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A Nyström method for a class of Fredholm integral equations on the real semiaxis *

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Abstract

A class of Fredholm integral equations of the second kind, with respect to the exponential weight function $w(x) = \exp(-(x^{-\alpha} + x^{\beta}))$, $\alpha > 0$, $\beta > 1$, on $(0, +\infty)$, is considered. The kernel $k(x, y)$ and the function $g(x)$ in such kind of equations,

$$f(x) - \mu \int_0^{+\infty} k(x, y) f(y) w(y) \mathrm{d}y = g(x), \quad x \in (0, +\infty),$$

can grow exponentially with respect to their arguments, when they approach to 0^+ and/or $+\infty$.

We propose a simple and suitable Nyström-type method for solving these equations. The study of the stability and the convergence of this numerical method in based on our results on weighted polynomial approximation and "truncated" Gaussian rules, recently published in *Acta Math. Hungar.*, **142** (2014), 167–198, and *IMA J. Numer. Anal.* **34** (2014), 1654–1685, respectively.

Moreover, we prove a priori error estimates and give some numerical examples. A comparison with other Nyström methods is also included.

Keywords: Fredholm integral equation; Nyström method; weighted polynomial approximation; Gaussian quadrature formula; orthogonal polynomials; truncation; error estimate.
MCS classification (2000): 65R10, 65D30, 65D32, 41A55.

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1 Introduction

The aim of this paper is to approximate the solution of integral equations of the form

$$f(x) - \mu \int_0^{+\infty} k(x,y) f(y) w(y) \mathrm{d}y = g(x)\,, \quad x \in (0,+\infty), \tag{1.1}$$

with the exponential weight function

$$w(x) = \exp\Big[-\Big(\frac{1}{x^\alpha} + x^\beta\Big)\Big], \quad \alpha > 0,\ \beta > 1, \tag{1.2}$$

and the parameter $\mu \in \mathbb{R}$. The kernel $(x,y) \mapsto k(x,y)$ and the function $x \mapsto g(x)$ can grow exponentially with respect to their arguments, when they approach to 0^+ and/or $+\infty$.

The weight functions similar to (1.2) have been considered in statistics. Following Stoyanov [20, §7.1] we mention here a simple example with the inverse Gaussian distribution (IG) with "easy" parameters, say $(1,1)$, i.e., a random variable $\theta \sim \mathrm{IG}$, with density function

$$w(x) = \begin{cases} \dfrac{\mathrm{e}}{\sqrt{2\pi}}\, x^{-3/2} \exp\Big[-\dfrac{1}{2}\Big(x + \dfrac{1}{x}\Big)\Big], & \text{if } \ x > 0, \\ 0, & \text{if } \ x \le 0. \end{cases}$$

In terms of the modified Bessel function of the second kind, its moments can be expressed in the form (cf. [17])

$$\int_0^{+\infty} x^k w(x)\, \mathrm{d}x = \mathrm{e}\sqrt{\frac{2}{\pi}}\, K_{k-1/2}(1), \quad k \in \mathbb{N}_0.$$

Also, this kind of weights on $(0,+\infty)$ were appeared in a consideration on expansions of confluent hypergeometric functions in terms of Bessel functions by Temme [21], as well as in the so-called Laurent-Hermite-Gauss quadrature rules investigated by Gustafson and Hagler [5] and Hagler [6].

Integral equations of the form (1.1), with proper assumptions on the kernel k and the function g, can occur in mathematical finance in computing distributions of geometrical brownian motion (see [3, 7, 8]).

However, as far as we know, the numerical treatment of this kind of integral equations does not appear in the literature. In this paper we are going to study integral equations of the form (1.1) in some suitable function space with weighted uniform metric and to approximate the solution by means of a Nyström interpolant. We will prove that this method is stable and convergent in the metric of the considered space. In order to prove the convergence of the method we will use some recent results on polynomial approximation with the weight w and related Gaussian quadrature rule, obtained by the authors in [15, 13, 14, 16].

Therefore, the results in this paper are new.

For the sake of completeness, we observe that in the weight w, given by (1.2), the $C^\infty-$function $\exp\left[-x^{-\alpha}\right]$ appears. Therefore one could think to introduce a new kernel function $k(x,y)\,\exp\left[-y^{-\alpha}\right]$ and, provided the function g fulfills some proper assumptions, to approximate the solution of equation (1.1) by using a Nyström interpolant based on Laguerre zeros, as in [10] (see also [4, 11, 9]). In Section 4 we will show that this procedure is in general more expensive and less precise. This fact is also one of the motivations of this paper.

The paper is structured as follows. In Section 2 we recall some basic facts and give some preliminary results. In Section 3 we introduce our numerical method and prove the main results. In Section 4 we will compare our method with the one based on Laguerre zeros. Finally, in Section 5 we show some numerical examples.

2 Basic facts and preliminary results

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ when $\mathcal{C}$ is independent of $a,b,\ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$.

Moreover, the symbols $\|\cdot\|_I$ and $\|\cdot\|$ will denote the uniform norm in some interval I and in $(0,+\infty)$, respectively.

Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

2.1 Function spaces with weighted uniform metric

Letting

$$u(x) = (1+x)^{\delta}\sqrt{w(x)}\,, \qquad \delta > \frac{1}{2}\,, \tag{2.1}$$

where

$$w(x) = \mathrm{e}^{-\left(x^{-\alpha}+x^{\beta}\right)}\,, \quad \alpha > 0\,, \beta > 1\,,$$

$x \in (0, +\infty)$, we introduce the function space

$$C_u := \left\{ f \in C^0(0,+\infty) : \lim_{x\to 0^+} f(x)u(x) = 0 = \lim_{x\to+\infty} f(x)u(x) \right\}, \tag{2.2}$$

with the norm

$$\|f\|_{C_u} := \|fu\| = \sup_{x\in(0,+\infty)} |f(x)u(x)|\ .$$

We emphasize that the space C_u contains functions, defined on the real semiaxis $(0,+\infty)$, which can grow exponentially both for $x \to 0^+$ and for $x \to +\infty$. Moreover, C_u is a Banach space.

