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Dino Gerardi
Lucas Maestri

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Dynamic Contracting with Limited Commitment and the Ratchet Effect*

Dino Gerardi† Lucas Maestri‡

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Abstract

We study dynamic contracting with adverse selection and limited commitment. A firm (the principal) and a worker (the agent) interact for potentially infinitely many periods. The worker is privately informed about his productivity and the firm can only commit to short-term contracts. The ratchet effect is in place since the firm has the incentive to change the terms of trade and offer more demanding contracts when it learns that the worker is highly productive.

As the parties become arbitrarily patient, the equilibrium outcome takes one of two forms. If the prior probability of the worker being productive is low, the firm offers a pooling contract and no information is ever revealed. In contrast, if this prior probability is high, the firm fires the unproductive worker at the very beginning of the relationship.

Keywords: Dynamic Contracting; Limited Commitment; Ratchet Effect.
JEL: D80; D82; D86.

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†Collegio Carlo Alberto, Università di Torino; dino.gerardi@carloalberto.org.
‡FGV/EPGE - Escola Brasileira de Economia e Finanças; maestri.lucas@gmail.com.

1 Introduction

Private information is pervasive in long-run relationships. Information revelation enhances efficiency as it helps in finding the best plans of action. However, parties involved in long-run relationships often fear that revealing their private information may worsen their future terms of trade. This problem is aggravated when the privately informed party (the agent) contracts with a party with a stronger bargaining position (the principal). These relationships are thus shaped by the principal's desire to elicit information and the agent's reluctance to reveal it.

The phenomenon above, known as the *ratchet effect*, is present in several real-life situations. Roy (1952) presents evidence that under the piece-rate system firms often worsen the workers' terms of trade after good performance. Similarly, Litwack (1991) documents how planners with small commitment power would use past outputs to establish future production targets and future managerial compensation, and explains its detrimental effect on incentives.

Our paper contributes to the literature on the ratchet effect by analyzing an infinite horizon contracting problem with short-term contracts. We consider the relationship between a worker and a firm as our main interpretation. In each period, the worker can produce a good of quality $q \in [0,1]$ at a cost that is linear in q. At the outset of the relationship, the worker is privately informed about his (persistent) marginal cost, which can be either low or high. We let p_0 denote the prior probability that the worker's cost is low. The firm can only commit to short-term contracts which indicate the payment that the worker is entitled to receive, in the current period, if he turns in a good of a certain specified quality. In each period in which the worker is employed, the firm offers a menu with finitely many contracts. Upon being offered a menu, the worker can either accept one contract in the menu or reject all contracts and end the relationship.

We show that when the discount factor is not too high the firm is able to extract the worker's private information independently of the value of the prior. In particular, if the prior p_0 is large the firm offers a *firing* menu in every period. A firing menu contains only one contract which specifies the efficient quality when the cost is low and yields a payoff equal to zero to the worker. The contract is accepted only by the low-cost worker. Thus, the firm learns the worker's cost in the first period by firing the high-cost worker.

When the prior is not too large, the firm employs a *sequentially screening* procedure.

The firm starts offering two contracts until it discovers the worker's cost, which happens in finitely many periods. During the screening procedure, the high-cost worker accepts the first contract while the low-cost worker randomizes between the two contracts. Once the screening is complete and the firm discovers the cost, the worker delivers the efficient quality and obtains zero payoff.

When the parties are sufficiently patient, the sequentially screening procedure described above is not feasible. To see why, consider the last period of the screening procedure in which the firm offers a menu that fully separates the two types of worker (in the sense that they accept two different contracts with probability one). However, for large discount factors, this is not possible. Indeed if there were separation, the low-cost worker could guarantee a large *future* payoff by mimicking the high-cost worker. Because of this it impossible to design two contracts that simultaneously satisfy the truthtelling constraints of the two types of worker. The low-cost worker can be prevented from imitating the high-cost worker only if the contract designed for him is very generous. But in this case the high-cost worker has an incentive to adopt the "take the money and run" strategy (i.e., accept the contract designed for the low-cost worker and then quit the relationship).[1]

The firm could, in principle, adopt more complex dynamic screening strategies. For instance, it could start offering menus in which different contracts are accepted with positive but different probabilities by both types of workers and then use this information in later periods to induce partial separation or even complete separation (through a firing menu). To investigate the feasibility and optimality of such strategies, we analyze the limiting outcome, as the parties become arbitrarily patient, of all perfect Bayesian equilibria.

We show that the limiting equilibrium outcome is unique and takes a very simple form. If the prior is below a certain threshold $\hat{p}$, then, in every period, the firm offers the most profitable contract that the high-cost worker is willing to accept. Both types of worker accept the contract (i.e., they pool) and there is no learning. In contrast, if the prior is above $\hat{p}$, the firm offers the firing menu and the high-cost worker quits the relationship without delay. In both cases, the limiting equilibrium allocation is inefficient.

Our results show that when the parties are sufficiently patient, the firm loses the ability to screen the worker without firing him when his cost is high. The driving forces behind our findings are similar to those which prevent full separation. When the discount factor is large, it is very costly for the firm to separate the two types of worker and continue the

[1] This result is reminiscent of Laffont and Tirole (1988).

relationship with both of them. A lasting relationship with the high-cost worker provides strong incentives to the low-cost worker to misrepresent his information. Using this fact, we show that the firm would not benefit from separating without firing even if this form of separation were feasible.

Our benchmark model assumes that the relationship ends automatically when the worker rejects all the contracts in the firm's menu. This modeling assumption captures situations in which upon disagreement the worker finds another job and becomes unavailable for the firm. Of course, one can also imagine situations in which the unemployed worker remains available to be rehired in the future. We therefore analyze an extension of the model that allows for rehiring. We study the infinitely repeated game in which, in each period, the firm proposes a menu of contracts from which the worker has to select at most one. We first show that the complete-information version of this game admits a folk theorem. Although the firm has the bargaining power to make offers, the worker can obtain large payoffs by credibly committing to reject unfavorable contracts. This is possible because the acceptance of unfavorable contracts by the worker triggers a continuation equilibrium in which the firm implements an efficient allocation that yields a zero payoff to the worker. We use these findings from the complete-information game to show that a version of the folk theorem holds for our model with rehiring.[2] In particular, when the parties are sufficiently patient, the firm can obtain a payoff arbitrarily close to the payoff of the optimal mechanism with commitment. These findings suggest that the labor-market structure may play an important role in the dynamics of incentive contracts.

This paper belongs to the literature on repeated adverse-selection with limited commitment pioneered by Freixas, Guesnerie, and Tirole (1985), Gibbons (1987), and Laffont and Tirole (1987, 1988). In these seminal papers, the parties interact for two periods. One of the main findings is that there is partial separation of the agent's types in the first period (i.e., the equilibrium is semi-pooling) and full separation in the second and final period. Therefore the outcome of two-period environments presents gradual information revelation. In contrast, our paper shows that when the relationship is infinitely repeated and the prior is low, the equilibrium allocation is close to a pooling allocation when the parties are patient.

Hart and Tirole (1988) analyze a dynamic model in which the seller makes a rental offer

[2] This finding is reminiscent of earlier contributions to repeated games with incomplete information and simultaneous moves (see Pęski (2008) and the references therein).

to the buyer in every period. The buyer's valuation for the good is private information and can take two values, both of which are larger than the seller's cost of producing the good. As the parties become sufficiently patient, the equilibrium allocation converges to the efficient allocation in which both types of buyer consume the good in every period. Note that for large values of the probability of the low valuation, this pooling allocation coincides with the seller's optimal mechanism under full commitment (i.e., lack of commitment is not detrimental to the seller's payoff). In a recent paper, Beccuti and Möller (2018) extend Hart and Tirole's analysis to the case in which the seller is more patient than the buyer. Halac (2012) studies a relational contract model in which the principal is privately informed about his outside option. When the uninformed party has the bargaining power, Coasian forces lead to a pooling outcome when the parties are sufficiently patient. Our work differs from these papers in two respects. First, in our model, the agent's private information is necessary to determine the best course of action and, therefore, pooling allocations are never optimal for the firm under full commitment. Second, we analyze environments in which the ratchet effect leads to inefficiencies.[3]

Our work is also related to the literature on renegotiation. The seminal paper by Laffont and Tirole (1990) analyzes a two-period model. Recently, Strulovici (2017) and Maestri (2017) study renegotiation in infinite horizon models. These studies find that equilibrium allocations become efficient as the parties become arbitrarily patient. In contrast, in our model the limit allocation is inefficient whenever the firing allocation is not a commitment solution.

Bhaskar (2014) studies learning in a dynamic model in which the principal and the agent are ex-ante symmetrically informed about the job's difficulty. When the agent's effort is unobservable, it is impossible for the principal to design a contract that induces an interior effort level in the first period. Bhaskar and Mailath (2017) consider a related dynamic model and show that inducing high effort becomes prohibitively costly for the principal as the parties become arbitrarily patient. Therefore, the ratchet effect imposes stringent constraints on the learning process of the relationship. In contrast, our paper assumes adverse-selection and no exogenous learning and concludes that the ratchet effect imposes constraints on the amount of private information that is revealed in a dynamic relationship.

[3] Our work analyzes the relationship between two infinitely lived players. In the context of political economy, several papers study the effects of limited commitment in repeated interactions between one principal and a continuum of privately informed agents (see, among others, Acemoglu, Golosov, and Tsyvinski (2010), Farhi, Sleet, Werning, and Yeltekin (2012), and Scheuer and Wolitzky (2016)).

There is also a connection between our paper and the literature on durable goods monopoly under limited commitment. Ausubel and Deneckere (1989) study a model in which the seller posts prices and obtain a folk theorem for the "no gap" case. In our context a folk theorem holds when rehiring is possible. Skreta (2006, 2015) analyzes more general selling mechanisms and shows that posting a price is the seller's optimal strategy. Of course, in these studies the relationship between the buyer and the seller ends as soon as the durable good is traded, while in our model the parties can make a new transaction in every period.

Finally, a number of authors have identified situations in which the ratchet effect is mitigated. Kanemoto and MacLeod (1992) argue that competition for secondhand workers guarantees the existence of efficient piece-rate contracts in long-term relationships. Carmichael and MacLeod (2000) show that if entry in a market is difficult, then it is possible to sustain cooperation between an infinitely lived firm and a stream of short lived worker. Our findings suggest that rehiring is another possible remedy to the ratchet effect.

