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# Heavily separable functors [☆]

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ABSTRACT

Prompted by an example related to the tensor algebra, we introduce and investigate a stronger version of the notion of separable functor that we call heavily separable. We test this notion on several functors traditionally connected to the study of separability.



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## Introduction

Given a field $\Bbbk$, the functor $\mathbf{P} : \mathrm{Bialg}_{\Bbbk} \to \mathrm{Vec}_{\Bbbk}$, assigning to a $\Bbbk$-bialgebra $B$ the $\Bbbk$-vector space of its primitive elements, admits a left adjoint $\mathbf{T}$, assigning to a vector space $V$ the tensor algebra $\mathbf{T}V$ endowed with its canonical bialgebra structure such that the elements in $V$ become primitive. By investigating the properties of the adjunction $(\mathbf{T}, \mathbf{P})$, together with its unit $\boldsymbol{\eta}$ and counit $\boldsymbol{\epsilon}$, we discovered that there is a natural retraction $\boldsymbol{\gamma} : \mathbf{PT} \to \mathrm{Id}$ of $\boldsymbol{\eta}$, i.e. $\boldsymbol{\gamma} \circ \boldsymbol{\eta} = \mathrm{Id}$, fulfilling the condition $\boldsymbol{\gamma}\boldsymbol{\gamma} = \boldsymbol{\gamma} \circ \mathbf{P}\boldsymbol{\epsilon}\mathbf{T}$. The existence of a natural retraction of the unit of an adjunction is, by Rafael Theorem, equivalent to the fact that the left adjoint is a separable functor. It is then natural to wonder if the above extra condition on the retraction $\boldsymbol{\gamma}$ corresponds to a stronger notion of separability. In the present paper, we show that an affirmative answer to this question is given by what we call a heavily separable (h-separable for short) functor and we investigate this notion in case of functors usually connected to the study of separability.

Explicitly, in Section 1 we introduce the concept of h-separable functor and we recover classical results in the h-separable case such as their behavior with respect to composition (Lemma 1.4). In Section 2, we obtain a Rafael type Theorem 2.1. As a consequence we characterize the h-separability of a left (respectively right) adjoint functor either with respect to the forgetful functor from the Eilenberg-Moore category of the associated monad (resp. comonad) in Proposition 2.3 or by the existence of an augmentation (resp. grouplike morphism) of the associated monad (resp. comonad) in Corollary 2.7.

Section 3 is devoted to the investigation of the h-separability of the induction functor $\varphi^*$ and of the restriction of scalars functor $\varphi_*$ attached to a ring homomorphism $\varphi : R \to S$. In Proposition 3.1, we prove that $\varphi^*$ is h-separable if and only if there is a ring homomorphism $E : S \to R$ such that $E \circ \varphi = \mathrm{Id}$. Characterizing whether $\varphi_*$ is h-separable (in this case we say that $S/R$ is h-separable) is more laborious. In Proposition 3.4, we prove that $S/R$ is h-separable if and only if it is endowed with what we call a *h*-separability idempotent, a stronger version of a separability idempotent. In Lemma 3.6 we show that the ring epimorphisms (by this we mean epimorphisms in the category of rings) provide particular examples of h-separability. Next we investigate the particular case when $\mathrm{Im}(\varphi) \subseteq Z(S)$, which holds e.g. when $R$ is commutative and $S$ is an $R$-algebra. In Theorem 3.11 we discover that, in this case, $S/R$ is h-separable if and only if $\varphi$ is a ring epimorphism. Moreover $S$ becomes commutative. As a consequence, in Proposition 3.12 we show that a h-separable algebra over a field $\Bbbk$ is necessarily trivial. In Proposition 3.18 we show that the twisted semigroup ring is h-separable over the base ring $R$ only in trivial cases. As a consequence, in Corollary 3.19 we show that the monoid ring and the matrix ring are h-separable over the base ring only in trivial cases.

In Section 4 we investigate the h-separability of the induction and forgetful functors attached to a coring. In particular the latter leads to the notion of h-coseparable coring. In Theorem 4.3 we characterize an h-coseparable $A$-coring $\mathcal{C}$ by the existence of a suitable $A$-bimodule map $\alpha : \mathcal{C} \otimes_A \mathcal{C} \to A$. In Theorem 4.4 we show that an h-coseparable coalgebra over a field $\Bbbk$ is necessarily trivial as a consequence of the analogous result for algebras we already proved. In Theorem 4.5, we establish that the induced functor attached to a coring is h-separable if and only if this coring has an invariant grouplike element. Then we investigate the h-separability of the induction functor and its right adjoint attached to a bimodule ${}_R\Sigma_S$ such that $\Sigma_S$ is finitely generated and projective. Our results on these functors are summarized in Theorem 4.8 and Theorem 4.9. As a consequence in Corollary 4.11 and Corollary 4.12 we obtain further characterizations of the h-separability of the functors $\varphi_*$ and $\varphi^*$ mentioned above. The latter corollary implies the h-coseparability of the Sweedler coring attached to a ring homomorphism $\varphi : R \to S$ that splits as a ring homomorphism.

Finally in Section 5 we provide a more general version of our starting example $(\mathbf{T}, \mathbf{P})$ involving monoidal categories and bialgebras therein.

## 1. Heavily separable functors

In this section we collect general facts about heavily separable functors.

**Definition 1.1.** For every functor $F : \mathcal{B} \to \mathcal{A}$ we set

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{B}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(FX, FY) : f \mapsto Ff$$

Recall that $F$ is called **separable** if there is a natural transformation

$$P_{-,-} := P^F_{-,-} : \mathrm{Hom}_{\mathcal{A}}(F-, F-) \to \mathrm{Hom}_{\mathcal{B}}(-, -)$$

such that $P_{X,Y} \circ F_{X,Y} = \mathrm{Id}$ for every $X, Y$ objects in $\mathcal{B}$.

We say that $F$ is **heavily separable** (**h-separable** for short) if it is separable and the $P_{X,Y}$'s make commutative the following diagram for every $X, Y, Z \in \mathcal{B}$.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(FX, FY) \times \mathrm{Hom}_{\mathcal{A}}(FY, FZ) & \xrightarrow{P_{X,Y} \times P_{Y,Z}} & \mathrm{Hom}_{\mathcal{B}}(X, Y) \times \mathrm{Hom}_{\mathcal{B}}(Y, Z) \\
\circ \downarrow & & \downarrow \circ \\
\mathrm{Hom}_{\mathcal{A}}(FX, FZ) & \xrightarrow{P_{X,Z}} & \mathrm{Hom}_{\mathcal{B}}(X, Z)
\end{array}$$

where the vertical arrows are the obvious compositions. On elements the above diagram means that $P_{X,Z}(f \circ g) = P_{Y,Z}(f) \circ P_{X,Y}(g)$.

**Remark 1.2.** We were tempted to use the word "strongly" at first, instead of "heavily", but a notion of "strongly separable functor" already appeared in the literature in connection with graded rings in [7, Definition 3.1].

**Remark 1.3.** The Maschke's Theorem for separable functors asserts that for a separable functor $F : \mathcal{B} \to \mathcal{A}$ a morphism $f : X \to Y$ splits (resp. cosplits) if and only if $F(f)$

does. Explicitly, if $F(f) \circ g = \mathrm{Id}$ (resp. $g \circ F(f) = \mathrm{Id}$) for some morphism $g$ then $f \circ P_{Y,X}(g) = \mathrm{Id}$ (resp. $P_{Y,X}(g) \circ f = \mathrm{Id}$). If $F(f) \circ g = \mathrm{Id}$ and $F(f') \circ g' = \mathrm{Id}$ for $f : X \to Y, f' : Y \to Z$, then $f \circ P_{Y,X}(g) = \mathrm{Id}$ and $f' \circ P_{Z,Y}(g') = \mathrm{Id}$ so that $f' \circ f \circ P_{Y,X}(g) \circ P_{Z,Y}(g') = \mathrm{Id}$ so that $P_{Y,X}(g) \circ P_{Z,Y}(g')$ is a section of $f' \circ f$. Since $F(f' \circ f) \circ g \circ g' = \mathrm{Id}$, we also have $f' \circ f \circ P_{Z,X}(g \circ g') = \mathrm{Id}$ so that $P_{Z,X}(g \circ g')$ is another section of $f' \circ f$. In general these two sections may differ but not in case $F$ is h-separable. Thus we get a sort of functoriality of the splitting. A similar remark holds for cosplittings. We thank J. Vercruysse for this observation.

**Lemma 1.4.** *Let $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors.*

i) *If $F$ and $G$ are h-separable so is $GF$.*
ii) *If $GF$ is h-separable so is $F$.*
iii) *If $G$ is h-separable, then $F$ is h-separable if and only if so is $GF$.*

**Proof.** i) By [15, Lemma 1] we know that $GF$ is separable with respect to $P^{GF}_{X,Y} := P^{F}_{X,Y} \circ P^{G}_{FX,FY}$. As a consequence, since $F$ and $G$ are h-separable, the following diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{A}}(GFX, GFY) \times \mathrm{Hom}_{\mathcal{A}}(GFY, GFZ) & \xrightarrow{P^{G}_{FX,FY} \times P^{G}_{FY,FZ}} & \mathrm{Hom}_{\mathcal{B}}(FX, FY) \times \mathrm{Hom}_{\mathcal{B}}(FY, FZ) & \xrightarrow{P^{F}_{X,Y} \times P^{F}_{Y,Z}} & \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \\
\circ \downarrow & & \downarrow \circ & & \downarrow \circ \\
\mathrm{Hom}_{\mathcal{A}}(GFX, GFZ) & \xrightarrow{P^{G}_{FX,FZ}} & \mathrm{Hom}_{\mathcal{B}}(FX, FZ) & \xrightarrow{P^{F}_{X,Z}} & \mathrm{Hom}_{\mathcal{C}}(X, Z)
\end{array}$$

commutes so that $GF$ is h-separable.

ii) By [15, Lemma 1] we know that $F_{X,Y}$ cosplits naturally through $P^{F}_{X,Y} := P^{GF}_{X,Y} \circ G_{FX,FY}$. On the other hand, since $G$ is a functor and $GF$ is h-separable the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{B}}(FX, FY) \times \mathrm{Hom}_{\mathcal{B}}(FY, FZ) & \xrightarrow{G_{FX,FY} \times G_{FY,FZ}} & \mathrm{Hom}_{\mathcal{A}}(GFX, GFY) \times \mathrm{Hom}_{\mathcal{A}}(GFY, GFZ) & \xrightarrow{P^{GF}_{X,Y} \times P^{GF}_{Y,Z}} & \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \\
\circ \downarrow & & \downarrow \circ & & \downarrow \circ \\
\mathrm{Hom}_{\mathcal{B}}(FX, FZ) & \xrightarrow{G_{FX,FZ}} & \mathrm{Hom}_{\mathcal{A}}(GFX, GFZ) & \xrightarrow{P^{GF}_{X,Z}} & \mathrm{Hom}_{\mathcal{C}}(X, Z)
\end{array}$$

so that $F$ is h-separable.

iii) It follows trivially from i) and ii). □

**Remark 1.5.** The present remark was pointed out by J. Vercruysse. If the functor $F : \mathcal{B} \to \mathcal{A}$ is a split monomorphism, meaning that there is a functor $G : \mathcal{A} \to \mathcal{B}$ such that $GF = \mathrm{Id}$, then $F$ is h-separable. This follows by setting $P_{X,Y} := G_{X,Y}$ as in Definition 1.1. It can also be proved by means of Lemma 1.4,ii).

**Lemma 1.6.** *A full functor is faithful if and only if it is h-separable.*

**Proof.** Let $F : \mathcal{B} \to \mathcal{A}$ be a full functor. If $F$ is faithful we have that the canonical map $F_{X,Y} : \mathrm{Hom}_{\mathcal{B}}(X,Y) \to \mathrm{Hom}_{\mathcal{A}}(FX,FY)$ is invertible so that we can take $P_{X,Y} := F^{-1}_{X,Y}$. Since $F$ is a functor, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}(X,Y) \times \mathrm{Hom}_{\mathcal{B}}(Y,Z) & \xrightarrow{F_{X,Y} \times F_{Y,Z}} & \mathrm{Hom}_{\mathcal{A}}(FX,FY) \times \mathrm{Hom}_{\mathcal{A}}(FY,FZ) \\ \circ \downarrow & & \downarrow \circ \\ \mathrm{Hom}_{\mathcal{B}}(X,Z) & \xrightarrow{F_{X,Z}} & \mathrm{Hom}_{\mathcal{A}}(FX,FZ) \end{array}$$

Reversing the horizontal arrows we obtain that $F$ h-separable.