For $1 \le r \in \mathbb{Z}$, we define the Sobolev-type spaces

$$W_r = W_r^{\infty}(u) = \left\{ f \in C_u :\ f^{(r-1)} \in AC(0,+\infty),\ \|f^{(r)}\varphi^r u\| < \infty \right\},$$

with the norm

$$\|f\|_{W_r} = \|fu\| + \|f^{(r)}\varphi^r u\|\,,$$

$\varphi(x) = \sqrt{x}$.

In order to define further function spaces, we introduce the following moduli of smoothness. For each $f \in C_u$ $r \ge 1$ and $0 < t < t_0$, we set

$$\Omega_{\varphi}^r(f,t)_u = \sup_{0<h\le t} \left\| \Delta_{h\varphi}^r(f)\, u \right\|_{\mathcal{I}_h(c)}\ ,$$

where

$$\mathcal{I}_h(c) = \left[h^{1/(\alpha+1/2)}, \frac{c}{h^{1/(\beta-1/2)}}\right],$$

$c > 1$ is a fixed constant, and

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + (r-i)h\varphi(x)\right), \quad \varphi(x) = \sqrt{x}\,.$$

We remark that the behavior of this modulus of smoothness is independent on the constant c (see [14]).

Then we define the complete rth modulus of smoothness by

$$\begin{aligned}\omega^r_\varphi(f,t)_u = \Omega^r_\varphi(f,t)_u \;&+\; \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{(0,t^{2/(2\alpha+1)}]} \\ &+\; \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{[c\,t^{-2/(2\beta-1)},+\infty)},\end{aligned}$$

with $c > 1$ a fixed constant.

For any $f \in W_r$, $r \geq 1$ and $t < t_0$, we have (see [14])

$$\Omega^r_\varphi(f,t)_u \leq \mathcal{C} \inf_{0<h\leq t} h^r \|f^{(r)}\varphi^r u\|_{\mathcal{I}_h(c)}\,, \tag{2.3}$$

where $\mathcal{C}$ is independent of f and t.

By means of the main part of the modulus of smoothness, we can define the Zygmund-type spaces

$$Z_s := Z^\infty_{s,r}(u) = \{f \in C_u : \; M_s(f) < \infty\}\,,$$

where

$$M_s(f) := \sup_{t>0} \frac{\Omega^r_\varphi(f,t)_u}{t^s}\,, \quad r > s\,, \quad s \in \mathbb{R}^+\,,$$

with the norm

$$\|f\|_{Z_s} = \|f\|_{L^p_u} + M_s(f)\,.$$

We remark that, in the definition of Z_s, the main part of the rth modulus of smoothness $\Omega^r_\varphi(f,t)_u$ can be replaced by the complete modulus $\omega^r_\varphi(f,t)_u$, as shown in [14].

2.2 Weighted polynomial approximation

Let us denote by

$$E_m(f)_u = \inf_{P\in\mathbb{P}_m} \|(f-P)\,u\|_p$$

the error of best weighted polynomial approximation of a function $f \in C_u$.

The following Jackson, weak Jackson and Stechkin inequalities have been proved in [14].

Theorem 1. *For any $f \in C_u$ and $m > r \geq 1$, we have*

$$E_m(f)_u \leq \mathcal{C}\,\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_u,$$

where $a_m \sim m^{1/\beta}$. Moreover, assuming $\Omega_\varphi^r(f,t)_u\,t^{-1} \in L^1[0,1]$,

$$E_m(f)_u \leq \mathcal{C}\int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(f,t)_u}{t}\,\mathrm{d}t\,, \qquad r < m\,.$$

Finally, for any $f \in C_u$ we get

$$\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_u \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \sum_{i=0}^{m}\left(\frac{i}{\sqrt{a_i}}\right)^r \frac{E_i(f)_u}{i+1}\,.$$

In any case $\mathcal{C}$ is independent of m and f.

In particular, by Theorem 1 and (2.3), for any $f \in W_r$ we get

$$E_m(f)_u \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f^{(r)}\varphi^r u\| \tag{2.4}$$

and, for any $f \in Z_s$, we have

$$E_m(f)_u \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s M_s(f)\,. \tag{2.5}$$

Moreover, the following equivalences (see [14])

$$f \in C_u \quad\Leftrightarrow\quad \lim_{t\to 0}\omega_\varphi(f,t)_u = 0 \quad\Leftrightarrow\quad \lim_{m\to\infty} E_m(f)_u = 0 \tag{2.6}$$

will be useful in the sequel.

2.3 Gaussian rules

Let $\{p_m(w)\}_m$ be the sequence of orthonormal polynomials related to $w(x) = \mathrm{e}^{-x^{-\alpha}-x^{\beta}}$. The zeros of $p_m(w)$ are located as follows

$$\varepsilon_m < x_1 < x_2 < \cdots < x_m < a_m \,,$$

where the Mhaskar–Rahmanov-Saff numbers a_m and ε_m fulfill $a_m \sim m^{1/\beta}$ and $\varepsilon_m \sim \left(\sqrt{a_m}/m\right)^{2/(2\alpha+1)}$.

For any continuous function f the Gaussian rule related to the weight w is given by

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{m} \lambda_k(w) f(x_k) + e_m\left(f\right) \,, \tag{2.7}$$

where x_k are the zeros of $p_m(w)$ and $\lambda_k(w)$ are the Christoffel numbers.