The rest of the paper is organized as follows. We present the model in Section 2. In Section 3, we briefly discuss the mechanism design problem with commitment. In Section 4 we show existence of equilibria and provide conditions under which all private information is revealed. Section 5 contains the main result which completely characterizes the unique equilibrium outcome when the parties are arbitrarily patient. In Section 6, we analyze the extension of the model in which rehiring is possible. Section 7 concludes. Most proofs are relegated to a number of appendices.

2 The Model

We study a dynamic principal-agent model with adverse selection and short-term contracts. We interpret the model as the relationship between a firm and a worker.

The worker has private information about his (persistent) type, which is equal to L with prior probability $p_0 \in (0,1)$, and equal to H with probability $1-p_0$. The firm and the worker interact for potentially infinitely many periods. In each period, the worker of type $i = H, L$ can produce a good of quality $q \in [0,1]$ at the cost $\theta_i q$, where $0 < \theta_L < \theta_H$. We refer to the low type L (high type H) as the low (high) cost worker. We write $\Delta\theta := \theta_H - \theta_L$. The worker bears an additional cost $\alpha \geqslant 0$ in every period in which he interacts with the firm. The cost α can be interpreted as the per-period payoff of an outside option available to the worker if he ends the relationship.

The firm's valuation of a good of quality q is $v(q)$. The function $v : [0,1] \to \mathbb{R}_+$ is twice continuously differentiable, increasing, strictly concave, and satisfies $v(0) = 0$.[4]

Both parties' preferences are linear in money. When the worker produces a good of quality q and the firm makes a transfer equal to x, the payoff of type $i = H, L$ is $x - \theta_i q - \alpha$, while the firm's payoff is $v(q) - x$.

We let q_i^*, $i = H, L$, denote the efficient quality produced by type i:

$$q_i^* = \arg\max_{q \in [0,1]} \left(v(q) - \theta_i q \right)$$

To make the problem interesting, we assume

$$v(q_H^*) - \theta_H q_H^* - \alpha > 0$$

This assumption guarantees that the firm prefers hiring the high-cost worker over collecting its outside option, which yields a payoff equal to zero. Moreover, we assume that $q_H^* \in (0,1)$ and, therefore $q_L^* > q_H^*$.[5] In this case, the efficient allocation varies with the worker's type.

The firm and the worker play the following game. At the beginning of period $t = 0, 1, \ldots$, the firm offers a menu m_t of contracts to the worker. Each contract is of the form (x_t, q_t) and specifies the transfer x_t paid by the firm and the quality $q_t \in [0,1]$ that the worker must produce. We assume that the quality is verifiable and, thus, each contract is enforceable. After receiving the menu m_t, the worker has two options: (i) selecting a contract from the menu; (ii) rejecting all the contracts and quitting the relationship. In the first case, the game moves to the next period $t + 1$. In the second case, the game ends and both parties obtain a continuation payoff equal to zero. The parties discount future payoffs at the common discount factor $\delta \in (0,1)$.

We let $\mathcal{M} = \bigcup_{j=1}^{M} (\mathbb{R} \times [0,1])^j$ denote the set of available menus, where $M \in \{2, 3, \ldots\}$ is the exogenous largest number of contracts that a menu can contain. The restriction

[4] The concavity of $v(\cdot)$ guarantees that the firm's screening problem in the proof of Proposition 1 is well behaved. The concavity also allows us to derive a number of useful bounds in the proof of Proposition 2. Finally, the assumption $v'(0) < \infty$ implies that for large values of the prior, the solution to the mechanism design problem with commitment is to fire the high type (see Section 3). This is used in the proof of the main result.

[5] In particular, we use this assumption in the proof of Proposition 1 to construct a sequence of separating contracts.

$M \geqslant 2$ guarantees that the menus can contain two contracts (so that it is possible for the firm to separate the two types of worker). When the firm offers the menu m_t, the set of actions available to the worker is $m_t \cup \{\emptyset\}$, where $\emptyset$ denotes the choice of rejecting all the contracts in m_t and quitting. We let a_t denote the agent's decision in period t.

For every $t = 1, 2, \ldots$, a period-t (non-final) public history $h^t = (m_0, a_0, \ldots, m_{t-1}, a_{t-1})$ consists of all the menus offered by the firm in previous periods $\tau = 0, \ldots, t-1$, as well as all the worker's decisions, provided that he never chose to quit (i.e., $a_\tau \neq \emptyset$ for every $\tau = 0, \ldots, t-1$). We let $H^0 = \{h^0\}$ denote the set containing the empty history h^0. We write H^t for the set of all period-t public histories. Finally, $\mathcal{H} = \cup_{t=0,1,\ldots} H^t$ is set of all (non-final) public histories.

A behavior strategy σ^F for the firm is a sequence $\left\{\sigma_t^F\right\}$, where σ_t^F is a function from H^t into $\Delta(\mathcal{M})$, mapping the history h^t into a (possibly random) menu. A behavior strategy $\left(\sigma^H, \sigma^L\right)$ for the worker is a sequence $\left\{\left(\sigma_t^H, \sigma_t^L\right)\right\}$, where σ_t^i, $i = H, L$, associates to every pair $(h^t, m_t) \in H^t \times \mathcal{M}$ a probability distribution over the set $m_t \cup \{\emptyset\}$. We write $\sigma = \left(\sigma^F, \sigma^H, \sigma^L\right)$ for a strategy profile. Finally, we let $\mu = \{\mu(h^t), \mu(h^t, m_t)\}_{h^t \in \mathcal{H}, m_t \in \mathcal{M}}$ denote the firm's system of beliefs, where $\mu(h^t)$ $(\mu(h^t, m_t))$ represents the probability that the firm assigns, at the history h^t (h^t, m_t), to the event that the worker's type is equal to L.

Our solution concept is perfect Bayesian equilibrium (PBE or equilibrium henceforth), formally defined below.

Definition 1 *A PBE of our game is a strategy profile σ and a system of beliefs μ such that:*

i) σ is sequentially rational given μ;

ii) for every history $(h^t, m_t) \in H^t \times \mathcal{M}$, $\mu(h^t, m_t) = \mu(h^t)$;

iii) for every $(h^t, m_t) \in H^t \times \mathcal{M}$ and for every $a_t \in m_t \cup \{\emptyset\}$, if

$$\left(1 - \mu\left(h^t\right)\right) \sigma_t^H\left(a_t | h^t, m_t\right) + \mu\left(h^t\right) \sigma_t^L\left(a_t | h^t, m_t\right) > 0$$

then the belief $\mu(h^t, m_t, a_t)$ is derived from $\mu(h^t)$ according to Bayes' rule:

$$\mu\left(h^t, m_t, a_t\right) = \frac{\mu(h^t)\, \sigma_t^L(a_t | h^t, m_t)}{(1 - \mu(h^t))\, \sigma_t^H(a_t | h^t, m_t) + \mu(h^t)\, \sigma_t^L(a_t | h^t, m_t)}$$

In addition to sequential rationality and Bayesian updating whenever possible (i.e., including off-path histories (h^t, m_t, a_t) that are reached with positive probability given

(h^t, m_t)), the concept of PBE imposes the "no signaling what you don't know" condition (Fudenberg and Tirole (1991)) in the sense that the firm does not revise its belief after proposing a menu.

Given a strategy profile σ and a system of beliefs μ, for each history h^t we let $V_F(h^t; (\sigma, \mu))$ denote the firm's continuation payoff at h^t. We also let $\mathbb{T} \in \mathbf{N} \cup \{\infty\}$ denote the random period in which the relationship terminates (we set $\mathbb{T} = \infty$ if the worker remains employed forever). Then we have:

$$V_F(h^t; (\sigma, \mu)) := \mathbb{E}_{(\sigma,\mu)} \left[(1-\delta) \sum_{\tau=t}^{\mathbb{T}-1} \delta^{\tau-t} \left(v(q_\tau) - x_\tau \right) \mid h^t \right]$$

where $\mathbb{E}_{(\sigma,\mu)}[Y|h^t]$ represents the conditional expected value (given h^t) of the random variable Y given the strategy profile σ and the system of beliefs μ. Analogously, for every history h^t we let $W_i(h^t; (\sigma, \mu))$ denote the expected continuation payoff at h^t of the worker of type $i = H, L$. We have:

$$W_i \left(h^t; (\sigma, \mu) \right) := \mathbb{E}_{(\sigma,\mu)} \left[(1-\delta) \sum_{\tau=t}^{\mathbb{T}-1} \delta^{\tau-t} \left(x_\tau - \theta_i q_\tau - \alpha \right) \mid i, h^t \right]$$

Here and in what follows, we use $\mathbf{N} = \{0, 1, \ldots\}$ to denote the set of nonnegative integers and adopt the convention that $\sum_{\tau=t}^{t-1} \delta^{\tau-t} = 0$.

To simplify the notation, we omit the argument (σ, μ) and write $V_F(h^t)$ and $W_i(h^t)$ when there is no ambiguity. We also use $V_F(h^t, m_t)$ and $W_i(h^t, m_t)$, $i = H, L$, to denote the firm and worker's payoff at the history (h_t, m_t).

For $i = H, L$, and $q \in [0, 1]$, we let

$$\pi_i (q) := v (q) - \theta_i q - \alpha$$

denote the firm's profits when the quality is q, the worker is of type i and the firm pays the reservation wage $\theta_i q + \alpha$. Therefore, $\pi_i (q_i^*)$ represents the highest level of profits that the firm can achieve from the interaction with type i. Clearly, $\pi_L (q_L^*) > \pi_H (q_H^*)$, and we let $\hat{p} \in (0, 1)$ be defined by $\pi_H (q_H^*) = \hat{p} \pi_L (q_L^*)$.

We conclude this section with a simple result that provides a lower bound to the firm's payoff under any PBE.

Lemma 1 *Fix a PBE (σ, μ). For every history $h^t \in \mathcal{H}$, we have:*

$$V_F(h^t; (\sigma, \mu)) \geqslant \max \left\{ \pi_H (q_H^*), \mu \left(h^t \right) \pi_L (q_L^*) \right\}$$

Proof of Lemma 1.