Conversely, if $F$ is h-separable it is in particular separable whence faithful. □

**Lemma 1.7.** *A functor isomorphic to a h-separable functor is h-separable.*

**Proof.** Let $\alpha : F \to G$ be an isomorphism of functors where $G : \mathcal{B} \to \mathcal{A}$ is h-separable with respect to $P^G_{X,Y}$. Let $\tau_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(FX,FZ) \to \mathrm{Hom}_{\mathcal{A}}(GX,GZ)$ be defined by $\tau_{X,Y}(f) := \alpha_Y \circ f \circ \alpha_X^{-1}$. Then $F$ is h-separable with respect to $P^G_{X,Y} \circ \tau_{X,Y}$. □

## 2. Heavily separable adjoint functors

In this section we investigate h-separable functors which are adjoint functors.

**Theorem 2.1** *(Rafael type theorem). Let $(L, R, \eta, \epsilon)$ be an adjunction where $L : \mathcal{B} \to \mathcal{A}$.*

(i) *$L$ is h-separable if and only if there is a natural transformation $\gamma : RL \to \mathrm{Id}_{\mathcal{B}}$ such that $\gamma \circ \eta = \mathrm{Id}$ and*

$$\gamma\gamma = \gamma \circ R\epsilon L. \tag{1}$$

(ii) *$R$ is h-separable if and only if there is a natural transformation $\delta : \mathrm{Id}_{\mathcal{A}} \to LR$ such that $\epsilon \circ \delta = \mathrm{Id}$ and*

$$\delta\delta = L\eta R \circ \delta. \tag{2}$$

**Proof.** The second part of the statement follows from the first one by duality, thus we only have to establish (i).

First recall that, by Rafael Theorem [18, Theorem 1.2], the functor $L$ is separable if and only if there is a natural transformation $\gamma : RL \to \mathrm{Id}_{\mathcal{B}}$ such that $\gamma \circ \eta = \mathrm{Id}$. Namely, given $P_{-,-}$ one defines $\gamma$ by

$$\gamma X = P_{RLX,X}(\epsilon LX) \tag{3}$$

so that, by naturality of $P_{-,-}$, for every $f : LX \to LY$, one has

$$\gamma Y \circ Rf \circ \eta X = P_{RLY,Y}\left(\epsilon LY\right) \circ Rf \circ \eta X = P_{X,Y}\left(\epsilon LY \circ LRf \circ L\eta X\right)$$
$$= P_{X,Y}\left(f \circ \epsilon LX \circ L\eta X\right) = P_{X,Y}(f).$$

Conversely, given $\gamma$, for every $f : LX \to LY$, one defines

$$P_{X,Y}(f) := \gamma Y \circ Rf \circ \eta X. \tag{4}$$

This correspondence between $P_{-,-}$ and $\gamma$ is clearly bijective.

Assume that (1) holds. Then, for all $f \in \mathrm{Hom}_{\mathcal{A}}\left(LX, LY\right)$ and $g \in \mathrm{Hom}_{\mathcal{A}}\left(LY, LZ\right)$, we have

$$\begin{aligned} P_{Y,Z}\left(g\right) \circ P_{X,Y}\left(f\right) &\overset{(4)}{=} \gamma Z \circ Rg \circ \eta Y \circ \gamma Y \circ Rf \circ \eta X \\ &\overset{\text{nat.}\gamma}{=} \gamma Z \circ \gamma RLZ \circ RLRg \circ RL\eta Y \circ Rf \circ \eta X \\ &\overset{(1)}{=} \gamma Z \circ R\epsilon LZ \circ RLRg \circ RL\eta Y \circ Rf \circ \eta X \\ &= \gamma Z \circ Rg \circ R\epsilon LY \circ RL\eta Y \circ Rf \circ \eta X \\ &= \gamma Z \circ Rg \circ Rf \circ \eta X \overset{(4)}{=} P_{X,Z}\left(g \circ f\right) \end{aligned}$$

so that $P_{Y,Z}\left(g\right) \circ P_{X,Y}\left(f\right) = P_{X,Z}\left(g \circ f\right)$ and hence $L$ is h-separable. Conversely, if the latter condition holds for every $X, Y, Z$ and $f, g$ as above, we have

$$\gamma\gamma X = \gamma X \circ \gamma RLX \overset{(3)}{=} P_{RLX,X}\left(\epsilon LX\right) \circ P_{RLRLX,RLX}\left(\epsilon LRLX\right)$$
$$= P_{RLRLX,X}\left(\epsilon LX \circ \epsilon LRLX\right) \overset{(4)}{=} \gamma X \circ R\left(\epsilon LX \circ \epsilon LRLX\right) \circ \eta RLRLX$$
$$= \gamma X \circ R\epsilon LX \circ R\epsilon LRLX \circ \eta RLRLX = \gamma X \circ R\epsilon LX$$

so that (1) holds. □

**Remark 2.2.** Note that the equality (1) means that, for every $B \in \mathcal{B}$, $\gamma B$ coequalizes the parallel pair $(R\epsilon LB, RL\gamma B)$. If we add the condition $\gamma \circ \eta = \mathrm{Id}$, as in Theorem 2.1, we get that the pair $(B, \gamma B)$ belongs to ${}_{RL}\mathcal{B}$, i.e. the Eilenberg-Moore category of the monad $(RL, R\epsilon L, \eta)$ and hence the following right-hand side diagram is a split coequalizer. In fact, given a monad $(T, \mu : TT \to T, \eta : \mathrm{Id} \to T)$ on the category $\mathcal{B}$ and an object $(X, \xi : TX \to X)$ in the Eilenberg-Moore category ${}_{T}\mathcal{B}$, of this monad, by [4, Lemma 4.4.3 ] the following left-hand side diagram is a split coequalizer.


In particular this coequalizer is absolute, i.e. preserved by every functor defined on $\mathcal{B}$.

A similar remark holds for $\delta$ as in Theorem 2.1 in connection with the Eilenberg-Moore category $\mathcal{B}^{LR}$ of the comonad $(LR, L\eta R, \epsilon)$.

**Proposition 2.3.** *Let $(L, R)$ be an adjunction.*

- $(i)$ *The functor $L$ is h-separable if and only if the forgetful functor $U : {}_{RL}\mathcal{B} \to \mathcal{B}$ is a split epimorphism i.e. there is a functor $\Gamma : \mathcal{B} \to {}_{RL}\mathcal{B}$ such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$.*
- $(ii)$ *The functor $R$ is h-separable if and only if the forgetful functor $U : \mathcal{B}^{LR} \to \mathcal{B}$ is a split epimorphism i.e. there is a functor $\Gamma : \mathcal{B} \to \mathcal{B}^{LR}$ such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$.*

**Proof.** We just prove $(i)$, the proof of $(ii)$ being similar. By Theorem 2.1, $L$ is h-separable if and only if there is a natural transformation $\gamma : RL \to \mathrm{Id}_{\mathcal{B}}$ such that $\gamma \circ \eta = \mathrm{Id}$ and (1) holds. For every $B \in \mathcal{B}$, define $\Gamma B := (B, \gamma B)$. Then $\Gamma B \in {}_{RL}\mathcal{B}$, as observed in Remark 2.2. Moreover any morphism $f : B \to C$ fulfills $f \circ \gamma B = \gamma C \circ RLf$ by naturality of $\gamma$. This means that $f$ induces a morphism $\Gamma f : \Gamma B \to \Gamma C$ such that $U\Gamma f = f$. We have so defined a functor $\Gamma : \mathcal{B} \to \mathcal{B}_{RL}$ such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$.

Conversely, let $\Gamma : \mathcal{B} \to {}_{RL}\mathcal{B}$ be a functor such that $U \circ \Gamma = \mathrm{Id}_{\mathcal{B}}$. Then, for every $B \in \mathcal{B}$, we have that $\Gamma B = (B, \gamma B)$ for some morphism $\gamma B : RLB \to B$. Since $\Gamma B \in \mathcal{B}_{RL}$ we must have that $\gamma B \circ \eta B = B$ and $\gamma B \circ RL\gamma B = \gamma B \circ R\epsilon LB$. Given a morphism $f : B \to C$, we have that $\Gamma f : \Gamma B \to \Gamma C$ is a morphism in ${}_{RL}\mathcal{B}$, which means that $f \circ \gamma B = \gamma C \circ RLf$ i.e. $\gamma := (\gamma B)_{B \in \mathcal{B}}$ is a natural transformation. By the foregoing $\gamma \circ \eta = \mathrm{Id}$ and (1) holds. □

**Corollary 2.4.** *Let $(L, R)$ be an adjunction.*

- $(i)$ *Assume that $R$ is strictly monadic (i.e. the comparison functor is an isomorphism of categories). Then the functor $L$ is h-separable if and only if $R$ is a split epimorphism.*
- $(ii)$ *Assume that $L$ is strictly comonadic (i.e. the cocomparison functor is an isomorphism of categories). Then the functor $R$ is h-separable if and only if $L$ is a split epimorphism.*

**Proof.** We just prove $(i)$, the proof of $(ii)$ being similar. Since the comparison functor $K : \mathcal{A} \to \mathcal{B}_{RL}$ is an isomorphism of categories and $U \circ K = R$, we get that $R$ is a split epimorphism if and only if $U$ is a split epimorphism. By Proposition 2.3, this is equivalent to the h-separability of $L$. □

**Remark 2.5.** Corollary 2.4 will allow us to show, in Remark 5.3, that the tensor algebra functor $T : \mathrm{Vec}_{\Bbbk} \to \mathrm{Alg}_{\Bbbk}$ is separable but not h-separable.

**Definition 2.6.** Following [11, Section 4] we say that a **grouplike morphism** for a comonad $(C, \Delta : C \to CC, \epsilon : C \to \mathrm{Id})$ is a natural transformation $\delta : \mathrm{Id} \to C$ such that $\epsilon \circ \delta = \mathrm{Id}$

and $\delta\delta = \Delta \circ \delta$. Dually an **augmentation** for a monad $(M, m : MM \to M, \eta : \mathrm{Id} \to M)$ is a natural transformation $\gamma : M \to \mathrm{Id}$ such that $\gamma \circ \eta = \mathrm{Id}$ and $\gamma\gamma = \gamma \circ m$.

An immediate consequence of the previous definition and Theorem 2.1 is the following result.

**Corollary 2.7.** *Let $(L, R, \eta, \epsilon)$ be an adjunction.*

*(i) $L$ is h-separable if and only if the monad $(RL, R\epsilon L, \eta)$ has an augmentation.*
*(ii) $R$ is h-separable if and only if the comonad $(LR, L\eta R, \epsilon)$ has a grouplike morphism.*

## 3. Heavily separable ring homomorphisms

Let $\varphi : R \to S$ be a ring homomorphism. The induction functor $\varphi^* := S \otimes_R (-) : R\text{-Mod} \to S\text{-Mod}$ is the left adjoint of the restriction of scalars functor $\varphi_* : S\text{-Mod} \to R\text{-Mod}$. This section is devoted to the h-separability of these functors.

**Proposition 3.1.** *Let $\varphi : R \to S$ be a ring homomorphism. Then the induction functor $\varphi^*$ is h-separable if and only if there is a ring homomorphism $E : S \to R$ such that $E \circ \varphi = \mathrm{Id}$.*

**Proof.** By [8, Theorem 27, page 100], we know that $\varphi^*$ is separable if and only if there is a morphism of $R$-bimodules $E : S \to R$ such that $E(1_S) = 1_R$. Given $\gamma$ for $\varphi^*$ as in Theorem 2.1, one defines $E(s) := (\gamma R)(s \otimes_R 1_R)$. Given $E$, one defines $\gamma M : S \otimes_R M \to M : s \otimes_R m \mapsto E(s)m$, for every $M \in R\text{-Mod}$, such that $\gamma \circ \eta = \mathrm{Id}$ where the unit $\eta$ is defined by $\eta M : M \to S \otimes_R M : m \mapsto 1_S \otimes_R m$. All natural transformations $\gamma$ such that $\gamma \circ \eta = \mathrm{Id}$ are of this form: this is checked by the naturality of $\gamma$ applied to the left $R$-module map $f_m : R \to M : r \mapsto rm$ available for all left $R$-module $M$ and $m \in M$.

The equality (1) rewrites as $E(x)E(y)m = E(xy)m$ for every $x, y \in S$ and $m \in M$, for every $M \in R\text{-Mod}$. Thus (1) is equivalent to ask that $E$ is multiplicative. Summing up $\varphi^*$ is h-separable if and only if there is a morphism of $R$-bimodules $E : S \to R$ which is a ring homomorphism. This is equivalent to ask that $E : S \to R$ is a ring homomorphism such that $E \circ \varphi = \mathrm{Id}$. □

It is known that $\varphi_*$ is separable if and only if $S/R$ is separable (see [15, Proposition 1.3]) if and only if it admits a separability idempotent. We are so lead to the following definition.

