification come from general AdS/CFT arguments and the fact that in examples one sees that it works. For example, the quiver gauge theories in [18] were *deduced* from the above identification of certain set of special baryon operators with pairs (Σ, L).

The second comment to make is that the set of baryon operators of the form $\mathcal{B}(\Sigma, L)$ is very small. Indeed, there are obvious generalisations of the construction outlined above. For example, rather than wrap a single D3-brane on Σ, we may wrap n D3-branes. The worldvolume gauge theory is then a $U(n)$ gauge theory, and presumably supersymmetry again requires the connection to be flat. A flat $U(n)$ connection is determined by its holonomies, which define a homomorphism

$$\rho : \pi_1(\Sigma) \to U(n) \ . \tag{6.4}$$

Gauge transformations act by conjugation, and thus the flat $U(n)$ connections are in 1-1 correspondence with

$$\mathrm{Hom}(\pi_1(\Sigma) \to U(n))/\mathrm{conjugation} \ . \tag{6.5}$$

For example, if $\pi_1(\Sigma) \cong \mathbb{Z}_r$ then the number of flat $U(n)$ connections on Σ is given by the number of partitions of n into r non-negative integers:

$$n = \sum_{i=1}^{r} k_i, \qquad k_i \in \{0, 1, 2, \ldots\} \ . \tag{6.6}$$

This is easy to see: there are r irreducible representations $\mathcal{R}_i$ of $\mathbb{Z}_r$, which are all one-dimensional. If we identify $\mathbb{Z}_r$ with the group of rth roots of unity then a root $\zeta \in \mathbb{Z}_r \subset U(1) \subset \mathbb{C}$ is sent to

$$\mathcal{R}_i : \zeta \to \zeta^i \ , \tag{6.7}$$

where we may regard $i \in \{1, \ldots, r\}$. An n-dimensional representation of $\mathbb{Z}_r$ may then be constructed from the r-vector $\mathbf{k} = \{k_i\}_{i=1}^r$. Specifically,

$$\mathcal{R}_{\mathbf{k}} : \zeta \to \mathrm{diag}(\zeta^1, \ldots, \zeta^1, \zeta^2, \ldots, \zeta^2, \ldots\ldots, \zeta^r, \ldots, \zeta^r) \in U(n) \tag{6.8}$$

where ζ^i occurs k_i times. Notice that all orderings of the entries in (6.8) are equivalent under conjugation. Indeed, a little thought shows that all flat $U(n)$ connections on Σ, using the identification (6.5), may be written in the form (6.8).

These D3-brane states may be interpreted naturally in the field theory as follows. We have r torsion line bundles L_i and thus r BPS baryon operators $\mathcal{B}(\Sigma, L_i)$. The BPS baryon operators form a ring, and thus we may multiply them. n D3-branes wrapped on Σ have n times the non-anomalous baryonic charge and R-charge of a single D3-brane wrapped on Σ. The candidate dual baryon operators are thus given by

$$\mathcal{B}(\Sigma, \mathbf{k}) \equiv \prod_{i=1}^{r} \mathcal{B}(\Sigma, L_i)^{k_i} \ . \tag{6.9}$$

where in order to have the correct non-anomalous charges we precisely require (6.6) to hold.

We may also consider Σ that have more than one connected component, say $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_k$, where each Σ_i is connected. If the Σ_i are all pairwise disjoint then presumably these may be treated precisely as above. However, we may also consider self-intersecting D3-branes, where $\Sigma_i \cap \Sigma_j \neq \emptyset$ for $i \neq j$. Understanding the effective theory on such a D3-brane, and thus counting its supersymmetric configuations, seems quite challenging. For example, there may be massless degrees of freedom, coming from massless strings between each component, associated to the intersection.

However, even this does not exhaust all baryonic operators that one may construct in the gauge theory. Presumably, the complete spectrum may be obtained by quantising the moduli space of all BPS D3-branes [106], which include *time-dependent*, rather than static, wrapped D3-branes. In this paper for simplicity we restrict attention to static singley-wrapped D3-branes on a compact smooth connected Σ. As we shall see, understanding the one-point functions of such operators in our non-conformal backgrounds is already quite challenging.

6.2 Baryon condensates: outline of the prescription

In the remainder of the paper we present a prescription for computing the VEVs of baryon operators which may be represented by a pair of data consisting of a smooth supersymmetric three-submanifold Σ, and a torsion line bundle L on Σ. The first step in performing any holographic computation is to extend the data from the boundary ($r = \infty$ in $\mathrm{AdS}_5 \times Y$) to the "bulk" (essentially $\mathbb{R}^4 \times X$). One must then identify an object, depending on the boundary data, that has appropriate transformation properties under the symmetries of the problem. Given that a baryonic particle is dual to a D3-brane with worldvolume $\mathbb{R}_t \times \Sigma$, a natural candidate for computing the VEV of the operator that creates such a particle is a *Euclidean D3-brane* that wraps a divisor $D \subset X$ with boundary $\partial D = \Sigma \subset Y$. More precisely, one should perform a path integral for such a Euclidean D3-brane, in a given background geometry, with fixed boundary conditions:

$$\langle \mathcal{B}(\Sigma, L) \rangle \;=\; \int_{\partial D = \Sigma} \mathcal{D}\Psi \exp(-S_{ED3}) \;\approx\; \sum \exp(-S_{ED3}^{\mathrm{on-shell}})\;. \tag{6.10}$$

In other words, we compute the partition function for a non-compact D3-brane in the background supergravity solution, where the boundary conditions for the D3-brane are held fixed.

An analogous prescription is applied in the case of computing expectation values of Wilson loop operators [107, 108]. Here one is instructed to compute the on-shell action of a Euclidean string with worldsheet whose boundary is the loop itself. For baryon operators, this idea was first proposed[32] in the context of a warped resolved conifold model in [30], in which case the worldvolume gauge field is zero. The purpose of the remainder of this paper will be to make the rough formula (6.10) more precise. The calculation that we will describe computes the semi-classical approximation to the partition function of a Euclidean D3-brane. This leads to the saddle-point sum on the right hand side of (6.10). In principle one should also compute the one-loop contributions to this saddle-point approximation. However, our main focus here is on understanding the worldvolume gauge field instantons, and also the coupling of RR fields to the D3-brane. In particular the one-loop terms do not involve the RR fields.

Since the D3-brane worldvolume is *non-compact*, the action is not invariant under gauge transformations of the RR fields. More precisely, the phase is not a gauge-invariant object since it will depend on the choice of reference gauge on the boundary for the background RR fields. However, even in the classical gauge theory the overall phase of a particular baryon operator is not physical. It clearly makes no sense to ask what the phase is of some baryon operator $\mathcal{O}$, since by acting with a baryonic symmetry this operator is equivalent to $e^{i\alpha}\mathcal{O}$ for any constant phase α. Physically there is no way to fix this ambiguity. However, it *does* make sense to ask what the *relative* phase of the VEV of a baryon operator is at different points in the VMS $\mathscr{M}$, since the above ambiguity cancels. In the gravity calculation, the condensate in (6.10) of course depends on the particular gravity background, which is a point in the gravity moduli space $p \in \mathscr{M}^{\mathrm{grav}}$; specifying p involves specifying a complex manifold X, a Kähler class for the asymptotically conical Calabi-Yau metric g_X on X, the positions of the stacks of N D3-branes, and the RR fields and B field. We are then more precisely

[32] See also [109].

computing $\langle \mathcal{B}(\Sigma, L)\rangle_p$. However, since the overall phase is not physical, the object of interest is really the relative phase

$$\arg\langle \mathcal{B}(\Sigma, L)\rangle_{p,p_0} \;=\; \arg\langle \mathcal{B}(\Sigma, L)\rangle_p - \arg\langle \mathcal{B}(\Sigma, L)\rangle_{p_0} \;, \tag{6.11}$$

where p_0 is any fixed choice of generic (smooth) point $p_0 \in \mathscr{M}^{\mathrm{grav}}$. One of our main results is that the quantity (6.11) is in fact gauge-invariant. The key point will be to show that under gauge transformations the "bare" condensate $\langle \mathcal{B}(\Sigma, L)\rangle_p$ transforms via terms which depend only on the boundary data, which then cancel in (6.11).

In [22] we gathered some preliminary evidence for the validity of the general prescription (6.10). In particular, we showed that the right hand side transforms with the correct phase under gauge transformations of C_4 of the type (5.1), which are dual to the non-anomalous $U(1)^{b_3(Y)}$ baryonic symmetries. Specifically,

$$\delta\,\langle \mathcal{B}(\Sigma, L)\rangle \;=\; \exp(i\beta_I Q^I[\mathcal{B}(\Sigma, L)])\,\langle \mathcal{B}(\Sigma, L)\rangle \;. \tag{6.12}$$

We also explained that the logarithmically divergent part of the Born-Infeld action

$$S_{BI} \;=\; T_3 \int_D \mathrm{d}^4\sigma \sqrt{\det g_D} H \;, \tag{6.13}$$

is in general proportional to the R-charge (6.3), and hence also conformal dimension, of the baryon operator $\mathcal{B}(\Sigma, L)$, as expected from the AdS/CFT dictionary. This conformal dimension is given by

$$\Delta(\Sigma) \;=\; \frac{N\pi \mathrm{vol}(\Sigma)}{2\mathrm{vol}(Y)} \;. \tag{6.14}$$

To obtain a finite contribution from (6.13) one can define the following quantity

$$S^{\text{finite}}_{BI} \;=\; \lim_{r_c\to\infty}\left[T_3 \int_{D_{r_c}} \mathrm{d}^4\sigma \sqrt{\det g_D} H - T_3 L^4 \int_\Sigma \mathrm{dvol}[\Sigma]\;\log r_c\right] \;, \tag{6.15}$$

where D_{r_c} is a cut-off compact four-manifold with boundary, such that $\lim_{r_c\to\infty} D_{r_c} = D$ and $\partial D = \Sigma$. This definition is in the spirit of holographic renormalisation (see *e.g.* [110]). Indeed, we have subtracted a "counterterm" that depends only on the boundary data, and in particular is covariant (it is simply the integral of the volume form of Σ). In the following we will not discuss (6.15) any further, but instead focus our attention on the reminding part of the on-shell action. As we shall explain, this is finite and therefore contributes multiplicatively to the baryon condensate.

6.3 Supersymmetric wordvolume instantons

In order to compute the *on-shell* Euclidean action one must solve the equation of motion for the gauge-invariant two-form $M = 2\pi\alpha' F - B$ on the D3-brane worldvolume. We focus on the contribution of supersymmetric D-branes for which one obtains certain non-linear instanton equations for M. These were investigated in [111] for Euclidean D-branes in a Calabi-Yau manifold, as well as other special holonomy manifolds.

In the presence of general fluxes and warp factors, the analysis becomes significantly more complicated. However, it was shown in [112] that the resulting equations are a rather natural extension of the flux-less equations, when expressed in terms of generalised calibrations. In the present paper, we are interested in Type IIB backgrounds that are warped Calabi-Yau geometries (3.8). In this case the κ-symmetry analysis for Euclidean D-branes essentially carries over [113] from the original treatment in [111]. The equations (for a general Euclidean D$(2n-1)$-brane) may be written as

$$\begin{aligned} \mathrm{e}^{i\omega - M}|_{2n} &= \mathrm{e}^{i\theta}\frac{\sqrt{\det(h+M)}}{\sqrt{\det h}}\mathrm{dvol}_{2n} && (6.16)\\ i_k\Omega_X \wedge \mathrm{e}^{i\omega - M} &= 0 \qquad k = 1,2,3\ . && (6.17)\end{aligned}$$

Here

$$\omega = H^{1/2}\omega_X \tag{6.18}$$

where H is the warp factor in (3.8), and ω_X, Ω_X are the Kähler form and holomorphic three-form of the Calabi-Yau (X, g_X), respectively. The symbol i_k denotes contraction with a complex vector field $\partial/\partial z^k$.