By contradiction, suppose that there exist a PBE (σ, μ), a history h^t, and $\varepsilon > 0$ such that

$$V_F(h^t; (\sigma, \mu)) < \max\left\{\pi_H\left(q_H^*\right), \mu\left(h^t\right)\pi_L\left(q_L^*\right)\right\} - \varepsilon$$

Suppose that $\pi_H\left(q_H^*\right) > \mu\left(h^t\right)\pi_L\left(q_L^*\right)$. If the firm offers the menu $\left\{\left(\theta_H q_H^* + \alpha + \frac{\varepsilon}{2}, q_H^*\right)\right\}$ in every period $t, t+1, \ldots$ (notice that both types strictly prefer to accept the contract in the menu rather than quit the relationship), then its continuation payoff will be equal to

$$\pi_H\left(q_H^*\right) - \frac{\varepsilon}{2} > V_F(h^t; (\sigma, \mu))$$

which is a contradiction.

Similarly, if $\pi_H\left(q_H^*\right) \leqslant \mu\left(h^t\right)\pi_L\left(q_L^*\right)$, the firm can guarantee a continuation payoff at least equal to

$$\mu\left(h^t\right)\left(\pi_L\left(q_L^*\right) - \frac{\varepsilon}{2}\right) > V_F(h^t; (\sigma, \mu))$$

by offering the menu $\left\{\left(\theta_L q_L^* + \alpha + \frac{\varepsilon}{2}, q_L^*\right)\right\}$ in every period $t, t+1, \ldots$ (in equilibrium, the low type must accept the contract in the menu). ■

Intuitively, the following two options are always available to the firm. The first option is to stop learning and offer $\left(\theta_H q_H^* + \alpha, q_H^*\right)$, the most profitable contract in the class of contracts that are accepted by both types of worker. The second option is to fire the high-cost worker and interact only with the low-cost worker. In this case, the most profitable contract is $\left(\theta_L q_L^* + \alpha, q_L^*\right)$.

3 The Commitment Allocation

It is useful to start the analysis by quickly reviewing the benchmark model in which the firm can fully commit to a sequence of menus $(m_0, m_1, \ldots)$. This provides an upper bound to the firm's profits in the game with limited commitment. It is well known that the solution to the firm's commitment problem is to replicate the optimal static mechanism (see, for example, Chapter 1 in Laffont and Tirole, 1993).

The optimal static mechanism takes two slightly different forms depending on whether $\alpha = 0$ or $\alpha > 0$. First, assume that $\alpha = 0$. In this case, there exists a critical value $p^C \in (\hat{p}, 1)$ such that if the prior p_0 is weakly larger than p^C, then the optimal menu (with commitment) is unique and equal to $\{(\theta_L q_L^* + \alpha, q_L^*)\}$.[6] The low-cost worker accepts the contract in the menu while the high-cost worker rejects it. Thus, the firm's profits are equal to $p_0 \pi_L(q_L^*)$.

On the other hand, if $p_0 < p^C$, then the unique optimal menu is

$$\left\{\left(x_H^C, q_H^C\right), \left(x_L^C, q_L^C\right)\right\} = \left\{\left(\theta_H q_H^C + \alpha, q_H^C\right), \left(\theta_L q_L^* + \Delta\theta q_H^C + \alpha, q_L^*\right)\right\} \tag{1}$$

for some $q_H^C \in (0, q_H^*)$. The high-cost worker accepts the first contract and obtains a payoff equal to zero. The low-cost worker is indifferent between the two contracts (therefore, he obtains a payoff equal to $\Delta\theta q_H^C$) and accepts the second contract. In this case, the firm's commitment profits are equal to:

$$p_0\left[v(q_L^*) - \theta_L q_L^* - \Delta\theta q_H^C - \alpha\right] + (1 - p_0)\left[v\left(q_H^C\right) - \theta_H q_H^C - \alpha\right].$$

We now turn to the case $\alpha > 0$. As in the first case, there exists a critical value of the prior $p^C \in (0, 1)$. If $p_0 > p^C$ the optimal mechanism is unique and equal to $\{(\theta_L q_L^* + \alpha, q_L^*)\}$. If $p_0 < p^C$, the unique optimal menu is $\left\{\left(x_H^C, q_H^C\right), \left(x_L^C, q_L^C\right)\right\}$ as in equation (1). Finally, if $p_0 = p^C$, then there are two optimal deterministic mechanisms: $\{(\theta_L q_L^* + \alpha, q_L^*)\}$ and $\left\{\left(x_H^C, q_H^C\right), \left(x_L^C, q_L^C\right)\right\}$ as in equation (1). In addition, if $p_0 = p^C$ there is a continuum of optimal random mechanisms, since any randomization between the two optimal deterministic mechanism is also an optimal mechanism.

Suppose that $p_0 < p^C$ or that $p_0 = p^C$ and $\alpha > 0$. It is immediate to see that in the dynamic game with limited commitment it is impossible to implement, in every period, the optimal mechanism of the form $\left\{\left(x_H^C, q_H^C\right), \left(x_L^C, q_L^C\right)\right\}$. This is because, according to Lemma 1, the firm's continuation payoff must be equal to $\pi_i(q_i^*)$ as soon as the firm discovers that the worker is of type i.[7]

It is also easy to see that for $p_0 \geqslant p^C$, the firm's payoff in any PBE must be equal to $p_0 \pi_L(q_L^*)$ (it cannot be smaller because of Lemma 1, and it cannot be larger because $\{(\theta_L q_L^* + \alpha, q_L^*)\}$ is an optimal mechanism with commitment). Therefore, if $p_0 \geqslant p^C$ all PBE share the following feature. The high type quits the relationship in the first period, while the low type accepts the contract $(\theta_L q_L^* + \alpha, q_L^*)$ in every period.

[6] To see why $p^C > \hat{p}$, let $V_F^C(p)$ be the commitment payoff of the firm when the prior is p. It is straightforward to show that $V_F^C(\cdot)$ is strictly increasing. If $p^C \leqslant \hat{p}$, then we obtain the following contradiction: $V_F^C(p^C) = p^C \pi_L(q_L^*) \leqslant \hat{p}\pi_L(q_L^*) = \pi_H(q_H^*) = V_F^C(0)$.

[7] Notice that $\pi_H\left(q_H^C\right) < \pi_H(q_H^*)$ since $q_H^C < q_H^*$. Also $v(q_L^*) - \theta_L q_L^* - \alpha - \Delta\theta q_H^C < \pi_L(q_L^*)$ since $q_H^C > 0$.

4 Existence and Learning

In this section, we show the existence of PBE for generic values of the parameters. We also identify the conditions under which the firm is able to learn the worker's type (with and without firing). In particular, if the parties are impatient, then learning is possible for any prior. In contrast, if the parties are sufficiently patient, then learning takes place only when the firm is willing to fire the high-cost worker.

We start with a general result that holds in every PBE: the low-cost worker's relationship with the firm lasts forever. Formally, we have the result below. We say that a certain property holds for almost all the menus offered by the firm at h^t if $\sigma_t^F(h^t)$ assigns probability one to the set of menus satisfying the property.

Lemma 2 *Fix a PBE* (σ, μ) *and an arbitrary history* h^t*. For almost all the menus* m_t *offered by the firm at* h^t*, we have*

$$\sum_{(x_t,q_t)\in m_t} \sigma_t^L\left((x_t,q_t)\,|h^t, m_t\right) = 1$$

To see why Lemma 2 is true, suppose that there are a PBE (σ, μ) and a history h^t at which the low type rejects all the contracts in the firm's menu with positive probability. This implies that the interaction with the high type must yield a strictly positive continuation payoff to the firm, otherwise its continuation payoff would be strictly smaller than $\mu(h^t)\,\pi_L(q_L^*)$, contradicting Lemma 1. Clearly, a strictly positive continuation payoff is possible only if the high type is expected to deliver a strictly positive (discounted) quality in the future. This and the individual rationality of the high type's behavior imply that the low type's decision to quit is not optimal, as he can guarantee a strictly positive payoff by imitating the high-cost worker at h^t and in every future period.

Suppose that the firm is interested in separating the two types and learning the worker's cost. This requires the existence of two decisions, one of which is taken only by the high type while the other is taken only by the low type. After observing the first (second) decision, the firm becomes convinced that the worker's cost is high (low).

In the light of Lemma 2, there are two ways in which separation can take place in equilibrium. One possibility is *separation with firing*: the high type quits the relationship and one of the contracts in the firm's menu is accepted only the low type. The other possibility is *separation with employment*: one contract in the firm's menu is accepted only by the high type, while another contract is accepted only by the low type.

Our next result shows that separation with employment cannot occur for large values of the discount factor.

Lemma 3 *Suppose that $\delta > \hat{\delta} := \frac{1}{1+q_H^*}$ and let (σ, μ) be an arbitrary PBE of the game. It is impossible to find a history (h^t, m_t) (on or off-path) satisfying the following properties:*

i) $\mu(h^t) \in (0,1)$;

ii) there exists a contract (x_H, q_H) in m_t for which $\sigma_t^H((x_H, q_H)|h^t, m_t) > 0$ and $\sigma_t^L((x_H, q_H)|h^t, m_t) = 0$;

iii) there exists a contract (x_L, q_L) in m_t for which $\sigma_t^L((x_L, q_L)|h^t, m_t) > 0$ and $\sigma_t^H((x_L, q_L)|h^t, m_t) = 0$.

By contradiction, suppose that at h^t the belief is nondegenerate and the firm's menu contains a contract (x_i, q_i) that is accepted (with positive probability) only by the type $i = H, L$. Following the acceptance of this contract, the firm's belief will assign probability one to the type i.[8] Furthermore, in equilibrium, the type i will select the efficient contract $(\theta_i q_i^* + \alpha, q_i^*)$ in every period after t. It is then easy to see that if the discount factor is sufficiently large, it is impossible to find two contracts, (x_H, q_H) and (x_L, q_L), to satisfy the two incentive compatibility constraints. If $\delta > \hat{\delta}$, for any pair of contracts $((x_H, q_H), (x_L, q_L))$, either the low type prefers to imitate the high type (at h^t and in every future period), or the high type has an incentive to adopt the "take the money and run" strategy (i.e., the strategy of accepting the generous contract (x_L, q_L) and then quitting).

We are now ready to state the main result of this section, which establishes (generic) existence of PBE.

Proposition 1 *For generic values of the parameters, there exists a PBE.*

The proof of Proposition 1 (in Appendix B) shows how to construct a PBE for all values of δ outside a set of discount factors which can contain at most two elements (the values of these two elements depend on the primitives θ_H, θ_L, α, $v(\cdot)$).[9] For the remainder of the paper, we assume that the discount factor δ does not belong to this (possibly empty) set.