In order to introduce our numerical method for solving equation (1.1), we are going to consider a "truncated" gaussian rule. Fixed $\theta \in (0,1)$, we define two indexes $j_1 = j_1(m)$ and $j_2 = j_2(m)$ as

$$\varepsilon_m < \varepsilon_{\theta m} \le x_{j_1} < \cdots < x_{j_2} \le a_{\theta m} < a_m \,.$$

To be more precise, with $\theta \in (0,1)$, j_1 and j_2 are such that

$$x_{j_1} = \max_{1\le k\le m} \left\{x_k \;:\; x_k \le \varepsilon_{\theta m}\right\}, \quad x_{j_2} = \min_{1\le k\le m} \left\{x_k \;:\; x_k \ge a_{\theta m}\right\}, \tag{2.8}$$

and, if $\{x_k \;:\; x_k \le \varepsilon_{\theta m}\}$ or $\{x_k \;:\; x_k \ge a_{\theta m}\}$ is empty, we set $x_{j_1} = x_1$ or $x_{j_2} = x_m$, respectively.

Then we consider the following "truncated" Gaussian rule

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \sum_{i=j_1}^{j_2} \lambda_i(w) f(x_i) + e_m^*(f) \tag{2.9}$$

and for any $f \in C_{u^2}$ we have (see [15])

$$|e_m^*(f)| \le \mathcal{C}\left\{E_M(f)_{u^2} + \mathrm{e}^{-cm^{\nu}} \left\|fu^2\right\|\right\}, \tag{2.10}$$

where

$$M = \left\lfloor \left(\frac{\theta}{\theta+1} \right) m \right\rfloor, \quad \nu = \left(1 - \frac{1}{2\beta} \right) \frac{2\alpha}{2\alpha+1}, \quad \mathcal{C} \neq \mathcal{C}(m,f), \quad \text{and} \quad c \neq c(m,f).$$

In particular, recalling the results in Section 2.2, for any $f \in W_r(u^2)$, we get

$$|e_m^*(f)| \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \|f\|_{W_r(u^2)}$$

and, for any $f \in Z_s(u^2)$, we have

$$|e_m^*(f)| \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^s \|f\|_{Z_s(u^2)} .$$

2.4 Compactness of linear operators in C_u

Let $A : C_u \to C_u$ be a linear operator. Then, following the Hausdorff definition, A is compact in C_u if and only if the limit condition

$$\lim_{m\to\infty} \sup_{\|f\|_{C_u}=1} E_m(Af)_u = 0 \tag{2.11}$$

holds. Taking into account (2.6), condition (2.11) can be rewritten in terms of moduli of smoothness (see [22, pp. 44, 93–94]) as follows

$$\lim_{t\to 0} \sup_{\|f\|_{C_u}=1} \omega_\varphi(Af,t)_u = 0 .$$

Coming back to equation (1.1), let us consider the operator K defined by

$$(Kf)(x) = \mu \int_0^{+\infty} k(x,y) f(y) w(y) \mathrm{d}y , \quad x \in (0,+\infty) . \tag{2.12}$$

Then, letting $k(x,y) = k_y(x) = k_x(y)$, since

$$\omega_\varphi(Kf,t)_u \leq |\mu| \|fu\| \sup_{y\in(0,+\infty)} \omega_\varphi(k_y,t)_u u(y) \int_0^{+\infty} \frac{\mathrm{d}y}{(1+y)^{2\delta}} ,$$

if

$$u(y) k_y \in C_u \quad \text{uniformly w.r.t.} \quad y, \tag{2.13}$$

then the operator K in C_u.

In an analogous way, the sequence of operators $\{A_m\}_m$ in C_u is collectively compact, i.e., the set

$$S = \{A_m f \in C_u \ : \ m \geq 1 \quad \text{and} \quad \|fu\| \leq 1\}$$

is relatively compact in C_u, if and only if the limit condition

$$\lim_{N\to\infty} \sup_{\|f\|_{C_u}=1} \sup_{m\in\mathbb{N}} E_N(A_m f)_u = 0$$

holds, namely if and only if

$$\lim_{t\to 0} \sup_{\|f\|_{C_u}=1} \sup_{m\in\mathbb{N}} \omega_\varphi(A_m f, t)_u = 0\,.$$

In particular, for the sequence of operators

$$(K_m f)(x) = \mu \sum_{i=j_1}^{j_2} \lambda_i(w) k(x, x_i) f(x_i)\,, \tag{2.14}$$

obtained by applying the "truncated" Gaussian rule (2.9) to $(Kf)(x)$ given by (2.12), it is not difficult to show that the collective compactness follows from the assumption

$$u(x)k_x \in C_u \quad \text{uniformly w.r.t.} \quad x\,. \tag{2.15}$$

3 The numerical method

Let us now introduce our numerical method for solving equation (1.1), i.e.,

$$f(x) - \mu \int_0^{+\infty} k(x,y) f(y) w(y) \mathrm{d}y = g(x)\,, \quad x \in (0, +\infty),$$

where $\mu \in \mathbb{R}$,

$$w(y) = \mathrm{e}^{-y^{-\alpha} - y^\beta}\,, \quad \alpha > 0\,,\ \beta > 1\,,$$

the given functions k and g can grow exponentially (w.r.t. x, y) when $x \to 0^+$ and/or $x \to +\infty$. Denoting the identity operator by I and the integral operator by K, we can rewrite this equation as

$$(I - K)\, f = g\,.$$

With u given by (2.1), we are going to study the equation (1.1) in the space C_u defined in Section 2.1. Under the assumptions (2.13), the Fredholm alternative holds true. So, if $\ker(I-K)=\{0\}$, equation (1.1) admits unique solution $f^*\in C_u$ for any fixed $g\in C_u$.

