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Solution theory to semilinear hyperbolic stochastic partial differential equations with polynomially bounded coefficients

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11 **Abstract**

We study mild solutions of a class of stochastic partial differential equations, involving operators with polynomially bounded coefficients. We consider semilinear equations under suitable hyperbolicity hypotheses on the linear part. We provide conditions on the initial data and on the stochastic terms, namely, on the associated spectral measure, so that mild solutions exist and are unique in suitably chosen functional classes. More precisely, function-valued solutions are obtained, as well as a regularity result.

12 *Keywords:* Semilinear stochastic hyperbolic partial differential equations,
13 Variable coefficients, Fourier integral operators

14 *2010 MSC:* Primary: 35L10, 60H15; Secondary: 35L40, 35S30

15 **1. Introduction**

16 The stochastic partial differential equations (SPDEs in the sequel) that we
17 consider in the present paper are of the general form

$$L(t, x, \partial_t, \partial_x)u(t, x) = \gamma(t, x, u(t, x)) + \sigma(t, x, u(t, x))\dot{\Xi}(t, x), \quad (1.1)$$

18 where L is a linear partial differential operator that contains derivatives with
19 respect to time ($t \in \mathbb{R}$) and space ($x \in \mathbb{R}^d$, $d \geq 1$) variables, γ and σ , respectively
20 the drift term and the diffusion coefficient, are real-valued functions, subject
21 to certain regularity conditions, Ξ is a random noise term white in time and
22 colored in space, and u is an unknown stochastic process called *solution* of the
23 SPDE. The equations (1.1) are semilinear: the only possible non-linearities are

24 on the right-hand side, and not in the operator L . In Subsection 1.1 below we
 25 will describe in more detail the conditions we impose on the operator L , the
 26 most important one being (a notion of) hyperbolicity; in Subsection 1.2 we will
 27 describe in detail the noise we consider.

28 Since the sample paths of the solution u are in general not in the domain
 29 of the operator L , in view of the singularity of the random noise, we rewrite
 30 (1.1) in its corresponding integral (i.e., *weak*) form and look for *mild solutions*
 31 of (1.1), that is, stochastic processes $u(t, x)$ satisfying

$$\begin{aligned}
 u(t, x) = v_0(t, x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \gamma(s, y, u(s, y)) dy ds \\
 &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y, u(s, y)) \dot{\Xi}(s, y) dy ds,
 \end{aligned}
 \tag{1.2}$$

32 where:

- 33 - v_0 is a deterministic term, taking into account the initial conditions;
- 34 - Λ is a suitable kernel, associated with the fundamental solution of the
 35 linear partial differential equation (linear PDE in the sequel) $Lu = 0$;
- 36 - the first integral in (1.2) is of deterministic type, while the second is a
 37 stochastic integral.

38 Note that both integrals in (1.2) contain a slight abuse of notation, since $\Lambda(t, s, x, y)$
 39 is, in general, a distribution with respect to the variables $(x, y) \in \mathbb{R}^{2d}$. Given
 40 the commonly wide usage of such so-called *distributional integrals*, we will also
 41 often adopt here this notation in the representation of our class of mild solutions
 42 to (1.1).

43 The kind of solution u we can construct for equation (1.1) depends on the
 44 approach we employ to make sense of the stochastic integral appearing in (1.2).
 45 In the present paper we follow the Da Prato-Zabczyk approach (see [19]), which
 46 consists in associating an Hilbert space valued Brownian motion with the ran-
 47 dom noise. One can then define the stochastic integral as an infinite sum of
 48 Itô integrals with respect to one-dimensional Brownian motions. This leads to
 49 solutions involving random functions taking values in suitable functional spaces.
 50 To our best knowledge, the most general result of existence and uniqueness of
 51 a function-valued solution to hyperbolic SPDEs is given in [28], where the au-
 52 thor considers a semilinear stochastic wave equation having a uniformly elliptic
 53 second order operator A in place of the Laplacian, with uniformly bounded
 54 coefficients depending on $x \in \mathbb{R}^d$, $d \geq 1$. There, sufficient conditions on the
 55 stochastic term $\dot{\Xi}$ and on the coefficients of A are given, in order to find a
 56 unique function-valued solution using semigroup theory. In the present paper
 57 we show existence and uniqueness of a function-valued solution to a wider class
 58 of *semilinear weakly hyperbolic* SPDEs, with *possibly unbounded coefficients* de-
 59 pending on $(t, x) \in [0, T] \times \mathbb{R}^d$, $d \geq 1$, see Subsection 1.1 below.

60 We recall that an alternative approach to give meaning to (1.1) is the one
 61 by Walsh and Dalang (see [10, 17, 34]), where the stochastic integral in (1.2)
 62 is defined as a stochastic integral with respect to a martingale measure derived
 63 from the random noise $\dot{\Xi}$. With this alternative approach one obtains a so-
 64 called *random-field solution*, that is, a solution u defined as a map associating a
 65 random variable to each $(t, x) \in [0, T_0] \times \mathbb{R}^d$, where $T_0 > 0$ is the time horizon of
 66 the equation. It is well known that in many cases the two approaches lead to the
 67 same solution u (in some sense) of an SPDE, see [18] for a precise comparison.

68 In [2, 7] we have constructed random-field solutions for arbitrary order, *lin-*
 69 *ear* weakly hyperbolic SPDEs with possibly unbounded coefficients, smoothly
 70 depending on $(t, x) \in [0, T] \times \mathbb{R}^d$. That construction cannot work for non-linear
 71 equations of the form (1.1). Indeed, the stationarity condition $\Lambda = \Lambda(t-s, x-y)$
 72 would be needed, but such condition (fulfilled by SPDEs with constant coeffi-
 73 cients) cannot be assumed if we want to deal with general linear operators L
 74 in (1.1), that is, admitting variable coefficients. We conclude comparing the
 75 function-valued solutions to (1.1) obtained in the present paper, in the special
 76 case of the linear equations, with the random-field solutions of the same equation
 77 found in [2].

78 We remark that in the present paper, as well as in [2, 7], the main tools used
 79 to construct and study the solutions, namely, pseudodifferential and Fourier
 80 integral operators, come from microlocal analysis, within the so-called *SG* (or
 81 *scattering*) calculus (see [12, 21, 27]). To our best knowledge, in [7] their full
 82 potential has been rigorously applied for the first time within the solution the-
 83 ory of hyperbolic SPDEs. Other applications of these operators in the context
 84 of S(P)DEs can be found in [33], where S(P)DEs are investigated in the frame-
 85 work of function-valued solutions by means of pseudodifferential operators, and
 86 in [25], where a program for employing Fourier integral operators in stochastic
 87 structural analysis is described. We are not aware of any other systematic ap-
 88 plication of microlocal and Fourier integral operators techniques. In particular,
 89 concerning the analysis of weakly semilinear hyperbolic SPDEs with unbounded
 90 coefficients, we provide it here. As it is customary for the classes of the associ-
 91 ated deterministic PDEs, we are interested in both the smoothness, as well as
 92 the decay at spatial infinity, of the solutions. Here we prove an analog of such
 93 *global regularity* properties, employing suitable *weighted Sobolev spaces*, namely,
 94 the so-called Sobolev-Kato spaces.

95 1.1. The equations we consider

96 As mentioned above, we study semilinear SPDEs (1.1) whose partial differen-
 97 tial operators L have coefficients in $(t, x) \in [0, T] \times \mathbb{R}^d$ that may admit a poly-
 98 nomial growth as $|x| \rightarrow \infty$. Namely, we treat *hyperbolic equations* of arbitrary
 99 order $m \in \mathbb{N}$ of the form (1.1), whose coefficients are defined on the whole space
 100 \mathbb{R}^d , with

$$L = D_t^m - \sum_{j=1}^m A_j(t, x, D_x) D_t^{m-j}, \quad A_j(t, x, D) = \sum_{|\alpha| \leq j} a_{\alpha j}(t, x) D_x^\alpha, \quad (1.3)$$

101 where $m \geq 1$, $a_{\alpha j} \in C^\infty([0, T], C^\infty(\mathbb{R}^d))$ for $|\alpha| \leq j$, $j = 0, \dots, m$, and, for all
 102 $k \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^d$, there exists a constant $C_{jk\alpha\beta} > 0$ such that

$$|\partial_t^k \partial_x^\beta a_{\alpha j}(t, x)| \leq C_{jk\alpha\beta} \langle x \rangle^{|\alpha| - |\beta|}, \quad (1.4)$$

103 for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $0 \leq |\alpha| \leq j$, $1 \leq j \leq m$, where $\langle x \rangle := \sqrt{1 + |x|^2}$.
 104 The hyperbolicity of L means that the symbol $\mathcal{L}_m(t, x, \tau, \xi)$ of the SG -principal
 105 part of L , defined here below, satisfies

$$\mathcal{L}_m(t, x, \tau, \xi) := \tau^m - \sum_{j=1}^m \sum_{|\alpha|=j} a_{\alpha j}(t, x) \xi^\alpha \tau^{m-j} = \prod_{j=1}^m (\tau - \tau_j(t, x, \xi)), \quad (1.5)$$

106 with $\tau_j(t, x, \xi)$ real-valued, $\tau_j \in C^\infty([0, T]; S^{1,1}(\mathbb{R}^d))$, $j = 1, \dots, m$. The latter
 107 means that, for any $\alpha, \beta \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0$, there exists a constant $C_{jk\alpha\beta} > 0$ such
 108 that

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta \tau_j(t, x, \xi)| \leq C_{jk\alpha\beta} \langle x \rangle^{1 - |\alpha|} \langle \xi \rangle^{1 - |\beta|}, \quad (1.6)$$

109 for $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$, $j = 1, \dots, m$; we shall refer to (1.6) saying that $\tau_j(t)$ is
 110 a symbol of class $S^{1,1}(\mathbb{R}^{2d})$, see Section 3 below for the precise definition of the
 111 so-called SG -classes of symbols $S^{m,\mu}(\mathbb{R}^d)$, $(m, \mu) \in \mathbb{R}^2$, and the corresponding
 112 class of pseudodifferential operators. The real solutions $\tau_j = \tau_j(t, x, \xi)$, $j =$
 113 $1, \dots, m$, of the equation $\mathcal{L}_m(t, x, \tau, \xi) = 0$ with respect to τ are usually called
 114 *characteristic roots* of the operator L .

115 **Definition 1.1.** We say that (1.3) is *weakly hyperbolic with roots of constant*
 116 *multiplicities* if the real-valued characteristic roots in (1.5) can be divided into
 117 n groups ($1 \leq n \leq m$) of distinct and separated roots, in the sense that,
 118 possibly after a reordering of the τ_j , $j = 1, \dots, m$, there exist $l_1, \dots, l_n \in \mathbb{N}$
 119 with $l_1 + \dots + l_n = m$ and n sets

$$G_1 = \{\tau_1 = \dots = \tau_{l_1}\}, G_2 = \{\tau_{l_1+1} = \dots = \tau_{l_1+l_2}\}, \dots, G_n = \{\tau_{m-l_n+1} = \dots = \tau_m\},$$

120 satisfying, for a constant $C > 0$,

$$\tau_j \in G_p, \tau_k \in G_q, p \neq q, 1 \leq p, q \leq n \Rightarrow |\tau_j(t, x, \xi) - \tau_k(t, x, \xi)| \geq C \langle x \rangle \langle \xi \rangle \quad (1.7)$$

121 for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$. The number $l = \max_{j=1, \dots, n} l_j$ is the *maximum*
 122 *multiplicity of the roots of \mathcal{L}_m* .

123 Notice that, in the case $n = 1$, we have only one group of m coinciding roots,
 124 that is, \mathcal{L}_m admits a single real root of multiplicity m , while for $n = m$ we say
 125 that the operator is strictly hyperbolic; the most famous example of a strictly
 126 hyperbolic operator is given by the wave operator.

127 **Example 1.2.** An example of a weakly hyperbolic operator L with roots of
 128 constant multiplicities is given by

$$L = (D_t^2 - \langle x \rangle^2 \langle D \rangle^2)^2 = D_t^4 - 2 \langle x \rangle^2 \langle D \rangle^2 D_t^2 + \langle x \rangle^4 \langle D \rangle^4 + \text{Op}(p), \quad x \in \mathbb{R}^d,$$

129 $p \in S^{3,3}(\mathbb{R}^d)$, where, for $c \in S^{m,\mu}(\mathbb{R}^d)$, $\text{Op}(c)$ denotes the pseudodifferential
 130 operator with symbol c , see Section 3. The *SG*-principal symbol of L is here
 131 $L_4(x, \tau, \xi) = (\tau^2 - \langle x \rangle^2 \langle \xi \rangle^2)^2$, with *separated* roots $\tau_{\pm}(x, \xi) = \pm \langle x \rangle \langle \xi \rangle$, both of
 132 *multiplicity 2*.

133 **Definition 1.3.** We say that (1.3) is *weakly hyperbolic with involutive roots* if
 134 the real-valued characteristic roots in (1.5) satisfy

$$\begin{aligned} [D_t - \text{Op}(\tau_j(t)), D_t - \text{Op}(\tau_k(t))] = & \text{Op}(a_{jk}(t)) (D_t - \text{Op}(\tau_j(t))) & (1.8) \\ & + \text{Op}(b_{jk}(t)) (D_t - \text{Op}(\tau_k(t))) + \text{Op}(c_{jk}(t)), \end{aligned}$$

135 for some $a_{jk}, b_{jk}, c_{jk} \in C^\infty([0, T], S^{0,0}(\mathbb{R}^d))$, $j, k = 1, \dots, m$.

136 **Remark 1.4.** Recall that roots of constant multiplicities are always involutive,
 137 see, e.g., [2] for a proof. The converse statement is not true in general, as shown
 138 in [24]: the operator

$$L = (D_t + tD_{x_1} + D_{x_2})(D_t - (t - 2x_2)D_{x_1}), \quad x \in \mathbb{R}^2,$$

139 is a weakly hyperbolic operator with involutive roots of non-constant multiplic-
 140 ities.

141 1.2. The stochastic noise

142 Here we describe the class of stochastic noises that we allow in our frame-
 143 work. Consider a distribution-valued Gaussian process $\{\Xi(\phi); \phi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times$
 144 $\mathbb{R}^d)\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with mean zero and covariance
 145 functional given by

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} (\phi(t) * \tilde{\psi}(t))(x) \Gamma(dx) dt, \quad (1.9)$$

146 where $\tilde{\psi}(t, x) := \psi(t, -x)$, $*$ is the convolution operator and Γ is a nonnegative,
 147 nonnegative definite, tempered measure on \mathbb{R}^d . Then, Théorème XVIII in [31,
 148 Chapter VII] implies that there exists a nonnegative tempered measure μ on
 149 \mathbb{R}^d such that $\mathcal{F}\mu = \hat{\mu} = \Gamma$. \mathcal{F} and $\hat{\cdot}$ denote the Fourier transform given, for
 150 functions $f \in L^1(\mathbb{R}^d)$, by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx. \quad (1.10)$$

In (1.10), $x \cdot \xi$ denotes the inner product in \mathbb{R}^d , and the Fourier transform
 is extended to tempered distributions $T \in \mathcal{S}'(\mathbb{R}^d)$ by the relation $\langle \mathcal{F}T, \phi \rangle =$
 $\langle T, \mathcal{F}\phi \rangle$, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. By Parseval's identity, the right-hand side of (1.9)
 can be rewritten as

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} [\mathcal{F}\phi(t)](\xi) \cdot \overline{[\mathcal{F}\psi(t)](\xi)} \mu(d\xi) dt.$$

151 The tempered measure Γ is usually called *correlation measure*. The tempered
 152 measure μ such that $\Gamma = \hat{\mu}$ is usually called *spectral measure*.

153 *1.3. The results we get*

154 We consider the SPDE (1.1) with L as in (1.3), (1.5),(1.7) and Ξ an $\mathcal{S}'(\mathbb{R}^d)$ -
 155 valued Gaussian process with correlation measure Γ and spectral measure μ ad
 156 described here above. We derive conditions on the coefficients of L , on the right-
 157 hand side terms γ and σ , and on the spectral measure μ (hence, on Ξ), such
 158 that there exists a unique function-valued (mild) solution to the corresponding
 159 Cauchy problem. The Cauchy data are going to be taken in Sobolev-Kato spaces

$$H^{z,\zeta}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{z,\zeta} = \|\langle \cdot \rangle^z \langle D \rangle^\zeta u\|_{L^2} < \infty\}, \quad (z, \zeta) \in \mathbb{R}^2. \quad (1.11)$$

160 The coefficients γ, σ will be chosen in suitable classes of Lipschitz functions,
 161 denoted by $\text{Lip}_{\text{loc}}(z, \zeta, r, \rho)$. Namely, for suitable $z, \zeta, r, \rho \in \mathbb{R}$, $r, \rho \geq 0$, we say
 162 that a function g belongs to $\text{Lip}(z, \zeta, r, \rho)$ if it is measurable and satisfies, for
 163 every $t \in [0, T]$,

$$\begin{aligned} \|g(t, \cdot, w)\|_{z,\zeta} &\leq C(t)(1 + \|w\|_{z+r,\zeta+\rho}) \quad \forall w \in H^{z+r,\zeta+\rho}(\mathbb{R}^d), \\ \|g(t, \cdot, w) - g(t, \cdot, v)\|_{z,\zeta} &\leq C(t)\|w - v\|_{z+r,\zeta+\rho} \quad \forall w, v \in H^{z+r,\zeta+\rho}(\mathbb{R}^d). \end{aligned}$$

164 More generally, we say that $g \in \text{Lip}_{\text{loc}}(z, \zeta, r, \rho)$ if the stated properties hold
 165 true for $w, v \in U$, with U a suitable open subset of $H^{z+r,\zeta+\rho}(\mathbb{R}^d)$. The precise
 166 description of the assumptions on σ and γ are postponed to Section 4, while
 167 we immediately give two examples of diffusion coefficients σ which fulfill the
 168 requested hypotheses.

169 **Example 1.5.** Let $\sigma(t, x, u) = u^2$. Then, σ is an admissible non-linearity for
 170 the equations we consider. More generally, we allow $\sigma(t, x, u) = u^n$, $n \in \mathbb{N}$,
 171 $n > 2$.