**Definitions 3.2.** 1) $S/R$ is **h-separable** if the functor $\varphi_* : S\text{-Mod} \to R\text{-Mod}$ is h-separable.

2) A **heavy separability idempotent** (**h-separability idempotent** for short) of $S/R$ is an element $\sum_i a_i \otimes_R b_i \in S \otimes_R S$ such that $\sum_i a_i \otimes_R b_i$ is a separability idempotent, i.e.

$$\sum_i a_i b_i = 1, \qquad \sum_i s a_i \otimes_R b_i = \sum_i a_i \otimes_R b_i s \quad \text{for every } s \in S, \tag{5}$$

which moreover fulfills

$$\sum_{i,j} a_i \otimes_R b_i a_j \otimes_R b_j = \sum_i a_i \otimes_R 1_S \otimes_R b_i. \tag{6}$$

**Remark 3.3.** Note that a h-separability idempotent $e := \sum_i a_i \otimes_R b_i$ is exactly a grouplike element in the Sweedler's coring $\mathcal{C} := S \otimes_R S$ such that $se = es$ for every $s \in S$ i.e. which is invariant. Note that $1_S \otimes_R 1_S$ is always a grouplike element in $\mathcal{C}$ but it is not invariant in general.

**Proposition 3.4.** *$S/R$ is h-separable if and only if it has a h-separability idempotent.*

**Proof.** We observed that $\varphi_*$ is the right adjoint of the induction functor $\varphi^* := S \otimes_R (-)$. Recall that $S/R$ is separable if and only if the map $S \otimes_R S \to S$ splits as an $S$-bimodule map. The splitting is uniquely determined by a so-called separability idempotent i.e. an element $\sum_i a_i \otimes_R b_i \in S \otimes_R S$ such that (5) hold. Using this element we can define $\delta : \mathrm{Id} \to \varphi^* \varphi_*$ such that $\epsilon \circ \delta = \mathrm{Id}$ by $\delta M : M \to S \otimes_R M : m \mapsto \sum_i a_i \otimes_R b_i m$. This natural transformation satisfies (2) if and only if

$$(S \otimes_R \delta M) \circ \delta M = (S \otimes_R \eta M) \circ \delta M.$$

Let us compute separately the two terms of this equality on any $m \in M$,

$$\begin{aligned}(S \otimes_R \delta M)(\delta M)(m) &= \sum_i a_i \otimes_R (\delta M)(b_i m) = \sum_{i,j} a_i \otimes_R a_j \otimes_R b_j b_i m \\ &\overset{(5)}{=} \sum_{i,j} a_i \otimes_R b_i a_j \otimes_R b_j m, \\ (S \otimes_R \eta M)(\delta M)(m) &= \sum_i a_i \otimes_R (\eta M)(b_i m) = \sum_i a_i \otimes_R 1_S \otimes_R b_i m.\end{aligned}$$

Thus $\delta$ satisfies (2) if and only if (6) holds true. □

**Corollary 3.5.** *Let $\varphi : R \to S$ and $\psi : S \to T$ be ring homomorphisms.*

1) *If $T/S$ and $S/R$ are h-separable so is $T/R$.*
2) *If $T/R$ is h-separable so is $T/S$.*
3) *If $S/R$ is h-separable then $T/S$ is h-separable if and only if so is $T/R$.*

**Proof.** It follows by Definition 3.2, Lemma 1.4 and the equality $\varphi_* \circ \psi_* = (\psi \circ \varphi)_*$. □

**Lemma 3.6.** *Let $\varphi : R \to S$ be a ring homomorphism. The following are equivalent.*

(*i*) *The map $\varphi$ is a ring epimorphism (i.e. an epimorphism in the category of rings);*

(*ii*) *the multiplication* $m : S \otimes_R S \to S$ *is invertible;*
(*iii*) $1_S \otimes_R 1_S$ *is a separability idempotent for* $S/R$;
(*iv*) $1_S \otimes_R 1_S$ *is a h-separability idempotent for* $S/R$.

*If these equivalent conditions hold true then* $S/R$ *is h-separable.*
*Moreover* $1_S \otimes_R 1_S$ *is the unique separability idempotent for* $S/R$.

**Proof.** $(i) \Leftrightarrow (ii)$ follows by [19, Proposition XI.1.2 page 226].

$(i) \Leftrightarrow (iii)$ follows by [19, Proposition XI.1.1 page 225].

$(iii) \Leftrightarrow (iv)$ depends on the fact that $1_S \otimes_R 1_S$ always fulfills (6).

By Proposition 3.4, $(iv)$ implies that $S/R$ is h-separable.

Let us check the last part of the statement. If $\sum_i a_i \otimes_R b_i$ is another separability idempotent, we get

$$\sum_i a_i \otimes_R b_i = \sum_i a_i 1_S \otimes_R 1_S b_i = 1_S \otimes_R 1_S \sum_i a_i b_i \overset{(5)}{=} 1_S \otimes_R 1_S. \quad \square$$

**Example 3.7.** We now give examples of ring epimorphisms $\varphi : R \to S$.

1) Let $S$ be a multiplicative closed subset of a commutative ring $R$. Then the canonical map $\varphi : R \to S^{-1}R$ is a ring epimorphism, cf. [3, Proposition 3.1]. More generally we can consider a perfect right localization of $R$ as in [19, page 229].

2) Consider the ring of matrices $\mathrm{M}_n(R)$ and the ring $\mathrm{T}_n(R)$ of $n \times n$ upper triangular matrices over a ring $R$. Then the inclusion $\varphi : \mathrm{T}_n(R) \to \mathrm{M}_n(R)$ is a ring epimorphism. In fact, given ring homomorphisms $\alpha, \beta : \mathrm{M}_n(R) \to B$ that coincide on $\mathrm{T}_n(R)$ then they coincide on all matrices. To see this we first check that $\alpha(E_{ij}) = \beta(E_{ij})$ for all $i > j$,

$$\begin{aligned}\alpha(E_{ij}) &= \alpha(E_{ij}E_{jj}) = \alpha(E_{ij})\alpha(E_{jj}) = \alpha(E_{ij})\beta(E_{jj}) = \alpha(E_{ij})\beta(E_{ji}E_{ij}) \\ &= \alpha(E_{ij})\beta(E_{ji})\beta(E_{ij}) = \alpha(E_{ij})\alpha(E_{ji})\beta(E_{ij}) = \alpha(E_{ij}E_{ji})\beta(E_{ij}) \\ &= \alpha(E_{ii})\beta(E_{ij}) = \beta(E_{ii})\beta(E_{ij}) = \beta(E_{ii}E_{ij}) = \beta(E_{ij}).\end{aligned}$$

Thus $\alpha(E_{ij}) = \beta(E_{ij})$ for every $i, j$. Now, given $r \in R$ we have

$$\alpha(rE_{ij}) = \alpha(rE_{ii}E_{ij}) = \alpha(rE_{ii})\alpha(E_{ij}) = \beta(rE_{ii})\beta(E_{ij}) = \beta(rE_{ii}E_{ij}) = \beta(rE_{ij}).$$

As a consequence $\alpha(M) = \beta(M)$ for all $M \in \mathrm{M}_n(R)$ as desired.

3) Any surjective ring homomorphism $\varphi : R \to S$ is trivially a ring epimorphism.

**Remark 3.8.** A kind of dual to Lemma 3.6, establishes that a $\Bbbk$-coalgebra homomorphism $\varphi : C \to D$ is a coalgebra monomorphism if and only if the induced functor $\mathcal{M}^C \to \mathcal{M}^D$ is full, see [14, Theorem 3.5]. Since this functor is always faithful, by Lemma 1.6 it is in this case h-separable.

**Proposition 3.9.** *Let $\varphi : R \to S$ be a ring homomorphism. Then $S/R$ is h-separable if and only if $S/\varphi(R)$ is h-separable.*

**Proof.** Write $\varphi = i \circ \overline{\varphi}$ where $i : \varphi(R) \to S$ is the canonical inclusion and $\overline{\varphi} : R \to \varphi(R)$ is the corestriction of $\varphi$ to its image $\varphi(R)$. By Lemma 3.6, we have that $\varphi(R)/R$ is h-separable. By Corollary 3.5, $S/R$ is h-separable if and only if $S/\varphi(R)$ is h-separable. □

The well-known Maschke theorem establishes that the group ring, if the group is finite and the characteristic does not divide the cardinality of the group, is separable. It is likewise well-known that the ring of matrices is separable, see e.g. [9, Example II, page 41]. In Corollary 3.19, we will show that they are both h-separable only in trivial cases.

### *3.1. Heavily separable algebras*

**3.10.** Let $R$ be a commutative ring, let $S$ be a ring and let $Z(S)$ be its center. We recall that $S$ is said to be an $R$-algebra, or that $S$ is an algebra over $R$, if there is a unital ring homomorphism $\varphi : R \to S$ such that $\varphi(R) \subseteq Z(S)$. In this case we set

$$r \cdot s = \varphi(r) \cdot s \quad \text{for every } r \in R \text{ and } s \in S.$$

Since $Im(\varphi) \subseteq Z(S)$, we have $r \cdot s = s \cdot r$ for every $r \in R$ and $s \in S$ and

$$r \cdot 1_S = \varphi(r) \cdot 1_S = \varphi(r) \cdot \varphi(1_R) = \varphi(r \cdot 1_R) = \varphi(r) \quad \text{for every } r \in R$$

so that $R1_S = Im(\varphi) \subseteq Z(S)$.

**Theorem 3.11.** *Let $\varphi : R \to S$ be a ring homomorphism such that $\varphi(R) \subseteq Z(S)$ (with $R$ not necessarily commutative). Then $S/R$ is h-separable if and only if the canonical map $\varphi : R \to S$ is a ring epimorphism. Moreover if one of these conditions holds, then $S$ is commutative.*

**Proof.** First we prove the statement for $R$ commutative (i.e. $S$ is an $R$-algebra).

($\Rightarrow$). Let $\sum_i a_i \otimes_R b_i$ be an h-separability idempotent. Since $\varphi(R) \subseteq Z(S)$, we get that the map $\tau : S \otimes_R S \to S \otimes_R S, \tau(a \otimes_R b) = b \otimes_R a$, is well-defined and left $R$-linear. Hence we can apply $S \otimes_R \tau$ on both sides of (6) to get $\sum_{i,j} a_i \otimes_R b_j \otimes_R b_i a_j = \sum_i a_i \otimes_R b_i \otimes_R 1_S$. By multiplying, we obtain $\sum_{i,j} a_i b_j \otimes_R b_i a_j = \sum_i a_i b_i \otimes_R 1_S$. By (5), we get

$$\sum_{i,j} a_i b_j \otimes_R b_i a_j = 1_S \otimes_R 1_S. \tag{7}$$

By (5) and using the map $\tau$, we get that $\sum_t a_t s b_t \in Z(S)$, for all $s \in S$. Therefore we have

$$s = 1_S \cdot 1_S \cdot s \overset{(7)}{=} \sum_{i,j} a_i b_j b_i a_j s = \sum_{i,j} a_i \left(b_j\right) b_i \left(a_j\right) s \left(1_S\right)$$

$$\overset{(6)}{=} \sum_{i,j,t} a_i b_j b_i \left(a_t s b_t\right) a_j = \sum_{i,j,t} a_i b_j b_i a_j \left(a_t s b_t\right) \overset{(7)}{=} \sum_t a_t s b_t \in Z\left(S\right).$$

We have so proved that $S \subseteq Z(S)$ and hence $S$ is commutative. Now, we compute

$$\sum_i a_i \otimes_R b_i \overset{(5)}{=} \sum_{i,j} a_i a_j b_j \otimes_R b_i \overset{S=Z(S)}{=} \sum_{i,j} a_j a_i b_j \otimes_R b_i \overset{(5)}{=} \sum_{i,j} a_i b_j \otimes_R b_i a_j$$

so that $\sum_i a_i \otimes_R b_i = 1_S \otimes_R 1_S$ by (7). We conclude by Lemma 3.6.

($\Leftarrow$) It follows by Lemma 3.6.

Let us come back to the general case when $R$ is not necessarily commutative. By Proposition 3.9, $S/R$ is h-separable if and only if $S/\varphi(R)$ is h-separable. Since $\varphi(R) \subseteq Z(S)$, it is commutative and hence, by the previous part of the proof we get that $S/\varphi(R)$ is h-separable if and only if the canonical inclusion $\varphi(R) \hookrightarrow S$ is a ring epimorphism. Since the map $R \to \varphi(R) : r \mapsto \varphi(r)$ is surjective, we get that $\varphi(R) \hookrightarrow S$ is a ring epimorphism if and only if $\varphi$ is a ring epimorphism. □

The following result establishes that there is no non-trivial h-separable algebra over a field $\Bbbk$.