[8] Notice that in a PBE, the beliefs must satisfy this condition at all histories, including those that are off-path.

[9] Our formal argument does not cover the values of δ at which the mapping $V^1 : [0,1] \to \mathbb{R}$ defined in equation (18) (see Appendix B) satisfies simultaneously $V^1(0) = \pi_H(q_H^*)$ and $\partial_+ V^1(0) = 0$. We show that there can be at most two such values of δ.

The equilibrium that we construct satisfies a number of properties. First, the equilibrium is "almost Markovian" in the sense that the parties' behavior in period t depends on the firm's belief and their actions in period $t-1$ (the history up to period $t-2$ affects the behavior in period t only through the belief). Second, the high type plays a pure strategy and his equilibrium payoff is equal to zero. Third, the menu proposed by the firm (at any history) contains at most two contracts. Finally, the firm adopts a deterministic behavior at on-path histories.

We now introduce some definitions to illustrate our equilibrium. First, we say that there is a *pooling* allocation if the firm offers the menu $\{(\theta_H q_H^* + \alpha, q_H^*)\}$ in every period and both types accept the contract $(\theta_H q_H^* + \alpha, q_H^*)$ (with probability one). We also say that there is a *firing* allocation if the firm offers the menu $\{(\theta_L q_L^* + \alpha, q_L^*)\}$ in every period, the high type quits in the first period, and the low type accepts the contract $(\theta_L q_L^* + \alpha, q_L^*)$ in every period. Finally, we say there is a *sequentially screening* allocation if the firm offers a menu with two contracts in every period in which its belief is nondegenerate. Furthermore, the high type accepts the first contract with probability one, while the low type accepts the second contract with strictly positive probability (if this probability is less than one, the low type randomizes between the two contracts). Therefore, in a sequentially screening allocation either the firm learns that the worker's type is low or it becomes more confident that the worker's type is high.

To illustrate our construction it is convenient to distinguish between the case $\delta \leqslant \hat{\delta}$ and the case $\delta > \hat{\delta}$. We start with the first case. We assume (without loss) that the firm offers the menu $\{(\theta_H q_H^* + \alpha, q_H^*)\}$ when the belief p is equal to zero. Also, the firm offers the menu $\{(\theta_L q_L^* + \alpha, q_L^*)\}$ when $p \geqslant p^C$. For any belief $p \in \left(0, p^C\right)$ we first give the following three options to the firm: i) offering a pooling menu, i.e. a menu with one contract that is accepted by both types; ii) offering a firing menu, i.e. a menu with one contract that induces separation with firing (i.e., the low type accepts the contract while the high type quits); iii) offering a menu with two contracts to induce separation with employment (this means that, with probability one, the two types choose different contracts).

Clearly, the optimal pooling menu is $\{(\theta_H q_H^* + \alpha, q_H^*)\}$, while the optimal firing menu is $\{(\theta_L q_L^* + \alpha, q_L^*)\}$. In case iii), we choose the two contracts to maximize the firm's payoff subject to the incentive compatibility (IC) and the individual rationality (IR) constraints. Notice that after separation with employment, the firm's belief is either zero or one. In both cases, the firm's behavior is known and we can compute the two types' continuation payoffs. As in the standard mechanism design problem with commitment, the optimal menu in case iii) is such that both the low type's IC constraint and the high type's IR constraint are binding.

We construct the firm's value function $V(\cdot;1)$ and the low type's payoff correspondence $\Phi(\cdot;1)$ when the firm is forced to choose one of the three options above.[10] We take $V(\cdot;1)$ and $\Phi(\cdot;1)$ as given and offer the firm the possibility of *probabilistic* separation with employment. This means that the firm offers two contracts. The high type accepts the first contract with probability one, while the low type randomizes between the contracts. After this round of probabilistic separation, the firm is again forced to use the three options above and, therefore, the parties continuation payoffs are given by $V(\cdot;1)$ and $\Phi(\cdot;1)$. In the probabilistic separation phase, we select the two contracts and the low type's behavior (i.e., the probability of accepting each contract) to maximize the firm's payoff subject, of course, to the IC and IR constraints. As usual, the solution to the optimization problem satisfies the low type's IC constraint with equality and, therefore, randomizing between the two contracts is indeed optimal for the low-cost worker.

The possibility of probabilistic separation defines a new value function $V(\cdot;2)$ and a new payoff correspondence $\Phi(\cdot;2)$. If $V(p;2) = V(p;1)$ for every $p \in [0,1]$, then we stop the process as the firm does not benefit from probabilistic separation. On the other hand, if $V(p;2) > V(p;1)$ (it is also easy to construct examples for which this is the case), then we allow for an additional round of probabilistic separation with employment.

We continue the process (allowing, at each iteration, for a new round of probabilistic separation) until we find a fixed point $(V(\cdot), \Phi(\cdot))$. We show that for generic values of the parameters a fixed point exists and is achieved after finitely many iterations. Moreover, our proof shows that if we fix the parameters $(\theta_H, \theta_L, \alpha, v(\cdot))$, then there is T such that for generic discount factors smaller than $1 - q_H^*/q_L^*$, the number of iterations is smaller than T (see Corollary 1 below for the implications of this result).

The pair $(V(\cdot), \Phi(\cdot))$ allows us to construct a simple equilibrium. For each belief p, the parties behave according to the solution of the firm's optimization problem (which yields the payoff $V(p)$ to the firm). The solution consists of the optimal menu and the worker's behavior. In particular, if the optimal menu is $\{(\theta_H q_H^* + \alpha, q_H^*)\}$, then both types accept the contract. If the optimal menu is $\{(\theta_L q_L^* + \alpha, q_L^*)\}$, then only the low type accepts the

[10] For some values of the beliefs p, the solution to the firm's problem is not unique and different solutions generally yield different payoffs to the low type (hence we use the correspondence $\Phi(\cdot;1)$).

contract. Finally, if the optimal menu contains two contracts, then the high type accepts the first contract (with probability one) while the low type accepts the second contract with probability in $(0,1]$ (this probability is part of the solution to the optimization problem).

The proof of Proposition 1 also specifies the parties' off-path behavior and shows that unilateral deviations are not profitable.

We conclude the discussion of the case of a low discount factor pointing out a property of our equilibria. Fix a PBE (σ,μ). We say that there is *full learning* by period t if for any $t' \geqslant t$ and for any *on-path* public history $h^{t'}$, the belief $\mu\left(h^{t'}\right)$ is either zero or one. This means that all the uncertainty about the worker's ability is resolved by period t.

Our construction shows that when the parties are not too patient, the firm never chooses the pooling allocation. Depending on the value of the prior, the firm prefers either the sequentially screening allocation (if the prior is low) or the firing allocation (if the prior is high). In both cases, there is full learning. Moreover, our proof shows that the number of periods until the worker completely reveals his private information is uniformly bounded.[11] Formally, we have the following result.

Corollary 1 *Fix the parameters $(\theta_H,\theta_L,\alpha,v(\cdot))$. There exists $T\in\{1,2,\ldots\}$ such that for any prior p_0 and for generic values of the discount factor smaller than $1-q_H^*/q_L^*$, there exists a PBE with full learning by period T.*

Finally, we point out that as δ shrinks to zero, the firm's equilibrium payoff converges to the payoff of the optimal mechanism with commitment. It is easy to check that this is a general property that holds for all PBE.[12]

We now turn to the case the case $\delta > \hat{\delta}$. Recall that in this case separation with employment is not feasible. Therefore, the firm is unable to implement a sequentially screening allocation. As a result, it chooses between the pooling and the firing allocation. The equilibrium that we construct takes a very simple form. If the prior is weakly larger than $\hat{p}$, the firm offers the optimal firing menu $\{(\theta_L q_L^*+\alpha, q_L^*)\}$ in every period and the high-cost worker quits in the first period. Thus, the equilibrium is with full learning by period one. In contrast, if the prior is smaller than $\hat{p}$, the firm offers the optimal pooling menu $\{(\theta_H q_H^*+\alpha, q_H^*)\}$ in every period and never updates (along the equilibrium path) its belief.

[11] This follows from the argument provided at the end of the proof of Lemma 9.

[12] For brevity, we omit the proof of this simple finding.

The analysis in this section shows that when the parties are sufficiently patient the ratchet effect has a strong impact on equilibrium behavior. In particular, it suggests that for sufficiently large values of δ, the firm can learn the worker's cost only by firing the high type. As we will see in the next section, this is not a special feature of our equilibrium but a much more general result.

5 Limit Uniqueness

In the last section, we constructed a very simple equilibrium for the case in which the parties are sufficiently patient ($\delta > \hat{\delta}$). This equilibrium implements the pooling allocation when the prior is smaller than $\hat{p}$ and the firing allocation when the prior is larger than $\hat{p}$. Our construction relies on the fact that a sequentially screening allocation is not feasible when the discount factor is above $\hat{\delta}$. However, the firm could, in principle, employ more complex dynamic screening strategies. For instance, one could imagine an equilibrium in which two or more contracts in the firm's menu are accepted with positive but different probabilities by the two types (in this case, the firm's belief could increase without jumping to one, as happens in a sequentially screening allocation). This raises the question of whether there are other equilibrium outcomes in addition to the one identified in Section 4. Moreover, can the firm do better than just offering the optimal pooling menu or the optimal firing menu?

We show that in the limit, as the parties become arbitrarily patient, there exists a unique equilibrium outcome. This outcome coincides with the equilibrium outcome in Section 4 (for the case $\delta > \hat{\delta}$). First, consider the case $p_0 > \hat{p}$. In the limit, as δ goes to one, the equilibrium allocation is firing and the high type quits the relationship without delay. In contrast, if $p_0 < \hat{p}$, the limiting equilibrium allocation is pooling and there is no learning.[13] Proposition 2 provides a formal characterization of the limiting outcome. Recall that $\mathbb{T}$ denotes the random time at which the worker quits the relationship.