172 **Example 1.6.** A right-hand side explicitly depending on $(t, x) \in [0, T] \times \mathbb{R}^d$
 173 and u , which is admissible for the equations we consider, is

$$\sigma(t, x, u) = \langle x \rangle^{l-m} \cdot \tilde{\sigma}(t, u), \quad (1.12)$$

174 where l is the maximum multiplicity of the roots and $\tilde{\sigma}$ is regular in time, satisfies
 175 suitable mapping properties with respect to the Sobolev-Kato spaces, and is
 176 (uniformly, locally) Lipschitz-continuous with respect to the second variable,
 177 see Definition 4.2 and Example 4.13 below for the precise conditions.

178 To our best knowledge, a diffusion coefficient of the rather general form
 179 (1.12) has never been systematically treated in the literature, except in [30],
 180 where, for $m = 2$, it has been incorporated in a certain model equation by
 181 means of ad-hoc techniques.

Example 1.7. More generally, a routine extension of the theory developed in
 the present paper allows for a stochastic term of the very general form

$$\sigma(t, x, u, D_x u, \dots, D_x^\alpha u), \quad |\alpha| \leq m - 1$$

182 in the right-hand side of (1.1). The only difference consists in the form of the
 183 Lipschitzianity assumptions and the corresponding mapping properties, see again
 184 Section 4.

185 We state here below the main result of the paper, whose precise formulation
 186 is given in Theorem 4.8. As customary for weakly hyperbolic operators, to
 187 achieve well-posedness we need to assume that the lower order terms of L satisfy
 188 (an adapted form of) a Levi condition (see (A.24) and Corollary A.13). This
 189 allows to give an explicit expression for the distribution $\Lambda(t, s)$ in terms of
 190 kernels of suitable Fourier integral operators, see (A.26). We work under an
 191 hypothesis of Lipschitz continuity for the nonlinearities in the right-hand side
 192 (see Definition 4.2 and Remark 4.3).

193 **Main Theorem.** *Consider the Cauchy problem for the SPDE (1.1) with L a*
 194 *weakly hyperbolic operator with roots of constant multiplicity, that is, L satisfies*
 195 *(1.3), (1.5), (1.7). Assume, for the spectral measure associated with Ξ , that*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) < \infty, \quad (1.13)$$

196 *where l is the maximum multiplicity of the roots of \mathcal{L}_m , $1 \leq l \leq m$. Moreover,*
 197 *assume that L is of Levi type and that $\gamma, \sigma \in \text{Lip}_{\text{loc}}(z, \zeta, m - l, 0)$, $z, \zeta \in \mathbb{R}$.*
 198 *Then, there exists a time horizon $0 < T_0 \leq T$ such that, for any choice of*
 199 *$u_j \in H^{z+m-1-j, \zeta+m-1-j}(\mathbb{R}^d)$, $0 \leq j \leq m-1$, the Cauchy problem admits a*
 200 *unique solution $u \in L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta}(\mathbb{R}^d))$ satisfying (1.2), where the*
 201 *first integral is a Bochner integral, and the second integral is understood as*
 202 *the stochastic integral of a suitable $H^{z+m-l, \zeta}(\mathbb{R}^d)$ -valued stochastic process with*
 203 *respect to the stochastic noise Ξ .*

204 Notice that the more general are the assumptions on L (i.e., the larger is
 205 l), the smallest is the class of the stochastic noises that we can allow to get a
 206 function-valued solution. Our main Theorem extends the results of [28] to the
 207 case of general higher order hyperbolic equations with coefficients in (t, x) , not
 208 uniformly bounded with respect to x and with roots that may coincide.

209 **Remark 1.8.** In Corollary 4.10 we explicitly write the result we get in the
 210 limit case $l = 1$, corresponding to strictly hyperbolic equations. We remark
 211 that in this case L automatically satisfies the Levi condition. Moreover, when
 212 $m = 2$, $l = 1$, and Γ is absolutely continuous, condition (1.13) reduces to the
 213 well-known condition $\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < \infty$, needed for existence and uniqueness
 214 of a solution to the stochastic wave equation.

215 We conclude the paper with a result concerning operators with involutive
 216 characteristics. We show that

217 *if L is weakly hyperbolic with involutive roots and $\int_{\mathbb{R}^d} \mu(d\xi) < \infty$, then,*
 218 *under suitable assumptions on γ, σ and the Cauchy data, there exists a unique*
 219 *function-valued solution to the Cauchy problem associated with the SPDE (1.1),*

220 see Theorem 4.14 for the precise statement. Notice that the condition on the
 221 spectral measure for the latter case coincides with (1.13) in the case $l = m$, and
 222 that all such conditions coincide when $m = 1$.

223 *1.4. Tools we employ*

224 The main tools for proving existence and uniqueness of solutions to (1.1)
225 will be the calculus of Fourier integral operators with symbols in the so-called
226 *SG* classes. Such symbols classes have been introduced in the '70s by H.O.
227 Cordes (see, e.g. [12]) and C. Parenti [27] (see also the *scattering calculus* by
228 R. Melrose, e.g. [21]).

229 Applications of the *SG* FIOs theory to *SG*-hyperbolic Cauchy problems
230 were initially given in [14, 16]. Many authors have, since then, expanded the
231 *SG* FIOs theory and its applications to the solution of hyperbolic problems in
232 various directions. To mention a few, see, e.g., M. Ruzhansky, M. Sugimoto
233 [29], E. Cordero, F. Nicola, L Rodino [11], and the references quoted there and
234 in [5].

235 In [5], Cauchy problems for general *SG*-hyperbolic first order systems have
236 been studied, constructing their fundamental solution $\{E(t, s)\}_{0 \leq s \leq t \leq T}$. The
237 existence of the fundamental solution provides, via Duhamel's formula, exist-
238 ence and uniqueness of the solution to the system, for any given Cauchy data
239 in the weighted Sobolev spaces $H^{z, \zeta}(\mathbb{R}^d)$, $(z, \zeta) \in \mathbb{R}^2$. A remarkable feature,
240 typical for these classes of hyperbolic problems, is the *well-posedness with loss*
241 *of decay/increase of growth at infinity*, see [3, 4, 16].

242 There are various techniques to switch from a Cauchy problem for an *SG*-
243 hyperbolic operator L of order $m \geq 2$ to a Cauchy problem for a first order
244 system, see, e.g., [1, 12, 14, 24]. In the approach we follow here, which is the
245 same used in [1, 16], one of the key results for this aim is an adapted version
246 of the so-called Mizohata Lemma of Perfect Factorization, see Proposition A.12
247 and Lemma A.15 in the Appendix¹. To construct the fundamental solution
248 of the operator L involved in (1.1), through the fundamental solution of the
249 associated first order system, we need, on one hand, to perform compositions
250 between pseudo-differential operators and Fourier integral operators of *SG* type,
251 using the theory developed in [13], and, on the other hand, compositions between
252 Fourier integral operators of *SG* type with possibly different phase functions.
253 The latter can be achieved using the composition results obtained in [5]. The
254 proof of the main theorems of the paper employs such fundamental solution,
255 together with the application of a fixed point scheme in suitable functional
256 spaces.

257 *1.5. Organization of the paper*

258 To provide a presentation of our results as self-contained as possible, for
259 the convenience of the reader, we provide (at different levels of detail) various
260 preliminaries from the existing literature, as described below.

261 In Section 2 we recall some notions about stochastic integration with respect
262 to Hilbert space-valued processes and the corresponding concept of function-
263 valued solution, following [19].

¹See also [20, 22, 23], for the original version of such results.

264 In Section 3 we give a description of the tools coming from microlocal analysis
 265 that we use for the construction of the fundamental solution of weakly hyperbolic
 266 with polynomially bounded coefficients.

267 In Section 4 we focus on the semilinear hyperbolic SPDE (1.1), (1.3), (1.5),
 268 and in Theorem 4.8 we study existence and uniqueness of a function-valued
 269 solution under the assumption of weak hyperbolicity with roots of constant
 270 multiplicity (1.7). Notice again that the case of strict hyperbolicity (the one of
 271 the waves) reduces to the special case $l = 1$ of Theorem 4.8, and needs no Levi
 272 condition. We give sufficient conditions on the coefficients, on the noise and
 273 on the right-hand side of (1.1) such that there exists a unique mild function-
 274 valued solution of the corresponding Cauchy problem. The key result to achieve
 275 existence and uniqueness of the solution is Lemma 4.6, which is a further main
 276 result in the present paper. We also prove, in Theorem 4.14, a similar result
 277 under the assumption of weak hyperbolicity with involutive roots (1.8). Finally,
 278 we make a comparison between the function-valued solutions obtained here, in
 279 the special case of linear equations, with the random-field solutions found in [2].

280 Some additional details about the tools we employ, coming from the micro-
 281 local approach to the solution of hyperbolic Cauchy problems for PDEs and
 282 systems associated with operators with polynomially bounded coefficients, see
 283 [2, 5, 12, 13, 14, 16], are summarized in the Appendix.

284 1.6. Notation

285 Throughout this article, we let $\langle a \rangle := (1 + |a|^2)^{1/2}$ for all $a \in \mathbb{R}^d$, and
 286 we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_*^d := \mathbb{R}^d \setminus \{0\}$. Also, α and β will generally de-
 287 note multiindices, with their standard arithmetic operations. As usual, we will
 288 denote partial derivatives with ∂ , and set $D = -i\partial$, i being the imaginary
 289 unit, which is convenient when dealing with Fourier transformations. We will
 290 denote by $C^m(X)$, $C_0^m(X)$, $\mathcal{S}(X)$, $\mathcal{D}(X)$, $\mathcal{S}'(X)$ and $\mathcal{D}'(X)$, the m -times con-
 291 tinuously differentiable functions, the m -times continuously differentiable func-
 292 tions with compact support, the Schwartz functions, the test functions space
 293 $C_0^\infty(X)$, the tempered distributions and the distributions on some finite or
 294 infinite-dimensional space X , respectively. Usually, $C > 0$ will denote a generic
 295 constant, whose value can change from line to line without further notice. When
 296 operator composition is considered, we will usually insert the symbol \circ when the
 297 notation $\text{Op}(b)$ and/or $\text{Op}_\varphi(a)$, for pseudodifferential and Fourier integral op-
 298 erators, respectively, are adopted for both factors, as well as in some situations
 299 where parameter-dependent operators occurs, for the sake of clarity. When at
 300 least one of the operators involved in the product of composition is denoted by
 301 a single capital letter, and when no confusion can occur, we will, as custom-
 302 ary, omit the symbol \circ completely, and just write, e.g., PQ , RD_t , etc. Finally,
 303 $A \asymp B$ means that the estimates $A \lesssim B$ and $B \lesssim A$ hold true, where $A \lesssim B$
 304 means that $|A| \leq c \cdot |B|$, for a suitable constant $c > 0$.

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318 **2. Stochastic integration.**

319 The mild formulation (1.2) is the way in which we understand the SPDE
 320 (1.1). In fact, we call (*mild*) *function-valued solution to (1.1)* an $L^2(\Omega)$ -family
 321 of random variables $u(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$, jointly measurable, satisfying
 322 the stochastic integral equation (1.2) where the last term in the right-hand side
 323 is understood within the theory of stochastic integrals taking value in Hilbert
 324 spaces.

325 In this section we recall some of the main results of the theory of stochastic
 326 integration with respect to cylindrical Wiener processes. Also, we recall the
 327 definition of the Hilbert space \mathcal{H} which will be suitable for our purposes of
 328 function-valued solutions to SPDEs. For the latter, we follow the exposition in
 329 [18].

330 **Definition 2.1.** Let Q be a self-adjoint, nonnegative definite and bounded
 331 linear operator on a separable Hilbert space H . An H -valued stochastic process
 332 $W = \{W_t(h); h \in H, t \geq 0\}$ is called a *cylindrical Wiener process on H* on the
 333 complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if the following conditions are fulfilled:

- 334 1. for any $h \in H$, $\{W_t(h); t \geq 0\}$ is a one-dimensional Brownian motion with
 335 variance $t\langle Qh, h \rangle_H$;
 336 2. for all $s, t \geq 0$ and $g, h \in H$,

$$\mathbb{E}[W_s(g)W_t(h)] = (s \wedge t)\langle Qg, h \rangle_H.$$

337 If $Q = Id_H$, then W is called a standard cylindrical Wiener process.

338 Let \mathcal{F}_t be the σ -field generated by the random variables $\{W_t(h); 0 \leq s \leq$
 339 $t, h \in H\}$ and the \mathbb{P} -null sets. The predictable σ -field is then the σ -field in
 340 $[0, T] \times \Omega$ generated by the sets $\{(s, t] \times A, A \in \mathcal{F}_t, 0 \leq s < t \leq T\}$.

341 We define H_Q to be the completion of the Hilbert space H endowed with
 342 the inner product

$$\langle g, h \rangle_{H_Q} := \langle Qg, h \rangle_H,$$

343 for $g, h \in H$. In the sequel, we let $\{v_k\}_{k \in \mathbb{N}}$ be a complete orthonormal basis of
 344 H_Q . Then, the stochastic integral of a predictable, square-integrable stochastic
 345 process with values in H_Q , $u \in L^2([0, T] \times \Omega; H_Q)$, is defined as

$$\int_0^t u(s) dW_s := \sum_{k \in \mathbb{N}} \langle u, v_k \rangle_{H_Q} dW_s(v_k).$$

346 In fact, the series in the right-hand side converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and its sum
 347 does not depend on the chosen orthonormal system $\{v_k\}_{k \in \mathbb{N}}$. Moreover, the Itô
 348 isometry

$$\mathbb{E} \left[\left(\int_0^t u(s) dW_s \right)^2 \right] = \mathbb{E} \left[\int_0^t \|u(s)\|_{H_Q}^2 ds \right]$$

349 holds true for any $u \in L^2([0, T] \times \Omega; H_Q)$. For more on one-dimensional inte-
 350 gration, see, e.g., [26].

351 This notion of stochastic integral can also be extended to operator-valued
 352 integrands. Let U be a separable Hilbert space and define $L_2^0 := L_2(H_Q, U)$ the
 353 set of Hilbert-Schmidt operators from H_Q to U . With this we can define the
 354 space of integrable processes (with respect to W) as the set of \mathcal{F} -measurable
 355 processes in $L^2([0, T] \times \Omega; L_2^0)$. Since one can identify the Hilbert-Schmidt op-
 356 erators $L_2(H_Q, U)$ with $U \otimes H_Q^*$, one can define the stochastic integral for any
 357 $u \in L^2([0, T] \times \Omega; L_2^0)$ coordinatewise in U . Moreover, it is possible to establish
 358 an Itô isometry, namely,

$$\mathbb{E} \left[\left\| \int_0^t u(s) dW_s \right\|_U^2 \right] := \int_0^t \mathbb{E} [\|u(s)\|_{L_2^0}^2] ds. \quad (2.1)$$

359 The stochastic noise introduced in Subsection 1.2 can be rewritten in terms
 360 of a cylindrical Wiener process. The space $\mathcal{C}_0^\infty(\mathbb{R}^d)$, with pre-inner product

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \mathcal{F} \phi(\xi) \overline{\mathcal{F} \psi(\xi)} \mu(d\xi),$$

361 can be completed to

$$\mathcal{H} := \overline{\mathcal{C}_0^\infty(\mathbb{R}^d)}^{\langle \cdot, \cdot \rangle_{\mathcal{H}}},$$

362 see [18, Lemma 2.4]. Then, $(\mathcal{H}; \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a real separable Hilbert space. We
 363 also set

$$\mathcal{H}_T := L^2([0, T]; \mathcal{H}).$$

364 Then, [18, Proposition 2.5] states the following result.

365 **Proposition 2.2.** *For $t \geq 0$ and $\phi \in \mathcal{H}$, set $W_t(\phi) = W(1_{[0, t]}(\cdot)\phi(\cdot))$. Then,
 366 the process $W = \{W_t(\phi), t \geq 0, \phi \in \mathcal{H}\}$ is a standard cylindrical Wiener process
 367 on \mathcal{H} (where we recall that “standard” here means assuming $Q = Id_{\mathcal{H}}$).*

368 **3. Microlocal analysis for linear operators with polynomially bounded**
 369 **coefficients**

370 We first recall some basic definitions and facts about the so-called *SG*-
 371 calculus of pseudodifferential and Fourier integral operators, through standard
 372 material appeared, e.g., in [5] and elsewhere (sometimes with slightly different
 373 notational choices). We include in the Appendix some additional details about
 374 the theory of hyperbolic linear operators in this context, to give a presentation
 375 as self-contained as possible.

376 The class $S^{m,\mu} = S^{m,\mu}(\mathbb{R}^d)$ of *SG* symbols of order $(m, \mu) \in \mathbb{R}^2$ is given by
 377 all the functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the property that, for any multiindices
 378 $\alpha, \beta \in \mathbb{N}_0^d$, there exist constants $C_{\alpha\beta} > 0$ such that the conditions

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m-|\alpha|} \langle \xi \rangle^{\mu-|\beta|}, \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad (3.1)$$

379 hold true, see, e.g., [12, 21, 27] for details. For $m, \mu \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, $a \in S^{m,\mu}$, the
 380 quantities

$$\|a\|_\ell^{m,\mu} = \max_{|\alpha+\beta| \leq \ell} \sup_{x, \xi \in \mathbb{R}^d} \langle x \rangle^{-m+|\alpha|} \langle \xi \rangle^{-\mu+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \quad (3.2)$$

381 are a family of seminorms, defining the Fréchet topology of $S^{m,\mu}$.

382 The corresponding classes of pseudodifferential operators $\text{Op}(S^{m,\mu}) = \text{Op}(S^{m,\mu}(\mathbb{R}^d))$
 383 are given by

$$(\text{Op}(a)u)(x) = (a(\cdot, D)u)(x) = (2\pi)^{-d} \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) d\xi, \quad a \in S^{m,\mu}(\mathbb{R}^d), u \in \mathcal{S}(\mathbb{R}^d), \quad (3.3)$$

extended by duality to $\mathcal{S}'(\mathbb{R}^d)$. The operators in (3.3) form a graded algebra
 with respect to composition, i.e.,

$$\text{Op}(S^{m_1, \mu_1}) \circ \text{Op}(S^{m_2, \mu_2}) \subseteq \text{Op}(S^{m_1+m_2, \mu_1+\mu_2}).$$

384 The symbol $c \in S^{m_1+m_2, \mu_1+\mu_2}$ of the composed operator $\text{Op}(a) \circ \text{Op}(b)$, $a \in$
 385 S^{m_1, μ_1} , $b \in S^{m_2, \mu_2}$, admits the asymptotic expansion

$$c(x, \xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi), \quad (3.4)$$

386 which implies that the symbol c equals $a \cdot b$ modulo $S^{m_1+m_2-1, \mu_1+\mu_2-1}$.