In order to approximate the solution of (1.1), we are going to use a Nyström method based on the "truncated" Gaussian rule defined in Section 2.3. To this end, we introduce the sequence of operators $\{K_m\}_m$,

$$(K_m f)(x)=\mu\sum_{i=j_1}^{j_2}\lambda_i(w)k(x,x_i)f(x_i) \tag{3.1}$$

which is obtained by applying the "truncated" Gaussian rule (2.9) to $(Kf)(x)$ given by (2.12). Then we are going to solve in C_u the equations

$$f_m(x)-(K_m f_m)(x)=g(x)\,,\quad m=1,2,\dots\,. \tag{3.2}$$

Multiplying both sides of (3.2) by $u(x)$, collocating at the quadrature knots and letting $a_i=(f_m u)(x_i)$, $b_i=(gu)(x_i)$, $i=j_1,\dots,j_2$, for $m=1,2,\dots$, we obtain the linear systems of equations

$$a_h-\mu\sum_{i=j_1}^{j_2}\lambda_i(w)k(x_h,x_i)\frac{u(x_h)}{u(x_i)}a_i=b_h,\quad h=j_1,\dots,j_2,$$

in the unknowns a_h, i.e.,

$$\sum_{i=j_1}^{j_2}\left[\delta_{ih}-\mu\lambda_i(w)k(x_h,x_i)\frac{u(x_h)}{u(x_i)}\right]a_i=b_h\,,\quad h=j_1,\dots,j_2\,. \tag{3.3}$$

If (3.3) is unisolvent and $(a^*_{j_1},\dots,a^*_{j_2})^T$ is its solution, then, by (3.2) and (3.1), we can define the Nyström interpolant

$$f^*_m(x)=\mu\sum_{i=j_1}^{j_2}\frac{\lambda_i(w)}{u(x_i)}k(x,x_i)a^*_i+g(x) \tag{3.4}$$

which we will be an approximation of the solution f^* of equation (1.1) in C_u-metric.

Notice that, due to the choice of the "truncated" Gaussian rule in place of the ordinary Gaussian rule (2.7), the matrix of coefficients of the system of equations (3.3), in notation $V_m^{(j_1,j_2)}$, has dimension $j_2 - j_1 + 1$ instead of m and this produces a reduction of the computational cost.

Let us prove the stability and convergence of our method.

Theorem 2. *Let u be the weight in (2.1). Assume*

(i) $u(y)k_y \in C_u$ *uniformly w.r.t. y;*
(ii) $u(x)k_x \in C_u$ *uniformly w.r.t. x;*
(iii) $g \in C_u$.

If $\ker(I - K) = \{0\}$, *the system of equations (3.3) is unisolvent and well-conditioned.*

Moreover, f_m^* converges to f^* in C_u and

$$\|(f_m^* - f^*)\, u\| \leq \mathcal{C} \sup_{x\in(0,+\infty)} u(x) \left\{ E_M \left(f^* k_x\right)_{u^2} + \mathrm{e}^{-cm^\nu} \left\| f^* k_x u^2 \right\| \right\} \tag{3.5}$$

where

$$M = \left\lfloor \left(\frac{\theta}{\theta+1} \right) m \right\rfloor, \quad \nu = \left(1 - \frac{1}{2\beta} \right) \frac{2\alpha}{2\alpha+1}, \quad \mathcal{C} \neq \mathcal{C}(m, f^*), \quad c \neq c(m, f^*).$$

Proof. As already mentioned in Section 2.4, from the assumption (i), i.e., (2.13), the compactness of the operator $K : C_u \to C_u$ follows. So the Fredholm alternative holds for equation (1.1) and, if $\ker(I - K) = \{0\}$, equation (1.1) admits unique solution $f^* \in C_u$.

Now, using (2.10), we have

$$\|(Kf - K_m f)\, u\| \leq \mathcal{C} \sup_{x\in(0,+\infty)} \left\{ E_M (f k_x)_{u^2} + \mathrm{e}^{-cm^\nu} \left\| f k_x u^2 \right\| \right\}, \tag{3.6}$$

i.e., the sequence $\{K_m\}_m$ strongly converges to the operator K.

Moreover, since $\{K_m\}_m$ is collectively compact by (ii), i.e., (2.15), it follows that

$$\lim_m \|(K - K_m)K_m\|_{C_u \to C_u} = 0$$

and, using [1, Theorem 4.1.1] or [19, Theorem 2.1], for $m \geq m_0$, the operators $(I - K_m)^{-1}$ exist and

$$\begin{aligned} \|(I-K_m)^{-1}\|_{C_u \to C_u} &\leq \frac{1 + \|(I-K)^{-1}\|_{C_u \to C_u} \|K_m\|_{C_u \to C_u}}{1 - \|(I-K)^{-1}\|_{C_u \to C_u} \|(K-K_m)K_m\|_{C_u \to C_u}} \\ &\leq \mathcal{C} < +\infty. \end{aligned}$$

Then, proceeding as in [1, pp. 112–113], we deduce that the matrix $V_m^{(j_1,j_2)}$ of the coefficients of system (3.3) is well conditioned, i.e.,

$$\operatorname{cond}\left(V_m^{(j_1,j_2)}\right) \leq \operatorname{cond}(I - K_m) \leq \mathcal{C} < \infty\,, \quad \mathcal{C} \neq \mathcal{C}(m)\,.$$

Finally, the error estimate (3.5) immediately follows by (3.6). □

From (3.5) we deduce that the order of convergence of our method depends on the smoothness of the kernel k and the solution f^* of equation (1.1). Now, we want to show a more explicit error estimate, depending on the smoothness of the known functions k and g. In particular, from Theorem 2 we deduce the following corollary.

Corollary 3. *Let the assumptions of Theorem 2 be replaced by*

(a) $u(y)k_y \in W_r(u)$ *uniformly w.r.t. y;*
(b) $u(x)k_x \in W_r(u)$ *uniformly w.r.t. x;*
(c) $g \in W_r(u)$.

Then, for m sufficiently large, we have

$$\|(f_m^* - f^*)\,u\| = O\left(\left(\frac{\sqrt{a_m}}{m}\right)^r\right)\,,$$

where the constants in "O" are independent of m and f^*.