Moreover, it is shown in [113] that in the presence of a non-trivial warp factor H the phase θ takes the fixed value $\mathrm{e}^{i\theta} = -1$. When $n = 2$, the case in which we are interested, D must be a *divisor*, holomorphically embedded in X, and the equations for M read

$$M_- = 0\ , \qquad\qquad \omega \wedge M = 0\ ,$$

where recall that M_- is the real part of a type $(2,0)$-form. These are in fact the usual instanton equations. M is a primitive $(1,1)$-form, which on a Kähler four-manifold (D, g_D) is equivalent to being anti-self-dual on D

$$*_4\, M = -M\ . \tag{6.19}$$

The metric h induced on D via its embedding into the spacetime (3.8) is conformal to the Kähler metric g_D on D induced via the embedding of D into (X, g_X). Specifically,

$$h = H^{1/2} g_D . \tag{6.20}$$

However, the Hodge star operator is conformally invariant when acting on middle-dimensional forms. Thus the above equations may be viewed as saying that M is harmonic anti-self-dual on (D, g_D).

6.4 The worldvolume gauge field

Let us now discuss in more detail the worldvolume gauge field M. As explained in section 6.2, given a three-submanifold Σ we first need to pick an asymptotically conical divisor D, such that $\partial D = \Sigma$. We will impose the following topological conditions on Σ and D:

$$b_1(\Sigma) = 0 \ , \quad H_1(D; \mathbb{Z}) = 0 \ , \quad H^2(D; \mathbb{C}) \cong H^{1,1}(D) \ . \tag{6.21}$$

These assumptions will simplify our computations later. In fact these conditions are not too restrictive, since they hold for any toric divisor D, with boundary Σ, in a smooth toric 3-fold variety X. For example, in this case Σ is necessarily a Lens space. We also assume that

$$D \text{ is a spin manifold .}$$

This is certainly more restrictive. We impose it simply so that the worldvolume gauge field is related to a genuine line bundle[33] $\mathcal{L}$ on D, rather than a Spin^c structure. Having made such a choice of D, one needs to extend the torsion line bundle L on Σ to a line bundle $\mathcal{L}$ over D, whilst also solving the instanton equations (6.19) described in the previous subsection. In the remainder of this subsection we explain how to solve this problem.

The supersymmetry conditions imply that $M|_\Sigma = 0$, and thus the worldvolume line bundle L is indeed a torsion line bundle on Σ. Notice this implies that

$$2\pi\alpha' F \mid_\Sigma = B \mid_\Sigma \ . \tag{6.22}$$

In section 3.3 we made a fixed choice of background B field on Y, and thus $B \mid_\Sigma$ is also a fixed closed two-form. The curvature two-form $F \mid_\Sigma$ of L is thus not flat, but is

[33]This assumption may presumably be lifted without altering our overall conclusions.

rather related to the B field via (6.22). In fact, we shall argue momentarily that M must be a harmonic two-form that is L^2-normalisable on (D, g_D). It then follows from the asymptotic expansion at large radius (*cf.* appendix A) that indeed $M = 0$ on Σ. More precisely, this may be rephrased as the statement $\lim_{r_c\to\infty} M\mid_{\partial D_{r_c}}= 0$.

The first problem is whether or not we may extend, topologically, the line bundle L on $\Sigma = \partial D$ over D itself; if it does not, the instanton does not exist. The extendability of the line bundle is determined by the long exact cohomology sequence for $(D, \partial D = \Sigma)$:

$$\begin{aligned}\cdots \qquad &\longrightarrow H^1(\Sigma;\mathbb{Z}) \longrightarrow H^2(D,\Sigma;\mathbb{Z}) \xrightarrow{f} H^2(D;\mathbb{Z}) \xrightarrow{i^*} \\ &\longrightarrow H^2(\Sigma;\mathbb{Z}) \longrightarrow H^3(D,\Sigma;\mathbb{Z}) \longrightarrow \quad \cdots \ .\end{aligned} \tag{6.23}$$

Here f is the forgetful map that forgets that a class is relative, and $i : \Sigma \hookrightarrow D$ is the inclusion map. Since $b_1(\Sigma) = 0$ by assumption (6.21), the universal coefficients theorem implies that $H^1(\Sigma;\mathbb{Z}) = 0$. By Poincaré duality, $H^3(D,\Sigma;\mathbb{Z}) \cong H_1(D;\mathbb{Z}) = 0$, where the latter is again by assumption (6.21). Exactness of the sequence (6.23) then implies that every element of $H^2(\Sigma;\mathbb{Z})$ lifts to an element of $H^2(D;\mathbb{Z})$. In fact,

$$H^2(\Sigma;\mathbb{Z}) \ \cong\ H^2(D;\mathbb{Z})/f(H^2(D,\Sigma;\mathbb{Z}))\ . \tag{6.24}$$

Concretely, this means that the line bundle L over Σ always extends over D to a line bundle $\mathcal{L}$ with first Chern class $c_1(\mathcal{L}) \in H^2(D;\mathbb{Z})$. Moreover, the extension is unique up to adding to $c_1(\mathcal{L})$ an element $f(c)$, where c is any element in $H^2(D,\Sigma;\mathbb{Z})$. In fact, even more is true. Since $H_1(D;\mathbb{Z})$ is trivial, again the universal coefficients theorem says that $H^2(D;\mathbb{Z})$ is torsion-free, and is thus a lattice. Similarly, $H^2(D,\Sigma,\mathbb{Z}) \cong H^2_{\mathrm{cpt}}(D;\mathbb{Z})$ is also a lattice. The pairing

$$H^2_{\mathrm{cpt}}(D;\mathbb{Z}) \times H^2(D;\mathbb{Z}) \to H^4_{\mathrm{cpt}}(D;\mathbb{Z}) \cong \mathbb{Z} \tag{6.25}$$

given by cup product and integral over D says that

$$\Lambda \ =\ H^2_{\mathrm{cpt}}(D;\mathbb{Z})\ , \qquad \Lambda^* \ =\ H^2(D;\mathbb{Z}) \tag{6.26}$$

are *dual* lattices.

Having chosen an extension of L to a line bundle $\mathcal{L}$ over D, we have now fixed uniquely the cohomology class of M, namely

$$[M] \ =\ \iota^*[B] + (2\pi)^2\alpha' c_1(\mathcal{L}) \ \in\ H^2(D;\mathbb{R})\ . \tag{6.27}$$

Recall that the background B field is flat and that different B field moduli are described by the group $H^2(X,Y;\mathbb{R})$. More precisely, we pick any flat extension B° of $B\mid_Y$ over X, and then any other flat B field with the same gauge at infinity is

$$B \;=\; B^\circ + B^\flat \tag{6.28}$$

where $B^\flat$ represents a class in $H^2(X,Y;\mathbb{R})$. In particular, $[B]\in H^2(X;\mathbb{R})$, and thus also $\iota^*[B]\in H^2(D;\mathbb{R})$, are determined by the moduli.

We must now solve the instanton equations (6.19) for M in the cohomology class (6.27). Furthermore, M must be chosen to be square-integrable, $M\in\mathcal{H}^2_{L^2}(D,g_D)$. To see this, notice that for κ-symmetric configurations, the BI part of the on-shell Euclidean D3-brane action

$$S_{BI} \;=\; T_3\int_D \mathrm{d}^4\sigma\sqrt{\det(h+M)}\ , \tag{6.29}$$

may be simplified upon using equation (6.16) [111]. In particular, using the relation (6.18) and specialising to linear instantons (6.19), the action (6.29) becomes

$$S_{BI} \;=\; T_3\int_D \mathrm{d}^4\sigma\sqrt{\det g_D}\left(H+\frac{1}{4}\mathrm{Tr}_{g_D}M^2\right)\ , \tag{6.30}$$

where we used anti-self-duality of M to rewrite the $M\wedge M$ term as the pointwise square norm on (D,g_D)

$$\|M\|^2_{g_D} \;=\; \frac{1}{2}\mathrm{Tr}_{g_D}M^2\ . \tag{6.31}$$

As we recalled earlier, the first term in (6.30) is logarithmically divergent at infinity. This divergence is physical, as it gives information on the conformal dimension of a baryon operator in the dual CFT [22]. Therefore, it is natural to require that the integral of $\mathrm{Tr}_{g_D}M^2$ does not affect this conformal dimension, justifying the requirement that M is L^2-normalisable.

We may now easily argue that $[M]$ may indeed be represented by an L^2 harmonic two-form $M\in\mathcal{H}^2_{L^2}(D,g_D)$. We apply once again the results (2.34) of [66] to an asymptotically conical divisor (D,g_D), of real dimension four. In particular, in the case at hand the long exact sequence (6.23), when tensored with the reals $\mathbb{R}$, implies that

$$H^2(D;\mathbb{R}) \overset{f^{-1}}{\cong} H^2(D,\Sigma;\mathbb{R})\cong H^2_{\mathrm{cpt}}(D;\mathbb{R}) \tag{6.32}$$

is an isomorphism of vector spaces. Concretely, this means that every de Rham cohomology class on D is represented by a compactly supported cohomology class. That is, given $[M] \in H^2(D;\mathbb{R})$ there is a compactly supported class $[M]_{\rm cpt} \in H^2_{\rm cpt}(D;\mathbb{R})$ such that $f([M]_{\rm cpt}) = [M]$. The middle isomorphism in (2.34) then shows that every element of $H^2(D;\mathbb{R})$ is represented by a *unique* L^2 harmonic two-form.

To conclude, we need to show that $M \in \mathcal{H}^2_{L^2}(D, g_D)$ is type $(1,1)$ and primitive in order to satisfy the instanton equations (6.19), and that moreover $M|_\Sigma = 0$ (notice that L^2-normalisability does not *a priori* imply this). The arguments are analogous to those presented in subsection 4.4. Firstly, note that if M is harmonic and L^2-normalisable, then $M \wedge \omega_D$ is an L^2 harmonic four-form on D. However, from (2.34) we see that $\mathcal{H}^4_{L^2}(D, g_D) \cong H^4(D;\mathbb{R}) = 0$, which implies that any such four-form must be zero. This proves that M must be primitive. We then also require that M be of Hodge type $(1,1)$. This follows from a similar argument to that presented in subsection 4.4: on a Kähler manifold $M_\pm$ are separately harmonic if M is. Since all the H^2 cohomology of D is of type $(1,1)$ by assumption in (6.21) *i.e.* $H^2(D;\mathbb{C}) \cong H^{1,1}(D)$, it follows from (2.34) that $M_- = 0$ and thus M is of type $(1,1)$. Thus we have proven that there always exists a unique solution to the instanton equations.

Finally, looking at Table 4 in appendix A in the case that $p = n = 2$, one learns that the leading term in the large r expansion of M is a closed and co-closed mode of type III$^-$, namely

$$M_0 \;=\; r^{-\sqrt{\mu}}\mathrm{d}\beta_\mu - \sqrt{\mu}r^{-1-\sqrt{\mu}}\mathrm{d}r \wedge \beta_\mu \tag{6.33}$$

where β_μ is a one-form on Σ obeying

$$\Delta_\Sigma\, \beta_\mu \;=\; \mu\, \beta_\mu \tag{6.34}$$

with $\mu > 0$. This shows that $M = 0$ on Σ.

6.5 A topological action for M

Having explained how to solve for a supersymmetric gauge field M in a given cohomology class $[M] \in H^2(D;\mathbb{R})$, we now begin our discussion of the on-shell D3-brane action, evaluated on such solutions. Let us consider the combined Born-Infeld and Chern-Simons parts of the action that depend on M. These pair naturally to construct the complex action

$$S[M] \;=\; i\mu_3 \left[\frac{\tau}{2}\int_D M \wedge M + \int_D M \wedge C_2\right] \,, \tag{6.35}$$

where $\tau = C_0 + i\exp(-\phi)$ is the axion-dilaton. Recall also from section 3.3 that the background C_2 field is flat, of the form

$$C_2 \;=\; C_2^{\circ} + C_2^{\flat} \tag{6.36}$$

where C_2° is a fixed flat C_2 field on X inducing a fixed gauge choice $C_2^{\circ}\mid_Y = C_2^Y$ on Y, and $C_2^{\flat}$ represents a class in $H^2(X,Y;\mathbb{R})$. In particular, C_2^Y determines a choice of marginal coupling in $H^2(Y;\mathbb{R})/H^2_{\mathrm{free}}(Y;\mathbb{Z})$, and a background C_2 determines a cohomology class $[C_2] \in H^2(X;\mathbb{R})$.

In this section we will show that the action (6.35) is a *topological* invariant: that is, it depends only on the topological classes $[M], \iota^*[C_2] \in H^2(D;\mathbb{R})$. In the following subsection we will investigate more fully the dependence of the on-shell action on the various background fields.