Proposition 2 ***I)*** *Fix $p_0 > \hat{p}$ and consider a sequence of discount factors $\{\delta_n\}_{n=1}^{\infty}$ converging to one. For every $n = 1, 2, \ldots$, let (σ_n, μ_n) be a PBE of the game with discount factor δ_n. Then we have:*

[13] In the case in which the prior p_0 is equal to $\hat{p}$, the limiting equilibrium outcome is not uniquely pinned down as there are PBE implementing the pooling allocation, the firing allocation and convex combinations of such allocations.

i) $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[\delta_n^{\mathbb{T}} \mid H\right] = 1$;

ii) $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{t=0}^{\mathbb{T}-1} \delta_n^t \left|q_t - q_L^*\right| |L\right] = 0$;

iii) $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{t=0}^{\mathbb{T}-1} \delta_n^t \left(x_t - \theta_L q_L^* - \alpha\right) |L\right] = 0.$

II) *Fix $p_0 < \hat{p}$ and consider a sequence of discount factors $\{\delta_n\}_{n=1}^{\infty}$ converging to one. For every $n = 1, 2, \ldots,$ let (σ_n, μ_n) be a PBE of the game with discount factor δ_n. Then we have:*

i) $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[\delta_n^{\mathbb{T}}\right] = 0$;

ii) $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{t=0}^{\mathbb{T}-1} \delta_n^t \left|q_t - q_H^*\right|\right] = 0$;

iii) *For $i = H, L$,* $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{t=0}^{\mathbb{T}-1} \delta_n^t \left(x_t - \theta_H q_H^* - \alpha\right) |i\right] = 0.$

The rest of the section provides the proof of Proposition 2 which consists of several steps. We outline in detail each step and relegate some technical arguments to Appendix C. To simplify the exposition, in this section and in Appendix C we assume that $\alpha > 0$ (some steps of the proof are more involved when $\alpha = 0$). In Appendix D, we show how to modify the proof to deal with the case $\alpha = 0$.

Any equilibrium (σ, μ) must satisfy the following two properties (among others). First, at every history h^t the firm's continuation payoff must be at least equal to $\max\left\{\pi_H\left(q_H^*\right), \mu\left(h^t\right)\pi_L\left(q_L^*\right)\right\}$. Second, at any point of the relationship the low type must prefer to follow his strategy rather than mimicking the high type from that point onwards. We show that when the prior is above $\hat{p}$ and the discount factor δ is close to one, only the allocations that are close (according to the metric implicit in the statement of Proposition 2) to the firing allocation can simultaneously satisfy the two properties mentioned above. In contrast, if $p_0 < \hat{p}$ and δ is close to one, only the allocations that are close to the pooling allocation can satisfy the two properties.

We start with the following simple observation. To prove Proposition 2 it is enough to restrict attention to equilibria in which (i) the firm's strategy in the first period is pure (i.e., the firm does not randomize among different menus at $t = 0$); and (ii) the high type's equilibrium payoff is equal to zero.

To see why restriction (i) is without loss, suppose that $\left(\left(\sigma^F, \sigma^H, \sigma^L\right), \mu\right)$ is a PBE and m_0 is a menu offered with positive probability by the firm at $t = 0$ ($\sigma_0^F\left(m_0|h_0\right) > 0$).

Let $\tilde{\sigma}^F$ be the strategy which is identical to σ^F in every period except the first one in which the firm offers the menu m_0 with certainty ($\tilde{\sigma}_0^F(m_0|h_0) = 1$). It is immediate to see that $\left(\left(\tilde{\sigma}^F, \sigma^H, \sigma^L\right), \mu\right)$ is also a PBE. Furthermore, the outcome of the equilibrium $\left(\left(\tilde{\sigma}^F, \sigma^H, \sigma^L\right), \mu\right)$ coincides with the continuation outcome of $\left(\left(\sigma^F, \sigma^H, \sigma^L\right), \mu\right)$ after the firm proposes the contract m_0. Therefore, if there exists a sequence of equilibria which violate Proposition 2, then there also exists a sequence of equilibria which violate the proposition and satisfy restriction (i).

We now turn to restriction (ii). Suppose that (σ, μ) is a PBE in which the firm offers the menu m_0 (with probability one) in the first period and which yields a strictly positive payoff $W_H(h^0; (\sigma, \mu))$ to the high type. Then it is possible to construct a new PBE $(\tilde{\sigma}, \tilde{\mu})$ which is outcome equivalent to (σ, μ) except for the fact that the first-period transfers are uniformly decreased by $(1-\delta)^{-1} W_H(h^0; (\sigma, \mu))$. In other words, in the first period the firm replaces every contract (x_0, q_0) in the menu m_0 with the contract $\left(x_0 - (1-\delta)^{-1} W_H(h^0; (\sigma, \mu)), q_0\right)$.[14] Finally, notice that the first two results in Proposition 2 (Part I and Part II) do not depend on the the equilibrium transfers while the third result follows from the first two (for Part I, this is verified in the end of Section 5.1, while for Part II this is verified in the end of Appendix C).

The next result summarizes our initial findings.

Claim 1 *In the rest of the proof of Proposition 2, it is without loss of generality to restrict attention to equilibria* (σ, μ) *in which the firm's strategy in the first period is pure and* $W_H(h^0; (\sigma, \mu)) = 0$.

Consider a PBE which yields a zero payoff to the high type. If all the contracts accepted with positive probability by the high type take the form $(\theta_H q + \alpha, q)$, for some $q \in [0, 1]$, then we can express the parties' payoff in terms of the qualities delivered by the worker and the analysis is somewhat simplified. In general, there is no guarantee that in equilibrium the high type breaks even with every contract when his initial payoff is equal to zero. However, our next result shows that it is still possible to express the parties' payoffs in terms of the qualities. In particular, we are interested in the payoffs of the firm and the

[14] In the new equilibrium $(\tilde{\sigma}, \tilde{\mu})$, each type of the worker accepts the contract $\left(x_0 - (1-\delta)^{-1} W_H(h^0; (\sigma, \mu)), q_0\right)$ with the same probability with which he accepts the contract (x_0, q_0) in the original equilibrium (σ, μ).

low type, both when he follows his strategy and when he mimics the high type (recall the two properties above that any equilibrium must satisfy).

Lemma 4 *Fix a PBE* (σ, μ) *and let* h^t *be a history such that* $\mu(h^t) < 1$ *and* $W_H(h^t; (\sigma, \mu)) = 0$. *Fix* $p \in (\mu(h^t), 1)$ *and let* $\tilde{\mathbb{T}} \in \mathbf{N} \cup \{\infty\}$ *denote the random time that stops the play at the first history* $\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)$ *at which the menu* $m_{\tilde{\mathbb{T}}}$ *contains a contract* $(x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})$ *accepted with positive probability and for which* $\mu\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}, (x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})\right) \geqslant p$ *(we set* $\tilde{\mathbb{T}} = \infty$ *if the event does not occur in finite time). Then we have:*

$$V_F\left(h^t\right) = \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\tilde{\mathbb{T}}-1}\delta^{\tau-t}\pi_H(q_\tau) + \mathbb{I}_{\{\tilde{\mathbb{T}}<\infty\}}\delta^{\tilde{\mathbb{T}}-t}\left(V_F\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right) + W_H\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)\right)|h^t\right]$$

$$W_L\left(h^t\right) = \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\tilde{\mathbb{T}}-1}\delta^{\tau-t}\Delta\theta q_\tau + \mathbb{I}_{\{\tilde{\mathbb{T}}<\infty\}}\delta^{\tilde{\mathbb{T}}-t}\left(W_L\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right) - W_H\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)\right)|h^t, L\right]$$

$$W_{LH}\left(h^t\right) \geqslant \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\tilde{\mathbb{T}}-1}\delta^{\tau-t}\Delta\theta q_\tau - \mathbb{I}_{\{\tilde{\mathbb{T}}<\infty\}}\delta^{\tilde{\mathbb{T}}-t}W_H\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)|h^t, H\right]$$

where $W_{LH}(h^t)$ *denotes the low type's continuation payoff at* h^t *if he mimics the high type (at* h^t *and at any history that follows).*

The argument behind this result is simple and consists in a change in the timing of transfers. To see how this works, consider a history h^t at which $W_H(h^t) = 0$. Let m_t be a menu offered (with positive probability) by the firm at h^t and let (x_t, q_t) be a contract accepted by the high type. Let h^{t+1} denote the history $(h^t, m_t, (x_t, q_t))$. If $W_H(h^{t+1}) = 0$, then we clearly have $x_t = \theta_H q_t + \alpha$. If instead $W_H(h^t, m_t, (x_t, q_t)) > 0$, then we increase the transfer x_t by the amount $\frac{\delta}{(1-\delta)}W_H(h^{t+1})$. Clearly, the new transfer is equal to $\theta_H q_t + \alpha$. At the same time, for every menu m_{t+1} offered at h^{t+1}, we decrease all the transfers of the contracts in m_{t+1} by the amount $\frac{1}{(1-\delta)}W_H(h^{t+1}, m_{t+1})$. These changes leave the parties' continuation payoffs unchanged. We repeat this procedure in period $t+1, \ldots, \tilde{\mathbb{T}}-1$.

5.1 High Belief Case: $p > \hat{p}$

We proceed with the analysis of the game when the prior is above $\hat{p}$. Recall that that when the prior belongs to $\left[p^C, 1\right]$ the unique equilibrium outcome is the firing allocation. Our goal is to show that this result extends in the limit, as δ goes to one, if $p_0 > \hat{p}$. The statement of Proposition 2 defines the notion of closeness to the firing allocation. A related and useful notion of closeness is brought by our next definition, where we define firing regions. When the firm's belief falls in a firing region, both the expected discounted length of the firm's relationship with the high type as well as the low type's continuation payoff vanish as the parties become arbitrarily patient.

Definition 2 *The interval $[p, 1]$ is a firing region if there exist $\bar{K} > 0$ and $\bar{\delta} < 1$ such that the following holds. Fix $\delta > \bar{\delta}$ and an arbitrary PBE (σ, μ). Consider a history h^t at which $\mu(h^t) \geqslant p$. Then we have:*

i) $\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}|h^t, H\right]$, the expected discounted time until the high type quits the relationship, is bounded by $\bar{K}(1-\delta)$;

ii) $V_F(h^t;(\sigma,\mu), H)$, the firm's continuation payoff at the history h^t conditional on type H, is bounded by $\bar{K}(1-\delta)$;

iii) $W_L(h^t;(\sigma,\mu))$, the low type's continuation payoff at the history h^t, is bounded by $\bar{K}(1-\delta)$.

Our next result bounds the expected length of the relationship and the parties' payoff when the firm's menu contains a contract that leads to a firing region.