387 The residual elements of the calculus are operators with symbols in

$$S^{-\infty, -\infty} = S^{-\infty, -\infty}(\mathbb{R}^d) = \bigcap_{(m, \mu) \in \mathbb{R}^2} S^{m, \mu}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d}),$$

388 that is, those having kernel in $\mathcal{S}(\mathbb{R}^{2d})$, continuously mapping $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.
 389 For any $a \in S^{m,\mu}$, $(m, \mu) \in \mathbb{R}^2$, $\text{Op}(a)$ is a linear continuous operator from
 390 $\mathcal{S}(\mathbb{R}^d)$ to itself, extending to a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself,

391 and from $H^{z,\zeta}(\mathbb{R}^d)$ to $H^{z-m,\zeta-\mu}(\mathbb{R}^d)$, where $H^{z,\zeta}(\mathbb{R}^d)$, $(z, \zeta) \in \mathbb{R}^2$, denotes the
392 Sobolev-Kato (or *weighted Sobolev*) space defined in (1.11), with the naturally
393 induced Hilbert norm. When $z \geq z'$ and $\zeta \geq \zeta'$, the continuous embedding
394 $H^{z,\zeta} \hookrightarrow H^{z',\zeta'}$ holds true. It is compact when $z > z'$ and $\zeta > \zeta'$. Since
395 $H^{z,\zeta} = \langle \cdot \rangle^z H^{0,\zeta} = \langle \cdot \rangle^z H^\zeta$, with H^ζ the usual Sobolev space of order $\zeta \in \mathbb{R}$, we
396 find $\zeta > k + \frac{d}{2} \Rightarrow H^{z,\zeta} \hookrightarrow C^k$, $k \in \mathbb{N}_0$.

397 **Remark 3.1.** Notice that in [28] the author uses the space

$$L_\omega^2 := \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \sqrt{\omega}u \in L^2(\mathbb{R}^d)\},$$

398 where $\omega(x) \in \mathcal{S}(\mathbb{R}^d)$ is a strictly positive even function such that for $|x| \geq 1$ we
399 have $\omega(x) = e^{-|x|}$. The weight ω can be substituted by $\omega(x) = \langle x \rangle^{-2z}$, $z > 0$,
400 with corresponding space

$$L_\omega^2 := \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \langle x \rangle^{-z}u \in L^2(\mathbb{R}^d)\},$$

401 coinciding with $H^{-z,0}(\mathbb{R}^d)$ in the notation above. In Section 4 we shall use the
402 $H^{z,\zeta}(\mathbb{R}^d)$ spaces to get a function-valued solution to (1.1).

403 One actually finds

$$\bigcap_{z,\zeta \in \mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{\infty,\infty}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d), \quad \bigcup_{z,\zeta \in \mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{-\infty,-\infty}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d), \quad (3.5)$$

404 as well as, for the space of *rapidly decreasing distributions*, see [6, 31],

$$\mathcal{S}'(\mathbb{R}^d)_\infty = \bigcap_{z \in \mathbb{R}} \bigcup_{\zeta \in \mathbb{R}} H^{z,\zeta}(\mathbb{R}^d). \quad (3.6)$$

405 Cordes introduced the class $\mathcal{O}(m, \mu)$ of the *operators of order* (m, μ) as
406 follows, see, e.g., [12].

407 **Definition 3.2.** A linear continuous operator $A: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ belongs to the
408 class $\mathcal{O}(m, \mu)$, $(m, \mu) \in \mathbb{R}^2$, of the operators of order (m, μ) if, for any $(z, \zeta) \in$
409 \mathbb{R}^2 , it extends to a linear continuous operator $A_{z,\zeta}: H^{z,\zeta}(\mathbb{R}^d) \rightarrow H^{z-m,\zeta-\mu}(\mathbb{R}^d)$.
410 We also define

$$\mathcal{O}(\infty, \infty) = \bigcup_{(m,\mu) \in \mathbb{R}^2} \mathcal{O}(m, \mu), \quad \mathcal{O}(-\infty, -\infty) = \bigcap_{(m,\mu) \in \mathbb{R}^2} \mathcal{O}(m, \mu).$$

411 **Remark 3.3.** 1. Trivially, any $A \in \mathcal{O}(m, \mu)$ admits a linear continuous ex-
412 tension $A_{\infty,\infty}: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$. In fact, in view of (3.5), it is enough to
413 set $A_{\infty,\infty}|_{H^{z,\zeta}(\mathbb{R}^d)} = A_{z,\zeta}$.
414 2. Theorem A.1 implies $\text{Op}(S^{m,\mu}(\mathbb{R}^d)) \subset \mathcal{O}(m, \mu)$, $(m, \mu) \in \mathbb{R}^2$.
415 3. $\mathcal{O}(\infty, \infty)$ and $\mathcal{O}(0, 0)$ are algebras under operator multiplication, $\mathcal{O}(-\infty, -\infty)$
416 is an ideal of both $\mathcal{O}(\infty, \infty)$ and $\mathcal{O}(0, 0)$, and $\mathcal{O}(m_1, \mu_1) \circ \mathcal{O}(m_2, \mu_2) \subset$
417 $\mathcal{O}(m_1 + m_2, \mu_1 + \mu_2)$.

418 We now introduce the class of SG -phase functions.

419 **Definition 3.4** (SG -phase function). A real valued function $\varphi \in C^\infty(\mathbb{R}^{2d})$ be-
 420 longs to the class \mathfrak{P} of SG -phase functions if it satisfies the following conditions:

- 421 1. $\varphi \in S^{1,1}(\mathbb{R}^d)$;
 422 2. $\langle \varphi'_x(x, \xi) \rangle \asymp \langle \xi \rangle$ as $|(x, \xi)| \rightarrow \infty$;
 423 3. $\langle \varphi'_\xi(x, \xi) \rangle \asymp \langle x \rangle$ as $|(x, \xi)| \rightarrow \infty$.

For any $a \in S^{m,\mu}$, $(m, \mu) \in \mathbb{R}^2$, $\varphi \in \mathfrak{P}$, the SG FIOs are defined, for $u \in \mathcal{S}(\mathbb{R}^n)$, as

$$(\text{Op}_\varphi(a)u)(x) = (2\pi)^{-d} \int e^{i\varphi(x,\xi)} a(x, \xi) \widehat{u}(\xi) d\xi, \quad (3.7)$$

and

$$(\text{Op}_\varphi^*(a)u)(x) = (2\pi)^{-d} \iint e^{i(x \cdot \xi - \varphi(y, \xi))} \overline{a(y, \xi)} u(y) dy d\xi. \quad (3.8)$$

424 Here the operators $\text{Op}_\varphi(a)$ and $\text{Op}_\varphi^*(a)$ are sometimes called SG FIOs of type
 425 I and type II, respectively, with symbol a and (SG -)phase function φ . Note
 426 that a type II operator satisfies $\text{Op}_\varphi^*(a) = \text{Op}_\varphi(a)^*$, that is, it is the formal
 427 L^2 -adjoint of the type I operator $\text{Op}_\varphi(a)$.

428 The analysis of SG FIOs started in [13], where composition results with the
 429 classes of SG pseudodifferential operators, and of SG FIOs of type I and type II
 430 with regular phase functions, have been proved. Also the basic continuity prop-
 431 erties in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ of operators in the class have been proved there, as
 432 well as a version of the Asada-Fujiwara $L^2(\mathbb{R}^d)$ -continuity, for operators $\text{Op}_\varphi(a)$
 433 with symbol $a \in S^{0,0}$ and regular SG -phase function $\varphi \in \mathfrak{P}_\delta$, see Definition
 434 3.6. The following theorem summarizes composition results between SG pseu-
 435 dodifferential operators and SG FIOs of type I that we are going to use in the
 436 present paper, see [13] for proofs and composition results with SG FIOs of type
 437 II.

Theorem 3.5. *Let $\varphi \in \mathfrak{P}$ and assume $b \in S^{m_1, \mu_1}(\mathbb{R}^d)$, $a \in S^{m_2, \mu_2}(\mathbb{R}^d)$, $(m_j, \mu_j) \in \mathbb{R}^2$, $j = 1, 2$. Then,*

$$\begin{aligned} \text{Op}(b) \circ \text{Op}_\varphi(a) &= \text{Op}_\varphi(c_1 + r_1) = \text{Op}_\varphi(c_1) \quad \text{mod } \text{Op}(S^{-\infty, -\infty}(\mathbb{R}^d)), \\ \text{Op}_\varphi(a) \circ \text{Op}(b) &= \text{Op}_\varphi(c_2 + r_2) = \text{Op}_\varphi(c_2) \quad \text{mod } \text{Op}(S^{-\infty, -\infty}(\mathbb{R}^d)), \end{aligned}$$

438 *for some $c_j \in S^{m_1+m_2, \mu_1+\mu_2}(\mathbb{R}^d)$, $r_j \in S^{-\infty, -\infty}(\mathbb{R}^d)$, $j = 1, 2$.*

439 To consider the composition of SG FIOs of type I and type II some more
 440 hypotheses are needed, leading to the definition of the classes \mathfrak{P}_δ and $\mathfrak{P}_\delta(\lambda)$ of
 441 regular SG -phase functions.

442 **Definition 3.6** (Regular SG -phase function). Let $\lambda \in [0, 1)$ and $\delta > 0$. A
 443 function $\varphi \in \mathfrak{P}$ belongs to the class $\mathfrak{P}_\delta(\lambda)$ if it satisfies the following conditions:

- 444 1. $|\det(\varphi''_{x\xi})(x, \xi)| \geq \delta, \forall(x, \xi);$
 445 2. the function $J(x, \xi) := \varphi(x, \xi) - x \cdot \xi$ is such that

$$\sup_{\substack{x, \xi \in \mathbb{R}^d \\ |\alpha + \beta| \leq 2}} \frac{|D_\xi^\alpha D_x^\beta J(x, \xi)|}{\langle x \rangle^{1-|\beta|} \langle \xi \rangle^{1-|\alpha|}} \leq \lambda. \quad (3.9)$$

446 If only condition (1) holds, we write $\varphi \in \mathfrak{P}_\delta$.

447 The result of a composition of SG FIOs of type I and type II with the same
 448 regular SG -phase functions is a SG pseudodifferential operator, see again [13].
 449 The continuity properties of regular SG FIOs on the Sobolev-Kato spaces can
 450 be expressed as follows, using the operators of order $(m, \mu) \in \mathbb{R}^2$ introduced
 451 above.

452 **Theorem 3.7.** *Let φ be a regular SG phase function and $a \in S^{m, \mu}(\mathbb{R}^d)$,*
 453 *$(m, \mu) \in \mathbb{R}^2$. Then, $\text{Op}_\varphi(a) \in \mathcal{O}(m, \mu)$.*

454 4. Function-valued solutions for semilinear SPDEs.

455 In this section we state and prove our main result of existence and uniqueness
 456 of a function-valued solution of the SPDE (1.1), under suitable assumptions of
 457 hyperbolicity for the operator L , see (1.3), (1.5). We work here with a class of
 458 operators with more general symbols than the (polynomial) ones appearing in
 459 (1.3). Namely, we consider operators of the form

$$L = D_t^m - \sum_{j=1}^m A_j(t, x, D_x) D_t^{m-j}, \quad (4.1)$$

460 where $A_j(t) = \text{Op}(a_j(t))$ are SG pseudo-differential operators with symbols
 461 $a_j \in C^\infty([0, T], S^{j, j})$, $1 \leq j \leq m$. Notice that, of course, (1.3) is a particular
 462 case of (4.1). The hyperbolicity condition on L becomes

$$\mathcal{L}_m(t, x, \tau, \xi) = \tau^m - \sum_{j=1}^m \tilde{A}_j(t, x, \xi) \tau^{m-j} = \prod_{j=1}^m (\tau - \tau_j(t, x, \xi)), \quad (4.2)$$

463 where \tilde{A}_j stands for the principal part of A_j , with characteristic roots $\tau_j(t, x, \xi) \in$
 464 \mathbb{R} , $\tau_j \in C^\infty([0, T]; S^{1, 1})$. Let us then consider the Cauchy problem

$$\begin{cases} Lu(t, x) = \gamma(t, x, u(t, x)) + \sigma(t, x, u(t, x)) \dot{\Xi}(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d \\ D_t^j u(0, x) = u_j(x), & x \in \mathbb{R}^d, 0 \leq j \leq m-1, \end{cases} \quad (4.3)$$

465 where L has the form (4.1), under conditions (4.2) and either (1.7) or (1.8).
 466 We also assume that $\gamma, \sigma : [0, +\infty) \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions,
 467 (at least locally-)Lipschitz-continuous, in our functional setting, with respect
 468 to the third variable, see Definition 4.2 and Theorem 4.8 below for the precise

469 hypotheses. Such assumptions are typical in semilinear problems. $\dot{\Xi}$ is the
 470 stochastic noise described in Subsection 1.2.

471 We are interested in finding conditions on L , on the stochastic noise $\dot{\Xi}$, and
 472 on $\sigma, \gamma, u_j, j = 0, \dots, m - 1$, such that (4.3) admits a unique function-valued
 473 solution of the form (1.2), following the stochastic integration theory presented
 474 in Section 2.

475 To this aim, we need first the distribution kernel Λ . Its construction for
 476 the weakly hyperbolic operators with roots of constant multiplicities is recalled,
 477 for the reader's convenience, in the Appendix (see also [2]), and consists of the
 478 following steps:

479 - reduction of the (formal) Cauchy problem

$$\begin{cases} Lu(t) = g(t) & t \in (0, T] \\ D_t^j u(0) = u_j, & 0 \leq j \leq m - 1, \end{cases} \quad (4.4)$$

480 where L is the operator in (4.3) and g is a short notation for the right-hand
 481 side, to an equivalent first order system;

482 - construction of the fundamental solution $E(t, s)$ for the system by The-
 483 orem A.6, and then of its (formal) solution, following Section 3 and the
 484 Appendix;

485 - construction of the distribution kernel Λ and of the (formal) solution to
 486 (4.4), in view of the equivalence of (4.4) and the corresponding first order
 487 system.

488 Notice that all the results on SG -hyperbolic differential operators recalled in
 489 Section 3 and the Appendix, in particular, Proposition A.12 and Lemma A.15,
 490 still hold true for SG -hyperbolic operators of the form (4.1). We adopt the same
 491 terminology and definitions also for this more general operators, with straight-
 492 forward modifications, where needed. In particular, the mentioned results imply
 493 that the distribution Λ is a finite sum of Schwartz kernels of Fourier integral
 494 operators with amplitudes of order $(l - m, l - m)$, see (A.26), (A.27).

495 Next, we need to understand the noise Ξ in terms of a canonically associated
 496 Hilbert space \mathcal{H}_Ξ , so that we can define the stochastic integral with respect to
 497 a cylindrical Wiener process on \mathcal{H}_Ξ . This is done in Subsection 4.1 here below.
 498 The conditions on the stochastic noise will be given on the spectral measure μ
 499 corresponding to the correlation measure Γ related to $\dot{\Xi}$.

500 Finally, in Subsection 4.2 we state and prove the first main result of this
 501 paper, namely Theorem 4.8. We will also prove in Theorem 4.14 a further
 502 result, for the involutive roots case, relying on the construction of the kernel Λ
 503 performed in [1]. In both situations, we can apply a fixed point technique, in
 504 view of the fundamental Lemma 4.6, which is the crucial step to achieve our
 505 claims.

506 **Remark 4.1.** With respect to the existing literature, in particular [28], we al-
 507 low here for general hyperbolic equations of higher orders, coefficients depending

508 both on time and space, and possibly with a polynomial growth with respect to
 509 x . We observe that in the strictly hyperbolic case, that is, for $l = 1$, the com-
 510 patibility condition (4.11) exactly corresponds, for $m = 2$, to the one obtained
 511 in [28].

512 *4.1. Admissible spectral measures for Hilbert space valued stochastic integrals.*

513 In this subsection we want to make sense of the stochastic integral appearing
 514 in (1.2) as a stochastic integral with respect to a cylindrical Wiener process on
 515 a Hilbert space, as described in Section 2. We know from (A.27) that, in the
 516 stochastic integral appearing in (1.2), Λ is the kernel of (a linear combination
 517 of) FIOs Z_{l-m} , with amplitudes of order $(l - m, l - m)$, where l stands for the
 518 maximum multiplicity of the characteristic roots ($l = 1$ in the case of a strictly
 519 hyperbolic operator, $1 < l \leq m$ in the constant multiplicities case). To give
 520 meaning to

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y, u(s, y)) \dot{\Xi}(s, y) dy ds = \int_0^t Z_{l-m}(t, s) \sigma(s, u(s)) d\Xi(s), \quad (4.5)$$

521 we first introduce the so-called Cameron-Martin space associated with Ξ . Given
 522 the Gaussian process Ξ described in Section 1.2, let us define

$$\mathcal{H}_\Xi = \{\widehat{\varphi\mu} : \varphi \in L^2_{\mu,s}(\mathbb{R}^d)\}, \quad (4.6)$$

523 where μ is the spectral measure associated with the noise Ξ , and $L^2_{\mu,s}$ is the
 524 space of symmetric functions in L^2_μ , i.e. $\check{\varphi}(x) = \varphi(-x) = \varphi(x)$, $x \in \mathbb{R}^d$, and
 525 $\int_{\mathbb{R}^d} |\varphi(x)|^2 \mu(dx) < \infty$. Clearly, $\mathcal{H}_\Xi \subset \mathcal{S}'(\mathbb{R}^d)$. The space \mathcal{H}_Ξ , endowed with
 526 the inner product

$$\langle \widehat{\varphi\mu}, \widehat{\psi\mu} \rangle_{\mathcal{H}_\Xi} := \langle \varphi, \psi \rangle_{L^2_\mu}, \quad \forall \varphi, \psi \in L^2_{\mu,s}(\mathbb{R}^d)$$

527 with corresponding norm

$$\|\widehat{\varphi\mu}\|_{\mathcal{H}_\Xi}^2 = \|\varphi\|_{L^2_\mu}^2$$

528 turns out to be a real separable Hilbert space, and it is the so-called "Cameron-
 529 Martin space" of Ξ , see [28, Propostition 2.1]. Thus, Ξ is a cylindrical Wiener
 530 process on $(\mathcal{H}_\Xi, \langle \cdot, \cdot \rangle_{\mathcal{H}_\Xi})$ which takes values in any Hilbert space \mathcal{U} such that
 531 the embedding $\mathcal{H}_\Xi \hookrightarrow \mathcal{U}$ is an Hilbert-Schmidt map.