More precisely, let $[M]_{\mathrm{cpt}} = f^{-1}[M] \in H^2_{\mathrm{cpt}}(D;\mathbb{R})$ denote the compactly supported version of $[M]$, and similarly $[C_2]_{\mathrm{cpt}} = f^{-1}[C_2] \in H^2_{\mathrm{cpt}}(D;\mathbb{R})$. Then we will show that for M the L^2 harmonic form constructed above we have

$$\int_D M\wedge M \;=\; [M]_{\mathrm{cpt}} \cup [M] \tag{6.37}$$

$$\int_D C_2\wedge M \;=\; \frac{1}{2}[M]_{\mathrm{cpt}} \cup [C_2] + \frac{1}{2}[C_2]_{\mathrm{cpt}} \cup [M]\ , \tag{6.38}$$

where the right hand side of these formulas denote the cup product

$$H^2_{\mathrm{cpt}}(D;\mathbb{R}) \times H^2(D;\mathbb{R}) \;\to\; H^4_{\mathrm{cpt}}(D;\mathbb{R}) \cong \mathbb{R}\ . \tag{6.39}$$

Consider (6.37) first. Let α denote any closed two-form representing $[M]$, and let α_{cpt} denote any closed compactly supported two-form representing $[M]_{\mathrm{cpt}}$. Consider the integral

$$\int_D M\wedge M - \int_D \alpha_{\mathrm{cpt}}\wedge\alpha \;=\; \int_D (M+\alpha_{\mathrm{cpt}})\wedge(M-\alpha) + \int_D M\wedge(\alpha-\alpha_{\mathrm{cpt}})\ . \tag{6.40}$$

Now

$$\alpha - M \;=\; \mathrm{d}\lambda\ , \qquad \alpha-\alpha_{\mathrm{cpt}} \;=\; \mathrm{d}\sigma \tag{6.41}$$

since $[M]=[\alpha]$ and $[\alpha] = f([\alpha]_{\mathrm{cpt}})$ by assumption. Since α_{cpt} is zero in a neighbourhood of infinity we have $\alpha = \mathrm{d}\sigma$ in this neighbourhood. More precisely, we may define $U = (r_0,\infty)\times\Sigma$; then for large enough r_0 we have $(\mathrm{d}\sigma - \alpha)\mid_U = 0$. We also have

$$\mathrm{d}(\sigma-\lambda)\mid_U = \; M\mid_U\ . \tag{6.42}$$

Recalling the asymptotic expansion (6.33), (6.34), we may thus take

$$(\sigma-\lambda)|_{\Sigma_{r_c}} \,=\, r_c^{-\sqrt{\mu}}\beta_\mu \ , \tag{6.43}$$

to leading order in r_c as $r_c\to\infty$. Note that we may also add $\mathrm{d}f$ to (6.43), where f is any function on U (not necessarily bounded as $r_c\to\infty$) – however, this drops out of the integral below since M is closed. Indeed, we then have

$$\int_D M\wedge M-\int_D \alpha_{\mathrm{cpt}}\wedge\alpha \,=\, \lim_{r_c\to\infty}\int_{\Sigma_{r_c}}(\sigma-\lambda)\wedge M \,=\, 0 \ , \tag{6.44}$$

where the last equality follows since both $M\,|_{\Sigma_{r_c}}\!\to 0$ and $(\sigma-\lambda)|_{\Sigma_{r_c}}\to 0$, as $r_c\to\infty$.

Now conisder (6.38). The discussion is analogous to that above. Given any $[C_2]\in H^2(D;\mathbb{R})$, there is a unique compactly supported class $[C_2]_{\mathrm{cpt}}\in H^2_{\mathrm{cpt}}(D;\mathbb{R})$ such that $f([C_2]_{\mathrm{cpt}})=[C_2]$. Let γ and γ_{cpt} be two-forms representing $[C_2]$ and $[C_2]_{\mathrm{cpt}}$, respectively, and consider the integrals

$$\begin{aligned}\int_D M\wedge C_2-\int_D\alpha_{\mathrm{cpt}}\wedge\gamma &= \\ &=\int_D(M+\alpha_{\mathrm{cpt}})\wedge(C_2-\gamma)+\int_D M\wedge\gamma-\int_D C_2\wedge\alpha_{\mathrm{cpt}} \end{aligned}\tag{6.45}$$

$$\begin{aligned}\int_D C_2\wedge M-\int_D\gamma_{\mathrm{cpt}}\wedge\alpha &= \\ &=\int_D(C_2+\gamma_{\mathrm{cpt}})\wedge(M-\alpha)+\int_D C_2\wedge\alpha-\int_D\gamma_{\mathrm{cpt}}\wedge M \ . \end{aligned}\tag{6.46}$$

Now we have

$$\gamma-C_2 \,=\, \mathrm{d}\nu \ , \tag{6.47}$$

thus the first terms on the right hand side may be evaluated by parts, giving

$$\begin{aligned}\int_D(M+\alpha_{\mathrm{cpt}})\wedge(C_2-\gamma) &= -\int_\Sigma M\wedge\nu \\ \int_D(C_2+\gamma_{\mathrm{cpt}})\wedge(M-\alpha) &= -\int_\Sigma C_2\wedge\lambda \ . \end{aligned}\tag{6.48}$$

As usual, we should understand the integrals on the right hand side of these expressions as a limit of integrals over Σ_{r_c}. Summing (6.45) and (6.46) we obtain

$$\begin{aligned}2\int_D C_2\wedge M-\int_D\gamma_{\mathrm{cpt}}\wedge\alpha-\int_D\alpha_{\mathrm{cpt}}\wedge\gamma &= \\ =\int_D M\wedge(\gamma-\gamma_{\mathrm{cpt}})-\int_\Sigma M\wedge\nu+\int_D C_2\wedge(\alpha-\alpha_{\mathrm{cpt}})-\int_\Sigma C_2\wedge\lambda \ . \end{aligned}\tag{6.49}$$

Now we have

$$\gamma - \gamma_{\rm cpt} \ = \ {\rm d}\zeta \ , \tag{6.50}$$

thus integrating again by parts, the second line in (6.49) reduces to

$$\int_\Sigma M \wedge (\zeta - \nu) + \int_\Sigma C_2 \wedge (\sigma - \lambda) \ . \tag{6.51}$$

We then use the fact that

$$M - \alpha_{\rm cpt} \ = \ {\rm d}(\sigma - \lambda) \ , \qquad\qquad C_2 - \gamma_{\rm cpt} \ = \ {\rm d}(\zeta - \nu) \ . \tag{6.52}$$

The argument for each term in (6.51) being zero is slightly different. Firstly, $\mathrm{d}(\zeta - \nu)$ is a well-defined smooth two-form on Σ, and thus $\zeta - \nu$ may be taken to be a smooth one-form; any exact part, divergent or otherwise, drops out of the integral. Since $M = 0$ on Σ, then the first integral in (6.51) is zero. Secondly, $(\sigma - \lambda)$ vanishes on Σ, proving that also the second integral in (6.51) is zero. In conclusion, we have shown that

$$\int_D C_2 \wedge M \ \ = \ \ \frac{1}{2}\int_D \gamma_{\rm cpt} \wedge \alpha + \frac{1}{2}\int_D \alpha_{\rm cpt} \wedge \gamma \ . \tag{6.53}$$

which is (6.38).

6.6 Gauge transformations of the action

We will now discuss the effect of various gauge transformations on the D3-brane action, extending the exposition in [22]. Because the worldvolume D is non-compact the discussion of gauge invariance is slightly subtle. Since the Born-Infeld part of the action is manifestly gauge-invariant, in the following we will focus on the Chern-Simons action:

$$S_{CS} \ = \ i\mu_3 \int_D \left[C_4 + M \wedge C_2 + C_0 \frac{1}{2} M \wedge M \right] + \frac{2\pi i}{48}\int_D C_0 \left[p_1(R_{TD}) - p_1(R_{ND}) \right] \tag{6.54}$$

where C_{2p} are the RR potentials. Recall that μ_3 is given by (2.13). The second term in (6.54) contains the curvature couplings in (2.29), where $p_1(R_{TD})$ and $p_1(R_{ND})$ denote Pontryagin curvature forms for the tangent bundle TD of D and its normal bundle ND in $\mathcal{M}$. We will postpone a discussion of this term until section 6.6.5.

Recall the discussion of background RR fields from section 3.3. We fix a gauge choice for the RR potentials on Y, which we may pull back to the UV boundary $\mathbb{R}^4 \times Y$. In particular this determines certain marginal couplings of the UV theory.

These RR potentials are then extended over X, or more precisely over spacetime $\mathcal{M} = \mathbb{R}^4 \times (X \setminus \{x_1, \ldots, x_m\})$, to potentials C_*° satisfying the relevant equations of motion. Here the subscript $*$ may take any of the values 0, 2 or 4, so C_* denotes any of C_0, C_2 or C_4. One may then add to these background fields any compactly supported flat RR field. These determine the flat form-field moduli discussed in section 3.3. We thus generally write

$$C_* \ = \ C_*^\circ + C_*^\flat \ . \tag{6.55}$$

We first show that, for *fixed* gauge at infinity, the on-shell D3-brane action is a well-defined function of the flat RR field moduli in (3.20). That is, the action is invariant under compactly supported small and large gauge transformations. It nevertheless certainly depends on C_*°, and in particular on the gauge choice this induces at infinity. However, we will then show that under *any* gauge transformation

$$C_*^\circ \ \to \ C_*^\circ + \mathrm{d}\lambda \ , \tag{6.56}$$

where $\mathrm{d}\lambda$ is unrestricted at infinity, the on-shell action changes by terms that depend *only* on the boundary data. The prescription for computing the relative phase of the condensate in (6.11) is then that the two terms on the right hand side should be computed with the same fixed background C_*°, inducing a fixed gauge choice at infinity. The two terms then certainly depend on this choice, as well as on the compactly supported cohomology classes of the flat fields $C_*^\flat$ in (6.55). However, if we change the choice of C_*° via a general gauge transformation (6.56), or similarly by large gauge transformations, the two terms will transform in the same way, since the change in the action depends only on the boundary data. This way the relative phase computed in (6.11) is independent of the background gauge choice of C_*°, and is also gauge invariant under compactly supported gauge transformations. Thus the relative phases, computed in this manner, depend only on the moduli that we described in section 3. We discuss small and large gauge transformations in turn.

6.6.1 Moduli: small gauge transformations

Consider the small gauge transformation

$$C_2 \to C_2 + \frac{2\pi}{\mu_1}\mathrm{d}\lambda \tag{6.57}$$

where λ is any one-form on $\mathcal{M}$ that vanishes on the UV boundary $\mathbb{R}^4 \times Y$. We will refer to such gauge transformations throughout as *compactly supported*. As explained in section 3.3, this transformation must be accompanied by a shift of the four-form potential

$$C_4 \to C_4 + \frac{2\pi}{\mu_1} B \wedge \mathrm{d}\lambda \tag{6.58}$$

leading to the change in the action

$$\delta S_{CS} \, = \, i\mu_3 \int_D \frac{2\pi}{\mu_1} \mathrm{d}\lambda \wedge \mathrm{e}^{2\pi\alpha' F} \, = \, i \int_\Sigma \lambda_\Sigma \wedge F \, = \, 0 \; . \tag{6.59}$$

Here $\lambda_\Sigma \equiv \lambda \mid_\Sigma = 0$ follows since $\Sigma \subset \mathbb{R}^4 \times Y$ and λ vanishes on the latter.

Now consider compactly supported small gauge transformations of C_4 *i.e.* such that the gauge generators vanish at infinity. A shift

$$C_4 \to C_4 + \frac{2\pi}{\mu_3} \mathrm{d}K \; . \tag{6.60}$$

leads to a change in the action

$$\delta S_{CS} \, = \, 2\pi i \int_D \mathrm{d}K \, = \, 2\pi i \int_\Sigma K_\Sigma \; . \tag{6.61}$$

But this integral vanishes since $K_{\mathbb{R}^4 \times Y} = 0$ and so in particular $K_\Sigma = 0$.