Proof. We note that the assumptions on the given functions imply $f^* \in W_r(u)$ and then $f^* k_x \in W_r(u^2)$. Hence, by (3.5) and (2.4), we get (see for

instance [12, Theorem 3.2])

$$\begin{aligned}\|(f_m^* - f^*)\,u\| &\le \mathcal{C} \sup_{x\in(0,+\infty)} u(x)\Big\{E_M\left(f^*k_x\right)_{u^2} + \mathrm{e}^{-cm^\nu}\left\|f^*k_xu^2\right\|\Big\}\\ &\le \mathcal{C} \sup_{x\in(0,+\infty)} u(x)\Big\{\|f^*u\|\,E_n\left(k_x\right)_u + \|k_xu\|\,E_n\left(f^*\right)_u\\ &\qquad\qquad +\mathrm{e}^{-cm^\nu}\,\|f^*u\|\,\|k_xu\|\Big\},\end{aligned}$$

with $n = \lfloor M/2 \rfloor$ and the corollary follows from (2.4). □

We note that, by (2.5), an analogous corollary holds if we replace the Sobolev spaces W_r by the Zygmund spaces Z_s.

4 Comparison with the Nyström method based on Laguerre zeros

The following observation is crucial. The integral

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \int_0^{\infty} f(x)\mathrm{e}^{-x^{-\alpha}-x^\beta}\,\mathrm{d}x \tag{4.1}$$

can be evaluated by means of the Gaussian rule related to the weight $w(x) = \mathrm{e}^{-x^{-\alpha}-x^\beta}$, i.e.,

$$G_m(w,f) = \sum_{k=1}^{m} \lambda_k(w) f(x_k)\,,$$

as described in Section 2.3. On the other hand, this integral can be rewritten as

$$\int_0^{\infty} \left[f(x)\mathrm{e}^{-x^{-\alpha}}\right] \mathrm{e}^{-x^\beta}\,\mathrm{d}x = \int_0^{\infty} \left[f(x)\mathrm{e}^{-x^{-\alpha}}\right] \sigma(x)\,\mathrm{d}x$$

and evaluated by using the Gaussian rule related to the Laguerre-type weight $\sigma(x) = \mathrm{e}^{-x^\beta}$, i.e.,

$$\overline{G}_m(\sigma,g) = \sum_{k=1}^{m} \lambda_k(\sigma) g(t_k) = \sum_{k=1}^{m} \lambda_k(\sigma) f(t_k)\mathrm{e}^{-t_k^{-\alpha}}\,,$$

where $g(x) = f(x)\mathrm{e}^{-x^{-\alpha}}$, $t_k = t_{m,k}(\sigma)$ are the zeros of the mth Laguerre-type polynomial $p_m(\sigma)$, satisfying

$$\frac{\mathcal{C}}{m^{2-1/\beta}} \le t_1 < \cdots < t_m < \mathcal{C}m^{1/\beta}\,,$$

and $\lambda_k(\sigma)$ are the corresponding Christoffel numbers (see, e.g., [11, 9]).

Now, considering the coefficients of the two Gaussian rules, we observe that for the first term of $G_m(w, f)$ we have

$$\lambda_1(w) \sim w(x_1)\Delta x_1 \sim \mathrm{e}^{-x_1^{-\alpha}}\Delta x_1 \sim \mathrm{e}^{-m^{\frac{\alpha(2\beta-1)}{\beta(2\alpha+1)}}},$$

whereas the first term of $\bar{G}_m(\sigma, g)$ fulfills

$$\lambda_1(\sigma)\mathrm{e}^{-t_1^{-\alpha}} \sim \sigma(t_1)\Delta t_1 \mathrm{e}^{-t_1^{-\alpha}} \sim \Delta t_1 \mathrm{e}^{-t_1^{-\alpha}} \sim \mathrm{e}^{-m^{\frac{\alpha(2\beta-1)}{\beta}}}.$$

This last quantity is much smaller than $\lambda_1(w)$ for large values of m and also smaller than the ordinary tolerance usually adopted in computation. Therefore a certain number $\eta = \eta(m)$ of summands of $\overline{G}_m(\sigma, g)$ do not give any contribution. So, if $G_m(w, f)$ computes the integral with a certain error, one could obtain the same precision using the Laguerre-type rule for larger values of m and with more evaluations of the function f. The following example confirms this fact.

Example 1. We apply the Gaussian quadrature rules w.r.t. the exponential weight $w(x) = \mathrm{e}^{-1/x^3-x^3}$ and the Laguerre weight $\sigma(x) = \mathrm{e}^{-x^3}$ for calculating

$$\int_0^{+\infty} \arctan\left(\frac{1+x}{4}\right) \mathrm{e}^{-1/x^3-x^3}\,\mathrm{d}x\,,$$

with $f(x) = \arctan\left(\frac{1+x}{4}\right)$ and $g(x) = \arctan\left(\frac{1+x}{4}\right)\mathrm{e}^{-1/x^3}$. This integral can be evaluated with a high precision using the Mathematica function `NIntegrate`.

In Table 1 we compare the relative errors obtained applying the two rules for increasing values m, working in double arithmetic precision. We note that underflow phenomena occurred in the case of Laguerre weights, while in the case of w the symbol "—" means that the required precision has already been obtained and the relative error is of the order of the machine epsilon.

We also want to observe that a similar argument applies a fortiori if we compare the two truncated Gaussian rule related to w and σ. In fact, in $G_m(w, f)$ we can drop some terms related to the zeros close to $\varepsilon(w)$ and some other terms related to the zeros close to $a_m(w)$, but in $\overline{G}_m(\sigma, g)$ we can

Table 1: Relative errors

m	relative error of $\overline{G}_m(\sigma, g)$	relative error of $G_m(w, f)$
2	3.077×10^{-1}	5.891×10^{-6}
7	1.222×10^{-2}	1.256×10^{-16}
30	9.005×10^{-7}	—
60	1.584×10^{-11}	—
110	6.984×10^{-16}	—

drop only some terms related to the largest zeros without loss of accuracy (see [11]).