532 The following Lemma 4.6 shows that the multiplication operator $\mathcal{H}_\Xi \ni \psi \mapsto$
 533 $Z_{l-m}(t, s) \sigma(s, u) \cdot \psi$ is Hilbert-Schmidt from \mathcal{H}_Ξ to $H^{z+m-l, \zeta}$, under suitable
 534 assumptions on σ . Therefore, (4.5) is well-defined as stochastic integral with
 535 respect to a cylindrical Wiener process on $(\mathcal{H}_\Xi, \langle \cdot, \cdot \rangle_{\mathcal{H}_\Xi})$ which takes values in
 536 $H^{z+m-l, \zeta}$.

537 **Definition 4.2.** The class $\text{Lip}(z, \zeta, r, \rho)$, for given $z, \zeta, r, \rho \in \mathbb{R}$, $r, \rho \geq 0$, consists
 538 of all measurable functions $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ such that there exists a
 539 real-valued, non negative, $C_t = C(t) \in C[0, T]$, fulfilling the following:

- 540 • for every $w \in H^{z+r, \zeta+\rho}(\mathbb{R}^d)$, $t \in [0, T]$, we have $\|g(t, \cdot, w)\|_{z, \zeta} \leq C(t)(1 +$
 541 $\|w\|_{z+r, \zeta+\rho})$;

542 • for every $w, v \in H^{z+r, \zeta+\rho}(\mathbb{R}^d)$, $t \in [0, T]$, we have $\|g(t, \cdot, w) - g(t, \cdot, v)\|_{z, \zeta} \leq$
 543 $C(t)\|w - v\|_{z+r, \zeta+\rho}$.

544 **Remark 4.3.** In Definition 4.2 we can actually relax the hypotheses, and ask
 545 that the stated properties hold for $w, v \in U$, with U a suitable open subset
 546 of $H^{w, \omega}(\mathbb{R}^d)$, for some $w \geq z + r$, $\omega \geq \zeta + \rho$ (typically, a sufficiently small
 547 neighbourhood of the initial data of the Cauchy problem). In this case, we
 548 indicate the corresponding set by $\text{Lip}_{\text{loc}}(z, \zeta, r, \rho)$.

549 **Remark 4.4.** Let $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable and $\zeta = \rho = 0$.
 550 Assume that there exists a real-valued, non negative, $C_t = C(t) \in C[0, T]$,
 551 satisfying

- 552 • for every $w \in \mathbb{R}$, $x \in \mathbb{R}^d$, $t \in [0, T]$, we have $|g(t, x, w)| \leq C(t)(|\kappa(x)| + |w|)$,
 553 for some $\kappa \in H^{z, \zeta}(\mathbb{R}^d)$, and
- 554 • for every $w, v \in \mathbb{R}$, $x \in \mathbb{R}^d$, $t \in [0, T]$, we have $|g(t, x, w) - g(t, x, v)| \leq$
 555 $C(t)|w - v|$.

Then, $g \in \text{Lip}(z, 0, r, 0)$. In fact, for some $C > 0$,

$$\begin{aligned} \|g(t, \cdot, w)\|_{z, 0}^2 &= \|\langle \cdot \rangle^z g(t, \cdot, w)\|_{L^2}^2 \leq C_t^2 \|\langle \cdot \rangle^z (|\kappa| + |w|)\|_{L^2}^2 \\ &\leq 2C_t^2 (\|\kappa\|_{z, 0}^2 + \|w\|_{z, 0}^2) \leq C^2 C_t^2 (1 + \|w\|_{z+r, 0})^2, \end{aligned}$$

556 and similarly for the Lipschitz continuity with respect to the third variable, cfr.
 557 [28].

Remark 4.5. Let $g(t, x, w) = w^n$, $n \in \mathbb{N}$. Then $g \in \text{Lip}_{\text{loc}}(z, \zeta, r, \rho)$, when
 $z, r, \rho \geq 0$, $\zeta > \frac{d}{2}$. In fact, when $w \in H^{z+r, \zeta+\rho}(\mathbb{R}^d)$ is such that $\|w\|_{z+r, \zeta+\rho} \leq R$,

$$\|w^n\|_{z, \zeta} \leq C \|w^n\|_{nz, \zeta} \leq C \|w\|_{z, \zeta}^n \leq \tilde{C} R^{n-1} \|w\|_{z+r, \zeta+\rho},$$

558 for the algebra properties of the Sobolev-Kato spaces, see e.g. [3, Proposition
 559 2.2].

560 **Lemma 4.6.** Let $Z_{l-m}(t, s)$ be a family of FIOs with amplitudes of order $(l -$
 561 $m, l - m)$, $0 \leq l \leq m$, parametrized by $0 \leq s \leq t \leq T$, and $\sigma \in \text{Lip}(z, \zeta, m - l, 0)$.
 562 If the spectral measure satisfies

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) < \infty, \quad (4.7)$$

(cfr (4.11)), then, for every $w \in H^{z+m-l, \zeta}(\mathbb{R}^d)$, the operator

$$\Phi(t, s) = \Phi_{l, m, \sigma, w}(t, s) : \psi \mapsto Z_{l-m}(t, s) \sigma(s, w) \psi$$

563 belongs to $L_0^2(\mathcal{H}_{\Xi}, H^{z+m-l, \zeta}(\mathbb{R}^d))$. Moreover, the Hilbert-Schmidt norm of $\Phi(t, s)$
 564 can be estimated by

$$\|\Phi(t, s)\|_{L_0^2(\mathcal{H}_{\Xi}, H^{z+m-l, \zeta})}^2 \leq C_{t, s}^2 (1 + \|w\|_{z+m-l, \zeta})^2 \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi),$$

565 for some $C_{t, s} > 0$.

566 **Remark 4.7.** Lemma 4.6 is the key result to prove Theorems 4.8 and 4.14. It
 567 is a generalization, for higher order equations and different functional spaces,
 568 of Lemma 2.2 in [28]. There, the author deals with the case $m = 2$ and $l = 1$,
 569 related to the wave equation, and works with a multiplication operator by a test
 570 function w , obtaining an estimate of the corresponding Hilbert-Schmidt norm
 571 involving a weighted L^2 norm of w .

572 *Proof of Lemma 4.6.* Let us fix an orthonormal basis $\{e_k\}_{k \in \mathbb{N}} = \{\widehat{f_k \mu}\}_{k \in \mathbb{N}}$ of
 573 \mathcal{H}_Ξ , where $\{f_k\}_{k \in \mathbb{N}}$ is an orthonormal basis in $L^2_{\mu,s}$. We compute

$$\begin{aligned}
 \|\Phi(t, s)\|_{L^2_0(\mathcal{H}_\Xi, H^{z+m-l, \zeta})}^2 &= \sum_{k \in \mathbb{N}} \|Z_{l-m}(t, s)\sigma(s, w)\widehat{f_k \mu}\|_{H^{z+m-l, \zeta}}^2 \\
 &= \sum_{k \in \mathbb{N}} \|\langle D \rangle^{l-m} \langle D \rangle^{m-l} \langle \cdot \rangle^{z+m-l} \langle D \rangle^\zeta Z_{l-m}(t, s)\sigma(s, w)\widehat{f_k \mu}\|_{L^2}^2 \\
 &= \sum_{k \in \mathbb{N}} \|\langle D \rangle^{l-m} \widetilde{Z}(t, s)\sigma(s, w)\widehat{f_k \mu}\|_{L^2}^2 \\
 &= (2\pi)^{-d} \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^d} \langle \xi \rangle^{2(l-m)} \left| \mathcal{F} \left(\widetilde{Z}(t, s)\sigma(s, w)\widehat{f_k \mu} \right) \right|^2 (\xi) d\xi \quad (4.8)
 \end{aligned}$$

574 with $\widetilde{Z}(t, s) = \langle D \rangle^{m-l} \langle \cdot \rangle^{z+m-l} \langle D \rangle^\zeta Z_{l-m}(t, s)$ family of FIOs of order (z, ζ) .
 575 Now, using the well-known fact that the Fourier transform of a product is the
 576 $(2\pi)^{-d}$ multiple of the convolution of the Fourier transforms, the property
 577 $f_k(-x) = f_k(x)$ (by the definition of $L^2_{\mu,s}$), that $\{f_k\}$ is an orthonormal system
 578 in L^2_μ , and Bessel's inequality, we get

$$\begin{aligned}
 (2\pi)^{-d} \sum_{k \in \mathbb{N}} \left| \mathcal{F} \left(\widetilde{Z}(t, s)\sigma(s, w)\widehat{f_k \mu} \right) \right|^2 (\xi) \\
 &= (2\pi)^{-2d} \sum_{k \in \mathbb{N}} \left| \mathcal{F} \left(\widetilde{Z}(t, s)\sigma(s, w) \right) * \widehat{f_k \mu} \right|^2 (\xi) \\
 &= (2\pi)^{-d} \sum_{k \in \mathbb{N}} \left| \mathcal{F} \left(\widetilde{Z}(t, s)\sigma(s, w) \right) * f_k \mu \right|^2 (\xi) \\
 &= (2\pi)^{-d} \sum_{k \in \mathbb{N}} \left| \int_{\mathbb{R}^d} \left[\mathcal{F} \left(\widetilde{Z}(t, s)\sigma(s, w) \right) \right] (\xi - \eta) f_k(\eta) \mu(d\eta) \right|^2 \\
 &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} \left| \mathcal{F} \left(\widetilde{Z}(t, s)\sigma(s, w) \right) \right|^2 (\xi - \eta) \mu(d\eta).
 \end{aligned}$$

579 Inserting this in (4.8), and using the continuity of \widetilde{Z} on Sobolev-Kato spaces we
 580 finally get:

$$\begin{aligned}
 \|\Phi(t, s)\|_{L^2_0(\mathcal{H}_\Xi, H^{z+m-l, \zeta})}^2 \\
 \leq (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \xi \rangle^{2(l-m)} \left| \mathcal{F} \left(\widetilde{Z}(t, s)\sigma(s, w) \right) \right|^2 (\xi - \eta) \mu(d\eta) d\xi \quad (4.9)
 \end{aligned}$$

$$\begin{aligned}
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \eta + \theta \rangle^{2(l-m)} \left| \mathcal{F} \left(\tilde{Z}(t, s) \sigma(s, w) \right) \right|^2 (\theta) \mu(d\eta) d\theta \\
&\leq (2\pi)^{-d} \left(\sup_{\theta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta + \eta \rangle^{2(l-m)} \mu(d\eta) \right) \int_{\mathbb{R}^d} \left| \mathcal{F} \left(\tilde{Z}(t, s) \sigma(s, w) \right) \right|^2 (\theta) d\theta \\
&= (2\pi)^{-d} \left(\sup_{\theta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta + \eta \rangle^{2(l-m)} \mu(d\eta) \right) \|\mathcal{F}(\tilde{Z}(t, s) \sigma(s, w))\|_{L^2}^2 \quad (4.10) \\
&\leq \left(\sup_{\theta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta + \eta \rangle^{2(l-m)} \mu(d\eta) \right) C_{t,s}^2 \|\sigma(s, w)\|_{z,\zeta}^2 \\
&\leq \left(\sup_{\theta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta + \eta \rangle^{2(l-m)} \mu(d\eta) \right) C_{t,s}^2 C_s^2 (1 + \|w\|_{z+m-l,\zeta})^2,
\end{aligned}$$

581 where $C_{t,s}$ stands for the norm in $\mathcal{L}(H^{z,\zeta}, H^{z,\zeta})$ of the FIO $\tilde{Z}(t, s) \langle D \rangle^{-\zeta} \langle x \rangle^{-z}$,
582 which, by Theorem 3.5, has amplitude of order $(0, 0)$. Since $\sigma \in \text{Lip}(z, \zeta, m -$
583 $l, 0)$, C_s is the constant in Definition 4.2. \square

584 4.2. Function-valued solutions for semilinear hyperbolic equations of arbitrary 585 order.

586 We are now ready to deal with existence and uniqueness of a function-valued
587 solution for the Cauchy problem (4.3) under conditions (4.2) and either (1.7) or
588 (1.8).

589 In Theorem 4.8 we study the weakly hyperbolic case with roots of constant
590 multiplicity; in the subsequent Corollary 4.10 we write down the corresponding
591 result in the particular case $l = 1$ of strictly hyperbolic SPDEs. In Theorem
592 4.14 we state a similar result for the involutive case.

593 **Theorem 4.8.** *Let us consider the Cauchy problem (4.3) for a hyperbolic SPDE*
594 *(1.1), where the partial differential operator L of the form (4.1) satisfies (4.2).*
595 *Moreover, assume that L is weakly SG-hyperbolic with constant multiplicities,*
596 *see Definition 1.1, and let l be the maximum multiplicity of the roots of \mathcal{L}_m . As-*
597 *sume also that L is of Levi type, that is, with the notation of Corollary A.13, it*
598 *satisfies (A.24). Suppose that $\gamma, \sigma \in \text{Lip}_{\text{loc}}(z, \zeta, m - l, 0)$, $z, \zeta \in \mathbb{R}$, in some suf-*
599 *ficiently small open subset $U \subset H^{z+m-1, \zeta+m-1}(\mathbb{R}^d) \hookrightarrow H^{z+m-l, \zeta}(\mathbb{R}^d)$. Finally,*
600 *assume for the spectral measure that*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) < \infty. \quad (4.11)$$

601 *Then, there exists a time horizon $0 < T_0 \leq T$ such that, for any choice of*
602 *$u_j \in H^{z+m-1-j, \zeta+m-1-j}(\mathbb{R}^d)$, $0 \leq j \leq m - 1$, $u_0 \in U$, the Cauchy problem*
603 *(4.3) admits a unique solution $u \in L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta}(\mathbb{R}^d))$ satisfying*

$$\begin{aligned}
u(t, x) &= v_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \gamma(s, y, u(s, y)) dy ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y, u(s, y)) \dot{\Xi}(s, y) dy ds \quad (4.12)
\end{aligned}$$

604 where $\Lambda(t, s)$ is the Schwartz kernel of $Z_{l-m}(t, s)$, a sum of FIOs with amplitudes
 605 of order $(l-m, l-m)$, explicitly obtained in (A.26), the first integral in (4.12) is a
 606 Bochner integral, and the second integral in (4.12) is understood as the stochastic
 607 integral of the $H^{z+m-l, \zeta}(\mathbb{R}^d)$ -valued stochastic process $Z_{l-m}(t, \cdot)\sigma(\cdot, u(\cdot))$ with
 608 respect to the stochastic noise Ξ , in the sense explained in Section 2.

609 **Remark 4.9.** Notice that the noise Ξ defines a cylindrical Wiener process on
 610 $(\mathcal{H}_\Xi(\mathbb{R}^d), \langle \cdot, \cdot \rangle_{\mathcal{H}_\Xi(\mathbb{R}^d)})$ with values in $H^{z+m-l, \zeta}(\mathbb{R}^d)$, by Lemma 4.6.

Corollary 4.10. *Let us consider the Cauchy problem (4.3) for a hyperbolic
 SPDE (1.1), where the partial differential operator L of the form (4.1) satisfies
 (4.2). Moreover, assume that L is strictly SG-hyperbolic, that is, \mathcal{L}_m satisfies
 (1.5) and the characteristic roots τ_j , $j = 1, \dots, m$, are distinct, in the sense that
 for a positive constant C we have*

$$|\tau_{j+1}(t, x, \xi) - \tau_j(t, x, \xi)| \geq C(x)\langle \xi \rangle \quad \forall (t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}, j = 1, \dots, m-1.$$

611 Suppose that $\gamma, \sigma \in \text{Lip}_{\text{loc}}(z, \zeta, m-1, 0)$, $z, \zeta \in \mathbb{R}$, in some sufficiently small
 612 open subset $U \subset H^{z+m-1, \zeta+m-1}(\mathbb{R}^d)$. Finally, assume for the spectral measure
 613 that

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-1}} \mu(d\xi) < \infty. \quad (4.13)$$

614 Then, there exists a time horizon $0 < T_0 \leq T$ such that, for any choice of
 615 $u_j \in H^{z+m-1-j, \zeta+m-1-j}(\mathbb{R}^d)$, $0 \leq j \leq m-1$, $u_0 \in U$, the Cauchy problem
 616 (4.3) admits a unique solution $u \in L^2([0, T_0] \times \Omega, H^{z+m-1, \zeta}(\mathbb{R}^d))$ satisfying

$$\begin{aligned} u(t, x) &= v_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \gamma(s, y, u(s, y)) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y, u(s, y)) \dot{\Xi}(s, y) dy ds \end{aligned} \quad (4.14)$$

617 where $\Lambda(t, s)$ is the Schwartz kernel of $Z_{1-m}(t, s)$, a sum of FIOs with ampli-
 618 tudes of order $(1-m, 1-m)$, explicitly obtained in (A.26), the first integral
 619 in (4.14) is a Bochner integral, and the second integral in (4.12) is under-
 620 stood as the stochastic integral of the $H^{z+m-1, \zeta}(\mathbb{R}^d)$ -valued stochastic process
 621 $Z_{1-m}(t, \cdot)\sigma(\cdot, u(\cdot))$ with respect to the stochastic noise Ξ , in the sense explained
 622 in Section 2.

623 **Remark 4.11.** Notice that, if the correlation measure Γ is absolutely continu-
 624 ous, then condition (4.13) is equivalent to

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^{m-1}} \mu(d\xi) < \infty, \quad (4.15)$$

625 see [28]. Condition (4.15) with $m = 2$ on the spectral measure is the one
 626 needed for the existence and uniqueness of both a function-valued solution and a
 627 random-field solution to a second order SPDE well-known in literature, namely,
 628 the stochastic wave equation.