6.6.2 Moduli: large gauge transformations

Now consider the large gauge transformation

$$C_2 \to C_2 + \frac{2\pi}{\mu_1} \sigma \tag{6.62}$$

where σ represents a class in $H^2_{\mathrm{free}}(\mathcal{M}, \partial\mathcal{M}; \mathbb{Z}) \cong H^2_{\mathrm{free}}(X, Y; \mathbb{Z})$. The net effect is the shift in the action

$$\delta S_{CS} \, = \, i\mu_3 \int_D \frac{2\pi}{\mu_1} \sigma \wedge \mathrm{e}^{2\pi\alpha' F} = 2\pi i \int_D \sigma \wedge \frac{F}{2\pi} \; . \tag{6.63}$$

However, since $[F]/2\pi \in H^2(D; \mathbb{Z})$ is quantised, the last expression may be understood as the cup product

$$\begin{array}{ccc} H^2(D, \Sigma; \mathbb{Z}) \times H^2(D; \mathbb{Z}) & \to & \mathbb{Z} \\ \left(\iota^* \sigma \;\; , \;\; \dfrac{[F]}{2\pi} \right) & \mapsto & \displaystyle\int_D \sigma \wedge \frac{F}{2\pi} \; . \end{array} \tag{6.64}$$

Here $\iota^* : H^2(X, Y; \mathbb{Z}) \to H^2(D, \Sigma; \mathbb{Z})$. Hence the action is invariant modulo $2\pi i\mathbb{Z}$.

Finally, consider large gauge transformations of C_4 that are compactly supported:

$$C_4 \to C_4 + \frac{2\pi}{\mu_3}\kappa \ . \tag{6.65}$$

Here κ is a closed compactly supported four-form with integral periods; that is, it represents a class in $H^4_{\text{free}}(X,Y;\mathbb{Z})$. Thus the exponentiated action is manifestly invariant since

$$\delta S_{CS} \ = \ 2\pi i\int_D \kappa \ = \ 2\pi i n \ \cong \ 0 \quad \mod 2\pi i\mathbb{Z} \ . \tag{6.66}$$

We have thus shown that the exponentiated on-shell D3-brane action is invariant under compactly supported gauge transformations of the RR fields.

6.6.3 Background choice: small gauge transformations

We now analyse the transformation properties of the D3-brane action under general small gauge transformations. This is completely straightforward. Consider the small gauge transformation

$$C_2 \to C_2 + \frac{2\pi}{\mu_1}\mathrm{d}\lambda \ . \tag{6.67}$$

Taking into account the corresponding transformation of C_4, the D3-brane action changes by

$$\delta S_{CS} \ = \ i\mu_3\int_D \frac{2\pi}{\mu_1}\mathrm{d}\lambda\wedge \mathrm{e}^{2\pi\alpha' F} \ = \ i\int_\Sigma \lambda_\Sigma \wedge F_\Sigma \ . \tag{6.68}$$

This of course depends only on the boundary data on $\Sigma\subset Y$. Note that $2\pi\alpha' F_\Sigma = B_\Sigma$.

Similarly,

$$C_4 \to C_4 + \frac{2\pi}{\mu_3}\mathrm{d}K \ . \tag{6.69}$$

leads to a change in the action

$$\delta S_{CS} \ = \ 2\pi i\int_D \mathrm{d}K \ = \ 2\pi i\int_\Sigma K_\Sigma \ , \tag{6.70}$$

which again trivially depends only on data at the boundary.

6.6.4 Background choice: large gauge transformations

We conclude by analysing the transformation properties of the D3-brane action under general large gauge transformations. This is less straightforward. Only the *exponentiated* action changes by terms depending only on the boundary data.

We begin with large gauge transformations of the axion. These may also be thought of as $SL(2;\mathbb{Z})$ transformations. Under the shift

$$C_0 \to C_0 + 1 \tag{6.71}$$

the action changes by

$$S_{CS} \;\to\; S_{CS} + i\mu_3 \int_D \mathrm{e}^{2\pi\alpha' F} \;, \tag{6.72}$$

so that

$$\delta S_{CS} = \frac{i}{4\pi}\int_D F \wedge F \;. \tag{6.73}$$

As we explain below, the change δS_{CS} in S_{CS} is thus given by the level $k = 1/2$ *Chern-Simons action* of the abelian connection A_Σ on the three-manifold Σ. This makes sense as an element of $i\mathbb{R}/2\pi\mathbb{Z}$ only when D is a spin manifold[34].

Let us briefly recall how the Chern-Simons action of (Σ, A_Σ) is defined. Suppose first that L is a topologically trivial line bundle over Σ on which A_Σ is a connection one-form. Thus A_Σ may be regarded as a globally-defined one-form on Σ, and there is no subtlety in defining the Chern-Simons action at level k:

$$S_{CS}(\Sigma, A_\Sigma) = \frac{ik}{2\pi}\int_\Sigma A_\Sigma \wedge \mathrm{d}A_\Sigma \;. \tag{6.74}$$

When the line bundle L is non-trivial, as it generally is in this paper, the definition of the Chern-Simons action for a connection A_Σ on L is more subtle. Let us begin by rewriting (6.74) in the case that L is trivial. If D is a four-manifold with boundary Σ, we may always extend A_Σ as a one-form A over D, and by Stokes' theorem we may write

$$S_{CS}(\Sigma, A_\Sigma) = \frac{ik}{2\pi}\int_D F \wedge F \tag{6.75}$$

[34]Recall that when D is not spin, the "gauge field" A is more precisely a Spin^c connection [114]. In this case the discussion is slightly modified, and the curvature couplings that we have ignored would be important in the analysis.

where $F = \mathrm{d}A$ is the curvature of A. Of course, the result is independent of the choice of extension of A_Σ over D. This formula, together with a non-trivial result in cobordism theory, is the key to defining $S_{CS}(\Sigma, A_\Sigma)$ in general. If Σ is an oriented three-manifold with a line bundle $L \to \Sigma$, then it is a non-trivial fact that there exists an oriented four-manifold D, with boundary $\Sigma = \partial D$, together with an extension $\mathcal{L}$ of L over D. Thus we may simply define the Chern-Simons action by the formula (6.75), where F is the curvature of any connection on $\mathcal{L}$ that restricts to the connection A_Σ on Σ. Of course, *a priori* this definition then depends on the choice of $(D, \mathcal{L})$. However, suppose that $(D', \mathcal{L}')$ is another such extension. Then the difference in Chern-Simons actions is given by

$$2\pi i k \int_W \frac{F}{2\pi} \wedge \frac{F}{2\pi} \ . \tag{6.76}$$

where W is the *compact* four-manifold $W = D \cup_\Sigma -D'$. Since $F/2\pi$ is integral, the difference in Chern-Simons actions is therefore an integer multiple of $2\pi i$, provided that $k \in \mathbb{Z}$. Thus (6.75) may be used to *define* the Chern-Simons action, regarded as an element of $i\mathbb{R}/2\pi\mathbb{Z}$.

When Σ is a spin three-manifold, which is always true when Σ is oriented, we may also define Chern-Simons theory at half-integer levels, $k \in \frac{1}{2}\mathbb{Z}$. Again, a key fact is that (Σ, L) always bounds a spin four-manifold with line bundle $(D, \mathcal{L})$. In this case the integral

$$\int_W \frac{F}{2\pi} \wedge \frac{F}{2\pi} \in 2\mathbb{Z} \tag{6.77}$$

is always even for W a compact spin four-manifold. Thus k may take half-integer values.

To summarise, a gauge transformation $C_0 \to C_0 + 1$ results in a change in the Chern-Simons term in the D3-brane action by the level 1/2 Chern-Simons action of (Σ, A_Σ). This is well-defined as an element of $i\mathbb{R}/2\pi\mathbb{Z}$, and thus the change in the exponentiated action $\exp(-\delta S_{CS})$ depends only on the boundary data. This analysis is particularly important when we come to consider summing over worldvolume instantons in the next subsection. In this case D is held fixed, but we precisely sum over different line bundles $\mathcal{L}$ on D extending L. The change in the phases of each term in the sum under $C_0 \to C_0 + 1$ are then all equal, modulo $2\pi i$.

Now consider the large gauge transformation

$$C_2 \to C_2 + \frac{2\pi}{\mu_1}\sigma \tag{6.78}$$

where σ is a closed two-form on $\mathcal{M}$ with integer periods. Thus σ represents a class $[\sigma] \in H^2_{\text{free}}(\mathcal{M};\mathbb{Z})$. In particular, σ defines a class $[\sigma]_X \in H^2_{\text{free}}(X;\mathbb{Z})$. The net effect is the shift

$$\delta S_{CS} = i\mu_3 \int_D \frac{2\pi}{\mu_1}\sigma \wedge \mathrm{e}^{2\pi\alpha' F} = i\int_D \sigma \wedge F \ . \tag{6.79}$$

The embedding $\Sigma \hookrightarrow \mathcal{M}$ gives a two-form σ_Σ with $[\sigma_\Sigma] \in H^2_{\text{free}}(\Sigma;\mathbb{Z}) \cong 0$. The integral in (6.79) may then be understood as a definition of the boundary quantity

$$i\int_\Sigma \sigma_\Sigma \wedge A_\Sigma \ . \tag{6.80}$$

The argument is similar to that for the Chern-Simons action above. For A_Σ a globally-defined connection one-form on a trivial line bundle L, the integral (6.80) is well-defined. We may then rewrite (6.80) by choosing any four-manifold D that bounds Σ, any extension σ_D of σ_Σ that is closed and has integer periods, and any extension A of A_Σ. Notice that σ_D exists by the same reasoning that $\mathcal{L}$ and F exist. Then Stokes' theorem implies that

$$i\int_\Sigma \sigma_\Sigma \wedge A_\Sigma = i\int_D \sigma_D \wedge F \ . \tag{6.81}$$

A non-trivial line bundle L may be extended to a line bundle $\mathcal{L}$ over D, with A a connection form on $\mathcal{L}$ extending A_Σ. Then (6.81) may be used as a definition of the left hand side. Any other D', $\sigma_{D'}$ may of course be used, and the difference between the two definitions is

$$i\int_W \sigma_W \wedge F \tag{6.82}$$

where $W = D \cup_\Sigma -D'$ and σ_W is obtained by gluing together σ_D and $\sigma_{D'}$, which recall agree on the gluing locus Σ. Since $[F]/2\pi$ and $[\sigma]_W$ are integral classes, this last integral is an integer multiple of $2\pi i$, and thus (6.81) is a well-defined definition of the left and side, modulo $2\pi i$.

Finally, consider large gauge transformations

$$C_4 \to C_4 + \frac{2\pi}{\mu_3}\kappa \tag{6.83}$$

where κ is a closed four-form on $\mathcal{M}$ with integer periods. Of course

$$\delta S_{CS} = 2\pi i \int_D \kappa_D \ . \tag{6.84}$$

If D' is any other extension of Σ then the difference

$$2\pi i \int_D \kappa_D - 2\pi i \int_{D'} \kappa_{D'} = 2\pi i \int_W \kappa_W \in 2\pi i \tag{6.85}$$

where as usual $D \cup_\Sigma -D'$ and κ_W is constructed by gluing κ_D and $\kappa_{D'}$ along Σ. This shows that (6.84) depends only on boundary data, modulo $2\pi i$.

	cpt supported		non-cpt supported	
	small	large	small	large
C_0	–	–	–	$\frac{i}{4\pi}\int_D F \wedge F$
C_2	0	$2\pi i\mathbb{Z}$	$i\int_\Sigma \lambda_\Sigma \wedge F$	$i\int_D \sigma_D \wedge F$
C_4	0	$2\pi i\mathbb{Z}$	$2\pi i \int_\Sigma K_\Sigma$	$2\pi i \int_D \kappa_D$

Table 3: Variation of the on-shell D3-brane action under gauge transformations of the RR fields. The integrals in the last column are invariants of the boundary data modulo $2\pi i\mathbb{Z}$. In particular, the top right hand entry is the level 1/2 Chern-Simons action for (Σ, A_Σ).

To summarise, the last two subsections have shown that $\exp(-S_{CS})$ changes by a quantity that depends only on boundary data, for *any* gauge transformation of the RR fields in the bulk. In contrast, the previous two subsections have shown that $\exp(-S_{CS})$ is *invariant* under any *compactly supported* gauge transformation of the RR fields in the bulk. This is summarised in Table 3.

6.6.5 Curvature terms

Finally, we turn to the curvature terms in (6.54). Recall that the first Pontryagin form of a real vector bundle E with curvature form R_E is given by

$$p_1(R_E) = -\frac{1}{8\pi^2}\mathrm{Tr}\, R_E \wedge R_E \ . \tag{6.86}$$

In the case at hand, E is either the tangent bundle of D or its normal bundle in the spacetime $\mathcal{M}$. The relevant connection in (2.29) is then the Levi-Civita connection of the induced metric h on D, or the induced connection on the normal bundle, respectively.

