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(Article begins on next page)
Perfect fluid geometries in Rastall’s cosmology

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Abstract

The equation of state of an ultrarelativistic perfect fluid is obtained as a necessary condition for a perfect fluid space-time in Rastall’s cosmology.

Key words: perfect fluid space-time; Rastall’s cosmology; ultrarelativistic fluid

1 Introduction

It is well known the importance of a perfect fluid description of gravitational sources in General Relativity [3]. It has been also stressed the relevance of compatibility of perfect-fluid solutions with modified or extended theories of gravity [1].

In this note we point out some cosmological features emerging from the request that the Rastall model [11] describe a perfect fluid geometry of space-time.

We shortly recall some features of the model introduced by Rastall in 1972; we refer to this paper for the relevant issues and stress that the chosen signature for the underlying Lorentzian manifold will be (−, +, +, +), in accordance with such a source.

Let then \( \text{Lor} \mathbf{X} \) be the open submanifold of the subbundle of symmetric tensors constituted by regular metrics with the above choice of the Lorentzian signature. An induced fibered chart on \( J^2 \text{Lor} \mathbf{X} \) has local coordinates

\[
(x^\alpha, g_{\alpha\beta}, g_{\alpha\beta\gamma}, g_{\alpha\beta\gamma\nu})
\]

and we can define other local coordinates \( g^{\mu\nu} \) and so on by the relation \( g_{\alpha\mu} g^{\mu\nu} = \delta^\nu_\alpha \).

As well known Einstein equations read

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_G R T_{\mu\nu},
\]

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where \( \kappa_{GR} = \frac{8\pi G}{c^4} \); the Ricci tensor is obtained from a metric connection, so that \( R_{\mu\nu} = \frac{1}{2} R_{j^2 g} \) and the scalar curvature \( R \) has to be intended as \( R(j^2 g) = g^{\alpha\beta} R_{\alpha\beta}(j^2 g) \). These equations naturally imply the conservation of energy-momentum tensor as a consequence of the Bianchi identities.

The Rastall model for gravity coupled with matter governed by an energy-momentum tensor satisfying (with \( \lambda \) a suitable non-null dimensional constant)

\[
\hat{T}_{\nu\mu} = \lambda R_{\nu\mu},
\]

where the comma denotes the partial derivative, and by field equations

\[
R_{\mu\nu} - \frac{1}{2} (1 - \frac{2}{4 - \kappa_r \lambda}) R g_{\mu\nu} = \kappa_r T_{\mu\nu},
\]

where \( \kappa_r \) is a dimensional constant to be determined in order to give the right Poisson equation in the static weak-field limit. This model introduces an energy-momentum tensor fulfilling the requirement \( \nabla_{\mu} \hat{T}^\mu_\nu = 0 \) without violating the Bianchi identities; for a 4-vector, say \( a_\mu \), \( \hat{T}^\mu_\nu = a_\mu \) and \( a_\mu \neq 0 \) on curved spacetime, but \( a_\mu = 0 \) on flat spacetime (in agreement with Special Relativity). Of course, in general, \( \kappa_r \neq \kappa_{GR} \). Taking the trace of Eqs.(1) gives us the so-called structural or master equation

\[
2 \kappa_r \lambda - 1 = \kappa_r \lambda - \frac{1}{4}.
\]

By this relation the Rastall equations (1) can be recast in the form of Einstein equations. Indeed, one can immediately write \( G_{\mu\nu} = \kappa_r S_{\mu\nu} \), where \( S_{\mu\nu} = T_{\mu\nu} - \frac{\kappa_r \lambda}{4 \kappa_r \lambda - 1} g_{\mu\nu} T \). By construction, this new energy-momentum tensor is conserved, i.e. \( S^\mu_\nu = 0 \). Furthermore, (if \( u_\mu \) is a 4-velocity with \( g^{\mu\nu} u_\mu u_\nu = -1 \)) by assuming \( \hat{T}_{\mu\nu} = -(\rho + p) u_\mu u_\nu - pg_{\mu\nu} \), i.e. the source is a perfect isentropic fluid with energy density \( \rho \) and pressure \( p \), we can explicitly work out an expression for \( S_{\mu\nu} \). The source turns out to be still a perfect fluid, provided we redefine its energy density and pressure. [11]

2 Perfect fluid space-times in Rastall’s cosmology

Rastall’s cosmology has been recently object of attention in scientific community; some papers are concerned with the querelle whether Rastall’s stress-energy tensor corresponds to an artificially isolated part of the physical conserved stress-energy or not [2, 7, 10, 15], some others are concerned with the derivation of a related variational principle (i.e. with the question how to get a - eventually modified - Rastall model which could be derived by a Lagrangian and physically different from Einstein gravity), see e.g. Refs. [12, 14, 13].