629 Moreover, the same condition (4.13) has been found in [7], looking for
630 random-field solutions to linear strictly hyperbolic equations with uniformly
631 bounded coefficients. The more general condition (4.11) is exactly the one ob-
632 tained in [2], looking for random-field solutions to linear hyperbolic SPDEs with
633 possibly unbounded variable coefficients. Thus, the class of the stochastic noises
634 we can deal with if we want to obtain either a function-valued or a random-field
635 solution of the Cauchy problem for an SPDE is described by (4.11) for all *SG*-
636 hyperbolic operators L . Condition (4.11) can be understood as a *compatibility*
637 *condition* between the noise and the equation: as the order of the equation
638 increases, we can allow for rougher stochastic noises Ξ ; as the maximum multi-
639 plicity of the roots decreases (i.e., as the regularity of the operator L increases),
640 we can allow for rougher stochastic noises Ξ .

641 We give here below a couple of examples of right-hand side that we can allow
642 in (4.3).

643 **Example 4.12.** Let $\sigma(t, u) = u^2$. Then, σ satisfies all the conditions required
644 in Theorem 4.8. More generally, we can allow also $\sigma(t, u) = u^n$, $n \in \mathbb{N}$, $n > 2$,
645 see Remark 4.5.

646 **Example 4.13.** A class of explicitly (t, x) -dependent nonlinear stochastic coef-
647 ficients which satisfy the requirements of Theorem 4.8 are those of the form

$$\sigma(t, x, u) = \langle x \rangle^{l-m} \cdot \tilde{\sigma}(t, u), \quad (4.16)$$

648 where $\tilde{\sigma} \in \text{Lip}_{\text{loc}}(z + m - l, \zeta, 0, 0)$. Indeed, the function σ in (4.16) fulfills the
649 assumptions of Theorem 4.8, being an element of $\text{Lip}_{\text{loc}}(z, \zeta, m - l, 0)$. In fact,
650 for every w in a sufficiently small subset $U \subset H^{z+m-l, \zeta}(\mathbb{R}^d)$, we have

$$\|\sigma(t, \cdot, w)\|_{z, \zeta} = \|\tilde{\sigma}(t, \cdot, w)\|_{z+m-l, \zeta} \leq C(t) (1 + \|w\|_{z+m-l, \zeta}),$$

651 and the verification of $\|\sigma(t, \cdot, w_1) - \sigma(t, \cdot, w_2)\|_{z, \zeta} \leq C(t) \|w_1 - w_2\|_{z+m-l, \zeta}$ fol-
652 lows similarly.

653 *Proof of Theorem 4.8.* To start, we follow the computations in the Appendix.
654 First, we perform a change of variable, defining the (nm) -dimensional vector
655 of unknowns W having entries given by (A.21). The equation $Lu(t) = g(t, u)$,
656 where formally $g(t, u) := \gamma(t, u) + \sigma(t, u)\dot{\Xi}(t)$, is then equivalent to the semilinear
657 hyperbolic system of first order (A.23) in the unknown W , with $g(t, u)$ in place
658 of $g(t)$. Such system has the form

$$\begin{cases} (D_t - \text{Op}(\kappa_1(t)) - \text{Op}(\kappa_0(t)))W(t) = F(t, W(t)) + G(t, W(t))\dot{\Xi}(t), & t \in [0, T], \\ W(0) = W_0, \end{cases} \quad (4.17)$$

659 with $\kappa_1 \in C^\infty([0, T], S^{1,1})$ real-valued and diagonal, $\kappa_0 \in C^\infty([0, T], S^{0,0})$, and
660 (nm) -dimensional vectors $F(t, W(t))$, $G(t, W(t))$ given by

$$F(t, W(t)) = \underbrace{(\tilde{F}(t, W), \dots, \tilde{F}(t, W(t)))^t}_{n \text{ times}}, \quad \tilde{F}(t, W(t)) = (\underbrace{0, \dots, 0}_{m-1 \text{ times}}, \gamma(t, W_1^{(1)}))^t,$$

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$$G(t, W(t)) = \underbrace{(\tilde{G}(t, W), \dots, \tilde{G}(t, W(t)))^t}_{n \text{ times}}, \quad \tilde{G}(t, W(t)) = (\underbrace{0, \dots, 0}_{m-1 \text{ times}}, \sigma(t, W_1^{(1)}))^t.$$

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We also have that $W_0 = \text{Op}(b)U_0$, with a $(mn \times m)$ -dimensional block-matrix symbol b with structure analogous to (A.25) and entries with the same orders, so that, by the assumptions of Theorem 4.8, we get $W_0 \in H^{z, \zeta}$.

By Theorem A.6 we can formally construct, via Duhamel's formula, the "mild solution" to (4.17):

$$W(t) = E(t, 0)W_0 + i \int_0^t E(t, s)F(s, W(s))ds + i \int_0^t E(t, s)G(s, W(s))d\Xi(s), \quad t \in [0, T_0],$$

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for a suitable $T_0 \in (0, T]$. Now, we go back to the equation (1.1) to get its (formal) solution u . By Lemma A.19, we know that $u(t)$ is the first entry of the vector $\text{Op}(\mathcal{Y}_n(t))W(t)$. Thus, as in (A.26), we obtain (formally)

$$\begin{aligned} u(t, x) &= v_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \gamma(s, y, u(s, y)) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y, u(s, y)) \dot{\Xi}(s, y) dy ds \\ &= v_0(t, x) + \int_0^t Z_{l-m}(t, s) \gamma(s, u(s)) ds + \int_0^t Z_{l-m}(t, s) \sigma(s, u(s)) \dot{\Xi}(s) ds, \end{aligned}$$

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where $v_0 \in \bigcap_{j \geq 0} C^j([0, T_0], H^{z+m-l-j, \zeta+m-l-j})$ depends on the Cauchy data, and $\Lambda \in C^\infty(\Delta_{T_0}, \mathcal{S}')$ is, for any $(t, s) \in \Delta_{T_0}$, the Schwartz kernel of the Fourier integral operator family Z_{l-m} , with amplitudes of order $(l-m, l-m)$. We then construct the map $u \rightarrow \mathcal{T}u$ on $L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta}(\mathbb{R}^d))$, defined as follows:

$$\begin{aligned} \mathcal{T}u(t) &:= v_0(t) + \int_0^t Z_{l-m}(t, s) \gamma(s, u(s)) ds + \int_0^t Z_{l-m}(t, s) \sigma(s, u(s)) dB_s \quad (4.18) \\ &:= v_0(t) + \mathcal{T}_1 u(t) + \mathcal{T}_2 u(t), \quad t \in [0, T_0], \end{aligned}$$

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where the last integral on the right-hand side is understood as the stochastic integral of the stochastic process $Z_{l-m}(t, \cdot) \sigma(\cdot, u(\cdot)) \in L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$ with respect to the cylindrical Wiener process $\{W_t(h)\}_{t \in [0, T], h \in H^{z+m-l, \zeta}}$ associated with the random noise $\Xi(t)$, which is well-defined by Lemma 4.6 and takes values in $H^{z+m-l, \zeta}$.

To prove that the solution (4.12) of the Cauchy problem (4.3) is indeed well-defined, we have to check that

$$\mathcal{T}: L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta}(\mathbb{R}^d)) \longrightarrow L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$$

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680

is well-defined, it is Lipschitz continuous on $L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$, and it becomes a contraction if we take T_0 small enough. Then, an application

681 of Banach's fixed point Theorem will provide existence of a unique solution
682 $u \in L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$ satisfying $u = \mathcal{T}u$, that is (4.12).
683 To verify that $\mathcal{T}u$ in (4.18) belongs to $L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$ for every
684 $u \in L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$ we notice that:
685 - $v_0 \in \bigcap_{j \geq 0} C^j([0, T_0], H^{z+m-l-j, \zeta+m-l-j}) \subset L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$;
686 - $\mathcal{T}_1 u$ is in $L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$; indeed, $\mathcal{T}_1 u(t)$ is defined as the Bochner inte-
687 gral on $[0, t]$ of the function $s \rightarrow Z_{l-m}(t, s)\gamma(s, u(s))$ with values in $L^2(\Omega, H^{z+m-l, \zeta})$,
688 and, by the properties of Bochner integrals, the continuity of $Z_{l-m}(t, s)$ on
689 Sobolev-Kato spaces, and the fact that $\gamma \in \text{Lip}(z, \zeta, m-l, 0)$, we have

$$\begin{aligned}
\|\mathcal{T}_1 u\|_{L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})}^2 &= \mathbb{E} \left[\int_0^{T_0} \|\mathcal{T}_1 u(t)\|_{z+m-l, \zeta}^2 dt \right] \\
&= \int_0^{T_0} \mathbb{E} \left[\left\| \int_0^t Z_{l-m}(t, s) (\gamma(s, u(s))) ds \right\|_{z+m-l, \zeta}^2 \right] dt \\
&\leq \int_0^{T_0} \int_0^t \mathbb{E} \left[\|Z_{l-m}(t, s) (\gamma(s, u(s)))\|_{z+m-l, \zeta}^2 \right] ds dt \\
&\leq \int_0^{T_0} \int_0^t C_{t,s}^2 \mathbb{E} \left[\|\gamma(s, u(s))\|_{z, \zeta+l-m}^2 \right] ds dt \\
&\leq \int_0^{T_0} \int_0^t C_{t,s}^2 C_s^2 \mathbb{E} \left[(1 + \|u(s)\|_{z+m-l, \zeta+l-m})^2 \right] ds dt \\
&\leq 2 \left(\max_{0 \leq s \leq t \leq T_0} C_{t,s}^2 C_s^2 \right) T_0 \int_0^{T_0} \left(1 + \mathbb{E} \left[\|u(s)\|_{z+m-l, \zeta}^2 \right] \right) ds \\
&= 2C_{T_0} T_0 (T_0 + \|u\|_{L^2([0, T_0] \times \Omega, H^{z+l-m, \zeta})}^2) < \infty;
\end{aligned}$$

690
691 - $\mathcal{T}_2 u$ is in $L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$, in view of the fundamental isometry (2.1),
692 Lemma 4.6 and the fact that the expectation can be moved inside and outside
693 time integrals, by Fubini's Theorem:

$$\begin{aligned}
\|\mathcal{T}_2 u\|_{L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})}^2 &= \mathbb{E} \left[\int_0^{T_0} \|\mathcal{T}_2 u(t)\|_{z+m-l, \zeta}^2 dt \right] \\
&= \int_0^{T_0} \mathbb{E} \left[\left\| \int_0^t Z_{l-m}(t, s) \sigma(s, u(s)) dW_s \right\|_{z+m-l, \zeta}^2 \right] dt \\
&= \int_0^{T_0} \int_0^t \mathbb{E} \left[\|Z_{l-m}(t, s) \sigma(s, u(s))\|_{L_0^2(\mathcal{H}_{\Xi, H^{z+m-l, \zeta}})}^2 \right] ds dt \\
&\leq \int_0^{T_0} \int_0^t \mathbb{E} \left[C_{t,s}^2 (1 + \|u(s)\|_{H^{z+m-l, \zeta}})^2 \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right] ds dt \\
&= \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right) \int_0^{T_0} \int_0^t C_{t,s}^2 \mathbb{E} \left[(1 + \|u(s)\|_{H^{z+m-l, \zeta}})^2 \right] ds dt
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right) \left(\max_{0 \leq s \leq t \leq T_0} C_{(t,s)}^2 \right) T_0 \left(T_0 + \int_0^{T_0} \mathbb{E} \left[\|u(s)\|_{z+m-l, \zeta}^2 \right] ds \right) \\
&= 2C_{T_0, m, l} T_0 (T_0 + \|u\|_{L^2([0, T_0] \times \Omega, H^{z+l-m, \zeta})}^2) < \infty.
\end{aligned}$$

694 Now, we take $u_1, u_2 \in L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})$ and compute

$$\begin{aligned}
&\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})}^2 \\
&\leq 2 \left(\|\mathcal{T}_1 u_1 - \mathcal{T}_1 u_2\|_{L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})}^2 + \|\mathcal{T}_2 u_1 - \mathcal{T}_2 u_2\|_{L^2([0, T_0] \times \Omega, H^{z+m-l, \zeta})}^2 \right) \\
&= 2 \int_0^{T_0} \mathbb{E} \left[\left\| \int_0^t Z_{l-m}(t, s) (\gamma(s, u_1(s)) - \gamma(s, u_2(s))) ds \right\|_{z+m-l, \zeta}^2 \right] dt \quad (4.19)
\end{aligned}$$

$$+ 2 \int_0^{T_0} \mathbb{E} \left[\left\| \int_0^t Z_{l-m}(t, s) (\sigma(s, u_1(s)) - \sigma(s, u_2(s))) dB_s \right\|_{z+m-l, \zeta}^2 \right] dt. \quad (4.20)$$

695 In the term (4.19) here above we can move the expectation and the $(z +$
696 $m - l, \zeta)$ -norm inside the integral with respect to s . Then, by continuity of
697 Z_{l-m} on Sobolev-Kato spaces, Definition 4.2, and the embedding $H^{z+m-l, \zeta} \hookrightarrow$
698 $H^{z+m-l, \zeta+l-m}$, we obtain

$$\begin{aligned}
&2 \int_0^{T_0} \mathbb{E} \left[\left\| \int_0^t Z_{l-m}(t, s) (\gamma(s, u_1(s)) - \gamma(s, u_2(s))) ds \right\|_{z+m-l, \zeta}^2 \right] dt \\
&\leq 2 \int_0^{T_0} \int_0^t \mathbb{E} \left[\|Z_{l-m}(t, s) (\gamma(s, u_1(s)) - \gamma(s, u_2(s)))\|_{z+m-l, \zeta}^2 \right] ds dt \\
&\leq 2 \int_0^{T_0} \int_0^t C_{t,s}^2 \mathbb{E} \left[\|\gamma(s, u_1(s)) - \gamma(s, u_2(s))\|_{z, \zeta+l-m}^2 \right] ds dt \\
&\leq 2 \int_0^{T_0} \int_0^t C_{t,s}^2 C_s^2 \mathbb{E} \left[\|u_1(s) - u_2(s)\|_{z+m-l, \zeta+l-m}^2 \right] ds dt \\
&\leq 2 \left(\max_{0 \leq s \leq t \leq T_0} C_{t,s}^2 C_s^2 \right) T_0 \int_0^{T_0} \mathbb{E} \left[\|u_1(s) - u_2(s)\|_{z+m-l, \zeta}^2 \right] ds \\
&= 2C_{T_0} T_0 \|u_1 - u_2\|_{L^2([0, T_0] \times \Omega, H^{z+l-m, \zeta})}^2.
\end{aligned}$$

699 To the term (4.20) we apply, here below, the fundamental isometry (2.1) to pass
700 from the first to the second line, formula (4.10) of Lemma 4.6 to pass from the
701 second to the third line, Definition 4.2 to pass from the third to the fourth line,
702 and finally get:

$$\begin{aligned}
&2 \int_0^{T_0} \mathbb{E} \left[\left\| \int_0^t Z_{l-m}(t, s) (\sigma(s, u_1(s)) - \sigma(s, u_2(s))) dB_s \right\|_{z+m-l, \zeta}^2 \right] dt \\
&= 2 \int_0^{T_0} \int_0^t \mathbb{E} \left[\|Z_{l-m}(t, s) (\sigma(s, u_1(s)) - \sigma(s, u_2(s)))\|_{L^2_0(\mathcal{H}_{\Xi, H^{z+m-l, \zeta}})}^2 \right] ds dt
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \int_0^{T_0} \int_0^t \mathbb{E} \left[C_{t,s}^2 \|\sigma(s, u_1(s)) - \sigma(s, u_2(s))\|_{H^{z,\zeta}}^2 \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right] ds dt \\
&\leq 2 \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right) \int_0^{T_0} \int_0^t C_{t,s}^2 C_s^2 \mathbb{E} \left[\|u_1(s) - u_2(s)\|_{z+m-l,\zeta}^2 \right] ds dt \\
&\leq 2C_{T_0} T_0 \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right) \|u_1 - u_2\|_{L^2([0,T_0] \times \Omega, H^{z+m-l,\zeta})}^2.
\end{aligned}$$

703 Summing up, we have proved that

$$\begin{aligned}
&\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{L^2([0,T_0] \times \Omega, H^{z,\zeta})}^2 \\
&\leq 2C_{T_0} T_0 \left(1 + \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right) \|u_1 - u_2\|_{L^2([0,T_0] \times \Omega, H^{z+m-l,\zeta})}^2,
\end{aligned}$$

704 that is, \mathcal{T} is Lipschitz continuous on $L^2([0, T_0] \times \Omega, H^{z+m-l,\zeta})$. Moreover, in
705 view of the assumption (4.11), if we take $T_0 > 0$ such that

$$2C_{T_0} T_0 \left(1 + \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) \right) < 1, \quad (4.21)$$

706 then \mathcal{T} becomes a strict contraction on $L^2([0, T_0] \times \Omega, H^{z+m-l,\zeta})$, and so it
707 admits a unique fixed point $u = \mathcal{T}u$. That is, there exists a unique, well-defined
708 solution of (4.3). To prove the estimate (4.21), it is sufficient to take T_0 small
709 enough, since the constant C_{T_0} is continuously dependent on T_0 . The proof is
710 complete. \square

711 4.3. The weakly hyperbolic case with involutive roots

712 We conclude the section with the statement of a result of existence and
713 uniqueness of a solution to the Cauchy problem (4.3) for the SPDE (1.1) in
714 the more general case of involutive roots, cfr. (1.8). With these even weaker
715 hyperbolicity assumption we can still switch from (4.3) to an equivalent first
716 order system (A.5), but at the price, as usual, of some further requirement
717 on the lower order terms of the operator L . Namely, we ask that L admits
718 a factorization (A.13) with symbols h_{jk} , $j = 1, \dots, m$, $k = 1, \dots, l_j$, such that
719 $h_{jk} \in C^\infty([0, T], S^{0,0})$. Notice that this is automatically true in the case of strict
720 hyperbolicity, and that only the request on the order of the symbols h_{jk} has to
721 be fulfilled in the case of hyperbolicity with constant multiplicities. We say, in
722 the present case, that L satisfies the *strong Levi condition*, or, equivalently, that
723 it is of *strong Levi type*. We state and discuss here below our further result,
724 under the hypothesis (1.8).