In this subsection we note that the curvature couplings evaluated at any two points p and p' in the same component of the supergravity moduli space (*i.e.* where the topology of X at these two points is the same) are in fact equal. Thus when we compute the relative phase of the condensate in (6.11), the curvature terms simply drop out.

To compute the on-shell action we have fixed a gauge for C_0, which means that $C_0 \in \mathbb{R}$ is a fixed real number. Choose points p and p' in the supergravity moduli space which have the same topology X for the Ricci-flat Kähler background – of course, the Kähler class, positions of the N D3-branes, and B field and RR field moduli may be different. However, in both cases Σ is extended to the same divisor $D \subset X$, and the difference in curvature couplings is thus

$$\frac{2\pi i C_0}{48}\left[\int_D [p_1(R_{ND}) - p_1(R_{TD})] - \int_D [p_1(R'_{ND}) - p_1(R'_{TD})]\right] . \tag{6.87}$$

Here R and R' denote the curvature forms in the two corresponding spacetimes $\mathcal{M}$, $\mathcal{M}'$. These depend on the metric g_X on X and also on the positions of the D3-branes. However, we may now define the double

$$\bar{D} \equiv D \cup_\Sigma -D . \tag{6.88}$$

The difference (6.87) is then

$$\frac{2\pi i C_0}{48}\int_{\bar{D}} [p_1(R_{N\bar{D}}) - p_1(R_{T\bar{D}})] \tag{6.89}$$

which is manifestly a topological invariant, since $\bar{D}$ is closed without boundary. One must be slightly careful in this argument, since the boundary Σ along which we glue is at infinite distance. However, one can simply cut off the integral at some large r_c, and glue the metrics and connections (smoothing appropriately) along $\Sigma_{r_c} = \partial D_{r_c}$. We may conveniently view $N\bar{D}$ as the normal bundle of $\bar{D}$ in the spacetime double

$$\bar{\mathcal{M}} = \mathcal{M} \cup_{\mathbb{R}^4\times Y} -\mathcal{M}' . \tag{6.90}$$

The key observation is that, due to its construction (6.88), $\bar{D}$ has an orientation-reversing diffeomorphism which sends a point in one copy of D to the corresponding point in the other copy. The fixed point set of this map is Σ. However, it is well-known that if $\bar{D}$ admits an orientation-reversing diffeomorphism, the Pontryagin number

$$p_1(\bar{D}) \equiv \int_{\bar{D}} p_1(R_{T\bar{D}}) \tag{6.91}$$

is zero. This is easy to see: the definition (6.86) is independent of orientation, whereas the fundamental class of $\bar{D}$ (and hence the integral) changes sign under a change of orientation. But any integral is diffeomorphism-invariant, hence the result. A similar result is true for the Pontryagin numbers of a vector bundle E over $\bar{D}$, *provided*[35]

[35] This is certainly not true in general.

the orientation-reversing diffeomorphism lifts to a bundle isomorphism of E. In the case at hand, the first Pontryagin class of E, which lives in $H^4(\bar{D};\mathbb{Z}) \cong \mathbb{Z}$, will then be invariant, and thus the Pontryagin number will change sign. The normal bundles in the two spacetimes are certainly isomorphic (although they have different curvature forms). Thus there is a natural bundle isomorphism of the normal bundle of the double that covers the orientation-reversing diffeomorphism, and it follows that

$$\int_{\bar{D}} p_1(R_{N\bar{D}}) \;=\; 0 \; . \tag{6.92}$$

Note that an alternative proof of the above would have been to use an APS index theorem argument, as in [115]. The idea would be to relate the curvature terms to an appropriate linear combination of indices of operators with APS boundary conditions. The APS index theorem would then relate the curvature terms to the indices, which would be topological invariants of D in X and thus fixed integers, and boundary terms.

6.7 Sum over gauge field instantons: theta functions

In section 6.5 we showed that $S[M]$ is a topological invariant, depending only on the cohomology classes $[M], \iota^*[C_2] \in H^2(D;\mathbb{R})$. Recall that we have

$$[M] \;=\; \iota^*[B] + (2\pi)^2\alpha' c_1(\mathcal{L}) \;\in\; H^2(D;\mathbb{R}) \; . \tag{6.93}$$

where $\mathcal{L}$ is a line bundle over D that restricts to L on $\Sigma = \partial D$.

However, for fixed L there are typically countably infinitely many $\mathcal{L}$ that extend L over D, and thus countably infinitely many instantons $\{M(\mathcal{L})\}$ with different topological classes $[M(\mathcal{L})] \in H^2(D;\mathbb{R})$. This infinite set may be characterised as follows. Recall that $\Lambda = H^2_{\mathrm{cpt}}(D;\mathbb{Z})$ and $\Lambda^* = H^2(D;\mathbb{Z})$ are dual lattices under the cup product

$$\Lambda \times \Lambda^* \;\to\; \mathbb{Z} \; , \tag{6.94}$$

and that there is a natural map

$$f : \Lambda \to \Lambda^* \tag{6.95}$$

that forgets that a class has compact support. Let $\mathcal{L}_0$ be any fixed extension of L over D, with $c_1(\mathcal{L}_0) \in \Lambda^*$. We then define

$$[M]_0 \;=\; \iota^*[B] + (2\pi)^2\alpha' c_1(\mathcal{L}_0) \;\in\; H^2(D;\mathbb{R}) \; , \tag{6.96}$$

so that the set of *all* gauge instantons that are asymptotic to the torsion line bundle L is given by

$$\Big\{[M(n)] = [M]_0 + (2\pi)^2\alpha' f(n) \mid n \in \Lambda\Big\} \ . \tag{6.97}$$

The D3-brane path integral thus naturally produces, for fixed choice of L, an *instanton sum*

$$\sum_{n\in\Lambda} \exp(-S[M(n)]) \ . \tag{6.98}$$

In order to obtain a more explicit expression for this sum it is convenient to introduce bases for the dual lattices. Let $\{e_i\}$ be a basis for Λ and $\{e_i^*\}$ be the dual basis for Λ^*, so that

$$\int_D e_i \wedge e_j^* \ = \ \delta_{ij} \ . \tag{6.99}$$

Here $i = 1, \ldots, b_2(D)$, where $b_2(D)$ is the second Betti number[36] of D. In this basis we may express the map (6.95) in terms of a matrix

$$f(e_i) \ = \ f_{ji}\, e_j^* \ , \tag{6.100}$$

where as usual a sum is understood over repeated indices. The matrix $\mathbf{f} = (f_{ij})$ is invertible and has integer coefficients. We now make some further definitions. Let

$$b \ = \ \frac{1}{2\pi\alpha'}\iota^*[B] \qquad\qquad c \ = \ \frac{1}{2\pi\alpha'}\iota^*[C_2] \tag{6.101}$$

$$a \ = \ b + 2\pi c_1(\mathcal{L}_0) \qquad\quad a(n) = a + 2\pi f(n) \ . \tag{6.102}$$

These are all elements of $H^2(D;\mathbb{R})$. For fixed L and background fields C_2, B and axion-dilaton τ, the instanton sum (6.98) may be written as

$$\begin{aligned} \mathcal{I}([C_2],[B],\tau,L) \ &= \ \sum_{n\in\Lambda} \exp(-S[M(n)]) \qquad\qquad (6.103) \\ &= \ \sum_{n\in\Lambda} \exp\left[-\frac{i}{4\pi}\Big(\tau\, a(n)_{\mathrm{cpt}} \cup a(n) + a(n)_{\mathrm{cpt}} \cup c + c_{\mathrm{cpt}} \cup a(n)\Big)\right] \end{aligned}$$

where

$$a(n)_{\mathrm{cpt}} = f^{-1}(a(n)) \ , \qquad\quad c_{\mathrm{cpt}} = f^{-1}(c) \ . \tag{6.104}$$

[36] Notice that our topological assumptions (6.21), together with the discussion in section 6.4, imply that all the degree two cohomology of D is represented by L^2 harmonic anti-self-dual two-forms. The assumptions (6.21) hold if D is a toric divisor, for example.

We may now expand the various forms in terms of the basis (6.99) as

$$a \;=\; a_i\, e_i^* \;, \qquad c \;=\; c_i\, e_i^* \;, \qquad n \;=\; n_i e_i \;, \tag{6.105}$$

where $a_i = b_i + 2\pi c_1(\mathcal{L}_0)_i$ and we may take

$$b_i \;=\; \frac{1}{2\pi\alpha'}\int_{S_i} B \qquad\qquad c_i \;=\; \frac{1}{2\pi\alpha'}\int_{S_i} C_2 \;. \tag{6.106}$$

where $\{S_i\}$ are a basis of two-cycles for $H_2(D;\mathbb{Z})$. A computation then shows that

$$a(n)_{\rm cpt} \cup a(n) \;=\; (f^{-1})_{ij}a_i a_j + 2\pi(f^{-1})_{ji}f_{jk}n_k a_i + 2\pi n_i a_i + (2\pi)^2 f_{ij}n_i n_j \tag{6.107}$$

and

$$a_{\rm cpt}(n) \cup c + c_{\rm cpt} \cup a(n) = (f^{-1})_{ij}(a_i c_j + c_i a_j) + 2\pi(f^{-1})_{ji}f_{jk}n_k c_i + 2\pi n_i c_i \;. \tag{6.108}$$

At this point, with a fixed basis and dual basis, we may view a, c and n as vectors in $\mathbb{R}^{b_2(D)}$, and the cup product as simply a dot product of vectors. In this notation, we may write the instanton sum as a product of two factors

$$\mathcal{I}([C_2],[B],\tau,L) \;=\; \mathcal{P}(\boldsymbol{c},\boldsymbol{b},\tau,L)\,\mathcal{Q}(\boldsymbol{c},\boldsymbol{b},\tau,L) \tag{6.109}$$

defined as

$$\mathcal{P}(\boldsymbol{c},\boldsymbol{b},\tau,L) \;\equiv\; \exp\left[-\frac{i}{2\pi}\boldsymbol{a}\cdot \mathbf{f}^{-1}_{\rm sym}\left(\frac{\tau}{2}\boldsymbol{a}+\boldsymbol{c}\right)\right] \tag{6.110}$$

$$\mathcal{Q}(\boldsymbol{c},\boldsymbol{b},\tau,L) \;\equiv\; \sum_{\mathbf{n}\in\mathbb{Z}^{b_2(D)}} \exp\left[-i\pi\tau\mathbf{n}\cdot\mathbf{f}\mathbf{n} - \frac{i}{2}\mathbf{n}(1+\mathbf{f}^T\mathbf{f}^{-1})(\tau\boldsymbol{a}+\boldsymbol{c})\right] \;. \tag{6.111}$$

Here we have defined

$$\mathbf{f}^{-1}_{\rm sym} \equiv \frac{1}{2}\left[\mathbf{f}^{-1} + (\mathbf{f}^{-1})^T\right] \;. \tag{6.112}$$

The sum in (6.111) precisely gives rise to a *Riemann theta function*. This is usually defined as

$$\theta[\mathbf{z},\mathbf{T}] = \sum_{\mathbf{n}\in\mathbb{Z}^r} \exp\left[2\pi i\left(\frac{1}{2}\mathbf{n}\cdot\mathbf{T}\,\mathbf{n} + \mathbf{n}\cdot\mathbf{z}\right)\right] \;. \tag{6.113}$$

Here $\mathbf{z}\in\mathbb{C}^r$ is a complex vector and $\mathbf{T}$ is a complex symmetric $r\times r$ matrix whose imaginary part is positive definite. The space of such matrices is denoted $\mathbb{H}_r$, and is known as the Siegel upper half-space. One requires the imaginary part of $\mathbf{T}$ to be positive definite in order that the sum in (6.113) converges. In fact, it then converges absolutely and uniformly on compact subsets of $\mathbb{C}^r \times \mathbb{H}_r$. Defining

$$\mathbf{T} \;=\; -\,\tau\,\mathbf{f}_{\rm sym} \qquad\qquad \mathbf{z} \;=\; -\,\frac{1}{4\pi}(1+\mathbf{f}^T\mathbf{f}^{-1})\,(\tau\boldsymbol{a}+\boldsymbol{c})\;\;. \tag{6.114}$$

we have that

$$\mathcal{Q}(\boldsymbol{c},\boldsymbol{b},\tau,L) \;=\; \theta[\mathbf{z},\mathbf{T}] \tag{6.115}$$

At first sight the expression (6.109) seems to depend on the choice of $\mathcal{L}_0$, which appears in $\boldsymbol{a}$ via equation (6.102). Of course, from the original definition of the instanton sum this cannot be true. Using the transformation properties of the theta function under shifts $\mathbf{z} \to \mathbf{z} + \mathbf{T}\mathbf{m} + \mathbf{k}$, with $\mathbf{m}, \mathbf{k} \in \mathbb{Z}^r$, one can in fact easily check that the right hand side of (6.109) is independent of the choice of $\mathcal{L}_0$, although each factor is separately not independent.