Let us now compare the convergence of the two Gaussian rules. To this aim, letting

$$v(x) = (1+x)^{\delta} w(x) = (1+x)^{\delta} \mathrm{e}^{-x^{-\alpha} - x^{\beta}}, \qquad \delta > 1,$$

$x \in (0, +\infty)$, we introduce the function space

$$C_v := \left\{ f \in C^0(0, +\infty) : \lim_{x \to 0^+} f(x)v(x) = 0 = \lim_{x \to +\infty} f(x)v(x) \right\},$$

with the norm

$$\|f\|_{C_v} := \|fv\| = \sup_{x \in (0, +\infty)} |f(x)v(x)| .$$

For more regular functions, we define the Sobolev-type spaces

$$W_r^{\infty}(v) = \left\{ f \in C_v : \ f^{(r-1)} \in AC(0, +\infty), \ \|f^{(r)} \varphi^r v\| < \infty \right\},$$

with the norm $\|f\|_{W_r^{\infty}(v)} = \|fv\| + \|f^{(r)} \varphi^r v\|$, where $\varphi(x) = \sqrt{x}$.

Then it is known that for any $f \in C_v$, the Gaussian rule $G_m(w, f)$ converges to the integral (4.1). Moreover, if $f \in W_r^{\infty}(v)$, $r \geq 1$, we have (see [15, 14])

$$\left| G_m(w, f) - \int_0^{\infty} f(x) w(x) \, \mathrm{d}x \right| \leq \frac{\mathcal{C}}{\left(m^{1 - 1/(2\beta)}\right)^r} \|f\|_{W_r^{\infty}(v)}.$$

The Laguerre–Gaussian rule deals with functions belonging to the space

$$C_{\bar{v}} := \left\{ g \in C^0[0, +\infty) : \lim_{x \to +\infty} g(x)\bar{v}(x) = 0 \right\},$$

with $\bar{v}(x) = (1+x)^{\delta}\mathrm{e}^{-x^{\beta}}$, $\delta > 1$,

$$\|g\|_{C_{\bar{v}}} := \|f\bar{v}\| = \sup_{x\in(0,+\infty)} |f(x)\bar{v}(x)| \,,$$

and/or to the Sobolev-type space

$$W_r^{\infty}(\bar{v}) = \left\{ g \in C_{\bar{v}} : \ g^{(r-1)} \in AC(0,+\infty), \ \|g^{(r)}\varphi^r\bar{v}\| < \infty \right\},$$

with

$$\|g\|_{W_r^{\infty}(\bar{v})} = \|gv\| + \|g^{(r)}\varphi^r\bar{v}\| \,,$$

$\varphi(x) = \sqrt{x}$ and $r \geq 1$. In analogy with the first Gaussian rule one has (see [11, 9])

$$(\forall g \in C_{\bar{v}}) \quad \overline{G}_m(\sigma, g) \to \int_0^{+\infty} g(x)\mathrm{e}^{-x^{\beta}}\,\mathrm{d}x \,, \quad m \to \infty \,, \tag{4.2}$$

and

$$\left| \overline{G}_m(\sigma, g) - \int_0^{+\infty} g(x)\mathrm{e}^{-x^{\beta}}\,\mathrm{d}x \right| \leq \frac{\mathcal{C}}{\left(m^{1-1/(2\beta)}\right)^r} \|g\|_{W_r^{\infty}(v)} \,. \tag{4.3}$$

Nevertheless, if $g(x) = f(x)\mathrm{e}^{-x^{-\alpha}}$ with $f \in C_v$ (i.e., in the case under consideration), the convergence relation (4.2) is true while the error estimate (4.3) is false in general. In fact, $f \in C_v$ implies $g \in C_{\bar{v}}$, but the norm $\|g\|_{W_r^{\infty}(\bar{v})}$ can be unbounded for $f \in W_r^{\infty}(v)$ (so, although inequality (4.3) holds). Therefore, the order of convergence of the Laguerre-Gaussian rule $\overline{G}_m(\sigma, g)$ is lower than the one of the rule $G_m(w, f)$.

From the previous observations we deduce that the Nyström interpolant obtained by approximating the integral

$$\int_0^{+\infty} \left[k(x,y)f(y)\mathrm{e}^{-y^{-\alpha}} \right] \mathrm{e}^{-y^{\beta}}\mathrm{d}y$$

by means of the Laguerre–Gaussian rule $\overline{G}_m(\sigma)$ will have a much larger number of summands with respect to the method proposed in this paper. This fact implies that the corresponding linear system will have a much larger order than the one in (3.3). We also want to emphasize that considering a

"truncated" version of $\overline{G}_m(\sigma)$ would not solve this problem, since it is due to the exponential behaviour of the integrand close to 0 and the "truncated" rule wold drop only the terms related to the largest zeros.

In conclusion, solving integral equations of the form (1.1) by using a Nyström method based on the Laguerre–Gaussian rule would require a larger computational cost (with possible underflow/overflow phenomena) and a lower order of convergence, as shown in the next section.

5 Numerical examples

In the following examples the exact solutions are unknown and the corresponding tables show only the behaviour of the Nyström interpolants. As in Section 4, all computations were performed in MATHEMATICA, Ver. 8.0. In particular, for constructing the corresponding Gaussian rules (2.7) we use a procedure given in [15] and the MATHEMATICA package `OrthogonalPolynomials` (cf. [2] and [18]), which is freely downloadable from the Web Site:

http://www.mi.sanu.ac.rs/~gvm/.

For the sake of brevity we omit the description of the numerical procedures for the computation of the zeros of $p_m(w)$, the Christoffel numbers and the Mhaskar–Rahmanov-Saff numbers ε_m and a_m. The interested reader can find all the details about these procedures in [15, pp. 1676–1680].

Example 2. We consider the Fredholm integral equation of the second kind

$$f(x) - \frac{1}{10}\int_0^{+\infty} \cosh\left(\frac{y+1}{x+1}\right) f(y)\mathrm{e}^{-y^{-3}-y^3}\,\mathrm{d}y = \sinh(x+3), \quad x \in (0,+\infty),$$

with $k(x,y) = \cosh((y+1)/(x+1))$, $w(x) = \mathrm{e}^{-x^{-3}-x^3}$, and $g(x) = \sinh(x+3)$. By (2.1) we choose the weight $u(x) = (1+x)\,\mathrm{e}^{-(x^{-3}+x^3)/2}$ and consider the equation in the space C_u given by (2.2). Since $\|K\|_{C_u\to C_u} < 1$ this equation admits a unique solution in C_u.