725 **Theorem 4.14.** *Let us consider the Cauchy problem (4.3) for an SPDE (1.1),*
726 *where the partial differential operator L of the form (4.1) satisfies the hyper-*
727 *bolicity hypothesis (4.2). Assume that L is SG-hyperbolic with involutive roots,*

728 that is, all the roots of the principal part \mathcal{L}_m of L are real-valued and form an
729 involutive system, in the sense of (1.8). Moreover, assume that L is of strong
730 Levi type. Suppose that $\gamma, \sigma \in \text{Lip}_{\text{loc}}(z, \zeta, 0, 0)$, $z, \zeta \in \mathbb{R}$, in some sufficiently
731 small open subset $U \subset H^{z+m-1, \zeta+m-1}(\mathbb{R}^d) \hookrightarrow H^{z, \zeta}(\mathbb{R}^d)$. Finally, assume that
732 the spectral measure satisfies the compatibility condition

$$\int_{\mathbb{R}^d} \mu(d\xi) < \infty. \quad (4.22)$$

733 Then, there exists a time horizon $0 \leq T_0 \leq T$ such that for any choice of
734 $u_j \in H^{z+m-1-j, \zeta+m-1-j}(\mathbb{R}^d)$, $0 \leq j \leq m-1$, $u_0 \in U$, the Cauchy problem
735 (4.3) admits a unique solution $u \in L^2([0, T_0] \times \Omega, H^{z, \zeta}(\mathbb{R}^d))$ satisfying

$$\begin{aligned} u(t, x) &= v_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \gamma(s, y, u(s, y)) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y, u(s, y)) \dot{\Xi}(s, y) dy ds, \end{aligned}$$

737 where $\Lambda(t, s)$ is obtained through the Schwartz kernels of Fourier integral opera-
738 tors with amplitudes of order $(0, 0)$, the first integral is a Bochner integral, and
739 the second integral is intended to be the stochastic integral of the $H^{z, \zeta}(\mathbb{R}^d)$ -valued
740 stochastic process $E_0(t, \cdot) \sigma(\cdot, u(\cdot))$ with respect to the stochastic noise Ξ .

741 **Remark 4.15.** Ξ defines a cylindrical Wiener process on $(\mathcal{H}_{\Xi}(\mathbb{R}^d), \langle \cdot, \cdot \rangle_{\mathcal{H}_{\Xi}(\mathbb{R}^d)})$
742 with values in $H^{z, \zeta}$, by Lemma 4.6.

743 *Proof of Theorem 4.14.* By the analysis in [1], we know that, also in this case,
744 using (A.26), the Cauchy problem (4.4) can be (formally) written as

$$\begin{aligned} u(t, x) &= v_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \gamma(s, y, u(s, y)) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y, u(s, y)) \dot{\Xi}(s, y) dy ds \\ &= v_0(t, x) + \int_0^t Z_0(t, s) \gamma(s, u(s)) ds + \int_0^t Z_0(t, s) \sigma(s, u(s)) \dot{\Xi}(s) ds, \end{aligned}$$

745 where $v_0 \in \bigcap_{j \geq 0} C^j([0, T_0], H^{z-j, \zeta-j})$ depends on the Cauchy data, and $\Lambda \in$
746 $C^\infty(\Delta_{T_0}, \mathcal{S}')$ is, for any $(t, s) \in \Delta_{T_0}$, the Schwartz kernel of the Fourier integral
747 operator family $Z_0(t, s)$, with amplitudes of order $(0, 0)$. Given the assumption
748 (4.22), identical to the case $l = m$ in the proof of Theorem 4.8, the result can
749 then be achieved through the same argument. \square

750 *4.4. Function-valued solutions and random-field solutions in the linear case.*

751 Consider now the special case of (4.3), with a SG-hyperbolic operator L
752 with constant multiplicities, where $\sigma(t, x, u(t, x)) = \sigma(t, x)$ and $\gamma(t, x, u(t, x)) =$

753 $\gamma(t, x), \gamma, \sigma \in C([0, T], H^{z, \zeta}), z \geq 0, \zeta > \frac{d}{2}, s \mapsto \mathcal{F}(\sigma)(s) = \nu_s \in L^2([0, T], \mathcal{M}_b),$
754 \mathcal{M}_b the space of complex-valued measures with finite total variation. That is,
755 we look at the Cauchy problem

$$\begin{cases} Lu(t, x) = \gamma(t, x) + \sigma(t, x)\dot{\Xi}(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d \\ D_t^j u(0, x) = u_j(x), & x \in \mathbb{R}^d, 0 \leq j \leq m-1, \end{cases} \quad (4.23)$$

756 for the linear SPDEs studied in [2]. Such (more restrictive) hypotheses imply
757 $\gamma, \sigma \in \text{Lip}(z, \zeta, r, \rho) \subset \text{Lip}_{\text{loc}}(z, \zeta, r, \rho)$ for any $r, \rho \geq 0$. In fact, recalling
758 Definition 4.2, trivially:

- 759 • for every $w \in H^{z+r, \zeta+\rho}, t \in [0, T], \|g(t, \cdot, w)\|_{z, \zeta} = \|g(t, \cdot)\|_{z, \zeta} \leq C(t)(1 +$
760 $\|w\|_{z+r, \zeta+\rho}),$ with $C(t) = \|g(t, \cdot)\|_{z, \zeta};$
- 761 • for every $w, v \in H^{z+r, \zeta+\rho}, t \in [0, T], \|g(t, \cdot, w) - g(t, \cdot, v)\|_{z, \zeta} \equiv 0 \leq$
762 $C(t)\|w - v\|_{z+r, \zeta+\rho}.$

763 Applying Theorem 4.8, we obtain the existence and uniqueness of a function-
764 valued solution for the linear Cauchy problem (4.23), which we here denote by
765 u_{fv} . Since in Theorem 4.12 of [2] we proved the existence and uniqueness of a
766 random-field solution of (4.23), which we here denote by u_{rf} , we now wish to
767 compare it with u_{fv} .

768 **Remark 4.16.** Notice that, in analogy with (4.12), u_{rf} satisfies

$$\begin{aligned} u_{\text{rf}}(t, x) &= v_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \gamma(s, y) dy ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y) \dot{\Xi}(s, y) dy ds. \end{aligned} \quad (4.24)$$

769 While the first two terms in the right-hand side of (4.24) clearly coincide with
770 the first two terms in the right-hand side of (4.12), the corresponding third,
771 stochastic terms in (4.12) and (4.24) are defined in different ways.

772 We now prove that a random-field solution of (4.23) is also a function-valued
773 solution.

774 **Proposition 4.17.** *Let u_{rf} and u_{fv} be the random-field solution and the function-*
775 *valued solution of (4.23), respectively, with L SG-hyperbolic with constant multi-*
776 *plicities, $\gamma, \sigma \in C([0, T], H^{z, \zeta}), z \geq 0, \zeta > \frac{d}{2}, s \mapsto \mathcal{F}(\sigma)(s) = \nu_s \in L^2([0, T], \mathcal{M}_b),$*
777 *\mathcal{M}_b the space of complex-valued measures with finite total variation. Then,*
778 *$u_{\text{rf}} = u_{\text{fv}} = u.$*

779 *Proof.* Our analysis in [2] shows that $\Lambda\sigma \in \mathcal{P}_0$, the completion of the class \mathcal{E} of
780 simple processes via the pre-inner product (defined for suitable f, g)

$$\langle f, g \rangle_0 = \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (f(s) * \tilde{g}(s))(x) \Gamma(dx) ds \right]$$

$$= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} [\mathcal{F}f(s)](\xi) \cdot \overline{[\mathcal{F}g(s)](\xi)} \mu(d\xi) ds \right].$$

781 By Proposition 3.12 in [18], it follows that the stochastic integrals of $\Lambda\sigma$ with
 782 respect to the martingale measure associated with $\tilde{\Xi}$ (considered in Section 4 of
 783 [2]), and with respect to the cylindrical Wiener process considered in Section 4
 784 are equal. This proves that $u_{\text{rf}} = u_{\text{fv}} = u$, as claimed. \square

785 **Appendix. Microlocal techniques for the solution of SG -hyperbolic**
 786 **problems for linear operators with polynomially bounded**
 787 **coefficients.**

788 We collect in this Appendix, for the convenience of the reader, some ad-
 789 ditional results concerning the SG -calculus and its applications to hyperbolic
 790 problems, which we mentioned along the main text. This material appeared,
 791 sometimes in slightly different form, in [5] and the references quoted therein.

792 *A.1. Boundedness and ellipticity*

793 The continuity property of the elements of $\text{Op}(S^{m,\mu})$ on the scale of spaces
 794 $H^{z,\zeta}(\mathbb{R}^d)$, $(m, \mu), (z, \zeta) \in \mathbb{R}^2$, is precisely expressed in the next Theorem A.1
 795 (see [12] and the references quoted therein for the result on more general classes
 796 of SG -symbols).

797 **Theorem A.1.** *Let $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m, \mu) \in \mathbb{R}^2$. Then, for any $(z, \zeta) \in \mathbb{R}^2$,
 798 $\text{Op}(a) \in \mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m, \zeta-\mu}(\mathbb{R}^d))$, and there exists a constant $C > 0$, de-
 799 pending only on d, m, μ, z, ζ , such that*

$$\|\text{Op}(a)\|_{\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m, \zeta-\mu}(\mathbb{R}^d))} \leq C \|a\|_{\left[\frac{d}{2}\right]+1}^{m,\mu}, \quad (\text{A.1})$$

800 where $[t]$ denotes the integer part of $t \in \mathbb{R}$.

801 The following characterization of the class $\mathcal{O}(-\infty, -\infty)$ is often useful, see
 802 [12].

803 **Theorem A.2.** *The class $\mathcal{O}(-\infty, -\infty)$ coincides with $\text{Op}(S^{-\infty, -\infty}(\mathbb{R}^d))$ and
 804 with the class of smoothing operators, that is, the set of all the linear continuous
 805 operators $A: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$. All of them coincide with the class of linear
 806 continuous operators A admitting a Schwartz kernel k_A belonging to $\mathcal{S}(\mathbb{R}^{2d})$.*

807 An operator $A = \text{Op}(a)$ and its symbol $a \in S^{m,\mu}$ are called *elliptic* (or
 808 $S^{m,\mu}$ -elliptic) if there exists $R \geq 0$ such that

$$C \langle x \rangle^m \langle \xi \rangle^\mu \leq |a(x, \xi)|, \quad |x| + |\xi| \geq R,$$

809 for some constant $C > 0$. If $R = 0$, a^{-1} is everywhere well-defined and smooth,
 810 and $a^{-1} \in S^{-m, -\mu}$. If $R > 0$, then a^{-1} can be extended to the whole of \mathbb{R}^{2d}

811 so that the extension \tilde{a}_{-1} satisfies $\tilde{a}_{-1} \in S^{-m, -\mu}$. An elliptic SG operator
 812 $A \in \text{Op}(S^{m, \mu})$ admits a parametrix $A_{-1} \in \text{Op}(S^{-m, -\mu})$ such that

$$A_{-1}A = I + R_1, \quad AA_{-1} = I + R_2,$$

813 for suitable $R_1, R_2 \in \text{Op}(S^{-\infty, -\infty})$, where I denotes the identity operator. In
 814 such a case, A turns out to be a Fredholm operator on the scale of functional
 815 spaces $H^{z, \zeta}(\mathbb{R}^d)$, $(z, \zeta) \in \mathbb{R}^2$.

816 The study of the composition of $M \geq 2$ SG FIOs of type I $\text{Op}_{\varphi_j}(a_j)$ with
 817 regular SG -phase functions $\varphi_j \in \mathfrak{P}_\delta(\lambda_j)$ and symbols $a_j \in S^{m_j, \mu_j}(\mathbb{R}^d)$, $j =$
 818 $1, \dots, M$, has been done in [5]. The result of such composition is still an SG -
 819 FIO with a regular SG -phase function φ given by the so-called *multi-product*
 820 $\varphi_1 \sharp \dots \sharp \varphi_M$ of the phase functions φ_j , $j = 1, \dots, M$, and symbol a as in Theorem
 821 A.3 here below.

822 **Theorem A.3.** *Consider, for $j = 1, 2, \dots, M$, $M \geq 2$, the SG FIOs of type*
 823 *$I \text{Op}_{\varphi_j}(a_j)$ with $a_j \in S^{m_j, \mu_j}(\mathbb{R}^d)$, $(m_j, \mu_j) \in \mathbb{R}^2$, and $\varphi_j \in \mathfrak{P}_\delta(\lambda_j)$ such that*
 824 *$\lambda_1 + \dots + \lambda_M \leq \lambda \leq \frac{1}{4}$ for some sufficiently small $\lambda > 0$. Then, there exists*
 825 *$a \in S^{m, \mu}(\mathbb{R}^d)$, $m = m_1 + \dots + m_M$, $\mu = \mu_1 + \dots + \mu_M$, such that, setting*
 826 *$\phi = \varphi_1 \sharp \dots \sharp \varphi_M$, we have*

$$\text{Op}_{\varphi_1}(a_1) \circ \dots \circ \text{Op}_{\varphi_M}(a_M) = \text{Op}_\phi(a).$$

827 *Moreover, for any $\ell \in \mathbb{N}_0$ there exist $\ell' \in \mathbb{N}_0$, $C_\ell > 0$ such that*

$$\|a\|_\ell^{m, \mu} \leq C_\ell \prod_{j=1}^M \|a_j\|_{\ell'}^{m_j, \mu_j}. \quad (\text{A.2})$$

828 Theorem A.3 is a corollary of the main Theorem in [5]. There, the *multi-*
 829 *product* of regular SG -phase functions is defined and its properties are studied,
 830 parametrices and compositions of regular SG FIOs with amplitude identically
 831 equal to 1 are considered, leading to the general composition $\text{Op}_{\varphi_1}(a_1) \circ \dots \circ$
 832 $\text{Op}_{\varphi_M}(a_M)$. It is needed for the determination of the fundamental solutions of
 833 the hyperbolic operators (1.3), involved in (1.1), in the case of involutive roots
 834 with non-constant multiplicities, see [1].

835 A.2. First order SG -hyperbolic linear systems

836 Here we summarize the main results concerning the analysis of Cauchy prob-
 837 lems for SG -hyperbolic linear systems with diagonal principal part, by means of
 838 the corresponding class of Fourier operators. Given a symbol $\varkappa \in C([0, T]; S^{1,1})$,
 839 set $\Delta_{T_0} = \{(s, t) \in [0, T_0]^2 : 0 \leq s \leq t \leq T_0\}$, $0 < T_0 \leq T$, and consider the
 840 eikonal equation

$$\begin{cases} \partial_t \varphi(t, s, x, \xi) = \varkappa(t, x, \varphi'_x(t, s, x, \xi)), & t \in [s, T_0], \\ \varphi(s, s, x, \xi) = x \cdot \xi, & s \in [0, T_0), \end{cases} \quad (\text{A.3})$$

841 with $0 < T_0 \leq T$. By an extension of the theory developed in [14], it is possible
 842 to prove that the following Proposition A.4 holds true.

843 **Proposition A.4.** For any small enough $T_0 \in (0, T]$, equation (A.3) admits a
 844 unique solution $\varphi \in C^1(\Delta_{T_0}, S^{1,1}(\mathbb{R}^d))$, satisfying $J \in C^1(\Delta_{T_0}, S^{1,1}(\mathbb{R}^d))$ and

$$\partial_s \varphi(t, s, x, \xi) = -\varkappa(s, \varphi'_\xi(t, s, x, \xi), \xi), \quad (\text{A.4})$$

845 for any $(t, s) \in \Delta_{T_0}$. Moreover, for every $\ell \in \mathbb{N}_0$ there exists $\delta > 0$, $c_\ell \geq 1$ and
 846 $\tilde{T}_\ell \in [0, T_0]$ such that $\varphi(t, s, x, \xi) \in \mathfrak{P}_\delta(c_\ell |t - s|)$, with $\|J\|_{2,\ell} \leq c_\ell |t - s|$ for all
 847 $(t, s) \in \Delta_{\tilde{T}_\ell}$.

848 **Remark A.5.** Of course, if additional regularity with respect to $t \in [0, T]$
 849 is fulfilled by the symbol \varkappa in the right-hand side of (A.3), this reflects in a
 850 corresponding increased regularity of the resulting solution φ with respect to
 851 $(t, s) \in \Delta_{T_0}$. Since here we are not dealing with problems concerning the t -
 852 regularity of the solution, we assume smooth t -dependence of the coefficients of
 853 L . Some of the results below will anyway be formulated in situations of lower
 854 regularity with respect to t .

855 Let us consider the Cauchy problem

$$\begin{cases} (D_t - \text{Op}(\kappa_1(t)) - \text{Op}(\kappa_0(t)))W(t) = Y(t), & t \in [0, T], \\ W(s) = W_0, & s \in [0, T], \end{cases} \quad (\text{A.5})$$

856 where the $(\nu \times \nu)$ -system is hyperbolic with diagonal principal part, that is:

- 857 - the matrix κ_1 satisfies $\kappa_1 \in C^\infty([0, T], S^{1,1})$, it is real-valued and diagonal,
 858 and each entry on the principal diagonal coincides with the value of one
 859 of the roots $\tau_j \in C^\infty([0, T]; S^{1,1})$, possibly repeated a number of times,
 860 depending on their multiplicities;
- 861 - the matrix κ_0 satisfies $\kappa_0 \in C^\infty([0, T], S^{0,0})$.

862 In analogy with the terminology introduced above, we will say that the system
 863 (A.5) is hyperbolic with constant multiplicities when the elements on the main
 864 diagonal of κ_1 are all distinct and satisfy (1.7). Similarly, we will say that the
 865 system is hyperbolic with involutive roots when they satisfy (1.8). We will also
 866 generally assume $W_0 \in H^{z,\zeta}$, $Y \in C([0, T], H^{z,\zeta})$, $(z, \zeta) \in \mathbb{R}^2$.