Note also that $\mathbf{f}_{\rm sym}$ is indeed negative definite. The argument for this traces back to the fact that for any $[M] \in H^2(D;\mathbb{R})$, which is represented by the vector $\mathbf{M}$ in the above basis, we have

$$\mathbf{M}\cdot\mathbf{f}^{-1}\mathbf{M} = [M]_{\rm cpt} \cup [M] = \int_D M\wedge M = -\int_D M \wedge *M \leq 0 \tag{6.116}$$

where $M \in \mathcal{H}^2_{L^2}(D, g_D)$ is the harmonic anti-self dual L^2-normalisable two-form that represents $[M] \in H^2(D;\mathbb{R})$. The inequality is strict provided $M \neq 0$. This shows that (the symmetric part of) $\mathbf{f}^{-1}$, and hence also the symmetric part of $\mathbf{f}$, is negative definite. This is precisely the condition required for the instanton sum to converge.

Notice that if $\mathbf{f}$ is symmetric the expressions simplify slightly. If in addition we formally set $\boldsymbol{a} = 0$, one obtains simply

$$\sum_{\mathbf{n}\in\mathbb{Z}^{b_2(D)}} \exp\Big[-i\pi\tau\mathbf{n}\cdot\mathbf{f}\,\mathbf{n} - i\mathbf{n}\cdot\boldsymbol{c}\Big]\;. \tag{6.117}$$

Interestingly, this sum has appeared recently as the partition function for fractional instantons [116, 117]. Indeed, these references obtain this result by computing a partition function that counts $U(1)$ SYM instantons with an "observable insertion".

6.8 Coupling to Goldstone and pseudo-Goldstone bosons

In this final subsection we collect various pieces together and present an expression for the gauge-invariant phase of the baryon condensate that we defined in (6.11). This will also give us the opportunity to discuss the dependence of this phase on the RR moduli fields. The phase of the "bare" condensate, evaluated at a point[37] $p_0 \in \mathscr{M}^{\mathrm{grav}}$ is

$$\arg\langle \mathcal{B}(\Sigma, L)\rangle_{p_0} \;=\; -\mu_3 \int_D C_4 - \frac{1}{2\pi}\boldsymbol{a}\cdot \mathbf{f}_{\mathrm{sym}}^{-1}\left(\frac{C_0}{2}\boldsymbol{a}+\boldsymbol{c}\right) + \arg\theta[\mathbf{z}, \mathbf{T}]\ , \qquad (6.118)$$

where recall that to determine a point p_0 in particular means choosing a B field and RR fields. Specifically, these enter into (6.118) through the definitions (6.114), (6.101), (6.102). For the relative phase (6.11) we then have, with a slight abuse of notation,

$$\arg\langle \mathcal{B}(\Sigma, L)\rangle_{p_0,p} \;=\; -\mu_3 \int_D [C_4(p) - C_4(p_0)] + \arg\frac{\mathcal{P}(p)}{\mathcal{P}(p_0)} + \arg\frac{\theta[p]}{\theta[p_0]}\ . \qquad (6.119)$$

This expression shows that the baryon condensate, as it currently stands, has a *definite charge* under the $U(1)^{b_2(X)}$ subgroup of baryonic symmetries associated to C_4. On the other hand, the theta function does not have a definite charge under the remaining $U(1)^{b_4(X)}$ subgroup of baryonic symmetries associated to C_2 (although recall we have shown that (6.119) is *invariant* under small and large gauge transformations of all RR fields, and in particular is a well-defined function of the C_2 moduli).

As we have explained in subsection 2.5, in the classical gauge theory the baryon operators form a ring graded by their charge under the full baryonic symmetry group $U(1)^{\chi-1}$. One may thus write a basis of baryon operators which have definite charge (the basis is homogeneous) under this symmetry group. In all known examples, the classical baryon operators dual to (Σ, L) indeed have definite charge under $U(1)^{\chi-1}$. For example, for the $Y^{p,q}$ theories [14] the baryon operators dual to (Σ, L) are determinants of the bifundamental fields, which thus carry charge ± 1 under precisely two $U(1)$ subgroups. These are simply the $U(1)$s of the head and tail gauge group of the corresponding bifundamental field, which in general are certainly anomalous. However, quantum mechanically, one expects that the vacuum expectation values of these operators should only have well-defined charges under *exact* global symmetries. In the gravity dual, the group $U(1)^{\chi-1}$ is identified with the RR field torus (3.20), with a $U(1)^{b_3(Y)}$ subgroup coming from non-compactly supported gauge transformations of the C_4 field. One thus expects the phases of the baryon VEVs to be linear precisely in these $b_3(Y)$ moduli.

However, as our calculation currently stands, the two sets of $b_4(X)$ anomalous symmetries enter the condensate calculation rather differently: the $b_4(X)$ moduli coming from C_4 behave in the same way as the $b_3(Y)$ moduli. We believe this is evidence

[37]We have omitted the curvature coupling, which cancels in (6.119) below.

for also summing over *disconnected* compact components $D_{\rm cpt}$ in the full condensate calculation. *A priori*, one should include these as contributions to the Euclidean path integral with fixed boundary conditions at infinity. The sum over such compact components would then break the asymmetry we have described above, giving the condensate a non-linear dependence also on the $b_4(X)$ modes associated to C_4.

This discussion may also be phrased in terms of the coupling of the phase of the condensate to the Goldstone and pseudo-Goldstone bosons. This generalises the discussion in [30]. The coupling may be obtained straighforwardly by considering two infinitesimally displaced points p_0 and $p_0 + \delta p$ in moduli space. Then the C_4 coupling in (6.119) gives

$$\delta\vartheta^M \int_D \Psi^M \qquad M = 1, \ldots, b_3(Y), b_3(Y)+1, \ldots, b_2(X) \ , \tag{6.120}$$

while the C_2 moduli $\delta\varphi^A$ clearly couple through a non-linear (p_0-dependent) expression. Notice that it is straightforward to show that Ψ^M is indeed integrable, using the boundary behaviour determined in section 4. Indeed, Ψ^M form a basis for $H^4_{\rm free}(X,Y;\mathbb{Z})$, and the coupling $\int_D \Psi^M$ is then topological.

Summing over *compact* four-cycles, the $b_4(X)$ pseudo-Goldstone modes of C_4, which are associated to classes in the image $H^4(X,Y;\mathbb{R}) \to H^4(X;\mathbb{R})$, would couple differently to each compact component. On the other hand, it is simple to see that the $b_3(Y)$ Goldstone modes $\delta\vartheta^I$ do not couple to the compact components, since

$$\delta\vartheta^I \int_{D_{\rm cpt}} \Psi^I \ = \ 0 \ . \tag{6.121}$$

This follows since by definition the Ψ^I are exact forms, which thus map to zero in $H^4(X;\mathbb{R})$. We thus see that also summing over $D_{\rm cpt}$ in the condensate calculation implies that only the $b_3(Y)$ Goldstone bosons couple linearly to the phase of the condensate, and that the two $b_4(X)$ sets of pseudo-Goldstone bosons associated to C_4 and C_2 are then treated more symmetrically.

7 Summary and discussion

7.1 Summary

In this section we summarise the constructions of the paper. We begin by recalling how one constructs a symmetry-breaking supergravity background, and describe the corresponding moduli space. We then summarise the prescription for computing baryon condensates in such a background.

7.1.1 Supergravity backgrounds

We first summarise how one constructs a supergravity background of section 3, and the moduli space of such vacua:

- The starting point is a Ricci-flat Kähler cone $(C(Y), g)$, together with a choice of flat form fields on the Sasaki-Einstein link (Y, g_Y). The latter means specifying a flat B field together with a point in the RR torus (3.7). By AdS/CFT, the corresponding AdS_5 background determines a dual four-dimensional SCFT, with the B field and RR fields determining the values of certain marginal couplings.

- We suppose that the underlying complex variety $Z = C(Y)$ above is such that it admits a crepant resolution $\pi : X \to Z$. By the conjecture in section 2.1, for each Kähler class in the Kähler cone of $H^2(X; \mathbb{R})$ there exists a unique asymptotically conical Ricci-flat Kähler metric g_X on X. This is known to be true in some cases, as discussed in section 2.1, and is a conjectural non-compact version of Yau's theorem.

- We pick m points $x_1, \ldots, x_m$ on X and place N_i pointlike D3-branes at each point, such that $\sum_{i=1}^m N_i = N$. Then one can always solve uniquely for the warp factor H in (3.10), as a sum of Green's functions on (X, g_X). In order that the supergravity approximation to string theory be valid one requires all N_i to be large.

- One picks particular differential form representatives of the B field and RR fields on Y, in their appropriate cohomology classes determined by the marginal couplings, and extends these over X such that they satisfy the supergravity equations of motion. The only non-flat field is C_4, whose field strength is given in terms of the warp factor H by (3.2). That the flat B field and C_2 field on Y may be extended as flat fields over X is a topological fact. The differential forms are denoted B°, C°_*. More precisely, B° and C°_2 may be defined once the resolution X is fixed, whereas C°_4 is a function of the D3-brane positions $x_1, \ldots, x_m$ and the metric on X. Thus we should write $C^\circ_4(\{x_i\}, [\omega_X])$, and choose solutions with fixed gauge $C^\circ_4 \mid_{\partial\mathcal{M}}$ at infinity.

- One may add to these background differential forms any flat field that is compactly supported, so that the gauge at infinity is fixed. We identify fields iff they differ by a compactly supported gauge transformation. This leads to the group $H^2(X,Y;\mathbb{R})/H^2_{\mathrm{free}}(X,Y;\mathbb{Z})$ classifying the space of such B fields, and the RR torus (3.20). These groups are clearly independent of the choice of fixed background forms B°, C°_*. In principle one can also turn on discrete torsion fields, which should be classified by K-theory.

- The moduli are then: a choice of crepant resolution X, a Kähler class in $H^2(X;\mathbb{R})$, the choice of where one puts the pointlike D3-branes, and the B field and RR field moduli described in the last item. The supergravity backgrounds describe[38] an RG flow from the UV SCFT dual to (Y,g_Y) to a product of $\mathcal{N}=4$ SYM theories with gauge groups $SU(N_i)$ in the IR, together with the Goldstone bosons of section 5.

7.1.2 Baryon condensates

The computation of baryon condensates in section 6 may then be summarised as follows:

- Our starting point is to pick a smooth[39] supersymmetric three-submanifold Σ together with a torsion line bundle L over Σ. A D3-brane wrapped over (Σ,L) is dual to a baryon operator $\mathcal{B}(\Sigma,L)$, whose condensate in one of the above vacua we would like to compute.

- The conjecture (6.10) is that the condensate $\langle\mathcal{B}(\Sigma,L)\rangle_p$, in a supergravity vacuum p described above, is given by a path integral over Euclidean D3-branes in the background with fixed boundary (Σ,L), at a (any) point in $\mathbb{R}^4$. In practice we may compute this semi-classically by evaluating the on-shell worldvolume action of such D3-branes. In this paper we have focused on the contribution of a particular asymptotically conical divisor D with boundary Σ. More generally one should presumably integrate over a moduli space[40] of such minimal surfaces

[38] One might also consider crepant *partial* resolutions, which if they admit Ricci-flat Kähler metrics with appropriate conical behaviour near the residual singularities would describe RG flows from the UV SCFT to more interesting (products of) SCFTs in the IR, together with some number of Goldstone bosons. Such backgrounds, where explicit Ricci-flat Kähler metrics are known, were studied in [22].

[39] More generally one can consider multiply-wrapped D3-branes, leading to flat non-abelian gauge bundles over Σ, or singular/intersecting Σ. These form a larger class of baryon operators, as discussed in section 6.1.