On the other hand, if we consider the weight the Laguerre-type weight $\tilde{u}(x) = (1+x)\,\mathrm{e}^{-x^3/2}$ and the associated function space $C_{\tilde{u}}$, this equation

admits a unique solution also in $C_{\tilde{u}}$. In Table 2 we compare the two associated Nyström methods, showing the correct decimal digits obtained in the Nyström interpolants at given points for the same values of m.

Using one of the two the Gaussian rules we obtain the corresponding Nyström interpolants $f_m^*(x)$, given by (3.4), and $\tilde{f}_m$. In Table 2 we give values of these interpolants at the points $x = 0.5$, $x = 1$ and $x = 5$. The same digits in $f_m^*(x)$ and $f_{25}^*(x)$ for $m = 5(5)20$ are bolded.

Table 2: Values of Nyström interpolants at $x = 0.5$, $x = 1$ and $x = 5$, for $m = 5(5)20$

m	$f_m^*(0.5)$ – exponential weight u	$\tilde{f}_m(0.5)$ – Laguerre weight $\tilde{u}$
5	**17.067206691**704378	**17.060**042557600486
10	**17.067206693043214**	**17.0679**2305565905
15	**17.067206693043214**	**17.0671**5565208371
20	**17.067206693043214**	**17.06720**0301955914
m	$f_m^*(1)$ – exponential weight u	$\tilde{f}_m(1)$ – Laguerre weight $\tilde{u}$
5	**27.676749186**738305735	**27.6713**848546487190
10	**27.676749187388134**070	**27.6773**392594205800
15	**27.676749187388135357**	**27.6767**086020279940
20	**27.676749187388135357**	**27.6767**435168185499
m	$f_m^*(5)$ – exponential weight u	$\tilde{f}_m(5)$ – Laguerre weight $\tilde{u}$
5	**1490.7310363**04753920402	**1490.7**275135882654744
10	**1490.731036305188948**542	**1490.731**492523212215
15	**1490.731036305188949804**	**1490.7310**0663711270
20	**1490.731036305188949804**	**1490.7310**314344204

Since the kernel and the solution in this case are very smooth, we see a very fast convegence of Nyström interpolants $f_m^*(x)$, so that $f_{25}^*(x)$ can be taken as a very well approximation of the exact solution $f^*(x)$. On the other hand, the Nyström interpolant based on Laguerre-type nodes converges more slowly.

In both cases the matrices of the related linear systems are well-conditioned. For instance the condition numbers of the matrices in (3.3) $V_m \equiv V_m^{(j_1,j_2)}$, with $j_1 = 1$ and $j_2 = m$, for $m = 5$, 10, 15 , 20 (in the infinity-norm) are 1.0218, 1.0260, 1.0271, 1.0284, respectively.

Example 3. We consider the Fredholm integral equation of the second kind

$$f(x) - \int_0^{+\infty} \cos(x+y) f(y) \mathrm{e}^{-y^{-3}-y^3}\, \mathrm{d}y = \mathrm{e}^{1/x^2}, \quad x \in (0, +\infty),$$

with $k(x,y) = \cos(x+y)$, $w(x) = \mathrm{e}^{-x^{-3}-x^3}$, and $g(x) = \mathrm{e}^{1/x^2}$. By (2.1) we choose the weight $u(x) = (1+x)\,\mathrm{e}^{-(x^{-3}+x^3)/2}$ and consider the equation in the space C_u given by (2.2). Since $\|K\|_{C_u \to C_u} < 1$ this equation admits a unique solution in C_u.

In this case the function g increases exponentially for $x \to 0^+$, so it does not belong to function spaces associated to generalized Laguerre weights. On the other hand, if we multiply both sides of the equation by e^{-1/x^2}, we obtain the equivalent equation

$$\tilde{f}(x) - \int_0^{+\infty} \left[\cos(x+y)\mathrm{e}^{y^{-2}-y^{-3}-x^{-2}}\right] \tilde{f}(y)\mathrm{e}^{-y^3}\, \mathrm{d}y = 1, \quad x \in (0, +\infty),$$

with $\tilde{f}(x) = f(x)\,\mathrm{e}^{-1/x^2}$. This last equation admits a unique solution in the space $C_{\tilde{u}}$, with $\tilde{u}(x) = (1+x)\,\mathrm{e}^{-x^3/2}$. In Table 3 we compare the two associated Nyström methods, showing the correct decimal digits obtained in the approximate solution at given points for the same values of m.

The method proposed in Section 3 is stable and the condition numbers of the matrices in (3.3) $V_m \equiv V_m^{(j_1,j_2)}$, with $j_1 = 1$ and $j_2 = m$, for $m = 10, 30, 50, 70$ (in the infinity-norm) are 1.0955, 1.1066, 1.1110, 1.1134, respectively. On the other hand, the method based on Laguerre zeros is less precise and applicable only for small values of m.

Example 4. Now we consider the equation

$$f(x) - \int_0^{+\infty} |\cos(x+y)|^{5/4} f(y)\mathrm{e}^{-y^{-2}-y^2}\, \mathrm{d}y = \frac{\mathrm{e}^{(x^3+8)/(4x)}}{x+4}, \quad x \in (0, +\infty),$$

with $k(x,y) = |\cos(x+y)|^{5/4}$, $w(x) = \mathrm{e}^{-x^{-2}-x^2}$, and

$$g(x) = \frac{\mathrm{e}^{(x^3+8)/(4x)}}{x+4}.$$

Table 3: Values of the approximate solution at $x = 0.5$, $x = 1$ and $x = 5$, for $m = 10(20)70$

m	$f_m^*(0.5)$ – exponential weight u	$\tilde{f}_m^*(0.5) \cdot \mathrm{e}^4$ – Laguerre weight $\tilde{u}$
10	**54.62464**753294463	**54.62**349877451629
30	**54.62466842**772536	**54.62466**806769319
50	**54.62466842781927**	**54.62466843**181872
70	**54.62466842781927**	Underflow occurred in computation