Lemma 5 *Suppose that $[p, 1]$ is a firing region. There exist $K > 0$ and $\tilde{\delta} < 1$ such that for every $\delta > \tilde{\delta}$ the following holds. Let (σ, μ) be a PBE and consider an arbitrary history h^t with $\mu(h^t) < p$. Suppose that at h^t the firm offers a menu m_t containing a contract $\left(x_t^L, q_t^L\right)$ accepted with positive probability and for which*

$$\mu\left(h^t, m_t, \left(x_t^L, q_t^L\right)\right) \geqslant p$$

Then we have:

i) $\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}|h^t, m_t, H\right]$, the expected discounted time until the high type quits the relationship, is bounded by $K(1-\delta)$;

ii) $V_F(h^t, m_t;(\sigma,\mu), H)$, the firm's continuation payoff at the history (h^t, m_t) conditional on type H, is bounded by $K(1-\delta)$;

iii) $W_L(h^t, m_t;(\sigma,\mu))$, the low type's continuation payoff at the history (h^t, m_t), is bounded by $K(1-\delta)$.

This result is closely connected to Lemma 3 which establishes that separation with employment cannot occur for large values of the discount factor. In fact, following the acceptance of a contract $\left(x_t^L, q_t^L\right)$ that leads to the firing region, the low type's continuation payoff is close to zero. Suppose the firm's relationship with the high type is long lasting. In this case, only a large transfer x_t^L can prevent the low type from mimicking the high type. But then it becomes profitable for the high type to accept the contract $\left(x_t^L, q_t^L\right)$ and then quit.

We are now ready to state the key result for the region of high beliefs.

Lemma 6 *For every* $\varepsilon \in (0, 1-\hat{p})$ *the interval* $[\hat{p}+\varepsilon, 1]$ *is a firing region.*

Proof of Lemma 6.

For every $p \geqslant \hat{p}$ let $f(p) \in [0, p-\hat{p}]$ be defined by

$$\frac{f(p)}{p}\pi'_H(0) + \left(1 - \frac{f(p)}{p}\right)\pi_H(q_H^*) = (p - f(p))\,\pi_L(q_L^*) \tag{2}$$

It is easy to check that the function $f : [\hat{p}, 1] \to [0, 1-\hat{p}]$ is strictly increasing and satisfies $f(\hat{p}) = 0$.[15]

The proof is by induction. We set $p(1) = p^C$ and $p(n) = p(n-1) - \frac{f(p(n-1))}{2}$ for $n = 2, 3, \ldots$. Clearly, the interval $[p(1), 1]$ is a firing region. We now prove the inductive step.

Claim 2 *Suppose that the interval* $[p, 1]$, $p \in (\hat{p}, 1)$, *is a firing region. Then* $\left[p - \frac{f(p)}{2}, 1\right]$ *is also a firing region.*

Fix δ and a PBE (σ, μ) and assume, without loss of generality (see Claim 1), that $W_H(h^0) = 0$. To simplify the exposition, we bound the expected length of the relationship (when the worker is of type H) and the parties' continuation payoffs at the empty history h^0. However, our bounds apply to any history h^t with $\mu(h^t) \in \left[p - \frac{f(p)}{2}, p\right)$.[16] It is convenient to start with the first property of a firing region and show that the expected discounted time until the high type quits is bounded by $K(1-\delta)$ (for δ large). We then use this result to establish the other two properties of a firing region.

[15] Recall that the function $\pi_H(\cdot)$ is concave and, therefore, $\pi'_H(0) > \pi'_H(0)\,q_H^* \geqslant \pi_H(q_H^*)$.

[16] This is because the continuation play starting at some history h^t is an equilibrium of the original game (when the prior is equal to the firm's belief at h^t).

We let $\tilde{\mathbb{T}} \in \mathbf{N} \cup \{\infty\}$ denote the random time that stops the play at the first history $\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)$ at which the menu $m_{\tilde{\mathbb{T}}}$ contains a contract $(x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})$ accepted with positive probability and for which $\mu\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}, (x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})\right) \geqslant p$.

Recall that $[p, 1]$ is a firing region and notice that properties i) and ii) in Lemma 5 immediately imply that the high type's continuation payoff at time $\tilde{\mathbb{T}}$ is close to zero when δ is large (in fact, $W_H\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)$ is bounded above by $W_L\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)$). Thus, it follows from Lemma 4 and Lemma 5 that there exist $K > 0$ and $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$ we can bound the parties' payoffs as follows:

$$
\begin{aligned}
V_F\left(h^0\right) \leqslant \bar{V}_F\left(h^0\right) &:= \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \pi_H\left(q_t\right) + \mathbb{I}_{\left\{\tilde{\mathbb{T}}<\infty\right\}}\delta^{\tilde{\mathbb{T}}}\mu\left(h^{\tilde{\mathbb{T}}}\right)\pi_L\left(q_L^*\right)\right] + K(1-\delta) \\
W_L\left(h^0\right) \leqslant \bar{W}_L\left(h^0\right) &:= \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \Delta\theta q_t | L\right] + K(1-\delta) \\
W_{LH}\left(h^0\right) \geqslant \underline{W}_{LH}\left(h^0\right) &:= \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \Delta\theta q_t | H\right] - K(1-\delta)
\end{aligned}
\tag{3}
$$

Furthermore, K and $\bar{\delta}$ are such that for every $\delta > \bar{\delta}$ the length of the high type's relationship is bounded as follows (this also follows from Lemma 5):

$$
\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t | h^0, H\right] \leqslant \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t | h^0, H\right] + K(1-\delta) \tag{4}
$$

Recall that for any equilibrium (σ, μ), $V_F\left(h^0\right)$ is bounded below by $p_0\pi_L\left(q_L^*\right)$ and $W_L\left(h^0\right)$ must be larger than $W_{LH}\left(h^0\right)$. This and the inequalities in (3) imply $\bar{V}_F\left(h^0\right) \geqslant p_0\pi_L\left(q_L^*\right)$ and $\bar{W}_L\left(h^0\right) \geqslant \underline{W}_{LH}\left(h^0\right)$. We now show that the last two inequalities can be simultaneously satisfied only if the expected discounted length of the high type's relationship shrinks to zero as δ goes to one. Formally, we will prove the following result.

Fact 1 *Fix $K > 0$ and $p > \hat{p}$. There exists $K' > 0$ such that, for every $p_0 \in \left[p - \frac{f(p)}{2}, p\right]$, for every δ, and for every PBE (σ, μ) (with $W_H\left(h^0; (\sigma, \mu)\right) = 0$) the following holds. Given K and (σ, μ) compute $\bar{V}_F\left(h^0\right), \bar{W}_L\left(h^0\right)$, and $\underline{W}_{LH}\left(h^0\right)$ as in (3). If $\bar{V}_F\left(h^0\right) \geqslant p_0\pi_L\left(q_L^*\right)$ and $\bar{W}_L\left(h^0\right) \geqslant \underline{W}_{LH}\left(h^0\right)$, then $\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t | h^0, H\right] \leqslant K'(1-\delta)$.*

To prove this fact, we apply mechanism-design and information-design techniques to our setting. We take a PBE (σ, μ) and construct a direct mediated mechanism that delivers the payoff $\bar{V}_F(h^0)$ to the firm, the payoff $\bar{W}_L(h^0)$ to the low type if he announces his type truthfully, and the payoff $\underline{W}_{LH}(h^0)$ to the low type if he lies about his type. The direct mechanism is as follows. The worker reveals his private information to a designer who, in turn, chooses an outcome and reports it to the firm. The outcome consists of a history $h^{\tilde{\mathbb{T}}}$ of the game and a message in $\{\mathfrak{m}_0, \mathfrak{m}_p\}$. In particular, if the worker announces the low type, then the designer chooses the outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ with probability $\Pr\left(h^{\tilde{\mathbb{T}}}|L\right)$, the probability of the history $h^{\tilde{\mathbb{T}}}$ when the worker's type is low and the parties play the equilibrium (σ, μ). On the other hand, if the worker announces the high type, then the designer chooses the outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ with probability $\Pr\left(h^{\tilde{\mathbb{T}}}|H\right)\frac{\mu\left(h^{\tilde{\mathbb{T}}}\right)(1-p)}{\left(1-\mu\left(h^{\tilde{\mathbb{T}}}\right)\right)p}$ and the outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_0\right)$ with probability $\Pr\left(h^{\tilde{\mathbb{T}}}|H\right)\left[1-\frac{\mu\left(h^{\tilde{\mathbb{T}}}\right)(1-p)}{\left(1-\mu\left(h^{\tilde{\mathbb{T}}}\right)\right)p}\right]$. The randomization between $\mathfrak{m}_p$ and $\mathfrak{m}_0$ is chosen in such a way that upon observing any outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ the firm's belief is equal to p. In fact, notice that

$$\frac{\Pr\left(L|h^{\tilde{\mathbb{T}}},\mathfrak{m}_p\right)}{\Pr\left(H|h^{\tilde{\mathbb{T}}},\mathfrak{m}_p\right)} = \frac{p_0\Pr\left(h^{\tilde{\mathbb{T}}}|L\right)}{(1-p_0)\Pr\left(h^{\tilde{\mathbb{T}}}|H\right)\frac{\mu\left(h^{\tilde{\mathbb{T}}}\right)(1-p)}{\left(1-\mu\left(h^{\tilde{\mathbb{T}}}\right)\right)p}} =$$

$$= \underbrace{\left(\frac{p_0\Pr\left(h^{\tilde{\mathbb{T}}}|L\right)}{(1-p_0)\Pr\left(h^{\tilde{\mathbb{T}}}|H\right)}\right)}_{\left(\frac{\mu\left(h^{\tilde{\mathbb{T}}}\right)}{1-\mu\left(h^{\tilde{\mathbb{T}}}\right)}\right)} \times \left(\frac{1-\mu\left(h^{\tilde{\mathbb{T}}}\right)}{\mu\left(h^{\tilde{\mathbb{T}}}\right)}\right) \times \left(\frac{p}{1-p}\right) = \frac{p}{1-p}$$

Clearly, upon observing any outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_0\right)$ the firm's belief is equal to zero.