**Proposition 3.12.** *Let $A$ be a h-separable algebra over a field $\Bbbk$. Then either $A = \Bbbk$ or $A = 0$.*

**Proof.** By Theorem 3.11, the unit $u : \Bbbk \to A$ is a ring epimorphism. By Lemma 3.6, we have that $A \otimes_{\Bbbk} A \cong A$ via multiplication. Since $A$ is h-separable over $\Bbbk$, it is in particular separable over $\Bbbk$. By [16, page 184], the separable $\Bbbk$-algebra $A$ is finite-dimensional. Thus, from $A \otimes_{\Bbbk} A \cong A$ we deduce that $A$ has either dimensional one or zero over $\Bbbk$. □

**Example 3.13.** $\mathbb{C}/\mathbb{R}$ is separable but not h-separable. In fact, by Proposition 3.12, $\mathbb{C}/\mathbb{R}$ is not h-separable. On the other hand $e = \frac{1}{2}(1 \otimes 1 - i \otimes i)$ is a separability idempotent (it is the only possible one). It is clear that $e$ is not a h-separability idempotent.

**Remark 3.14.** Let $A$ and $B$ be commutative rings and let $R = A \times B$ be their product. Denote by $p_A : R \to A$ and $p_B : R \to B$ the canonical projections. Then $A$ becomes an $R$-algebra via $p_A$ and $B$ becomes an $R$-algebra via $p_B$. Moreover $R = A \times B$ is their product in the category of $R$-algebras. Since $R/R$ is clearly h-separable, the product of $R$-algebras may be h-separable.

**Lemma 3.15.** *Let $A$ and $B$ be $R$-algebras and let $S = A \times B$ be their product in the category of $R$-algebras. Set $e_1 := (1_A, 0_B) \in S$ and $e_2 := (0_A, 1_B) \in S$. The following are equivalent.*

(*i*) *$S/R$ is h-separable.*
(*ii*) *$A/R$ and $B/R$ are h-separable and $e_1 \otimes_R e_2 = 0 = e_2 \otimes_R e_1$.*

**Proof.** First, by Theorem 3.11 and Lemma 3.6, the conditions $(i)$ and $(ii)$ can be replaced respectively by

- $1_S \otimes_R 1_S$ is a separability idempotent of $S/R$
- $1_A \otimes_R 1_A$ and $1_B \otimes_R 1_B$ are separability idempotents of $A/R$ and $B/R$ respectively and $e_1 \otimes_R e_2 = 0 = e_2 \otimes_R e_1$.

Note also that, if the first condition holds, then, for $i \neq j$ we get

$$e_i \otimes_R e_j = e_i 1_S \otimes_R 1_S e_j = 1_S \otimes_R 1_S e_i e_j = 0$$

so that $e_1 \otimes_R e_2 = 0 = e_2 \otimes_R e_1$. Thus the latter condition can be assumed to hold.

Denote by $p_A : S \to A$ and $p_B : S \to B$ the canonical projections ad by $i_A : A \to S$ and $i_B : B \to S$ the canonical injections.

Since $1_S = e_1 + e_2$ and $e_1 \otimes_R e_2 = 0 = e_2 \otimes_R e_1$, we get that $1_S \otimes_R 1_S = e_1 \otimes_R e_1 + e_2 \otimes_R e_2$. For every $s \in S$ we have

$$\begin{aligned} s \otimes_R 1_S &= se_1 \otimes_R e_1 + se_2 \otimes_R e_2 \\ &= s(1_A, 0_B) \otimes_R (1_A, 0_B) + s(0_A, 1_B) \otimes_R (0_A, 1_B) \\ &= (p_A(s), 0_B) \otimes_R (1_A, 0_B) + (0_A, p_B(s)) \otimes_R (0_A, 1_B) \\ &= (i_A \otimes_R i_A)(p_A(s) \otimes_R 1_A) + (i_B \otimes_R i_B)(p_B(s) \otimes_R 1_B) \end{aligned}$$

Similarly $1_S \otimes_R s = (i_A \otimes_R i_A)(1_A \otimes_R p_A(s)) + (i_B \otimes_R i_B)(1_B \otimes_R p_B(s))$.

As a consequence, for every $s \in S$, the equality $s \otimes_R 1_S = 1_S \otimes_R s$ holds if and only if

$$\begin{aligned} &(i_A \otimes_R i_A)(p_A(s) \otimes_R 1_A) + (i_B \otimes_R i_B)(p_B(s) \otimes_R 1_B) \\ &= (i_A \otimes_R i_A)(1_A \otimes_R p_A(s)) + (i_B \otimes_R i_B)(1_B \otimes_R p_B(s)) \end{aligned}$$

if and only if

$$p_A(s) \otimes_R 1_A = 1_A \otimes_R p_A(s) \qquad \text{and} \qquad p_B(s) \otimes_R 1_B = 1_B \otimes_R p_B(s).$$

Since $p_A$ and $p_B$ are surjective, we get that to require that $s1_S \otimes_R 1_S = 1_S \otimes_R 1_S s$ for every $s \in S$ is equivalent to require that

$$a1_A \otimes_R 1_A = 1_A \otimes_R 1_A a \qquad \text{and} \qquad b1_B \otimes_R 1_B = 1_B \otimes_R 1_B b$$

for every $a \in A, b \in B$. We have so proved that $1_S \otimes_R 1_S$ is a separability idempotent of $S/R$ if and only if $1_A \otimes_R 1_A$ and $1_B \otimes_R 1_B$ are separability idempotents of $A/R$ and $B/R$ under the assumption $e_1 \otimes_R e_2 = 0 = e_2 \otimes_R e_1$. $\square$

The following result is similar to [9, Corollary 1.7 page 44].

**Lemma 3.16.** *Let $R$ be a commutative ring. Let $A$ and $B$ be $R$-algebras. Then, if $B/R$ is h-separable, so is $(A \otimes_R B)/A$. As a consequence if both $A/R$ and $B/R$ are h-separable, so is $(A \otimes_R B)/R$.*

**Proof.** Since $B/R$ is h-separable, by Theorem 3.11, we have that the unit $u_B : R \to B$ is a ring epimorphism. By Lemma 3.6 this means that $1_B \otimes_R 1_B$ is a separability idempotent of $B/R$. Thus also $(1_A \otimes_R 1_B) \otimes_A (1_A \otimes_R 1_B)$ is a separability idempotent of $(A \otimes_R B)/A$.

As a consequence also $A \otimes_R u_B : A \otimes_R R \to A \otimes_R B$ is a ring epimorphism by the same lemma. If $A/R$ is h-separable, then the unit $u_A : R \to A$ is a ring epimorphism too. Thus the composition

$$R \xrightarrow{u_A} A \cong A \otimes_R R \xrightarrow{A \otimes_R u_B} A \otimes_R B,$$

i.e. the unit of $A \otimes_R B$, is an epimorphism. By Theorem 3.11, $(A \otimes_R B)/R$ is h-separable. □

The following result is due to the referee.

**Proposition 3.17.** *Let $R$ be a commutative ring and let $S$ be any $R$-algebra that is free as an $R$-module. Then $S/R$ is h-separable if and only if either $S = 0$ or $S \cong R$.*

**Proof.** Assume $S/R$ is h-separable. If $R = 0$ then $S = 0$ as $S$ is an $R$-algebra. If $R \neq 0$, by Krull's theorem we have a maximal ideal $I$ of $R$. Set $\Bbbk := R/I$. By Lemma 3.16, we deduce that $(\Bbbk \otimes_R S)/\Bbbk$ is h-separable. By Proposition 3.12, we conclude that $\Bbbk \otimes_R S = \Bbbk$ and hence the $R$-module $S$ is free of rank zero or one and hence either $S = 0$ or $S \cong R$. We have so proved that $S/R$ is h-separable implies $S = 0$ or $S \cong R$. The other implication is trivial. □

### *3.2. Heavily separable twisted semigroup ring*

Let $R$ be a ring, $G$ be a group and consider $RG$, the group ring. S. Caenepeel posed the following problem: to characterize whether $RG/R$ is h-separable. In order to give a complete answer to this question and also to the question whether the matrix ring is h-separable, we will use the following construction that can be found in [17].

Let $G$ be a semigroup and let $R$ be a ring with identity. A **twisting** from $G$ into $R$ is a map $\omega : G \times G \to Z(R)$ which satisfies $\omega(i,j)\,\omega(ij,t) = \omega(i,jt)\,\omega(j,t)$ for all $i,j,t \in G$. The **twisted semigroup ring** of $G$ over $R$ with twisting $\omega$, denoted by $R^\omega G$, is the $R$-ring with basis $G$ and multiplication defined, for all $i,j \in G$, by

$$i \cdot_\omega j := \omega(i,j)\, ij,$$

and extended by linearity. It follows easily that $R^\omega G$ is an associative $R$-ring (not necessarily with identity, so it is just an $R$-bimodule with an $R$-bilinear and associative multiplication). Since $RG$ is an $R$-bimodule via $r\left(\sum_{i\in G} r_i i\right) r' := \sum_{i\in G} r r_i r' i$, the multiplication becomes as follows

$$\left(\sum_{i\in G} r_i i\right)\cdot_\omega \left(\sum_{j\in G} r'_j j\right) := \sum_{i,j\in G} r_i r'_j \omega\left(i,j\right) ij.$$

**Proposition 3.18.** *Assume that the twisted semigroup ring $R^\omega G$ is a unitary $R$-ring (i.e. a monoid in the monoidal category of $R$-bimodules) with unit $\varphi : R \to R^\omega G$. If $(R^\omega G)/R$ is h-separable then either $R = 0$ or $|G| = 0, 1$.*

**Proof.** Set $1_{R^\omega G} := \varphi(1_R) := \sum_{i\in G} u_i i$ for some $u_i \in R$. Since $\varphi$ is $R$-bilinear we have $r\varphi(1_R) = \varphi(r) = \varphi(1_R) r$ and hence $\sum_{i\in G} r u_i i = \sum_{i\in G} u_i r i$. As a consequence $r u_i = u_i r$ for all $i \in G$ and $r \in R$. In other words $u_i \in Z := Z(R)$. Since the codomain of $\omega$ is exactly $Z$, it makes sense to consider the twisted semigroup ring $Z^\omega G$. By the foregoing $1_{R^\omega G} \in Z^\omega G$ so that $Z^\omega G$ is a unitary $Z$-ring.

Since $R^\omega G$ is a unitary $R$-ring we must have $\varphi(r)x = rx$ for all $r \in R, x \in R^\omega G$. In particular $\varphi(r)i = ri$ for all $i \in G$.

If $r \in \mathrm{Ker}(\varphi)$ and $i \in G$ we have $ri = \varphi(r)i = 0 = \varphi(0_R)i = 0_R i$ so that $r = 0_R$. Thus $\varphi$ is necessarily injective if $G \neq \emptyset$.

Assume $R^\omega G/R$ is h-separable. Let $e := \sum_{i,j\in G} r_{i,j} i \otimes_R j$ be a h-separability idempotent with $r_{i,j} \in R$ almost all zero. For all $r \in R$ we have $re = \sum_{i,j\in W} r r_{i,j} i \otimes_R j$ and

$$er = \sum_{i,j\in G} r_{i,j} i \otimes_R jr = \sum_{i,j\in G} r_{i,j} i \otimes_R rj = \sum_{i,j\in G} r_{i,j} ir \otimes_R j = \sum_{i,j\in G} r_{i,j} ri \otimes_R j.$$

From $re = er$ we get

$$\sum_{i,j\in G} r r_{i,j} i \otimes_R j = \sum_{i,j\in G} r_{i,j} r i \otimes_R j$$

Since the tensor product of free modules remains free, the set $\{i \otimes_R j \mid i, j \in G\}$ is a basis for $S \otimes_R S$ as a left $R$-module. As a consequence, the equality above implies $r r_{i,j} = r_{i,j} r$ for all $r \in R$. Thus $r_{i,j} \in Z$.