The fundamental solution, or *solution operator*, of (A.5) is a family

$$\{E(t, s) : (t, s) \in [0, T_0]^2\}, \quad 0 < T_0 \leq T$$

867 of linear continuous operators in the strong topology of $\mathcal{L}(H^{z,\zeta}, H^{z,\zeta})$, $(z, \zeta) \in$
 868 \mathbb{R}^2 , see [12]. In the cases of strict SG -hyperbolicity or of SG -hyperbolicity
 869 with constant multiplicities, such family can be explicitly expressed in terms
 870 of suitable (matrices of) SG FIOs of type I, modulo smoothing terms, see [14,
 871 16] and Subsection A.3 below. In the case of SG -hyperbolicity with variable
 872 multiplicities, it is, in general, a limit of a sequence of (matrices of) SG FIOs
 873 of type I, see [5]. A remarkable special case is the involutive roots one, where,
 874 again, $E(t, s)$ can be expressed as a finite linear combination of (matrices of)

875 SG FIOs of type I, modulo smoothing terms, see [1]. See, e.g., [20] and [32] for
 876 the results in the classical situations, where the variable x belongs to a bounded
 877 set.

878 In all the three cases mentioned above, the fundamental solution satisfies

$$\begin{cases} (D_t - \text{Op}(\kappa_1(t)) - \text{Op}(\kappa_0(t)))E(t, s) = 0, & (t, s) \in [0, T_0]^2, \\ E(s, s) = I, & s \in [0, T_0]. \end{cases} \quad (\text{A.6})$$

879 The fundamental solution of a first order SG-hyperbolic system with diago-
 880 nal principal part, $E(t, s)$, has the following properties, which actually hold for
 881 the broader class of symmetric first order system of the type (A.5), of which
 882 systems with real-valued, diagonal principal part are a special case, see [12], Ch.
 883 6, §3, and [14].

884 **Theorem A.6.** *Let the system (A.5) be hyperbolic with diagonal principal part*
 885 $\kappa_1 \in C^1([0, T], S^{1,1}(\mathbb{R}^d))$, *and lower order part* $\kappa_0 \in C^1([0, T], S^{0,0}(\mathbb{R}^d))$. *Then,*
 886 *for any choice of* $W_0 \in H^{z,\zeta}(\mathbb{R}^d)$, $Y \in C([0, T], H^{z,\zeta}(\mathbb{R}^d))$, *there exists a unique*
 887 *solution* $W \in C([0, T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([0, T], H^{z-1,\zeta-1}(\mathbb{R}^d))$ *of (A.5),* $(z, \zeta) \in$
 888 \mathbb{R}^2 , *given by Duhamel's formula*

$$W(t) = E(t, s)W_0 + i \int_s^t E(t, \vartheta)Y(\vartheta)d\vartheta, \quad t \in [0, T].$$

889 Moreover, the solution operator $E(t, s)$ has the following properties:

- 890 1. $E(t, s): \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is an operator belonging to $\mathcal{O}(0, 0)$, $(t, s) \in$
 891 $[0, T]^2$; its first order derivatives, $\partial_t E(t, s)$, $\partial_s E(t, s)$, exist in the strong
 892 operator convergence of $\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-1,\zeta-1}(\mathbb{R}^d))$, $(z, \zeta) \in \mathbb{R}^2$, and be-
 893 long to $\mathcal{O}(1, 1)$;
- 894 2. $E(t, s)$ is bounded and strongly continuous from $[0, T]_{ts}^2$ to $\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z,\zeta}(\mathbb{R}^d))$,
 895 $(z, \zeta) \in \mathbb{R}^2$; $\partial_t E(t, s)$ and $\partial_s E(t, s)$ are bounded and strongly continuous
 896 from $[0, T]_{ts}^2$ to $\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-1,\zeta-1}(\mathbb{R}^d))$, $(z, \zeta) \in \mathbb{R}^2$;
- 897 3. for $t, s, t_0 \in [0, T]$ we have

$$E(t_0, t_0) = I, \quad E(t, s)E(s, t_0) = E(t, t_0), \quad E(t, s)E(s, t) = I;$$

4. $E(t, s)$ satisfies, for $(t, s) \in [0, T]^2$, the differential equations

$$D_t E(t, s) - (\text{Op}(\kappa_1(t)) + \text{Op}(\kappa_0(t)))E(t, s) = 0, \quad (\text{A.7})$$

$$D_s E(t, s) + E(t, s)(\text{Op}(\kappa_1(s)) + \text{Op}(\kappa_0(s))) = 0; \quad (\text{A.8})$$

- 898 5. the operator family $E(t, s)$ is uniquely determined by the properties (1)-(3)
 899 here above, and one of the differential equations (A.7), (A.8).

900 **Corollary A.7.** 1. Under the hypotheses of Theorem A.6, $E(t, s)$ is invert-
 901 ible on $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$, and $H^{z,\zeta}(\mathbb{R}^d)$, $(z, \zeta) \in \mathbb{R}^2$, with inverse given by
 902 $E(s, t)$, $s, t \in [0, T]$.

903 2. If, additionally, one assumes $\kappa_1 \in C^m([0, T], S^{1,1}(\mathbb{R}^d))$, $\kappa_0 \in C^m([0, T], S^{0,0}(\mathbb{R}^d))$,
 904 $m \geq 2$, the partial derivatives $\partial_t^j \partial_s^k E(t, s)$ exist in strong operator conver-
 905 gence of $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, and $\partial_t^j \partial_s^k E(t, s) \in \mathcal{O}(j+k, j+k)$, $j+k \leq$
 906 m . Moreover, $\partial_t^j \partial_s^k E(t, s)$ is strongly continuous from $[0, T]_{ts}^2$ to every
 907 $\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-j-k, \zeta-j-k}(\mathbb{R}^d))$, $(z, \zeta) \in \mathbb{R}^2$, $j+k \leq m$.

908 In [5] we have proved the next Theorem A.8, concerning the structure of
 909 $E(t, s)$, in the spirit of the approach followed in [20].

910 **Theorem A.8.** *Under the same hypotheses of Theorem A.6, if T_0 is small*
 911 *enough, for every fixed $(t, s) \in \Delta_{T_0}$, $E(t, s)$ is a limit of a sequence of matrices of*
 912 *SG FIOs of type I, with regular phase functions $\varphi_{jk}(t, s)$ belonging to $\mathfrak{P}_\delta(c_h|t-s|)$,*
 913 *$c_h \geq 1$, of class C^1 with respect to $(t, s) \in \Delta_{T_0}$, and amplitudes belonging*
 914 *to $C^1(\Delta_{T_0}, S^{0,0}(\mathbb{R}^d))$.*

915 In the case of strict hyperbolicity, or, more generally, hyperbolicity with
 916 constant multiplicities, we can actually “decouple” the equations in (A.5) into
 917 n blocks of smaller dimensions, by means of the so-called *perfect diagonalizer*,
 918 an element of $C^\infty([0, T], \text{Op}(S^{0,0}))$. Thus, the solution of (A.5) can be reduced
 919 to the solution of n independent smaller systems. The principal part of the co-
 920 efficient matrix of each one of such decoupled subsystems admits then a single
 921 distinct eigenvalue of maximum multiplicity, so that it can be treated, essen-
 922 tially, like a scalar SG-hyperbolic equations of first order. Explicitely, see, e.g.,
 923 [14, 20],

924 **Theorem A.9.** *Assume that the system (A.5) is hyperbolic with constant mul-*
 925 *tiplicities ν_j , $j = 1, \dots, N$, $\nu_1 + \dots + \nu_n = \nu$, with diagonal principal part*
 926 *$\kappa_1 \in C^\infty([0, T], S^{1,1}(\mathbb{R}^d))$ and $\kappa_0 \in C^\infty([0, T], S^{0,0}(\mathbb{R}^d))$, both of them $(\nu \times \nu)$ -*
 927 *dimensional matrices. Then, there exist $(\nu \times \nu)$ -dimensional matrices $\omega \in$*
 928 *$C^\infty([0, T], S^{0,0}(\mathbb{R}^d))$ and $\tilde{\kappa}_0 \in C^\infty([0, T], S^{0,0}(\mathbb{R}^d))$ such that*

$$\det(\omega) \asymp 1 \Rightarrow \omega^{-1} \in C^\infty([0, T], S^{0,0}(\mathbb{R}^d)), \quad \tilde{\kappa}_0 = \text{diag}(\tilde{\kappa}_{01}, \dots, \tilde{\kappa}_{0n}),$$

929 $\tilde{\kappa}_{0j}(\nu_j \times \nu_j)$ -dimensional matrix, and

$$\begin{aligned} & (D_t - \text{Op}(\kappa_1(t)) - \text{Op}(\kappa_0(t)))\text{Op}(\omega(t)) - \text{Op}(\omega(t))(D_t - \text{Op}(\kappa_1(t)) - \text{Op}(\tilde{\kappa}_0(t))) \\ & \in C^\infty([0, T], \text{Op}(S^{-\infty, -\infty}(\mathbb{R}^d))). \end{aligned} \quad (\text{A.9})$$

930 In this situation, by an extension of the results in [14, 16], we can give an
 931 explicit form to the fundamental solution $E(t, s)$ in Theorem A.8, in terms of
 932 (smooth families of) SG FIOs of type I, modulo smoothing remainders. With
 933 the results of Theorem A.9 at hand, we solve, by means of the so-called *geomet-*
 934 *rical optics* (or FIOs) method, the system

$$\begin{cases} (D_t - \text{Op}(\kappa_1(t)) - \text{Op}(\tilde{\kappa}_0(t)))\tilde{E}(t, s) = 0, & t \in [0, T_0], \\ \tilde{E}(s, s) = I, & s \in [0, T_0]. \end{cases} \quad (\text{A.10})$$

935 Notice that the *approximate solution operator* $\tilde{A}(t, s)$, $(t, s) \in \Delta_{T_0}$, in terms of
 936 SG FIOs solves the corresponding operator problem up to smoothing remain-
 937 ders. Namely, the FIOs family $\tilde{A}(t, s)$ solves the system

$$\begin{cases} (D_t - \text{Op}(\kappa_1(t)) - \text{Op}(\tilde{\kappa}_0(t)))\tilde{A}(t, s) = \tilde{R}_1(t, s), & (t, s) \in \Delta_{T_0}, \\ \tilde{A}(s, s) = I + \tilde{R}_2(s), & s \in [0, T_0), \end{cases} \quad (\text{A.11})$$

where \tilde{R}_1 and \tilde{R}_2 are suitable smooth families of operators in $\mathcal{O}(-\infty, -\infty)$, coming from the solution method, see [12, 13, 14, 16, 20] for more details. It turns out that $\tilde{A}(t, s)$ belongs to $\mathcal{O}(0, 0)$ for any $(t, s) \in \Delta_{T_0}$. Explicitly,

$$\begin{aligned} \tilde{A}(t, s) &= \text{diag}(\tilde{A}^{(1)}(t, s), \dots, \tilde{A}^{(m)}(t, s)), \\ \tilde{A}^{(p)}(t, s) &= \text{diag}(\text{Op}_{\varphi_{\varpi_p(1)}(t, s)}(a_1^{(p)}(t, s)), \dots, \text{Op}_{\varphi_{\varpi_p(m)}(t, s)}(a_m^{(p)}(t, s))), p = 1, \dots, m, \end{aligned}$$

938 with phase functions $\varphi_j \in C^\infty(\Delta_{T_0}, \mathfrak{P}_\delta(\lambda))$, $\lambda = \lambda(T_0)$ suitably small, so-
 939 lutions of the eikonal equations (A.3) with τ_j in place of \varkappa , and symbols
 940 $a_j^{(p)} \in C^\infty(\Delta_{T_0}, S^{0,0})$, $p, j = 1, \dots, m$, see [14]. Solving the equations in (A.10)
 941 modulo smoothing terms is enough for our aims. Indeed, we have the following
 942 result (see [2] for its proof).

943 **Proposition A.10.** *Under the hypotheses (4.1), (4.2), let $A(t, s) = \text{Op}(\omega(t)) \circ$
 944 $\tilde{A}(t, s) \circ \text{Op}(\omega_{-1}(s))$, with $\tilde{A}(t, s)$ solution of (A.11), $(t, s) \in \Delta_{T_0}$, and $\text{Op}(\omega_{-1}(s))$
 945 *parametrix of the perfect diagonalizer* $\text{Op}(\omega(s))$, $s \in [0, T]$. Then, the solution
 946 $E(t, s)$ of (A.6) and the operator family $A(t, s)$ satisfy $E - A \in C^\infty(\Delta_{T_0}, \text{Op}(S^{-\infty, -\infty}(\mathbb{R}^d)))$.*

947 **Remark A.11.** Proposition A.10 means that the Schwartz kernels of E and A
 948 differ by a family of elements of $\mathcal{S}(\mathbb{R}^{2d})$, smoothly depending on $(t, s) \in \Delta_{T_0}$.

949 Using Proposition A.10, by repeated applications of Theorem 3.5, we finally
 950 obtain

$$E(t, s) = E_0(t, s) + R(t, s), \quad (t, s) \in \Delta_{T_0}, \quad (\text{A.12})$$

951 where

952 - E_0 is a $(nm \times nm)$ -dimensional matrix of operators in $\mathcal{O}(0, 0)$ given by

$$E_0(t, s) = \left(\sum_{p=1}^n \text{Op}_{\varphi_p(t, s)}(e_{pjk}(t, s)) \right)_{j, k=0, \dots, nm-1},$$

953 with the regular phase-functions $\varphi_p(t, s)$, solutions of the eikonal equations
 954 associated with τ_p , and symbols $e_{pjk}(t, s) \in S^{0,0}$, $j, k = 0, \dots, nm - 1$,
 955 $p = 1, \dots, n$, smoothly depending on $(t, s) \in \Delta_{T_0}$;

956 - R is a $(nm \times nm)$ -dimensional matrix of elements in $C^\infty(\Delta_{T_0}, \text{Op}(S^{-\infty, -\infty}))$,
 957 operators with kernel in $\mathcal{S}(\mathbb{R}^{2d})$, smoothly depending on $(t, s) \in \Delta_{T_0}$, that
 958 is,

$$R = (\text{Op}(r_{jk}(t, s)))_{j, k=0, \dots, nm-1},$$

959 with symbols $r_{jk} \in C^\infty(\Delta_{T_0}, S^{-\infty, -\infty})$, $j, k = 0, \dots, nm-1$, collecting the
 960 remainders of the compositions in $\text{Op}(\omega) \circ \tilde{A} \circ \text{Op}(\omega_{-1})$ and the difference
 961 $E - A$.

962 Achieving a similar result for systems with involutive roots is not straightfor-
 963 ward. In fact, in this case, the system cannot, in general, be diagonalized block
 964 by block, and a quite technical analysis is needed, see [1].

965 *A.3. Fundamental solution for SG-hyperbolic linear operators*

966 By the hyperbolicity hypotheses, as it will be explained below, to obtain the
 967 term depending on the initial conditions and the kernel Λ , associated with the
 968 linear operator in (1.1), it is enough to know the fundamental solution of first
 969 order systems with diagonal principal part. The next results are employed to
 970 switch from (4.4) to a first order linear system of the form (A.5).

971 **Proposition A.12.** *Let L be a hyperbolic operator with constant multiplicities*
 972 *l_j , $j = 1, \dots, n \leq m$. Denote by $\theta_j \in G_j$, $j = 1, \dots, n$, the distinct real roots of*
 973 *\mathcal{L}_m in (1.5). Then, it is possible to factor L as*

$$L = L_n \cdots L_1 + \sum_{j=1}^m \text{Op}(r_j(t)) D_t^{m-j} \quad (\text{A.13})$$

with

$$L_j = (D_t - \text{Op}(\theta_j(t)))^{l_j} + \sum_{k=1}^{l_j} \text{Op}(h_{jk}(t)) (D_t - \text{Op}(\theta_j(t)))^{l_j-k}, \quad (\text{A.14})$$

$$h_{jk} \in C^\infty([0, T], S^{k-1, k-1}(\mathbb{R}^d)), \quad r_j \in C^\infty([0, T], S^{-\infty, -\infty}(\mathbb{R}^d)), \quad j = 1, \dots, n, k = 1, \dots, l_j. \quad (\text{A.15})$$

974 The following corollary is an immediate consequence of Proposition A.12, and
 975 is proved by means of a reordering of the distinct roots θ_j , $j = 1, \dots, n$.

976 **Corollary A.13.** *Let ϖ_j , $j = 1, \dots, n$, denote the reordering of the n -tuple*
 977 *$(1, \dots, n)$, given, for $k = 1, \dots, n$, by*

$$\varpi_j(k) = \begin{cases} j+k-1 & \text{for } j+k \leq n+1, \\ j+k-n-1 & \text{for } j+k > n+1, \end{cases} \quad (\text{A.16})$$

978 *That is, for $n \geq 2$, $\varpi_1 = (1, \dots, n)$, $\varpi_2 = (2, \dots, n, 1)$, \dots , $\varpi_n = (n, 1, \dots, n -$
 979 $1)$. Then, under the same hypotheses of Proposition A.12, we have, for any
 980 $p = 1, \dots, n$,*

$$L = L_{\varpi_p(n)}^{(p)} \cdots L_{\varpi_p(1)}^{(p)} + \sum_{j=1}^m \text{Op}(r_j^{(p)}(t)) D_t^{m-j} \quad (\text{A.17})$$

981 with

$$L_j^{(p)} = (D_t - \text{Op}(\theta_j(t)))^{l_j} + \sum_{k=1}^{l_j} \text{Op}(h_{jk}^{(p)}(t)) (D_t - \text{Op}(\theta_j(t)))^{l_j-k}, \quad (\text{A.18})$$

982

$$h_{jk}^{(p)} \in C^\infty([0, T], S^{k-1, k-1}(\mathbb{R}^d)), j = 1, \dots, n, k = 1, \dots, l_j, \quad (\text{A.19})$$

$$r_j^{(p)} \in C^\infty([0, T], S^{-\infty, -\infty}(\mathbb{R}^d)), j = 1, \dots, m. \quad (\text{A.20})$$

983 **Remark A.14.** Of course, for $n = 1$, we only have the single “reordering”
984 $\varpi_1 = (1)$, $l_1 = l = m$, and

$$L = L_1^{(1)} + \sum_{j=1}^m \text{Op}(r_j^{(1)}(t)) D_t^{m-j}$$

with

$$L_1^{(1)} = (D_t - \text{Op}(\theta_1(t)))^m + \sum_{k=1}^m \text{Op}(h_{1k}^{(1)}(t)) (D_t - \text{Op}(\theta_1(t)))^{m-k},$$

$$h_{1k}^{(1)} \in C^\infty([0, T], S^{k-1, k-1}(\mathbb{R}^d)), k = 1, \dots, m, \quad r_j^{(1)} \in C^\infty([0, T], S^{-\infty, -\infty}(\mathbb{R}^d)), j = 1, \dots, m$$

985 With inductive procedures similar to those performed in [8, 9] and [23],
986 respectively, it is possible to prove the following Lemma A.15.