[40] For toric geometries note that there is a unique connected toric divisor $D\subset X$ with $\partial D=\Sigma$.

with boundary Σ, which also raises the issue of fermion zero modes and whether one should consider only connected D. We shall discuss these matters further in section 7.2. For now we focus on the contribution to the semi-classical evaluation of the path integral of a smooth connected divisor D.

- As shown in our previous paper [22], the part of the Born-Infeld action that is independent of the D3-brane worldvolume gauge field M has precisely the correct divergence at large r to interpret $\exp(-S_{D3})$ as the VEV of an operator with conformal dimension equal to that of the D3-brane wrapped on (Σ, L). One may also perform a simple holographic renormalisation of this part of the action. The condensate is identically zero if D contains any of the points x_i where the background D3-branes are placed [22].

- One must next extend the torsion line bundle L on Σ to a line bundle $\mathcal{L}$ on D. Given the topological assumptions (6.21), which for example hold for toric varieties, this is always possible. We have then shown that there is always a unique supersymmetric solution for the worldvolume gauge field M which is L^2 normalisable, for any extension $\mathcal{L}$. This ensures that the gauge field does not contribute to the conformal dimension result above (there is no renormalisation required), and that M is flat at infinity. Moreover, the on-shell action is a topological invariant.

- The imaginary part of the D3-brane action is described by the Chern-Simons terms. Even classically the overall phase of the VEV of a baryon operator is not physical; but the relative phase of the VEVs at different points in the moduli space *is* physical, and it is this quantity that we shall compute. Thus we must pick a base point p_0, which is a particular choice of smooth supergravity vacuum, and compute the phase of the on-shell D3-brane action in a vacuum p relative to the phase evaluated in the background p_0. In practice, we study the case in which both p and p_0 both lie in the same chamber C, meaning that $X \cong X_0$ are isomorphic.

- For fixed X and choice of fixed background fields B°, C_0°, C_2° and $C_4^\circ(\{x_i\}, [\omega_X])$, the invariance of the D3-brane action under compactly supported gauge transformations in section 6.6 implies that the Chern-Simons action is a well-defined function of the B field and RR field moduli. These moduli consist of a point in $H^2(X, Y; \mathbb{R})/H^2_{\text{free}}(X, Y; \mathbb{Z})$, and a point in the RR torus (3.20). The value of the Chern-Simons action certainly depends on the arbitrary choice of fixed background fields above. However, under *any* gauge transformation of the background fields, the exponentiated Chern-Simons action changes by terms that depend only on the boundary data. Since the boundary data is fixed and equal for all points in the moduli space, if one computes the difference of Chern-Simons actions, evaluated at any point p in the moduli space and a fixed point p_0, respectively, then this relative value is gauge-invariant modulo $2\pi i\mathbb{Z}$.

- Finally, the choice of $\mathcal{L}$ is far from unique: for a fixed L on Σ there are infinitely many extensions $\mathcal{L}(n)$ over D, labelled by a point in a lattice $n \in \Lambda \cong H^2(D, \Sigma; \mathbb{Z})$. Since there is a unique L^2 solution to the worldvolume gauge field equations of motion for each $\mathcal{L}(n)$, in the Euclidean path integral one naturally sums over the lattice Λ. This leads to a Riemann theta function, described in section 6.7.

7.2 Discussion

The results we have described in this paper leave a number of issues open to further study. In this final subsection we discuss some of the remaining problems.

Firstly, we encourage geometric analysts to prove the non-compact version of the Calabi conjecture in section 2.1. This is vital for the form of the supergravity moduli space we have described. Since submitting the first version of this paper to the archive, the conjecture has subsequently been proved in [45] in the case that the Kähler class is compactly supported. The general case in which $[\omega]_Y$ is non-zero is thus still open, although we believe[41] solving this is now just a technical problem. It would also be interesting to understand in more detail how the classical VMS of section 2.5 compares to the supergravity moduli space of section 3, especially in its global structure. For example, the B field is periodic[42], while the FI parameters, over which the classical vacuum moduli space $\mathcal{M}$ fibers, are real numbers.

The metric fluctuations of section 4.4 should certainly be studied properly, giving a more complete fluctuation analysis than we have presented in section 4. In particular the recent paper [87], which appeared whilst this article was being completed, will be very useful. The results in sections 4 and 5 relate to the Kaluza-Klein spectrum

[41]We thank C. van Coevering and A. Futaki for discussions on this issue.

[42]Recall that the conifold may be realised as the IR fixed point of an RG flow induced via mass perturbation of the $\mathcal{N} = 2$ A_1 orbifold theory, and the periodicity of the B field may be understood from a field theory point of view in terms of Seiberg duality of this theory [82]. Whether such an argument can be extended is not clear.

on general Sasaki-Einstein manifolds. It would be interesting to undertake a general investigation of these spectra, and also to obtain better control over the eigenvalues μ^A that arise in the asymptotic expansions. It is also important to study further the identification of massive vector multiplets in AdS_5 with anomalous baryonic currents that we discussed in section 5.3. In particular, the key point that needs to be addressed is whether these currents belong to the Kaluza-Klein spectrum of $\mathrm{AdS}_5 \times Y$, as suggested by the results of this paper, or whether they correspond to highly massive states, like the Konishi current of $\mathcal{N} = 4$ SYM. For instance, it would be interesting to see if it is possible to get a handle on these currents via a field theory calculation.

An outstanding problem is to understand precisely how baryons in the classical field theory are related to baryons, realised as wrapped D3-branes, in $\mathrm{AdS}_5 \times Y$. We have given some idea of how complicated the latter are in section 6.1, and we refer the reader back to that section for a reminder of the discussion. Particularly difficult to understand are D3-branes wrapped on singular (or intersecting) Σ, and time-dependent D3-branes. This is essentially a geometric problem. One would also like to understand how the anomalous part of classical baryonic charge group $U(1)^{\chi-1}$ is realised in terms of D3-branes wrapping Σ, with appropriate supersymmetric gauge bundles, on Y: a 1-1 mapping between baryon operators and D3-brane states implies there is such a realisation. Understanding this problem is probably a necessary prerequisite to calculating VEVs of more general baryon operators. In section 6.8 we have alluded to the fact that the full condensate probably involves also summing over compact components. It is also important to address fermion zero modes, which would give vanishing conditions. Another interesting question is whether there is any hope that the gravity condensates may be reproduced by a field theory calculation. In particular, it would be interesting to understand how theta functions may arise.

Finally, perhaps the most interesting remaining issue concerns the $2b_4(X)$ massless pseudo-Goldstone modes. As we have explained, these massless modes correspond to flat directions in the classical moduli space. This is different from the situation discussed in section 2.3, where the RR moduli are instead axions which get "eaten" by the worldvolume gauge fields, via a generalised Green-Schwarz mechanism. Notice that the existence of these massless fields may be also understood from the complementary point of view of Kaluza-Klein reduction on (warped) Calabi-Yau manifolds. In particular, $b_2(X)$ of them are *Kähler moduli* of the non-compact Calabi-Yau, complexified by the RR C_4 moduli, which are expected to be classically massless. In general, in Calabi-Yau compactifications a potential for massless modes can be generated by D-brane instantons wrapping compact cycles in X. Thus an instanton-induced superpotential may lift some of the moduli we have described. Understanding how such mechanisms may work in the context of AdS/CFT is clearly very interesting. Since instanton-induced effects are generally proportional to the on-shell instanton action, the same reasoning as in section 6.8 implies that the $b_3(Y)$ Goldstone bosons, which by Goldstone's theorem are certainly massless, do not couple to such D-brane instantons. Thus the $b_3(Y)$ massless fields should be massless after any such instanton effects are taken into account; the remaining massless modes we have found are not protected, and it would be interesting to try to understand if and how they may gain a (small) mass via D-brane instanton effects. Correspondingly, it would be nice to understand the realisation of this mechanism directly in the gauge theory.

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A Closed and co-closed forms on cones

In this appendix we study L^2 closed and co-closed forms on cones. If (W, g_W) is a compact Riemannian manifold then its cone $C(W) \cong \mathbb{R}^+ \times W$ has metric

$$\mathrm{d}\rho^2 + \rho^2 g_W \tag{A.1}$$

where $\rho > 0$. We use the coordinate ρ, rather than r, since for applications in the main text we will sometimes have $\rho = r$ but sometimes $\rho = 1/r$. We will correspondingly need to study forms that are L^2 on intervals of the form $[\rho_0, \infty)$ and $(0, \rho_0]$ for some (any) ρ_0 with $0 < \rho_0 < \infty$. We assume that $C(W)$ has even dimension $2n$, so that W has dimension $2n - 1$. The analysis below essentially follows that in [86, 118, 119].

Let θ be a p-form on $C(W)$ of the form

$$\theta \ = \ g(\rho)\alpha + f(\rho)\mathrm{d}\rho \wedge \beta \ . \tag{A.2}$$

Here α and β are pull-backs of forms on W, and thus are independent of ρ. One easily computes

$$* \, \theta \;=\; (-1)^p \rho^{2n-2p-1} g \mathrm{d}\rho \wedge *_W \alpha + \rho^{2n-2p+1} f *_W \beta \;, \tag{A.3}$$

and

$$\mathrm{d}\theta \;=\; g' \mathrm{d}\rho \wedge \alpha + g \mathrm{d}\alpha - f \mathrm{d}\rho \wedge \mathrm{d}\beta \tag{A.4}$$

$$\mathrm{d}^\dagger \theta \;=\; \frac{g}{\rho^2} \mathrm{d}^\dagger_W \alpha - \frac{f}{\rho^2} \mathrm{d}\rho \wedge \mathrm{d}^\dagger_W \beta - \left[f' + (2n-2p+1) \frac{f}{\rho} \right] \beta \;. \tag{A.5}$$

The Laplacian $\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d}$ acting on θ is then

$$\begin{aligned} \Delta\theta \;=\; & \left[-g'' - (2n-2p-1) \frac{g'}{\rho} \right] \alpha + \frac{g}{\rho^2} \Delta_W \alpha - \frac{2g}{\rho^3} \mathrm{d}\rho \wedge \mathrm{d}^\dagger_W \alpha \\ + \; & \left[-f'' + (2n-2p+1) \left(\frac{f}{\rho^2} - \frac{f'}{\rho} \right) \right] \mathrm{d}\rho \wedge \beta + \frac{f}{\rho^2} \mathrm{d}\rho \wedge \Delta_W \beta - \frac{2f}{\rho} \mathrm{d}\beta \;. \end{aligned} \tag{A.6}$$

Here $*_W$, $\mathrm{d}^\dagger_W = (-1)^p *_W \mathrm{d} *_W$ and Δ_W are the Hodge operator, codifferential and Laplacian on (W, g_W), respectively. Note this corrects the formula in [86].

An arbitrary p-form on $C(W)$ may be written

$$\theta \;=\; \alpha(\rho) + \mathrm{d}\rho \wedge \beta(\rho) \tag{A.7}$$

where $\alpha(\rho)$, $\beta(\rho)$ are forms on $W_\rho \subset C(W)$. For fixed ρ, we may expand $\alpha(\rho)$ and $\beta(\rho)$ in terms of eigenmodes of the Laplacian Δ_W

$$\alpha(\rho) \;=\; \sum_{\mu \in \mathrm{Spec} \Delta_W^{(p)}} g_\mu(\rho) \alpha_\mu \tag{A.8}$$

$$\beta(\rho) \;=\; \sum_{\lambda \in \mathrm{Spec} \Delta_W^{(p-1)}} f_\lambda(\rho) \beta_\lambda \tag{A.9}$$

where

$$\Delta_W^{(p)} \alpha_\mu \;=\; \mu \, \alpha_\mu \tag{A.10}$$

$$\Delta_W^{(p-1)} \beta_\lambda \;=\; \lambda \, \beta_\lambda \;. \tag{A.11}$$

We wish to classify harmonic p-forms θ on the cone that are both closed and co-closed.

Suppose first that $\beta(\rho) = 0$. $\mathrm{d}\theta = 0$ immediately gives[43] $g'_\mu \alpha_\mu = 0$ and $g_\mu \mathrm{d}\alpha_\mu = 0$ for each mode α_μ, which implies $g_\mu = c_\mu$ is constant. $\mathrm{d}^\dagger \theta = 0$ implies that $\mathrm{d}^\dagger_W \alpha_\mu = 0$.