Set $e' := \sum_{i,j\in G} r_{i,j} i \otimes_Z j \in Z^\omega G \otimes_Z Z^\omega G$. Note that the map $\alpha : Z^\omega G \otimes_Z Z^\omega G \to R^\omega G \otimes_R R^\omega G : i \otimes_Z j \mapsto i \otimes_R j$ is injective as $\{i \otimes_Z j \mid i, j \in G\}$ is a basis for $Z^\omega G \otimes_Z Z^\omega G$. For all $x = \sum_{t\in G} x_t t \in Z^\omega G$ we have

$$\alpha(xe') = \alpha\left(\sum_{i,j,t\in G} x_t t \cdot_\omega r_{i,j} i \otimes_Z j\right) = \alpha\left(\sum_{i,j\in G} x_t r_{i,j} \omega(t,i)\, ti \otimes_Z j\right)$$

$$= \sum_{i,j\in G} x_t r_{i,j}\omega\left(t,i\right) ti\otimes_R j = xe,$$

$$\alpha\left(e'x\right) = \alpha\left(\sum_{i,j,t\in G} r_{i,j} i\otimes_Z j\cdot_\omega x_t t\right) = \alpha\left(\sum_{i,j,t\in G} r_{i,j} i\otimes_Z x_t\omega\left(j,t\right) jt\right)$$

$$= \alpha\left(\sum_{i,j,t\in G} r_{i,j} x_t\omega\left(j,t\right) i\otimes_Z jt\right) = \sum_{i,j,t\in G} r_{i,j} x_t\omega\left(j,t\right) i\otimes_R jt = ex.$$

From $xe = ex$ and the injectivity of $\alpha$ we get that $xe' = e'x$. Moreover

$$m_{Z^\omega G}\left(e'\right) = \left(m_{R^\omega G}\circ\alpha\right)\left(e'\right) = m_{R^\omega G}\left(e\right) = 1$$

The map $\beta : Z^\omega G\otimes_Z Z^\omega G\otimes_Z Z^\omega G \to R^\omega G\otimes_R R^\omega G\otimes_R R^\omega G : i\otimes_Z j\otimes_Z k \mapsto i\otimes_R j\otimes_R k$ is also injective, by a similar argument on the basis. Since $e$ is a h-separability idempotent, we have

$$\begin{aligned}
&\beta\left(\sum_{i,j\in G}\sum_{i',j'\in G} r_{i,j} i\otimes_Z j\cdot_\omega\left(r_{i',j'}i'\right)\otimes_Z j'\right)\\
=\;&\beta\left(\sum_{i,j\in G}\sum_{i',j'\in G} r_{i,j} i\otimes_Z r_{i',j'}\omega\left(j,i'\right) ji'\otimes_Z j'\right)\\
\overset{r_{i,j}\in Z}{=}\;&\beta\left(\sum_{i,j\in G}\sum_{i',j'\in G} r_{i,j} r_{i',j'}\omega\left(j,i'\right) i\otimes_Z ji'\otimes_Z j'\right)\\
=\;&\sum_{i,j\in G}\sum_{i',j'\in G} r_{i,j} r_{i',j'}\omega\left(j,i'\right) i\otimes_R ji'\otimes_R j'\\
=\;&\sum_{i,j\in G}\sum_{i',j'\in G} r_{i,j} i\otimes_R j\cdot_\omega\left(r_{i',j'}i'\right)\otimes_R j'\\
=\;&\sum_{i,j\in G} r_{i,j} i\otimes_R 1_G\otimes_R j = \beta\left(\sum_{i,j\in G} r_{i,j} i\otimes_Z 1_G\otimes_Z j\right)
\end{aligned}$$

and hence we can cancel $\beta$ obtaining that $e'$ is a h-separability idempotent. Thus $\left(Z^\omega G\right)/Z$ is h-separable. Since $Z$ is commutative and $Z^\omega G$ is a free left $Z$-module with basis $G$, we deduce that either $Z^\omega G = 0$ or $Z^\omega G \cong Z$ by Proposition 3.17. Thus $Z = 0$ (and hence $R = 0$) or $|G| = 0, 1$. □

**Corollary 3.19.** *Let $R \neq 0$ be a ring.*

*1) Let $G$ be a monoid and let $RG$ be the monoid ring. If $RG/R$ is h-separable then $|G| = 1$.*

*2) Consider the matrix ring $\mathrm{M}_n\left(R\right)$, $n \geq 1$. If $\mathrm{M}_n\left(R\right)/R$ is h-separable then $n = 1$.*

**Proof.** It follows by Proposition 3.18 and the following observations.

1) If $G$ is a monoid and $\omega(i,j) = 1_R$ for all $i,j$, then $R^\omega G = RG$ is the ordinary monoid algebra.

2) Set $I := \{1,\ldots,n\}$. Then $G := I \times I$ is a semigroup where $(i,j)(s,t) := (i,t)$. Set $\omega((i,j),(s,t)) := \delta_{j,s} \in Z(R)$. Then

$$\left(\sum_{i,j} r_{i,j}(i,j)\right) \cdot_\omega \left(\sum_{s,t} r'_{s,t}(s,t)\right) = \sum_{i,j,s,t} r_{i,j} r'_{s,t} \omega((i,j),(s,t))(i,j)(s,t)$$
$$= \sum_{i,j,s,t} r_{i,j} r'_{s,t} \delta_{j,s}(i,t)$$

so that $R^\omega G$ is isomorphic to the ring of matrices $\mathrm{M}_n(R)$ through the assignment $(i,j) \leftrightarrow E_{ij}$. □

**Remark 3.20.** Note that the twisted semigroup ring $R^\omega G$ needs not to be a unitary $R$-ring in general. To see this, take the same semigroup $G = I \times I$ as in 2) of the proof of Corollary 3.19. Set $\omega((i,j),(s,t)) = 1_R$. If $R \neq 0$ then $R^\omega G$ has no identity unless $n = 1$. In fact, if $1_{R^\omega G} = \sum_{i,j} r_{i,j}(i,j)$ is an identity then

$$(s,t) = 1_{R^\omega G} \cdot_\omega (s,t) = \left(\sum_{i,j} r_{i,j}(i,j)\right) \cdot_\omega (s,t) = \sum_{i,j} r_{i,j}(i,t)$$

for all $s,t$. Thus $\sum_j r_{i,j} = \delta_{i,s}$. Since $s$ is arbitrary, if $n > 1$ we can take either $s = i$ or $s \neq i$. This leads to $1_R = 0_R$, a contradiction. Thus $n = 1$.

## 4. Heavily (co)separable corings

Consider an $A$-coring $\mathcal{C}$. The induction functor $R := (-) \otimes_A \mathcal{C}$: Mod-$A \to \mathcal{M}^{\mathcal{C}}$ is the right adjoint of the forgetful functor $L : \mathcal{M}^{\mathcal{C}} \to$ Mod-$A$, see e.g. [5, Lemma 3.1]. In this section we investigate the h-separability of these functors.

**Definition 4.1.** An $A$-coring $(\mathcal{C}, \Delta, \varepsilon)$ will be called **h-coseparable** if and only if the forgetful functor $L : \mathcal{M}^{\mathcal{C}} \to$ Mod-$A$ is h-separable.

**Remark 4.2.** Note that a h-coseparable coring is in particular coseparable by [5, Corollary 3.6].

**Theorem 4.3.** *An $A$-coring $(\mathcal{C}, \Delta, \varepsilon)$ is h-coseparable if and only if there is an $A$-bimodule map $\alpha : \mathcal{C} \otimes_A \mathcal{C} \to A$ such that $\alpha \circ \Delta = \varepsilon$ and for all $x,y,z \in \mathcal{C}$,*

$$\sum x_1 \alpha(x_2 \otimes_A y) = \sum \alpha(x \otimes_A y_1) y_2. \tag{8}$$

$$\sum \alpha\left(x \otimes_A y_1\right) \alpha\left(y_2 \otimes_A z\right)=\alpha\left(x \varepsilon(y) \otimes_A z\right). \tag{9}$$

**Proof.** By [5, Theorem 3.5], the forgetful functor $L : \mathcal{M}^{\mathcal{C}} \rightarrow \text{Mod-}A$ is separable if and only if there is an $A$-bimodule map $\alpha : \mathcal{C} \otimes_A \mathcal{C} \rightarrow A$ such that $\alpha \circ \Delta = \varepsilon$ and, for all $x, y \in \mathcal{C}$, the formula (8) holds true. We know that $L$ is h-separable if and only if there is $\gamma : RL \rightarrow \mathrm{Id}$, where $R := (-) \otimes_A \mathcal{C} : \text{Mod-}A \rightarrow \mathcal{M}^{\mathcal{C}}$ is the induction functor, such that $\gamma \circ \eta = \mathrm{Id}$ and (1) holds. Given $\gamma$ of this form we can set $\alpha := \varepsilon \circ \gamma\mathcal{C}$. The proof of [5, Theorem 3.5] shows that $\alpha \circ \Delta = \varepsilon$ and, for all $x, y \in \mathcal{C}$, the formula (8) holds true. Moreover $\gamma\mathcal{C}$ can be expressed in terms of $\alpha$ as $\gamma\mathcal{C}\left(x \otimes_A y\right) = \sum x_1 \alpha\left(x_2 \otimes_A y\right) = \sum \alpha\left(x \otimes_A y_1\right) y_2$. We compute

$$\begin{aligned}
(\varepsilon \circ(\gamma \gamma) \mathcal{C})\left(x \otimes_A y \otimes_A z\right) &= (\varepsilon \circ \gamma \mathcal{C} \circ R L \gamma \mathcal{C})\left(x \otimes_A y \otimes_A z\right)=\alpha\left(\gamma \mathcal{C} \otimes_A \mathcal{C}\right)\left(x \otimes_A y \otimes_A z\right) \\
&= \alpha\left(\sum \alpha\left(x \otimes_A y_1\right) y_2 \otimes_A z\right)=\sum \alpha\left(x \otimes_A y_1\right) \alpha\left(y_2 \otimes_A z\right)
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon \circ(\gamma \circ R \epsilon L) \mathcal{C})\left(x \otimes_A y \otimes_A z\right) &= (\varepsilon \circ \gamma \mathcal{C} \circ R \epsilon L \mathcal{C})\left(x \otimes_A y \otimes_A z\right) \\
&= \alpha\left(\epsilon L \mathcal{C} \otimes_A \mathcal{C}\right)\left(x \otimes_A y \otimes_A z\right)=\alpha\left(x \varepsilon(y) \otimes_A z\right)
\end{aligned}$$

In view of (1) evaluated on $\mathcal{C}$ we obtain (9).

Conversely, given $\alpha$ as in the statement, by the proof of [5, Theorem 3.5] we can define $\gamma : RL \rightarrow \mathrm{Id}$ by setting $\gamma N\left(n \otimes_A x\right) := \sum n_0 \alpha\left(n_1 \otimes_A x\right)$ for every $(N, \rho) \in \mathcal{M}^{\mathcal{C}}, n \in N, x \in \mathcal{C}$. Moreover $\gamma \circ \eta = \mathrm{Id}$. Let us show that (1) holds as well. Indeed we have

$$\begin{aligned}
((\gamma \gamma) N)\left(n \otimes_A x \otimes_A y\right) & = (\gamma N \circ R L \gamma N)\left(n \otimes_A x \otimes_A y\right) \\
& = (\gamma N)\left(\gamma N \otimes_A \mathcal{C}\right)\left(n \otimes_A x \otimes_A y\right) \\
& = (\gamma N)\left(\sum n_0 \alpha\left(n_1 \otimes_A x\right) \otimes_A y\right) \\
& = \sum\left(n_0 \alpha\left(n_1 \otimes_A x\right)\right)_0 \alpha\left(\left(n_0 \alpha\left(n_1 \otimes_A x\right)\right)_1 \otimes_A y\right) \\
& \overset{\rho \in \text{Mod-}A}{=} \sum n_0 \alpha\left(n_1 \alpha\left(n_2 \otimes_A x\right) \otimes_A y\right) \\
& \overset{(8)}{=} \sum n_0 \alpha\left(\alpha\left(n_1 \otimes_A x_1\right) x_2 \otimes_A y\right) \\
& = \sum n_0 \alpha\left(n_1 \otimes_A x_1\right) \alpha\left(x_2 \otimes_A y\right) \\
& \overset{(9)}{=} \sum n_0 \alpha\left(n_1 \varepsilon(x) \otimes_A y\right)
\end{aligned}$$

and

$$\begin{aligned}
((\gamma \circ R \epsilon L) N)\left(n \otimes_A x \otimes_A y\right) &= (\gamma N)\left(\epsilon L N \otimes_A \mathcal{C}\right)\left(n \otimes_A x \otimes_A y\right) \\
&= (\gamma N)\left(n \varepsilon(x) \otimes_A y\right)
\end{aligned}$$

$$= \sum (n\varepsilon(x))_0 \alpha((n\varepsilon(x))_1 \otimes_A y)$$
$$= \sum n_0 \alpha(n_1 \varepsilon(x) \otimes_A y). \quad \square$$