We study this model from the point of view of the compatibility of its geometry with a perfect fluid description of space-time. Indeed, as a gravity model it provides a non minimal coupling between metric and matter and we try to understand how this is related with this compatibility requirement.

Indeed it is well known that by suitably introducing a matter density \( \mu \), the energy density of space-time \( \bar{\rho} \) can be defined as a function of \( \mu \) satisfying
\( \dot{p} = \mu \dot{\rho} - \ddot{\rho} \) where the prime denotes derivation with respect to \( \mu \); (this is equivalent with asking whether we can derive the tensor \( R_{\mu\nu} \equiv \ddot{T}_{\mu\nu} \) from a variational principle, i.e. with looking for a Lagrangian density depending on \( \rho(\mu) \) from which \( \ddot{T}_{\mu\nu} \) can be derived variationally; such kind of space-times appear e.g. in the dual Lagrangian description of the Ricci tensor for barotropic perfect relativistic fluids [6], see also Refs. [4, 5, 8]).

We therefore have

\[
\ddot{T}_{\alpha\beta} = \ddot{\rho}(\mu)g_{\alpha\beta} - \mu \ddot{\rho}'(\mu)[g_{\alpha\beta} + u_\alpha u_\beta],
\]

with trace \( \ddot{T} = 4\ddot{\rho} - 3\mu \ddot{\rho}' \); moreover \( \ddot{p} = \mu^2 \frac{d \ddot{\rho}}{d\mu} = \mu \ddot{\rho}' - \ddot{\rho} \).

### 2.1 Ultrarelativistic equation of state from a perfect fluid geometry

By requiring the perfect fluid Ricci tensor to be also a Rastall’s geometry the following identity should hold true.

\[
\ddot{T}_{\alpha\beta} \equiv \ddot{\rho}(\mu)g_{\alpha\beta} - \mu \ddot{\rho}'(\mu)[g_{\alpha\beta} + u_\alpha u_\beta] =
\]

\[
= [\rho - \mu \rho' - \frac{1}{2} \frac{2\lambda - 1}{4\lambda - 1} (4\rho - 3\mu \rho')] g_{\alpha\beta} - \mu \rho' u_\alpha u_\beta \equiv
\]

\[
\equiv (\ddot{\rho} - \mu \ddot{\rho}') g_{\alpha\beta} - \mu \ddot{\rho}' u_\alpha u_\beta = -\ddot{\rho} g_{\alpha\beta} - (\ddot{\rho} + \ddot{\rho}') u_\alpha u_\beta. \tag{2}
\]

The above identity leads to

\[ \rho' = \ddot{\rho}. \]

By inserting this and deriving with respect to \( \mu \) we get

\[
\frac{2\lambda - 1}{4\lambda - 1} (4\rho' - 3(\mu \rho')') = 0.
\]

Let us now exclude the case \( 2\lambda - 1 = 0 \). We see that a necessary condition for the Ricci tensor of a nontrivial Rastall’s geometry of matter to be a perfect fluid is

\[ 4\rho' - 3(\mu \rho')' = 0. \]

i.e.

\[ \rho' - 3\rho' = 0. \]

The equation of state is therefore

\[ p = \frac{1}{3} \rho + \text{const.}(\mu), \]
i.e. if we denote $\alpha$ the integration constant (a constant function of $\mu$) (uniquely determined by initial data)

$$p - \alpha = \frac{1}{3}\rho$$

In particular it is easy to get that $-3\alpha = \rho - 3p = \bar{T}$, i.e. $\alpha = -\frac{1}{3}\bar{T}$.

By redefining the pressure $\pi = p - \alpha = p + \frac{1}{3}\bar{T}$ the equation of state is the one of an ultrarelativistic perfect fluid with pressure $\pi$ and energy density $\rho$.

$$\pi = \frac{1}{3}\rho \iff p = \frac{1}{3}(\rho - \bar{T})$$

i.e. with a rescaled pressure or equivalently a energy density rescaled by the trace of the energy-momentum tensor. This relation gives us a model with constant (with respect to the matter density $\mu$) energy-tensor trace $\bar{T}$ and consequently constant (with respect to $\mu$) scalar curvature.

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