[43] Here and in the rest of this appendix, a prime denotes derivative with respect to ρ.

Thus α_μ is both closed and co-closed on the link (W, g_W) and thus harmonic, and so $\mu = 0$.

Suppose instead that $\alpha(\rho) = 0$. $\mathrm{d}^\dagger\theta = 0$ implies that $\mathrm{d}_W^\dagger\beta_\lambda = 0$, while $\mathrm{d}\theta = 0$ implies that $\mathrm{d}\beta_\lambda = 0$. Thus again β_λ is harmonic on (W, g_W), and so $\lambda = 0$. The equation $\mathrm{d}^\dagger\theta = 0$ also implies

$$\rho f_0' + (2n - 2p + 1) f_0 \;=\; 0 \tag{A.12}$$

which has general solution

$$f_0 \;=\; c\rho^{-2n+2p-1} \;. \tag{A.13}$$

More generally, focusing on an eigenmode α_μ in the equation $\mathrm{d}\theta = 0$ gives $\alpha_\mu \propto \mathrm{d}\beta_\lambda$ for some λ. Without loss of generality we may take $\alpha_\mu = \mathrm{d}\beta_\lambda$. Applying Δ_W to this relation gives $\lambda = \mu$. We then have the relation

$$g_\mu' = f_\mu \;. \tag{A.14}$$

The equation $\mathrm{d}^\dagger\theta = 0$ implies either $f_\mu = 0$, in which case g_μ is constant and we reduce to the solution already discussed above, or else $\mathrm{d}_W^\dagger\beta_\mu = 0$. $\mathrm{d}^\dagger\theta = 0$ then implies

$$g_\mu\, \mathrm{d}_W^\dagger\alpha_\mu \;=\; \rho^2 \left[f_\mu' + (2n - 2p + 1)\frac{f_\mu}{\rho} \right] \beta_\mu \tag{A.15}$$

or, equivalently,

$$\rho^2 g_\mu'' + \rho(2n - 2p + 1) g_\mu' - \mu g_\mu \;=\; 0 \;. \tag{A.16}$$

This has general solution

$$g_\mu \;=\; c_+\rho^{p-n+\nu_p} + c_-\rho^{p-n-\nu_p} \tag{A.17}$$

where $c_\pm$ are constants and we have defined

$$\nu_p = \sqrt{(p-n)^2 + \mu} \;. \tag{A.18}$$

One might also worry that there is an additional solution when the two solutions in (A.17) coincide. This occurs when $\nu_p = 0$, which implies (since necessarily $\mu \geq 0$) $\mu = 0$, $n = p$, leading to the equation

$$\rho g_0'' + g_0' \;=\; 0 \tag{A.19}$$

which has general solution

$$g_0 \;=\; c_1 + c_2 \log \rho\ . \tag{A.20}$$

However, note that if $\mu = 0$ then α_0 is harmonic, and the relation $\alpha_0 = \mathrm{d}\beta_0$ is impossible by the Hodge decomposition on (W, g_W) (alternatively, β_0 is harmonic and thus closed, so $\alpha_0 = 0$).

To summarise, any closed and co-closed p-form on $C(W)$ may be written as a convergent sum of the following three types of modes

$$\text{(I)} \;:\; \alpha_0 \tag{A.21}$$

$$\text{(II)} \;:\; \rho^{-2n+2p-1}\mathrm{d}\rho \wedge \beta_0 \tag{A.22}$$

$$\text{(III)}^{\pm} \;:\; \rho^{p-n\pm\nu_p}\mathrm{d}\beta_\mu + (p - n \pm \nu_p)\rho^{p-n-1\pm\nu_p}\mathrm{d}\rho \wedge \beta_\mu\ . \tag{A.23}$$

Here α_0, β_0 are harmonic p-forms and $(p-1)$-forms, respectively, while β_μ in mode III is a co-closed $(p-1)$-form which is an eigenfunction of Δ_W with eigenvalue μ. Note that $\mu > 0$ necessarily for modes of type III.

It is straightforward to compute the pointwise square norms $\|\theta\|^2 = \frac{1}{p!}\theta_{i_1\ldots i_p}\theta^{i_1\ldots i_p}$ of the above modes. For a general p-form θ as in (A.2) one obtains

$$\|\theta\|^2 \;=\; \rho^{-2p}\Big[g^2\|\alpha\|^2_W + \rho^2 f^2 \|\beta\|^2_W\Big]\ . \tag{A.24}$$

Here $\|\cdot\|_W$ denotes the pointwise norm on (W, g_W). The pointwise square norms of the above modes are then given by a non-zero function on W times the function of ρ given in Table 4. Using these formulae it is a simple matter to determine which modes

mode	$\|\theta\|^2$	L^2_0	L^2_∞
I	ρ^{-2p}	$p < n$	$p > n$
II	ρ^{-4n+2p}	$p > n$	$p < n$
III$^+$	$\rho^{-2n+2\nu_p}$	yes	no
III$^-$	$\rho^{-2n-2\nu_p}$	no	yes

Table 4: Summary of the square-integrability of the various modes.

are L^2 near to $\rho = 0$ and $\rho = \infty$. Fix some ρ_0 with $0 < \rho_0 < \infty$. If the integral of the pointwise square norm of θ over $(0, \rho_0] \times W$ is finite then we shall say that θ is L^2_0. On the other hand, if the integral of the pointwise square norm of θ over $[\rho_0, \infty) \times W$ is finite then we shall say that θ is L^2_∞. The relevant integrals take the form

$$\int F(\rho)\rho^{2n-1}\mathrm{d}\rho\ . \tag{A.25}$$

In particular, if $F(\rho) = \rho^{-2n+\gamma}$ then F is integrable on $(0, \rho_0]$ iff $\gamma > 0$ and is integrable on $[\rho_0, \infty)$ iff $\gamma < 0$.

B Eigenvalues of Laplacians on (Y, g_Y)

In this section we derive some formulae relating the one-forms $\beta^{(1)A}$ in the main text, which are eigenforms of the Laplacian Δ_Y, to *scalar* eigenfunctions on (Y, g_Y). These formulae are used in section 5.3.

Suppose that ψ is an L^2 harmonic two-form on (X, g_X). As discussed in the main text, there is an asymptotic expansion of ψ with leading term

$$\psi \sim \mathrm{d}\left(r^{-1-\nu}\beta\right) \tag{B.1}$$

where β is a co-closed one-form on (Y, g_Y) satisfying

$$\Delta_Y \beta = \mu\,\beta \tag{B.2}$$

and

$$\nu = \sqrt{1+\mu}\ . \tag{B.3}$$

As argued in the main text, ψ is $(1,1)$ and primitive, namely

$$\omega_X \lrcorner \psi = 0\ , \tag{B.4}$$

where ω_X is the Kähler form on X. Since asymptotically

$$\omega_X \sim \omega_{C(Y)} = \frac{1}{2}\mathrm{d}(r^2\eta) \tag{B.5}$$

the equation (B.4) gives, from its leading term,

$$\mathrm{d}\beta \lrcorner \mathrm{d}\eta = 2(1+\nu)\beta \lrcorner \eta\ . \tag{B.6}$$

Defining the function

$$f = \beta \lrcorner \eta \tag{B.7}$$

one can also prove the identity

$$\Delta_Y f = \mu\,\beta \lrcorner \eta - \mathrm{d}\beta \lrcorner \mathrm{d}\eta\ . \tag{B.8}$$

This is proven using (B.2), together with the fact that η is a Killing one-form, and the Weitzenböck formula

$$\Delta_Y \beta \;=\; -\Delta^i \Delta_i \beta \,+\, \mathrm{Ric}_Y \cdot \beta \;. \tag{B.9}$$

On a Sasaki-Einstein five-manifold $\mathrm{Ric}_Y = 4g_Y$. Combining (B.6) and (B.8) one obtains

$$\Delta_Y f \;=\; E\, f \tag{B.10}$$

where

$$E \;=\; \mu - 2 - 2\sqrt{1+\mu} \;. \tag{B.11}$$

This last formula is used in section 5.3.

C Flat form fields

In this appendix we review the classification of flat form fields, up to gauge equivalence, on a spacetime $\mathcal{M}$. Such fields play an important role throughout the paper.

A flat $(p-1)$-form potential C has, by definition, field strength $G = \mathrm{d}C = 0$. Since it is the field strength G that generally enters the supergravity equations, one may typically turn on flat fields without altering the equations of motion. The potential C transforms under a form of gauge transformation via

$$C \to C + \mathrm{d}\lambda \tag{C.1}$$

where λ is any $(p-2)$-form on spacetime $\mathcal{M}$. In fact, more generally C also transforms under large gauge transformations

$$C \to C + \frac{2\pi}{\mu} a \tag{C.2}$$

where a is any closed $(p-1)$-form with *integral* periods. These reduce to (C.1) when the cohomology class of a is trivial. The constant μ depends on the normalisation of the potential, and may be determined from the Wess-Zumino couplings of an object coupling electrically to C. Specifically, the latter is given by

$$S_{WZ} = \mu \int C \;. \tag{C.3}$$

The transformations (C.2) then leave the exponentiated action $\exp(iS_{WZ})$ invariant. This leads to the group $H^{p-1}(\mathcal{M};\mathbb{R})/H^{p-1}_{\text{free}}(\mathcal{M};\mathbb{Z})$, classifying the space of closed potentials mod gauge transformations. Here $H^{p-1}_{\text{free}}(\mathcal{M};\mathbb{Z})$, which is the image of $H^{p-1}(\mathcal{M};\mathbb{Z})$ in $H^{p-1}(\mathcal{M};\mathbb{R})$, is the group of large gauge transformations.

However, in general not all flat form fields arise this way. The p-form field strength G satisfies a form of Dirac quantisation, and consequently defines an element of $H^p(\mathcal{M};\mathbb{Z})$. A flat p-form field on a spacetime $\mathcal{M}$ then lies in the kernel of the map $H^p(\mathcal{M};\mathbb{Z}) \to H^p(\mathcal{M};\mathbb{R})$. But this kernel is by definition the torsion component $H^p_{\text{tor}}(\mathcal{M};\mathbb{Z})$. Such a torsion p-form field strength is not described *globally* by a closed $(p-1)$-form potential. Indeed, the short exact coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0 \tag{C.4}$$

induces in a standard way the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \quad & H^{p-1}(\mathcal{M};\mathbb{Z}) \longrightarrow H^{p-1}(\mathcal{M};\mathbb{R}) \longrightarrow H^{p-1}(\mathcal{M};U(1)) \xrightarrow{\beta} \\ & H^p(\mathcal{M};\mathbb{Z}) \longrightarrow H^p(\mathcal{M};\mathbb{R}) \longrightarrow \cdots \end{aligned} \tag{C.5}$$

which implies that $H^p_{\text{tor}}(\mathcal{M};\mathbb{Z}) \cong \beta(H^{p-1}(\mathcal{M};U(1)))$. Here β is the so-called Bockstein map. In fact it is $H^{p-1}(\mathcal{M};U(1))$ which classifies, up to gauge equivalence, flat form fields with a field strength of degree p. An element of this group may be regarded as specifying the *holonomy* of the potential over closed $(p-1)$-cycles. Thus, if γ is a chain representing a $(p-1)$-cycle $[\gamma] \in H_{p-1}(\mathcal{M};\mathbb{Z})$, we may define the holonomy of the potential C over γ to be

$$\exp\left(i\mu \int_\gamma C\right) \ . \tag{C.6}$$

The holonomy of a flat potential defines a homomorphism $H_{p-1}(\mathcal{M};\mathbb{Z}) \to U(1)$. Since $U(1)$ is a divisible group, the group of such homomorphisms is $H^{p-1}(\mathcal{M};U(1))$. The long exact coefficient sequence (C.5) implies that in general the group $H^{p-1}(\mathcal{M};U(1))$ is disconnected, with the number of connected components being the number of elements in $H^p_{\text{tor}}(\mathcal{M};\mathbb{Z})$. Thus the discussion in the previous paragraph misses the flat fields that have torsion fluxes $[G] \in H^p_{\text{tor}}(\mathcal{M};\mathbb{Z})$, which are not described globally by a closed $(p-1)$-form potential C.

In this paper we shall largely not include the torsion flat fields in the discussion. An important exception to this is in section 6. Another reason for ignoring the torsion classes is that, although we are treating RR fields in terms of cohomology in this paper, more precisely they are classified by K-theory [120]. These differ in their torsion.

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