**Theorem 4.4.** *Let $C$ be a h-coseparable coalgebra over a field $\Bbbk$. Then $C = \Bbbk$ or $C = 0$.*

**Proof.** If $C$ is h-coseparable it is in particular coseparable. Thus $C$ is cosemisimple and hence $C$ is direct sum of simple subcoalgebras. Let $D$ be such a subcoalgebra and let $i : D \to C$ be the canonical inclusion. If $\alpha : C \otimes C \to \Bbbk$ is the map of Theorem 4.3, then $\alpha \circ (i \otimes i) : D \otimes D \to \Bbbk$ fulfills the same properties of $\alpha$ for $D$. Thus the same theorem ensures that $D$ is h-coseparable. Since $D$ is simple, it is finite-dimensional by the fundamental theorem for coalgebras. Thus the algebra $D^* = \mathrm{Hom}_{\Bbbk}(D, \Bbbk)$ is h-separable over $\Bbbk$. In fact, if we consider the functor $\Pi : \mathcal{M}^D \to D^*$-Mod that maps a comodule $(M, \rho)$ to the module $(M, \mu_\rho)$ where $\mu_\rho(f \otimes m) := \sum m_0 f(m_1)$ and we denote by $\varphi : \Bbbk \to D^*$ the unit of the algebra $D^*$, then $\varphi_* \circ \Pi = L : \mathcal{M}^D \to \Bbbk$-Mod is the forgetful functor. Since $D$ is finite-dimensional, the functor $\Pi$ is an isomorphism. Since $D$ is h-coseparable we also have that $L$ is h-separable. If $\Lambda$ is an inverse for $\Pi$, by Lemma 1.6 it is h-separable. Now $\varphi_* = \varphi_* \circ \Pi \circ \Lambda = L \circ \Lambda$ and the latter is h-separable by Lemma 1.4. Thus $\varphi_*$ is h-separable and hence $D^*/\Bbbk$ is h-separable. By Proposition 3.12, we get $D^* = \Bbbk, 0$. Hence $D = \Bbbk, 0$. Assume $C \neq 0$. Then $C$ is a group-like coalgebra of the form $\Bbbk G$ for some non-empty set $G \subseteq C$. The properties of $\alpha$ rewrite as follows for every $x, y, z \in G$.

$$\alpha(x \otimes x) = 1, \qquad x\alpha(x \otimes y) = \alpha(x \otimes y)y, \qquad \alpha(x \otimes y)\alpha(y \otimes z) = \alpha(x \otimes z).$$

From the equality in the middle and the fact that $\alpha(x \otimes y) \in \Bbbk$, by linear independence of the grouplike elements we obtain $\alpha(x \otimes y) = 0$ for $x \neq y$. Thus $\alpha(x \otimes y) = \delta_{x,y}\alpha(x \otimes x) = \delta_{x,y}$ whence the third equality becomes $\delta_{x,y}\delta_{y,z} = \delta_{x,z}$. If $G$ contains two elements $x \neq y$ and we take $z := x$, we get $\delta_{x,y}\delta_{y,x} = \delta_{x,x}$ i.e. $0_{\Bbbk} = 1_{\Bbbk}$ so that $\Bbbk = 0$ and hence $C = \Bbbk G = 0$, a contradiction. Therefore $|G| = 1$ and hence $C = \Bbbk$. $\square$

Consider an $A$-coring $\mathcal{C}$ and its set of invariant elements $\mathcal{C}^A = \{c \in \mathcal{C} \mid ac = ca,$ for every $a \in A\}$. In [5, Theorem 3.3], it is proved that the induction functor $R := (-) \otimes_A \mathcal{C}$: Mod-$A \to \mathcal{M}^{\mathcal{C}}$ is separable if and only if there is an invariant element $e \in \mathcal{C}^A$ such that $\varepsilon_{\mathcal{C}}(e) = 1$. Next result provides a similar characterization for the h-separable case.

**Theorem 4.5.** *Given an $A$-coring $\mathcal{C}$, the induction functor $R := (-) \otimes_A \mathcal{C}$ : Mod-$A \to \mathcal{M}^{\mathcal{C}}$ is h-separable if and only if $\mathcal{C}$ has an invariant grouplike element.*

**Proof.** Since $L \dashv R$, by Corollary 2.7, $R$ is h-separable if and only if the comonad $(LR, L\eta R, \epsilon)$ has a grouplike morphism. A grouplike morphism for this particular comonad is equivalent to an invariant grouplike element for the coring $\mathcal{C}$ i.e. an element $e \in \mathcal{C}^A$ such that $\varepsilon_{\mathcal{C}}(e) = 1$ and $\Delta_{\mathcal{C}}(e) = e \otimes_A e$. $\square$

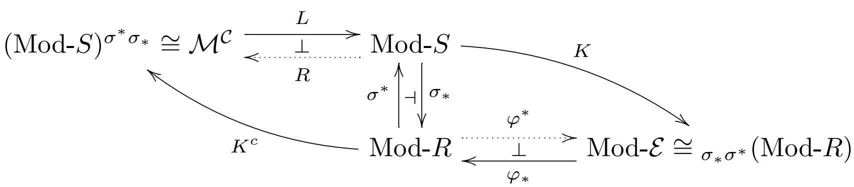
**Remark 4.6.** Let $\mathcal{C}$ be an $A$-coring. We recall that, by [5, Lemma 5.1], if $A$ is a right $\mathcal{C}$-comodule via $\rho_A : A \to A \otimes_A \mathcal{C}$, then $g = \rho_A(1_A)$ is a grouplike element of $\mathcal{C}$. Conversely if $g$ is a grouplike element of $\mathcal{C}$, then $A$ is a right $\mathcal{C}$-comodule via $\rho_A : A \to A \otimes_A \mathcal{C}$ defined by $\rho_A(a) = 1_A \otimes_A (g \cdot a)$. Moreover, if $g$ is a grouplike element of $\mathcal{C}$, then, by [5, page 404], $g$ is an invariant element if and only if $A = A^{\mathrm{co}\mathcal{C}} := \{a \in A \mid ag = ga\}$.

**Examples 4.7.** 1) Let $B$ be a bialgebra and let $A$ be a right $B$-comodule algebra via the trivial coaction $\rho : A \to A \otimes B : a \mapsto a \otimes 1_B$. Then, by [6, 33.2], we have that $\mathcal{C} := A \otimes B$ is an $A$-coring with coproduct $A \otimes \Delta_B$, counit $A \otimes \varepsilon$ and $A$-bimodule structure given by $a''(a \otimes b)a' = a''aa' \otimes b$. The corresponding grouplike element in $\mathcal{C}$ is $g = \rho(1_A) = 1_A \otimes 1_B$ which is invariant.

2) Let $A$ be a commutative ring and let $\mathcal{C}$ be an $A$-coalgebra, see [6, 1.1]. Then $\mathcal{C}$ is, in particular, an $A$-coring and any grouplike element in $\mathcal{C}$ is automatically invariant.

3) Let $A$ be an arbitrary ring and let $S$ be a set. Any $s \in S$ is an invariant grouplike element in the grouplike coring $\mathcal{C} = A^{(S)}$, see [6, 17.5].

The referee suggested to investigate the h-separability of the following functors. Let ${}_R\Sigma_S$ be a bimodule such that $\Sigma_S$ is finitely generated and projective. Let $\sigma^* := (-) \otimes_R \Sigma$ be the induction functor and let $\sigma_* := (-) \otimes_S \Sigma^*$ be its right adjoint where $\Sigma^* := \mathrm{Hom}_S(\Sigma, S)$. The Eilenberg-Moore category $(\text{Mod-}S)^{\sigma^*\sigma_*}$ comes out to be isomorphic to the category $\mathcal{M}^{\mathcal{C}}$ of right comodules over the comatrix $R$-coring $\mathcal{C} := \Sigma^* \otimes_R \Sigma$ defined in [10]. Dually, if we consider the endomorphism ring $\mathcal{E} := \mathrm{End}_S(\Sigma) \cong \Sigma \otimes_S \Sigma^*$ and the canonical morphism $\varphi : R \to \mathcal{E}$, defined by $\varphi(r)(s) = rs$ for all $r \in R$ and $s \in \Sigma$, then the Eilenberg-Moore category ${}_{\sigma_*\sigma^*}(\text{Mod-}R)$ comes out to be isomorphic to the category Mod-$\mathcal{E}$. Thus we have the following diagram where $K^c$ and $K$ are the cocomparison and the comparison functor respectively, $L \circ K^c = \sigma^*$ and $\varphi_* \circ K = \sigma_*$. Here $L \dashv R$ denote the same adjunction of the beginning of this section in the particular case of the comatrix coring.




Note that we have considered the right-hand version of the adjunction $\varphi^* \dashv \varphi_*$ in order to relate it to right comodules over $\mathcal{C}$.

**Theorem 4.8.** *In the above setting, the following assertions are equivalent.*

- $(i)$ $\sigma_*$ *is h-separable;*
- $(ii)$ $L$ *is a split epimorphism;*
- $(iii)$ $R$ *is h-separable;*
- $(iv)$ $\mathcal{C} := \Sigma^* \otimes_R \Sigma$ *has an invariant grouplike element;*
- $(v)$ $\sigma_*$ *is monadic and* $\mathcal{E}/R$ *is h-separable;*
- $(vi)$ $K$ *is an equivalence and* $\varphi_*$ *is h-separable.*

**Proof.** $(i) \Leftrightarrow (ii)$ follows by Proposition 2.3(ii).

$(ii) \Leftrightarrow (iii)$ follows by Corollary 2.4(ii), once observed that $L$ is strictly comonadic (the cocomparison functor of $L \dashv R$ is the identity).

$(iii) \Leftrightarrow (iv)$ follows by Theorem 4.5.

$(v) \Leftrightarrow (vi)$. By definition $\mathcal{E}/R$ is h-separable if and only if $\varphi_*$ is h-separable. On the other hand, $\sigma_*$ is monadic means that the comparison functor $K$ is an equivalence of categories.

$(vi) \Rightarrow (i)$. By Lemma 1.6, $K$ is h-separable. Now apply Lemma 1.4 to the equality $\varphi_* \circ K = \sigma_*$.

$(i) \Rightarrow (vi)$. Since $\sigma_*$ is h-separable, it is in particular separable. By the dual version of [13, Proposition 3.16], which can be applied since the category Mod-$S$ is Cauchy complete (every idempotent morphism splits) and the opposite of a Cauchy complete category is Cauchy complete, we have that $\sigma_*$ is monadic i.e. $K$ is an equivalence of categories. If $\Lambda$ denotes a quasi-inverse for $K$, then $\Lambda$ is h-separable by Lemma 1.6. We get $\varphi_* \cong \varphi_* \circ K \circ \Lambda = \sigma_* \circ \Lambda$ and the latter is h-separable by Lemma 1.4. Hence $\varphi_*$ is h-separable by Lemma 1.7. □

**Theorem 4.9.** *In the above setting, the following assertions are equivalent.*

- $(i)$ $\sigma^*$ *is h-separable;*
- $(ii)$ $\varphi_*$ *is a split epimorphism;*
- $(iii)$ $\varphi^*$ *is h-separable;*
- $(iv)$ *there is a ring homomorphism* $E : \mathcal{E} \to R$ *such that* $E \circ \varphi = \mathrm{Id}$*;*
- $(v)$ $\sigma^*$ *is comonadic and* $\mathcal{C}$ *is h-coseparable;*
- $(vi)$ $K^c$ *is an equivalence and* $L$ *is h-separable.*

**Proof.** $(i) \Leftrightarrow (ii)$ follows by Corollary 2.3(i)

$(ii) \Leftrightarrow (iii)$ follows by Corollary 2.4(i), once observed that $\varphi_*$ is always strictly monadic (the comparison functor of $\varphi^* \dashv \varphi_*$ is the identity).

$(iii) \Leftrightarrow (iv)$ follows by Proposition 3.1.

$(v) \Leftrightarrow (vi)$. By definition $\mathcal{C}$ is h-coseparable if and only if $L$ is h-separable. On the other hand, $\sigma^*$ is comonadic means that the cocomparison functor $K^c$ is an equivalence of categories.

$(vi) \Rightarrow (i)$. By Lemma 1.6, $K^c$ is h-separable. Now apply Lemma 1.4 to the equality $L \circ K^c = \sigma^*$.

$(i) \Rightarrow (vi)$. Since $\sigma^*$ is h-separable, it is in particular separable. By [13, Proposition 3.16], which can be applied since the category Mod-$R$ is Cauchy complete, we have that $\sigma^*$ is comonadic i.e. $K^c$ is an equivalence of categories. If $\Lambda$ denotes a quasi-inverse for $K$, then $\Lambda$ is h-separable by Lemma 1.6. We get $L \cong L \circ K^c \circ \Lambda = \sigma^* \circ \Lambda$ and the latter is h-separable by Lemma 1.4. Hence $L$ is h-separable by Lemma 1.7. □

**Remark 4.10.** The functor $\sigma_*$ is monadic, i.e. the comparison functor $K = (-) \otimes_S \Sigma^* :$ Mod-$S \to {}_{\sigma_*\sigma^*}$(Mod-$R$) $\cong$ Mod-$\mathcal{E}$ is an equivalence, provided $\Sigma_S$ is a progenerator (this follows by Morita Theory since $\mathcal{E} = \mathrm{End}_S(\Sigma)$, see [19, Proposition IV.10.7, page 108]).