We now turn to the payoffs of the firm and the low type. The firm's payoff depends only on the outcome and not on the message sent by worker to the designer. Consider an arbitrary history $h^{\tilde{\mathbb{T}}} = \left(m_0, (x_0, q_0), \ldots, m_{\tilde{\mathbb{T}}}, \left(x_{\tilde{\mathbb{T}}-1}, q_{\tilde{\mathbb{T}}-1}\right)\right)$. If the outcome is $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$, the firm's payoff is equal to:

$$(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t\pi_H(q_t) + \mathbb{I}_{\left\{\tilde{\mathbb{T}}<\infty\right\}}\delta^{\tilde{\mathbb{T}}}p\pi_L(q_L^*) + K(1-\delta)$$

If the outcome is $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_0\right)$, the firm's payoff is equal to:

$$(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \pi_H(q_t) + K(1-\delta)$$

It is immediate to check that if every type reveals his type truthfully, then the firm's expected payoff is equal to $\bar{V}_F(h^0)$.

Consider now the low type. His payoff depends both on the outcome and on the message that he sends to the designer. First, if the outcome is either $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ or $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_0\right)$, then the low type obtains a payoff equal to

$$(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \Delta\theta q_t$$

In addition, the low type obtains an extra payoff which is equal to $K(1-\delta)$ if he is honest, and equal to $-K(1-\delta)$ if he lies to the designer. It follows that the low type's expected payoff is equal to $\bar{W}_L(h^0)$ if he reveals his type truthfully, and equal to $\underline{W}_{LH}(h^0)$ if he lies to the mediator.

Recall that the low type's incentive compatibility constraint $\bar{W}_L(h^0) \geqslant \underline{W}_{LH}(h^0)$ is satisfied since it is slacker than the equilibrium constraint $W_L(h^0) \geqslant W_{LH}(h^0)$. Therefore, we say that the mechanism is incentive compatible (we assume that the high type is sincere).[17]

It is natural to ask why we introduced the messages $\mathfrak{m}_0$ and $\mathfrak{m}_p$ in the mechanism given that they do not affect the worker's payoffs and the firm is a passive player in this construction. In particular, there exists a payoff equivalent and incentive compatible mechanism in which the designer chooses only the history $h^{\tilde{\mathbb{T}}}$ (with probabilities that depend on the worker's report) and the firm's payoff is equal to

$$(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \pi_H(q_t) + \mathbb{I}_{\left\{\tilde{\mathbb{T}}<\infty\right\}}\delta^{\tilde{\mathbb{T}}}\mu\left(h^{\tilde{\mathbb{T}}}\right)\pi_L(q_L^*) + K(1-\delta)$$

The reason is that the additional messages allow us to classify all the histories $h^{\tilde{\mathbb{T}}}$ into two large classes, depending on whether they are associated to the message $\mathfrak{m}_0$ or to the

[17] We (weakly) enlarge the set of incentive compatible mechanisms by assuming sincere behavior of the high type.

message $\mathfrak{m}_p$. Recall that when the firm observes the message $\mathfrak{m}_z$, $z = 0, p$, its belief is equal to z. Thus, by the martingale property of the beliefs (see Aumann and Maschler (1995) and Kamenica and Gentzkow (2011)), we conclude that the probability of observing the message $\mathfrak{m}_0$ is equal to $\left(1 - \frac{p_0}{p}\right)$ while the probability of observing the message $\mathfrak{m}_p$ is equal $\frac{p_0}{p}$.

This allows us to rewrite the firm's payoff $\bar{V}_F\left(h^0\right)$ as follows:

$$\bar{V}_F\left(h^0\right) = \left(1 - \frac{p_0}{p}\right)\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \pi_H\left(q_t\right)|\mathfrak{m}_0\right] +$$

$$\frac{p_0}{p}\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \pi_H\left(q_t\right) + \mathbb{I}_{\left\{\tilde{\mathbb{T}}<\infty\right\}}\delta^{\tilde{\mathbb{T}}} p \pi_L\left(q_L^*\right)|\mathfrak{m}_p\right] + K\left(1-\delta\right)$$

We now consider the low type's incentive compatibility constraint. Fix an arbitrary outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ and let $\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ denote the ex-ante probability of the outcome. Recall that the firm's belief upon observing the outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ is equal to p. This immediately implies:

$$\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right) = \frac{p_0}{p}\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p|L\right) = \frac{1-p_0}{1-p}\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p|H\right)$$

We conclude that the outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ is reached with probability $\frac{p}{p_0}\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ when the worker announces that his type is low, and with probability $\frac{1-p}{1-p_0}\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_p\right)$ when the worker announces that his type is high.

Similarly, an outcome $\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_0\right)$ is reached with probability $\frac{1}{1-p_0}\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_0\right)$ if the worker announces the high type and with probability zero if the worker announces the low type ($\Pr\left(h^{\tilde{\mathbb{T}}}, \mathfrak{m}_0\right)$ denotes the ex-ante probability of the outcome).

Combining these observations, we can rewrite the low type's payoffs as follows:

$$\bar{W}_L\left(h^0\right) = \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \Delta\theta q_t|\mathfrak{m}_p\right] + K\left(1-\delta\right)$$

$$\underline{W}_{LH}\left(h^0\right) = \left(\tfrac{1}{1-p_0}\right)\left(1-\tfrac{p_0}{p}\right)\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t\Delta\theta q_t|\mathfrak{m}_0\right]+$$

$$\left(\tfrac{1-p}{1-p_0}\right)\left(\tfrac{p_0}{p}\right)\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t\Delta\theta q_t|\mathfrak{m}_p\right]-K(1-\delta)$$

We now construct an upper bound $\breve{V}_F\left(h^0\right)$ to $\bar{V}_F\left(h^0\right)$ and show that the expected discounted length of the high type's relationship is bounded by $K'(1-\delta)$ if $\breve{V}_F\left(h^0\right) \geqslant p_0\pi_L\left(q_L^*\right)$ and $\bar{W}_L\left(h^0\right) \geqslant \underline{W}_{LH}\left(h^0\right)$. Clearly, this will imply Fact 1.

For $z = 0, p$, we let

$$\Upsilon_z = \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t|\mathfrak{m}_z\right]$$

denote the expected discounted length of the relationship conditional on the message $\mathfrak{m}_z$, and let $\tilde{q}_z$ be defined by

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t q_t|\mathfrak{m}_z\right] = \Upsilon_z\tilde{q}_z$$

Using this, the concavity of the function $\pi_H(\cdot)$ and Jensen's inequality we obtain the desired bound on $\bar{V}_F\left(h^0\right)$:

$$\bar{V}_F\left(h^0\right) \leqslant \breve{V}_F\left(h^0\right) := \left(1-\tfrac{p_0}{p}\right)\Upsilon_0\pi_H\left(\tilde{q}_0\right)+\tfrac{p_0}{p}\left[\Upsilon_p\pi_H\left(\tilde{q}_p\right)+\left(1-\Upsilon_p\right)p\pi_L\left(q_L^*\right)\right]+K(1-\delta)$$

At the same time, we can express $\bar{W}_L\left(h^0\right)$ and $\underline{W}_{LH}\left(h^0\right)$ as:

$$\bar{W}_L\left(h^0\right) = \Upsilon_p\tilde{q}_p\Delta\theta+K(1-\delta)$$

$$\underline{W}_{LH}\left(h^0\right) = \left(\tfrac{1}{1-p_0}\right)\left(1-\tfrac{p_0}{p}\right)\Upsilon_0\tilde{q}_0\Delta\theta+\left(\tfrac{1-p}{1-p_0}\right)\left(\tfrac{p_0}{p}\right)\Upsilon_p\tilde{q}_p\Delta\theta-K(1-\delta)$$

Finally, using the definition of Υ_0 and Υ_p and inequality (4), we obtain the following bound to the length of the high type's relationship:

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t|h^0,H\right] \leqslant \left(\frac{1}{1-p_0}\right)\left(1-\frac{p_0}{p}\right)\Upsilon_0+\left(\frac{1-p}{1-p_0}\right)\left(\frac{p_0}{p}\right)\Upsilon_p+K(1-\delta)$$

The following claim concludes the proof of Fact 1.

Claim 3 *Fix $K > 0$ and $p > \hat{p}$. There exists $K' > 0$ such that, for every $p_0 \in \left[p - \frac{f(p)}{2}, p\right]$, for every δ, and for all $(\Upsilon_z, \tilde{q}_z) \in [0,1]^2$, $z = 0, p$, the inequalities*

$$\left(1 - \frac{p_0}{p}\right) \Upsilon_0 \pi_H (\tilde{q}_0) + \frac{p_0}{p} \left[\Upsilon_p \pi_H (\tilde{q}_p) + (1 - \Upsilon_p) p \pi_L (q_L^*)\right] + K (1 - \delta) \geqslant p_0 \pi_L (q_L^*) \quad (5)$$

$$\Upsilon_p \tilde{q}_p \Delta\theta + K (1 - \delta) \geqslant \left(\frac{1}{1 - p_0}\right) \left(1 - \frac{p_0}{p}\right) \Upsilon_0 \tilde{q}_0 \Delta\theta + \left(\frac{1 - p}{1 - p_0}\right) \left(\frac{p_0}{p}\right) \Upsilon_p \tilde{q}_p \Delta\theta - K (1 - \delta) \quad (6)$$

are simultaneously satisfied only if

$$\left(\frac{1}{1 - p_0}\right) \left(1 - \frac{p_0}{p}\right) \Upsilon_0 + \left(\frac{1 - p}{1 - p_0}\right) \left(\frac{p_0}{p}\right) \Upsilon_p + K (1 - \delta) \leqslant K' (1 - \delta) \quad (7)$$

Notice that inequalities (5) and (6) capture the constraints $\breve{V}_F (h^0) \geqslant p_0 \pi_L (q_L^*)$ and $\bar{W}_L (h^0) \geqslant \underline{W}_{LH} (h^0)$, while the left hand side of inequality (7) represents the upper bound to the expected discounted length of the high type's relationship.