m	$f_m^*(1)$ – exponential weight u	$\tilde{f}_m^*(1) \cdot \mathrm{e}$ – Laguerre weight $\tilde{u}$
10	**2.6340**588382504633	**2.63**37766902545059
30	**2.6340654174**597524	**2.634065**6449192323
50	**2.6340654174976796**	**2.634065**4201667359
70	**2.6340654174976796**	Underflow occurred in computation

m	$f_m^*(5)$ – exponential weight u	$\tilde{f}_m^*(5) \cdot \mathrm{e}^{0.04}$ – Laguerre weight $\tilde{u}$
10	**1.2543661133492144**	**1.25**31280168491409
30	**1.2543856822837072**	**1.2543**846501698619
50	**1.2543856823546275**	**1.254385682**3259046
70	**1.2543856823546275**	Underflow occurred in computation

We consider this equation in C_u, where $u(x) = (1+x)\,\mathrm{e}^{-(x^{-2}+x^2)/2}$. Since $\|K\|_{C_u \to C_u} < 1$ this equation admits a unique solution in C_u. By Theorem 2 and Corollary 3, since $u(y)k_y \in Z_{5/4}(u)$ uniformly w.r.t. y, $u(x)k_x \in Z_{5/4}(u)$ uniformly w.r.t. x, while g is a smooth function, we have

$$\|(f_m^* - f^*)\,u\| = O\left(\left(\frac{\sqrt{a_m}}{m}\right)^{5/4}\right) = O\left(m^{-15/16}\right),$$

taking into account that $a_m \sim m^{1/2}$.

Now, we apply the Gaussian quadratures for $m = 10(10)50$ and $m = 100(50)300$, with the corresponding truncation as in Example 8.2 in [15]. Following Table 2 from [15], we present here in Table 4 the indices j_2 and j_2 in "truncated sums" in (3.1) and (2.8) for $\theta = 1/20$, as well as the condition numbers of these reduced matrices $V_m^{(j_1,j_2)}$. Their dimensions are $j_2 - j_1 + 1$ instead of m as in the case of Gaussian formulae, dropping $c\,m^2$ terms, $c < 1$, in the matrix of coefficients in the system of linear equations.

The absolute errors of the corresponding weighted Nyström interpolants at

Table 4: Absolute errors of the weighted Nyström interpolants $f_m^*(x)$ at $x = 1/2, 1, 4, 8$, for $m = 10$, $m = 100$ and $m = 200$

m	(j_1, j_2)	$\mathrm{cond}(V_m^{(j_1,j_2)})$	$x = 1/2$	$x = 1$	$x = 4$	$x = 8$
10	$(1, 6)$	1.166	1.15(−4)	7.93(−4)	2.42(−6)	8.51(−17)
100	$(7, 45)$	1.212	3.59(−5)	6.60(−5)	1.11(−7)	1.61(−17)
200	$(11, 90)$	1.217	6.90(−7)	3.78(−6)	1.16(−8)	9.61(−19)

some selected x are also given in the same table (we have considered as exact the approximated solution obtained for $m = 300$). Numbers in parentheses indicate decimal exponents, e.g., 1.15(−4) means 1.15×10^{-4}. Moreover, the values of the corresponding Nyström interpolants at the same selected points x are given in Table 5.

Table 5: Values of Nyström interpolants $f_m^*(x)$ at $x = 1/2, 1, 4, 8$, for $m = 10$, $m = 100$ and $m = 200$

m	$x = 1/2$	$x = 1$	$x = 4$	$x = 8$
10	**12.98**424	**1.990**655	**11.32**452	**950832.875**282
100	**12.9846**8	**1.991**643	**11.32**594	**950832.875**892
200	**12.9848**8	**1.991**728	**11.3260**2	**950832.87602**6

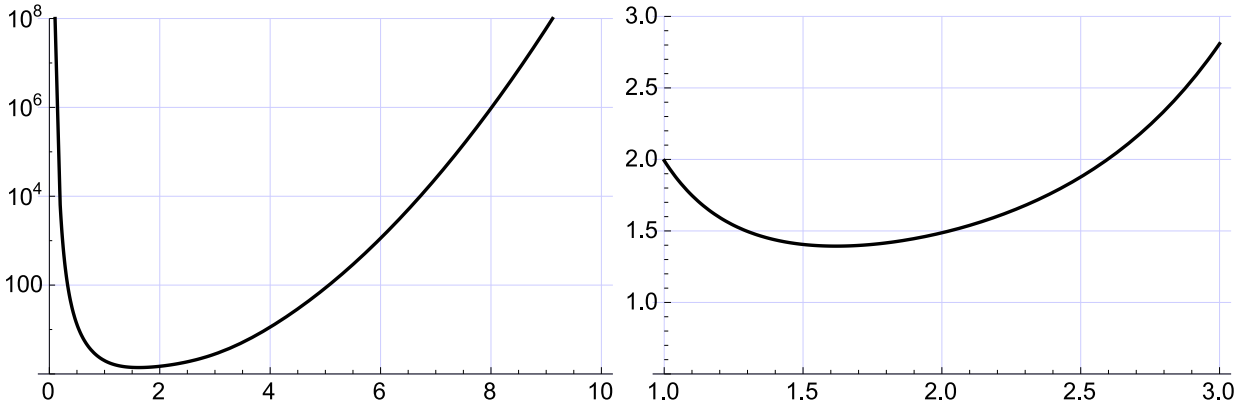

Figure 1: The Nyström interpolant $f_m^*(x)$ for $0 \le x \le 10$ (left) and for $1 \le x \le 3$ (right), when $m = 300$, $j_1 = 15$, $j_2 = 134$

The Nyström interpolant $f_{300}^*(x)$ obtained with $j_1 = 15$ and $j_2 = 134$, for $0 \le x \le 10$ and $1 \le x \le 3$ is displayed in Figure 1.

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