The functor $\sigma^*$ is comonadic, i.e. the cocomparison functor $K^c = (-)\otimes_R \Sigma :$ Mod-$R \to$ (Mod-$S$)$^{\sigma^*\sigma_*} \cong \mathcal{M}^{\mathcal{C}}$ is an equivalence, if $\Sigma$ is a faithfully flat left $R$-module, [10, Theorem 3.10].

Now let $\varphi : R \to S$ be a ring homomorphism. Consider $\varphi^* := (-) \otimes_R S :$ Mod-$R \to$ Mod-$S$ and the restriction of scalars functor $\varphi_*$. Take $\Sigma := {}_R S_S$ with right regular action and left action induced by $\varphi$. In this case $\mathcal{C} = S \otimes_R S$ is the Sweedler $S$-coring, $\mathcal{E} \cong S$, $K = \mathrm{Id}$ (i.e. $\varphi_*$ is always strictly monadic) and $K^c = (-) \otimes_R S$. Moreover ($S$-Mod)$^{\varphi^*\varphi_*}$ is (see e.g. [6, pages 252–253]) the category Desc$(S/R)$ = ($S$-Mod)$^{\varphi^*\varphi_*}$ of descent data associated to the ring homomorphism $\varphi$.

Let $R := (-)\otimes_S \mathcal{C} :$ Mod-$S \to \mathcal{M}^{\mathcal{C}}$ be the induction functor and let $L : \mathcal{M}^{\mathcal{C}} \to$ Mod-$S$ be the forgetful functor.

As a consequence of Theorem 4.8 and Theorem 4.9 we obtain the following corollaries.

**Corollary 4.11.** *In the above setting, the following assertions are equivalent.*

- $(i)$ *$\varphi_*$ is h-separable (i.e. $S/R$ is h-separable);*
- $(ii)$ *$L$ is a split epimorphism;*
- $(iii)$ *$R$ is h-separable;*
- $(iv)$ *$\mathcal{C}$ has an invariant grouplike element.*

**Corollary 4.12.** *In the above setting, the following assertions are equivalent.*

- $(i)$ *$\varphi^*$ is h-separable;*
- $(ii)$ *$\varphi_*$ is a split epimorphism;*
- $(iii)$ *there is a ring homomorphism $E : S \to R$ such that $E \circ \varphi = \mathrm{Id}$;*
- $(iv)$ *$\varphi^*$ is comonadic and the Sweedler $S$-coring is h-coseparable.*

**Remark 4.13.** Corollary 4.11 establishes a relation among the functors $\varphi_*$, $L$, $R$ and the Sweedler coring. In particular it retrieves Remark 3.3 from a different point of view.

**Remark 4.14.** The equivalence between *(iii)* and *(iv)* in Corollary 4.12, provides an analogue of [5, Corollary 3.7], once observed that $\varphi^*$ is comonadic provided that $S$ is a faithfully flat left $R$-module [8, Proposition 109]. By mimicking the proof of [5, Corollary 3.7], we can give a direct proof of the fact that the Sweedler $S$-coring $\mathcal{C} := S \otimes_R S$ is h-coseparable if $\varphi^*$ is h-separable. In fact, given a ring homomorphism $E : S \to R$ such that $E \circ \varphi = \mathrm{Id}$, we can define $\alpha : \mathcal{C} \otimes_S \mathcal{C} \to S$ by setting $\alpha\left((x \otimes_R y) \otimes_S (z \otimes_R t)\right) := xE(yz)t$, for all $x, y, z, t \in S$. One easily checks that $\alpha$ fulfills $\alpha \circ \Delta = \varepsilon$, (8) and (9), for all $x, y, z \in \mathcal{C}$.

## 5. Example on monoidal categories

In the present section $\mathcal{M}$ denotes a preadditive braided monoidal category such that

- $\mathcal{M}$ has equalizers and denumerable coproducts;
- the tensor products are additive and preserve equalizers and denumerable coproducts.

In view of the assumptions above, we can apply [12, Theorem 2, page 172] to get an adjunction $(T, \Omega)$ and [1, Theorem 4.6] to get an adjunction $(\mathbf{T}, \mathbf{P})$ as in the following diagram

$$\begin{array}{ccc} \mathrm{Bialg}(\mathcal{M}) & \xrightarrow{\ \mho\ } & \mathrm{Alg}(\mathcal{M}) \\ \mathbf{T}\uparrow\downarrow\mathbf{P} & & T\uparrow\downarrow\Omega \\ \mathcal{M} & \xrightarrow{\ \mathrm{Id}\ } & \mathcal{M} \end{array}$$

Here $\mathrm{Alg}(\mathcal{M})$ denotes the category of algebras (or monoids) in $\mathcal{M}$, $\mathrm{Bialg}(\mathcal{M})$ is the category of bialgebras (or bimonoids) in $\mathcal{M}$, the functors $\mho$ and $\Omega$ are the obvious forgetful functors and, by construction of $\mathbf{T}$, we have $\mho \circ \mathbf{T} = T$.

It is noteworthy that, since $\Omega$ has a left adjoint $T$, then $\Omega$ is strictly monadic (the comparison functor is a category isomorphism), see [2, Theorem A.6].

Let $V \in \mathcal{M}$. By construction $\Omega T V = \oplus_{n \in \mathbb{N}} V^{\otimes n}$, see [1, Remark 1.2]. Denote by $\alpha_n V : V^{\otimes n} \to \Omega T V$ the canonical inclusion and note it is natural in $V$. The unit $\eta : \mathrm{Id}_{\mathcal{M}} \to \Omega T$ of the adjunction $(T, \Omega)$ is defined by $\eta V := \alpha_1 V$ while the counit $\epsilon : T\Omega \to \mathrm{Id}$ is uniquely defined by

$$\Omega\epsilon\,(A, m, u) \circ \alpha_n A = m^{n-1} \text{ for every } n \in \mathbb{N} \tag{10}$$

where $m^{n-1} : A^{\otimes n} \to A$ denotes the iterated multiplication of an algebra $(A, m, u)$ defined by $m^{-1} = u, m^0 = \mathrm{Id}_A$ and, for $n \geq 2, m^{n-1} = m \circ \left(m^{n-2} \otimes A\right)$.

Denote by $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ the unit and counit of the adjunction $(\mathbf{T}, \mathbf{P})$.

Consider the natural transformation $\xi : \mathbf{P} \to \Omega\mho$ defined by

$$\mathbf{P} \xrightarrow{\eta\mathbf{P}} \Omega T\mathbf{P} = \Omega\mho\mathbf{T}\mathbf{P} \xrightarrow{\Omega\mho\boldsymbol{\epsilon}} \Omega\mho.$$

We have $\epsilon\mho \circ T\xi = \epsilon\mho \circ T\Omega\mho\boldsymbol{\epsilon} \circ T\eta\mathbf{P} = \mho\boldsymbol{\epsilon} \circ \epsilon T\mathbf{P} \circ T\eta\mathbf{P} = \mho\boldsymbol{\epsilon}$ i.e.

$$\epsilon\mho \circ T\xi = \mho\boldsymbol{\epsilon}. \tag{11}$$

Since $\xi$, in view of the adjunction $(T, \Omega)$, is uniquely determined by the latter equality, we get that $\xi$ is exactly the natural transformation of [1, Theorem 4.6], whose components are the canonical inclusions of the subobject of primitives of a bialgebra $B$ in $\mathcal{M}$ into $\Omega\mho B$.

Define the functor

$$(-)^+ : \mathrm{Bialg}\,(\mathcal{M}) \to \mathcal{M}$$

that assigns to every bialgebra $A$ the kernel $(A^+, \zeta A : A^+ \to \Omega\mho A)$ of the counit $\varepsilon_{\Omega\mho A} : \Omega\mho A \to \mathbf{1}$ of the underlying coalgebra of $A$ (i.e. the equalizer of $\varepsilon_{\Omega\mho A}$ and the zero morphism) and to every morphism $f$ the induced morphism $f^+$.

Since $\zeta A$ is natural in $A$ we get a natural transformation $\zeta : (-)^+ \to \Omega\mho$ which is by construction a monomorphism on components.

**Lemma 5.1.** *The natural transformation $\xi : \mathbf{P} \to \Omega\mho$ factors through the natural transformation $\zeta : (-)^+ \to \Omega\mho$ (i.e. there is $\widehat{\xi} : \mathbf{P} \to (-)^+$ such that $\xi = \zeta \circ \widehat{\xi}$) which is a monomorphism on components.*

**Proof.** Given $A \in \mathrm{Bialg}\,(\mathcal{M})$ we have that $\xi$ and $\zeta$ are defined by the following kernels.

$$\begin{array}{ccccc}
\mathbf{P}A & \xrightarrow{\xi A} & \Omega\mho A & \xrightarrow{(u_{\Omega\mho A}\otimes\Omega\mho A)l^{-1}_{\Omega\mho A}+(\Omega\mho A\otimes u_{\Omega\mho A})\circ r^{-1}_{\Omega\mho A}-\Delta_{\Omega\mho A}} & \Omega\mho A \otimes \Omega\mho A \\
\downarrow{\scriptstyle \widehat{\xi}A} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle m_{\mathbf{1}}(\varepsilon_{\Omega\mho A}\otimes\varepsilon_{\Omega\mho A})} \\
A^+ & \xrightarrow{\zeta A} & \Omega\mho A & \xrightarrow{\varepsilon_{\Omega\mho A}} & \mathbf{1}
\end{array}$$

Since the right square above commutes, there is a unique morphism $\widehat{\xi}A : \mathbf{P}A \to A^+$ such that $\zeta A \circ \widehat{\xi}A = \xi A$. The naturality of $\zeta A$ and $\xi A$ in $A$ implies the one of $\widehat{\xi}A$ so that $\zeta \circ \widehat{\xi} = \xi$. □

There is a unique morphism $\omega V : \Omega T V \to V$ such that

$$\omega V \circ \alpha_n V = \delta_{n,1}\mathrm{Id}_V, \text{ for every } n \in \mathbb{N}. \tag{12}$$

Given $f : V \to W$ a morphism in $\mathcal{M}$, by naturality of $\alpha_n$, we get for every $n \in \mathbb{N}$,

$$\omega W \circ \Omega T f \circ \alpha_n V = \omega W \circ \alpha_n W \circ f^{\otimes n} = \delta_{n,1} f^{\otimes n} = \delta_{n,1} f = f \circ \omega V \circ \alpha_n V$$

so that $\omega W \circ \Omega T f = f \circ \omega V$ which means that $\omega := (\omega V)_{V\in\mathcal{M}}$ is a natural transformation $\omega : \Omega T \to \mathrm{Id}_{\mathcal{M}}$.

**Lemma 5.2.** *The natural transformation* $\omega$ *fulfills* $\omega \circ \eta = \mathrm{Id}$ *and*

$$\omega\omega \circ \Omega T \zeta \mathbf{T} = \omega \circ \Omega \epsilon T \circ \Omega T \zeta \mathbf{T}. \tag{13}$$

**Proof.** We have

$$\omega V \circ \eta V = \omega V \circ \alpha_1 V \overset{(12)}{=} \mathrm{Id}_V$$

and hence $\omega \circ \eta = \mathrm{Id}$. Let us check (13). For every $V \in \mathcal{M}$ we compute

$$\begin{aligned} &\omega\omega V \circ \Omega T \zeta \mathbf{T} V \circ \alpha_n (-)^+ \mathbf{T} V \\ &= \omega V \circ \Omega T \omega V \circ \Omega T \zeta \mathbf{T} V \circ \alpha_n (-)^+ \mathbf{T} V = \omega V \circ \alpha_n V \circ (\omega V)^{\otimes n} \circ (\zeta \mathbf{T} V)^{\otimes n} \\ &= \delta_{n,1} (\omega V)^{\otimes n} \circ (\zeta \mathbf{T} V)^{\otimes n} = \delta_{n,1} \omega V \circ \zeta \mathbf{T} V. \end{aligned}$$

On the other hand

$$\begin{aligned} \omega V \circ \Omega \epsilon T V \circ \Omega T \zeta \mathbf{T} V \circ \alpha_n (-)^+ \mathbf{T} V &= \omega V \circ \Omega \epsilon T V \circ \alpha_n \Omega \mho \mathbf{T} V \circ (\zeta \mathbf{T} V)^{\otimes n} \\ &= \omega V \circ \Omega \epsilon T V \circ \alpha_n \Omega T V \circ (\zeta \mathbf{T} V)^{\otimes n} \\ &\overset{(10)}{=} \omega V \circ m_{\Omega T V}^{n-1} \circ (\zeta \mathbf{T} V)^{\otimes n}. \end{aligned}$$