Lemma A.15. *Under the same hypotheses of Proposition A.12, for all $k = 0, \dots, m-1$, it is possible to find symbols $\varsigma_{kpq} \in C^\infty([0, T], S^{k-q+l_p-n, k-q+l_p-n}(\mathbb{R}^d))$, $p = 1, \dots, n$, $q = 0, \dots, l_p - 1$, such that, for all $t \in [0, T]$,*

$$\theta^k = \sum_{p=1}^n \left[\sum_{q=0}^{l_p-1} \varsigma_{kpq}(t) (\theta - \theta_p(t))^q \right] \cdot \left[\prod_{\substack{1 \leq j \leq n \\ j \neq p}} (\theta - \theta_j(t))^{l_j} \right].$$

987 Let us denote by θ_j , $j = 1, \dots, n$, the distinct values of the roots τ_k , $k =$
988 $1, \dots, m$, and with ϖ_p , $p = 1, \dots, n$, the reorderings of the n -tuple $(1, \dots, n)$
989 defined in (A.16).

990 The equivalence of the Cauchy problems for the equation $Lu(t) = g(t)$ and
991 a 1×1 system (A.5) is then trivial for $m = 1$. For $m \geq 2$, we will now
992 define a (nm) -dimensional vector of unknown W and construct a corresponding
993 linear first order hyperbolic system, with diagonal principal part and constant
994 multiplicities, equivalent to $Lu(t) = g(t)$.

995 Let us set, for convenience, with the notation introduced in Corollary A.13,

$$l^{(p,k)} = \begin{cases} 0, & k = 0, \\ \sum_{1 \leq j \leq k} l_{\varpi_p(j)}, & 1 \leq k \leq n-1, \text{ if } n \geq 2, \\ m, & k = n, \end{cases}$$

$$L^{(p,k)} = \begin{cases} I, & k = 0, \\ L_{\varpi_p(k)}^{(p)} \cdots L_{\varpi_p(1)}^{(p)}, & 1 \leq k \leq n-1, \text{ if } n \geq 2, \end{cases}$$

996 $p = 1, \dots, n$, and define

$$W_{l^{(p,k)}+j+1}^{(p)}(t) = (D_t - \text{Op}(\theta_{\varpi_p(k+1)}(t)))^j L^{(p,k)} u(t) \quad (\text{A.21})$$

997 for $p = 1, \dots, n$, $k = 0, \dots, n-1$, $j = 0, \dots, l_{\varpi_p(k+1)} - 1$. Using Lemma A.15,
998 we can express the t derivatives of u in terms of the components of W from
999 (A.21). In fact:

1000 **Lemma A.16.** *Under the hypotheses of Lemma A.15, for all $k = 1, \dots, m-1$,
1001 $p = 1, \dots, n$, it is possible to find symbols $w_{kj}^{(p)} \in C^\infty([0, T], S^{j,j}(\mathbb{R}^d))$, $j =$
1002 $1, \dots, k$, such that, with the (nm) -dimensional vector W defined in (A.21),*

$$D_t^k u(t) = \sum_{j=1}^k \text{Op}(w_{kj}^{(p)}(t)) W_{k-j+1}^{(p)}(t) + W_{k+1}^{(p)}(t). \quad (\text{A.22})$$

1003 By the definition (A.21), we find the extension of (A.22) to $k = 0$ in the form
1004 $u(t) = W_1^{(p)}(t)$, $p = 1, \dots, n$. Using Lemma A.16 we see that (A.17), (A.21)
1005 and (A.22) give rise to a block diagonal linear system in the nm unknown
1006 $W_{l^{(p,k)}+j+1}^{(p)}(t)$ with blocks labeled by $p = 1, \dots, n$, of the type

$$\left\{ \begin{array}{l} \dots, \\ (D_t - \text{Op}(\theta_{\varpi_p(1)}(t))) W_{j+1}^{(p)}(t) = W_{j+2}^{(p)}(t), \quad j = 0, \dots, l_{\varpi_p(1)} - 2, \text{ if } l_{\varpi_p(1)} \geq 2, \\ (D_t - \text{Op}(\theta_{\varpi_p(1)}(t))) W_{l^{(p,1)}+1}^{(p)}(t) = - \sum_{k=1}^{l_{\varpi_p(1)}} \text{Op}(h_{\varpi_p(1)k}^{(p)}(t)) W_{l^{(p,1)}-k+1}^{(p)}(t) + W_{l^{(p,1)}+1}^{(p)}(t), \\ (D_t - \text{Op}(\theta_{\varpi_p(2)}(t))) W_{l^{(p,1)}+j+1}^{(p)}(t) = W_{l^{(p,1)}+j+2}^{(p)}(t), \quad j = 0, \dots, l_{\varpi_p(2)} - 2, \text{ if } l_{\varpi_p(2)} \geq 2, n \geq 2, \\ (D_t - \text{Op}(\theta_{\varpi_p(2)}(t))) W_{l^{(p,2)}+1}^{(p)}(t) = - \sum_{k=1}^{l_{\varpi_p(2)}} \text{Op}(h_{\varpi_p(2)k}^{(p)}(t)) W_{l^{(p,2)}-k+1}^{(p)}(t) + W_{l^{(p,2)}+1}^{(p)}(t), \text{ if } n \geq 2, \\ \dots, \\ (D_t - \text{Op}(\tau_{\varpi_p(n)}(t))) W_m^{(p)}(t) = - \sum_{k=1}^{l_{\varpi_p(n)}} \text{Op}(h_{\varpi_p(n)k}^{(p)}(t)) W_{m-k+1}^{(p)}(t) \\ \quad - \sum_{j=1}^{m-1} \left(\sum_{q=1}^{m-j} \text{Op}(r_j^{(p)}(t)) \circ \text{Op}(w_{m-j,q}^{(p)}(t)) W_{m-j-q+1}^{(p)}(t) + \text{Op}(r_j^{(p)}(t)) W_{m-j+1}^{(p)}(t) \right) \\ \quad - \text{Op}(r_m^{(p)}(t)) W_1^{(p)}(t) + g(t), \\ \dots \end{array} \right. \quad (\text{A.23})$$

1007 and equivalent, block by block, to the equation $Lu(t) = g(t)$.

1008 As it is very well-known in the usual hyperbolic theory, in the case of weak
 1009 hyperbolicity the principal term does not provide enough information, by it-
 1010 self, to imply well-posedness of the Cauchy problem. In other words, lower
 1011 order terms are also relevant in this case, and one needs to impose additional
 1012 conditions on them. We will then assume that L satisfies the *SG*-Levi condition

$$h_{jk}^{(p)} \in C^\infty([0, T], S^{0,0}(\mathbb{R}^d)), \quad p, j = 1, \dots, n, k = 1, \dots, l_j, \quad (\text{A.24})$$

1013 see Corollary A.13.

1014 **Remark A.17.** Let us observe that, indeed, (A.24) needs to be fulfilled only
 1015 for a single value of $p = 1, \dots, n$. Also, (A.24) is automatically fulfilled when L
 1016 is strictly *SG*-hyperbolic. If L satisfies (A.24) we will also say that L is of Levi
 1017 type.

1018 It is clear, in view of the calculus of *SG* pseudodifferential operators, the
 1019 fact that $r_j^{(p)} \in C^\infty([0, T], S^{-\infty, -\infty})$, $p = 1, \dots, n$, and the inclusions among
 1020 the *SG* symbols, that the system (A.23) is a hyperbolic first order linear system
 1021 of the form (A.5), where:

- the $(nm \times nm)$ -dimensional, block-diagonal matrix $\kappa_1 \in C^\infty([0, T], S^{1,1})$ is
 given by $\kappa_1 = \text{diag}(\kappa_{11}, \dots, \kappa_{1n})$, with each block defined by

$$\kappa_{1p} = \text{diag}(\underbrace{\theta_{\omega_p(1)}, \dots, \theta_{\omega_p(1)}}_{l_{\omega_p(1)} \text{ times}}, \underbrace{\theta_{\omega_p(2)}, \dots, \theta_{\omega_p(2)}}_{l_{\omega_p(2)} \text{ times}}, \dots, \underbrace{\theta_{\omega_p(n)}, \dots, \theta_{\omega_p(n)}}_{l_{\omega_p(n)} \text{ times}}), \quad p = 1, \dots, n;$$

1022 - the $(nm \times nm)$ -dimensional, block-diagonal matrix $\kappa_0 \in C^\infty([0, T], S^{0,0})$ is
 1023 given by $\kappa_0 = \text{diag}(\kappa_{01}, \dots, \kappa_{0m})$ with suitable matrices κ_{0p} having entries in
 1024 $C^\infty([0, T], S^{0,0})$, $p = 1, \dots, n$;
 1025 - the right-hand side is

1026

$$Y(t) = (\underbrace{G(t), \dots, G(t)}_{n \text{ times}})^t, \quad G(t) = (\underbrace{0, \dots, 0}_{m-1 \text{ times}}, g(t))^t.$$

1027 The initial data W_0 is obtained by $W_0 = \text{Op}(b)U_0$, with $U_0 = (u_0, \dots, u_{m-1})^t$
 1028 and a $(mn \times m)$ -dimensional block-matrix symbol b with the following structure:

$$b = \begin{pmatrix} b^{(1)} \\ \text{---} \\ \dots \\ \text{---} \\ b^{(n)} \end{pmatrix}, \quad b^{(p)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ b_{10}^{(p)} & 1 & 0 & 0 & \dots \\ b_{20}^{(p)} & b_{21}^{(p)} & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad p = 1, \dots, n, \quad (\text{A.25})$$

1029 and the $(m \times m)$ -dimensional matrices $b^{(p)}$ satisfying

1030 - if $m \geq 2$, $b_{jk}^{(p)} \in S^{j-k, j-k}$, $j > k$, $j = 1, \dots, m-1$, $k = 0, \dots, j-1$,

- 1031 - $b_{jj}^{(p)} = 1 \in S^{0,0}$, $j = 0, \dots, m-1$,
- 1032 - if $m \geq 2$, $b_{jk}^{(p)} = 0$, $j < k$, $j = 0, \dots, m-2$, $k = j+1, \dots, m-1$,
- 1033 $p = 1, \dots, m$.

Remark A.18. Consider, for instance, the case $n = 1$, that is, \mathcal{L}_m admits a unique real root $\theta_1 = \tau_1$ of maximum multiplicity $l = l_1 = m$. Then, there is a single “reordering” $\varpi_1 = (1)$, the vector W has m components, $W = (W_1^{(1)}, \dots, W_m^{(1)})$, and (A.23) consists of a single block of m equations. Namely, in view of Corollary A.13, assuming $n \geq 2$ and dropping everywhere the ⁽¹⁾ label, (A.21) reads, in this case,

$$\begin{aligned} W_1(t) &= u(t), \\ W_2(t) &= (D_t - \text{Op}(\tau_1(t)))u(t) = (D_t - \text{Op}(\tau_1(t)))W_1(t), \\ &\dots, \\ W_m(t) &= (D_t - \text{Op}(\tau_1(t)))^{m-1}u(t) = (D_t - \text{Op}(\tau_1(t)))W_{m-1}(t), \end{aligned}$$

while $Lu(t) = g(t)$ is then equivalent to

$$\begin{aligned} (D_t - \text{Op}(\tau_1(t)))^m u(t) + \sum_{k=1}^m \text{Op}(h_{1k}(t))(D_t - \text{Op}(\tau_1(t)))^{m-k} u(t) \\ + \sum_{j=1}^m \text{Op}(r_j(t))D_t^{m-j} u(t) = g(t) \\ \Leftrightarrow \\ (D_t - \text{Op}(\tau_1(t)))W_m(t) = - \sum_{k=1}^m \text{Op}(h_{1k}(t))W_{m-k+1}(t) \\ - \sum_{j=1}^{m-1} \left(\sum_{q=1}^{m-j} \text{Op}(r_j(t)) \circ \text{Op}(w_{m-j,q}(t))W_{m-j-q+1}(t) + \text{Op}(r_j(t))W_{m-j+1}(t) \right) \\ - \text{Op}(r_m(t))W_1(t) + g(t), \end{aligned}$$

1034 that is,

$$\left\{ \begin{array}{l} (D_t - \text{Op}(\tau_1(t)))W_1(t) = W_2(t) \\ \dots \\ (D_t - \text{Op}(\tau_1(t)))W_{m-1}(t) = W_m(t) \\ (D_t - \text{Op}(\tau_1(t)))W_m(t) = - \sum_{k=1}^m \text{Op}(h_{1k}(t))W_{m-k+1}(t) \\ - \sum_{j=1}^{m-1} \left(\sum_{q=1}^{m-j} \text{Op}(r_j(t)) \circ \text{Op}(w_{m-j,q}(t))W_{m-j-q+1}(t) + \text{Op}(r_j(t))W_{m-j+1}(t) \right) \\ - \text{Op}(r_m(t))W_1(t) + g(t), \end{array} \right.$$

1035 which has the form (A.5) with $Y(t) = (\underbrace{0, \dots, 0}_{m-1 \text{ times}}, g(t))^t$, as claimed, since $\kappa_1(t) =$
1036 $\text{diag}(\tau_1(t), \dots, \tau_1(t))$, while the coefficients of the components of W in the right-
1037 hand sides of the equations are all symbols of order $(0, 0)$, since $S^{-\infty, -\infty} \subset S^{0, 0}$.

1038 The next Lemma A.19 from [16], see also [8, 9] and [23], is the key result to
1039 achieve, from (A.12) and the expressions of E_0 and R , the correct regularity of
1040 u .

1041 **Lemma A.19.** *There exists a $(m \times mn)$ -dimensional matrix $\mathcal{Y}_n \in C^\infty([0, T_0], S^{0, 0}(\mathbb{R}^d))$
1042 such that the k -th row consists of symbols of order $(l - m + k, l - m + k)$,
1043 $k = 0, \dots, m - 1$, and*

$$\begin{pmatrix} u(t) \\ \dots \\ D_t^{m-1} u(t) \end{pmatrix} = \text{Op}(\mathcal{Y}_n(t))W(t), \quad t \in [0, T_0].$$

1044 Assume that $g \in C([0, T], H^{z, \zeta})$, $(z, \zeta) \in \mathbb{R}^2$. Then, the Cauchy problem
1045 for the first order system (A.5) with $s = 0$, equivalent to (4.4), fulfills all the
1046 assumptions of Theorem A.6. An application of Theorem A.6, together with
1047 (A.12) and Lemma A.19 initially gives

$$\begin{aligned} \begin{pmatrix} u(t) \\ \dots \\ D_t^{m-1} u(t) \end{pmatrix} &= [\text{Op}(\mathcal{Y}_n(t)) \circ (E_0(t, 0) + R(t, 0)) \circ \text{Op}(b)]U_0 \\ &+ i \int_0^t [\text{Op}(\mathcal{Y}_n(t)) \circ (E_0(t, s) + R(t, s))]Y(s)ds, \quad t \in [0, T_0]. \end{aligned}$$

1048 Then, taking into account that the only non-vanishing entries of Y coincide
1049 with g , computations with matrices, the structure of the entries of \mathcal{Y}_n and b ,
1050 and further applications of Theorem 3.5 give

$$\begin{aligned} u(t) &= \sum_{j=0}^{m-1} \left[\sum_{p=1}^n \text{Op}_{\varphi_p(t, 0)}(z_{pj}^0(t)) + \text{Op}(r_j^0(t)) \right] u_j \\ &+ i \int_0^t \left[\sum_{p=1}^n \text{Op}_{\varphi_p(t, s)}(z_p^1(t, s)) + \text{Op}(r^1(t, s)) \right] g(s)ds, \quad (\text{A.26}) \\ &= v_0(t) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, \cdot, y)g(s, y) dyds, \end{aligned}$$

1051 where

- 1052 - the phase functions φ_p are solution to the eikonal equations (A.3), with
1053 θ_p in place of \varkappa , $p = 1, \dots, n$;
- 1054 - $z_{pj}^0 \in C^\infty([0, T_0], S^{l-1-j, l-1-j})$, $p = 1, \dots, n$, $r_j^0 \in C^\infty([0, T_0], S^{-\infty, -\infty})$,
1055 $j = 0, \dots, m - 1$, so that $v_0 \in \bigcap_{j \geq 0} C^j([0, T_0], H^{z+m-l-j, \zeta+m-l-j})$;

1056 - $\Lambda \in C^\infty(\Delta_{T_0}, \mathcal{S}')$ is, for any $(t, s) \in \Delta_{T_0}$, the Schwartz kernel of the
 1057 operator

$$Z_{l-m}(t, s) = i \left[\sum_{p=1}^n \text{Op}_{\varphi_p(t,s)}(z_p^1(t, s)) + \text{Op}(r^1(t, s)) \right], \quad (\text{A.27})$$

1058 with $z_p^1 \in C^\infty(\Delta_{T_0}, \mathcal{S}^{l-m, l-m})$, $p = 1, \dots, m$, $r^1 \in C^\infty(\Delta_{T_0}, \mathcal{S}^{-\infty, -\infty})$, so
 1059 that also

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, \cdot, y) g(s, y) dy ds \in \bigcap_{j \geq 0} C^j([0, T_0], H^{z+m-l-j, \zeta+m-l-j}).$$

1060 Notice the usual abuse of notation, using the kernel $\Lambda(t, s)$ in the *distribu-*
 1061 *tional integral* in (A.26). By Proposition A.2, $\Lambda(t, s)$ differs by an element of
 1062 $C^\infty(\Delta_{T_0}, \mathcal{S})$ from the kernel of

$$\tilde{Z}_{l-m}(t, s) = i \sum_{p=1}^n \text{Op}_{\varphi_p(t,s)}(z_p^1(t, s)). \quad (\text{A.28})$$

1063 By the analysis in [1], in the case of involutive roots analogous formulae can
 1064 be obtained for u and Λ . Namely, the final expression (A.26) for u , $v_0 \in$
 1065 $\bigcap_{j \geq 0} C^j([0, T_0], H^{z-j, \zeta-j})$, as well as (A.27) and (A.28) with $l = m$, hold true.

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