Hence we have to check that

$$\delta_{n,1} \omega V \circ \zeta \mathbf{T} V = \omega V \circ m_{\Omega T V}^{n-1} \circ (\zeta \mathbf{T} V)^{\otimes n}.$$

For $n = 0$ we have

$$\delta_{0,1} \omega V \circ \zeta \mathbf{T} V = 0 = \omega V \circ \alpha_0 V = \omega V \circ u_{\Omega T V} = \omega V \circ m_{\Omega T V}^{-1} \circ (\zeta \mathbf{T} V)^{\otimes 0}.$$

For $n = 1$ we have

$$\delta_{1,1} \omega V \circ \zeta \mathbf{T} V = \omega V \circ \zeta \mathbf{T} V = \omega V \circ m_{\Omega T V}^{0} \circ (\zeta \mathbf{T} V)^{\otimes 1}.$$

For $n \geq 2$ we have $\delta_{n,1} \omega V \circ \zeta \mathbf{T} V = 0$. In order to prove that also $\omega \circ m_{\Omega T V}^{n-1} \circ (\zeta \mathbf{T} V)^{\otimes n} = 0$ we need first to give a different expression for $\omega V \circ m_{\Omega T V}$. To this aim, for every $m, n \in \mathbb{N}$, we compute (we use the identifications $V \otimes \mathbf{1} \cong V \cong \mathbf{1} \otimes V$)

$$\begin{aligned} &\omega V \circ m_{\Omega T V} \circ (\alpha_m V \otimes \alpha_n V) \\ &= \omega V \circ \alpha_{m+n} V = \delta_{m+n,1} \mathrm{Id}_V \\ &= \delta_{m,1} \delta_{n,0} \mathrm{Id}_{V \otimes \mathbf{1}} + \delta_{m,0} \delta_{n,1} \mathrm{Id}_{\mathbf{1} \otimes V} \\ &= r_V \circ (\delta_{m,1} \mathrm{Id}_V \otimes \delta_{n,0} \mathrm{Id}_{\mathbf{1}}) + l_V \circ (\delta_{m,0} \mathrm{Id}_{\mathbf{1}} \otimes \delta_{n,1} \mathrm{Id}_V) \\ &= r_V \circ (\omega V \otimes \varepsilon_{\Omega T V}) \circ (\alpha_m V \otimes \alpha_n V) + l_V \circ (\varepsilon_{\Omega T V} \otimes \omega V) \circ (\alpha_m V \otimes \alpha_n V) \\ &= (r_V \circ (\omega V \otimes \varepsilon_{\Omega T V}) + l_V \circ (\varepsilon_{\Omega T V} \otimes \omega V)) \circ (\alpha_m V \otimes \alpha_n V). \end{aligned}$$

Since the tensor products preserve denumerable coproducts, the equalities above yield the identity

$$\omega V \circ m_{\Omega TV} = r_V \circ (\omega V \otimes \varepsilon_{\Omega TV}) + l_V \circ (\varepsilon_{\Omega TV} \otimes \omega V) .$$

Using it, we obtain

$$\begin{aligned} &\omega V \circ m_{\Omega TV}^{n-1} \circ (\zeta \mathbf{T} V)^{\otimes n} \\ &= \omega V \circ m_{\Omega TV} \circ \left(m_{\Omega TV}^{n-2} \otimes \Omega TV\right) \circ (\zeta \mathbf{T} V)^{\otimes n} \\ &= (r_V \circ (\omega V \otimes \varepsilon_{\Omega TV}) + l_V \circ (\varepsilon_{\Omega TV} \otimes \omega V)) \circ \left(m_{\Omega TV}^{n-2} \otimes \Omega TV\right) \circ (\zeta \mathbf{T} V)^{\otimes n} \\ &= r_V \circ \left(\omega V \circ m_{\Omega TV}^{n-2} \otimes \varepsilon_{\Omega TV}\right) \circ (\zeta \mathbf{T} V)^{\otimes n} + l_V \circ \left(\varepsilon_{\Omega TV} \circ m_{\Omega TV}^{n-2} \otimes \omega V\right) \circ (\zeta \mathbf{T} V)^{\otimes n} \\ &= r_V \circ \left(\omega V \circ m_{\Omega TV}^{n-2} \circ (\zeta \mathbf{T} V)^{\otimes n-1} \otimes \varepsilon_{\Omega TV} \circ \zeta \mathbf{T} V\right) \\ &\quad + l_V \circ \left((\varepsilon_{\Omega TV} \circ \zeta \mathbf{T} V)^{\otimes n-1} \otimes \omega V \circ \zeta \mathbf{T} V\right) . \end{aligned}$$

The last two summands are zero as $\varepsilon_{\Omega TV} \circ \zeta \mathbf{T} V = \varepsilon_{\Omega \mho \mathbf{T} V} \circ \zeta \mathbf{T} V = 0$ by definition of $\zeta$. $\square$

**Remark 5.3.** As observed, the comparison functor $K : \mathrm{Alg}(\mathcal{M}) \to \mathcal{M}_{\Omega T}$ is an isomorphism of categories. By Corollary 2.4, $T$ is h-separable if and only if $\Omega : \mathrm{Alg}(\mathcal{M}) \to \mathcal{M}$ is a split epimorphism. Let us prove, by contradiction, that this is not the case. Assume that there is a functor $\Gamma : \mathcal{M} \to \mathrm{Alg}(\mathcal{M})$ such that $\Omega\Gamma = \mathrm{Id}$. Let $V \in \mathcal{M}$. Then $\Gamma V = (V, mV, uV)$ for some morphisms $mV : V \otimes V \to V$ and $uV : \mathbf{1} \to V$. Let $f : V \to V$ be the zero morphism. Then $\Gamma f : \Gamma V \to \Gamma V$ is an algebra morphism and $\Omega\Gamma f = f$. Thus $f$ is unitary i.e. $uV = f \circ uV = 0 \circ uV = 0$. Hence $\mathrm{Id}_V = mV \circ (V \otimes uV) \circ r_V^{-1} = 0$. As a consequence any morphism $h : V \to W$ would be zero as $h = h \circ \mathrm{Id}_V$ for every $V, W \in \mathcal{M}$. Hence $\mathrm{Hom}_{\mathcal{M}}(V, W) = \{0\}$. This happens only if all objects are isomorphic to the unit object $\mathbf{1}$, i.e. if the skeleton of $\mathcal{M}$ is the trivial monoidal category $(\mathcal{T}, \otimes, \mathbf{1})$, where $\mathrm{Ob}(\mathcal{T}) = \{\mathbf{1}\}$, $\mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, \mathbf{1}) = \{\mathrm{Id}_{\mathbf{1}}\}$ and the tensor product is given by $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$ and $\mathrm{Id}_{\mathbf{1}} \otimes \mathrm{Id}_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}}$. This is evidently a restrictive condition on $\mathcal{M}$. Thus, in general $T : \mathcal{M} \to \mathrm{Alg}(\mathcal{M})$ is not heavily separable. On the other hand the equality $\omega \circ \eta = \mathrm{Id}$ obtained in Lemma 5.2 means that the functor $T : \mathcal{M} \to \mathrm{Alg}(\mathcal{M})$ is separable.

As a particular case, we get that the functor $T : \mathrm{Vec}_{\Bbbk} \to \mathrm{Alg}_{\Bbbk}$ is separable but not h-separable.

**Theorem 5.4.** *Set $\boldsymbol{\gamma} := \omega \circ \xi \mathbf{T} : \mathbf{PT} \to \mathrm{Id}_{\mathcal{M}}$. Then $\boldsymbol{\gamma} \circ \boldsymbol{\eta} = \mathrm{Id}$ and $\boldsymbol{\gamma}\boldsymbol{\gamma} = \boldsymbol{\gamma} \circ \mathbf{P}\boldsymbol{\epsilon}\mathbf{T}$. Hence the functor $\mathbf{T} : \mathcal{M} \to \mathrm{Bialg}(\mathcal{M})$ is h-separable.*

**Proof.** We compute

$$\begin{aligned} \boldsymbol{\gamma} \circ \boldsymbol{\eta} &= \omega \circ \xi \mathbf{T} \circ \boldsymbol{\eta} \overset{\text{def. } \xi}{=} \omega \circ \Omega \mho \boldsymbol{\epsilon} \mathbf{T} \circ \eta \mathbf{PT} \circ \boldsymbol{\eta} \\ &= \omega \circ \Omega \mho \boldsymbol{\epsilon} \mathbf{T} \circ \Omega T \boldsymbol{\eta} \circ \eta = \omega \circ \Omega \mho \boldsymbol{\epsilon} \mathbf{T} \circ \Omega \mho \mathbf{T} \boldsymbol{\eta} \circ \eta = \omega \circ \eta = \mathrm{Id}. \end{aligned}$$

Moreover

$$\begin{aligned}
\Omega\epsilon\mho\circ\xi\mathbf{T}\xi &= \Omega\epsilon\mho\circ\Omega\mho\mathbf{T}\xi\circ\xi\mathbf{TP} \\
&\overset{\text{def.}\xi}{=} \Omega\epsilon\mho\circ\Omega T\Omega\mho\boldsymbol{\epsilon}\circ\Omega T\eta\mathbf{P}\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{TP}\circ\eta\mathbf{PTP} \\
&= \Omega\mho\boldsymbol{\epsilon}\circ\Omega\epsilon\mho\mathbf{TP}\circ\Omega T\eta\mathbf{P}\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{TP}\circ\eta\mathbf{PTP} \\
&= \Omega\mho\boldsymbol{\epsilon}\circ\Omega\mho\boldsymbol{\epsilon}\mathbf{TP}\circ\eta\mathbf{PTP} \\
&= \Omega\mho\boldsymbol{\epsilon}\circ\Omega\mho\mathbf{TP}\boldsymbol{\epsilon}\circ\eta\mathbf{PTP} \\
&= \Omega\mho\boldsymbol{\epsilon}\circ\eta\mathbf{P}\circ\mathbf{P}\boldsymbol{\epsilon}\overset{\text{def.}\xi}{=}\xi\circ\mathbf{P}\boldsymbol{\epsilon}
\end{aligned}$$

so that

$$\begin{aligned}
&\boldsymbol{\gamma\gamma} = \omega\omega\circ\xi\mathbf{T}\xi\mathbf{T} = \omega\omega\circ\Omega\mho\mathbf{T}\zeta\mathbf{T}\circ\xi\mathbf{T}\widehat{\xi}\mathbf{T} = \omega\omega\circ\Omega T\zeta\mathbf{T}\circ\xi\mathbf{T}\widehat{\xi}\mathbf{T} \\
&\overset{(13)}{=} \omega\circ\Omega\epsilon T\circ\Omega T\zeta\mathbf{T}\circ\xi\mathbf{T}\widehat{\xi}\mathbf{T} = \omega\circ\Omega\epsilon\mho\mathbf{T}\circ\xi\mathbf{T}\xi\mathbf{T} = \omega\circ\xi\mathbf{T}\circ\mathbf{P}\boldsymbol{\epsilon}\mathbf{T} = \boldsymbol{\gamma}\circ\mathbf{P}\boldsymbol{\epsilon}\mathbf{T} \quad \square
\end{aligned}$$

Finally, we would like to explain how the construction above works when $\mathcal{M}$ is the category $\mathrm{Vec}_{\Bbbk}$ of vector spaces over a field $\Bbbk$. In this case $\mathrm{Bialg}(\mathrm{Vec}_{\Bbbk}) = \mathrm{Bialg}_{\Bbbk}$ and $\mathbf{P} : \mathrm{Bialg}_{\Bbbk} \to \mathrm{Vec}_{\Bbbk}$ becomes the functor that assigns to a $\Bbbk$-bialgebra $B$ the $\Bbbk$-vector space of its primitive elements and acts on morphisms as the restriction on primitive elements. Its left adjoint $\mathbf{T}$, assigns to a vector space $V$ the tensor algebra $TV$, where $\Omega TV := \oplus_{n\in\mathbb{N}} V^{\otimes n}$, endowed with its canonical bialgebra structure where the elements in $V$ are primitive. To a linear map $f : V \to W$ it assigns the bialgebra map $Tf : TV \to TW$, where $\Omega Tf := \oplus_{n\in\mathbb{N}} f^{\otimes n}$. The linear map $\omega V : \Omega TV \to V$ defined by (12), is just the canonical projection onto $V$. The map $\boldsymbol{\gamma} V : \mathbf{PT}V \to V$ is the restriction of $\omega V$ to $\mathbf{PT}V$. It is known that the primitive elements in $\mathbf{PT}V$ are homogeneous, i.e. $\mathbf{PT}V = \oplus_{n\in\mathbb{N}}(\mathbf{PT}V \cap V^{\otimes n})$, see e.g. [20, 9.10.2]. Thus, $\boldsymbol{\gamma} V$ is given by the projection on the space of primitive elements of homogeneous degree 1.

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