The proof of Claim 3 is tedious and relegated to Appendix C. The logic behind this claim is better understood when one considers the problem of maximizing $\breve{V}_F (h^0)$ (with respect to Υ_z and $\tilde{q}_z$, $z = 0, p$) subject to the incentive compatibility constraint $\bar{W}_L (h^0) \geqslant \underline{W}_{LH} (h^0)$. Clearly, $\breve{V}_F (h^0)$ is maximized by setting Υ_0 equal to one, $\tilde{q}_0$ equal to q_H^*, and Υ_p equal to zero (recall that $p > \hat{p}$ and, therefore, $p\pi_L (q_L^*) > \pi_H (q_H^*) \geqslant \pi_H (\tilde{q}_p)$ for any $\tilde{q}_p$). However, this would violate the low type's incentive compatibility constraint. Hence the following trade-off emerges. To increase the firm's payoff by increasing Υ_0 and satisfying the incentive compatibility constraint (6) it is necessary to increase Υ_p as well, which decreases the firm's payoff. Notice that when the prior p_0 is close to p, $\left(1 - \frac{p_0}{p}\right) \Upsilon_0$ and $\tilde{q}_0$ have a small impact both on the firm's payoff and the constraint. In contrast, Υ_p has a small impact on the constraint, and a large (negative) impact on the firm's payoff. We conclude that for δ and p_0 sufficiently large, the optimal values of $\left(1 - \frac{p_0}{p}\right) \Upsilon_0$ and Υ_p must be close to zero. Therefore, if we could maximize the firms's payoff subject to $\bar{W}_L (h^0) \geqslant \underline{W}_{LH} (h^0)$ the solution would be close to a firing allocation, yielding a payoff close to $p_0 \pi_L (q_L^*)$. For the same reason, any allocation that satisfies $\bar{W}_L (h^0) \geqslant \underline{W}_{LH} (h^0)$ and that is not close to a firing allocation leads to a payoff for the firm smaller than $p_0 \pi_L (q_L^*)$, hence violating (5).

We have shown that the first property of a firing region holds: there exists $\breve{K} > 0$ such that $E_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t|h^0,H\right] \leq (1-\delta)\breve{K}$. We now turn to the remaining two properties. To verify the second property (the firm's payoff conditional on type H shrinks to zero weakly faster than $1-\delta$), notice that

$$V_F(h^0;H) \leqslant v(1)\,\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t|h^0,H\right] \leqslant v(1)\,\breve{K}(1-\delta)$$

Finally, we use the result above to bound the low type's continuation payoff $W_L(h^0)$ (third property). We have

$$p_0\pi_L(q_L^*) \leqslant V_F(h^0) \leqslant (1-p_0)\,V_F(h^0;H) + p_0\left(\pi_L(q_L^*) - W_L\left(h^0\right)\right)$$

which implies

$$W_L\left(h^0\right) \leqslant \frac{1-p_0}{p_0}V_F(h^0;H) < \frac{1-\hat{p}}{\hat{p}}V_F(h^0;H) \leqslant \frac{1-\hat{p}}{\hat{p}}v(1)\,\breve{K}(1-\delta)$$

This concludes the proof of Claim 2 and consequently of Lemma 6. ■

The argument above implies property i) of Part I of Proposition 2: $\lim_{n\to\infty} E_{(\sigma_n,\mu_n)}\left[\delta_n^{\mathbb{T}} \mid H\right] = 1$. Suppose that the rest of Part I of Proposition 2 does not hold. This means that either

$$\limsup_{n\to\infty} E_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{t=0}^{\mathbb{T}-1}\delta_n^t\,|q_t - q_L^*|\,|L\right] > 0$$

or (notice that the firm's payoff conditional on type L cannot be negative, otherwise the total payoff would be smaller than $\pi_H(q_H^*)$)

$$\limsup_{n\to\infty} E_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{t=0}^{\mathbb{T}-1}\delta_n^t\,(x_t - \theta_L q_L^* - \alpha)\,|L\right] > 0$$

Then it is possible to find a subsequence of equilibria for which the firm's payoff conditional on type L converges to a value smaller than $\pi_L(q_L^*)$. In light of property i), this would imply that the firm's limit payoff would be strictly smaller than $p_0\pi_L(q_L^*)$, a contradiction.

5.2 Low Belief Case: $p < \hat{p}$

We now turn to the case of a belief p smaller than $\hat{p}$. We start with a simple result which shows that the equilibrium belief cannot grow too quickly around $\hat{p}$ when the parties are sufficiently patient.

Lemma 7 *For every $\varepsilon > 0$ there exists $\bar{\delta} < 1$ satisfying the following. For every PBE (σ, μ) of a game in which $\delta > \bar{\delta}$, it is impossible to find a history h^t with $\mu(h^t) < \hat{p} - \varepsilon$ and at which the firm offers a menu m_t containing a contract (x_t, q_t) accepted with positive probability and such that $\mu(h^t, m_t, (x_t, q_t)) > \hat{p} + \varepsilon$.*

Recall that for every $\varepsilon > 0$, the interval $[\hat{p} + \varepsilon, 1]$ is a firing region. Therefore, it follows from Lemma 5 that if δ is close to one and the belief jumps from $\mu(h^t) < \hat{p} - \varepsilon$ to $\mu(h^t, m_t, (x_t, q_t)) > \hat{p} + \varepsilon$, the firm's continuation payoff at h^t must be close to $\mu(h^t)\pi_L(q_L^*)$. But then the firm's payoff would be smaller than $\pi_H(q_H^*)$ (since $\mu(h^t) < \hat{p} - \varepsilon$ and $\hat{p}\pi_L(q_L^*) = \pi_H(q_H^*)$), contradicting Lemma 1.

We now outline the proof of Part II of Proposition 2 (see Appendix C for the formal proof). Consider the first result which asserts that $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)}\left[\delta_n^{\mathbb{T}}\right] = 0$. By contradiction, let us assume that there exists a sequence $\{\delta_n, (\sigma_n, \mu_n)\}_{n=1}^{\infty}$ such that δ_n converges to one, (σ_n, μ_n) is a PBE of the game with discount factor equal to δ_n, and $\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)}\left[\delta_n^{\mathbb{T}}\right] = \xi > 0$. To ease the notation, in what follows we suppress the index n and write δ and (σ, μ) to denote an arbitrary element of the sequence. Without loss of generality (see Claim 1), we assume that the equilibrium (σ, μ) yields to the high type a payoff equal to zero.

Fix a small ε and let $\tilde{\mathbb{T}} \in \mathbf{N} \cup \{\infty\}$ denote the random time that stops the play at the first history $\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)$ at which the menu $m_{\tilde{\mathbb{T}}}$ contains a contract $(x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})$ accepted with positive probability and for which $\mu\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}, (x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})\right) \geqslant \hat{p} + \varepsilon$. It follows from Lemma 4, Lemma 5, and from the fact that $[\hat{p} + \varepsilon, 1]$ is a firing region that for δ close to one we can bound the firm's equilibrium payoff as follows:

$$V_F(h^0) \leqslant \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \pi_H(q_t) + \mathbb{I}_{\{\tilde{\mathbb{T}}<\infty\}}\delta^{\tilde{\mathbb{T}}}\mu\left(h^{\tilde{\mathbb{T}}}\right)\pi_L(q_L^*)\right] + \varepsilon \leqslant$$

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t \pi_H(q_t) + \mathbb{I}_{\{\tilde{\mathbb{T}}<\infty\}}\delta^{\tilde{\mathbb{T}}}(\hat{p}+\varepsilon)\pi_L(q_L^*)\right] + \varepsilon$$

where the second inequality holds because the belief at the history $h^{\tilde{\mathbb{T}}}$ is bounded above (by definition) by $\hat{p}+\varepsilon$.

Notice that we can take ε to be arbitrarily small. Therefore, since $\pi_H(q_H^*) = \hat{p}\pi_L(q_L^*)$, and q_H^* is the unique maximizer of $\pi_H(\cdot)$, the inequality above implies that

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t\left|q_H^*-q_t\right|\right]\approx 0 \tag{8}$$

for δ sufficiently large. If this were not the case, then the firm's payoff would be strictly smaller than $\pi_H(q_H^*)$ (again, for δ sufficiently large).

We now examine the relation between $\mathbb{E}_{(\sigma,\mu)}\left[\delta^{\mathbb{T}}\right]$ and $\mathbb{E}_{(\sigma,\mu)}\left[\delta^{\tilde{\mathbb{T}}}\right]$ when δ is close to one. First, at the history $\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)$ the expected discounted length of the relationship with the high type is close to zero (see Lemma 5 and recall that the interval $[\hat{p}+\varepsilon,1]$ is a firing region). Second, the firm's belief at $h^{\tilde{\mathbb{T}}}$ must be close to $\hat{p}$ (a value of $\mu\left(h^{\tilde{\mathbb{T}}}\right)$ far away from $\hat{p}$ would contradict Lemma 7). Finally, recall that for δ large, $\mathbb{E}_{(\sigma,\mu)}\left[\delta^{\mathbb{T}}\right]$ is close (by assumption) to ξ. Putting these observations together and using Bayes rule, we conclude that $\mathbb{E}_{(\sigma,\mu)}\left[\delta^{\tilde{\mathbb{T}}}\right]$ is close to $\frac{\xi}{1-\hat{p}}$ for δ close to one.

The last part of the proof analyzes the low type's incentives to follow the equilibrium strategy and derives a contradiction. Let $\Pr\left(h^{\tilde{\mathbb{T}}}\right)$ denote th (ex-ante) probability of reaching the history $h^{\tilde{\mathbb{T}}}$ and assume that the firm's belief $\mu\left(h^{\tilde{\mathbb{T}}}\right)$ is close to $\hat{p}$. Bayes' rule tells us that $h^{\tilde{\mathbb{T}}}$ is reached with probability close to $\frac{\hat{p}}{p_0}\Pr\left(h^{\tilde{\mathbb{T}}}\right)$ if the low type follows his strategy σ^L, and with probability close to $\frac{1-\hat{p}}{1-p_0}\Pr\left(h^{\tilde{\mathbb{T}}}\right)$ if he mimics the high type and plays the strategy σ^H.

Using the definition of $\tilde{\mathbb{T}}$ (and the fact that $[\hat{p}+\varepsilon,1]$ is a firing region) we can approximate the low type's payoffs as follows. For δ close to one, σ^L yields a payoff close to

$$\begin{gathered}\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t\Delta\theta q_t|L\right]\approx\Delta\theta q_H^*\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t|L\right]\approx\\ \Delta\theta q_H^*\left(1-\mathbb{E}_{(\sigma,\mu)}\left[\delta^{\tilde{\mathbb{T}}}|L\right]\right)\approx\Delta\theta q_H^*\left(1-\frac{\hat{p}}{p_0}\frac{\xi}{1-\hat{p}}\right)\end{gathered} \tag{9}$$

where the first relation follows from (8), while the third relation is a consequence of Bayes' rule.

Similarly, the strategy σ^H yields a payoff close to

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t\Delta\theta q_t|H\right]\approx\Delta\theta q_H^*\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\tilde{\mathbb{T}}-1}\delta^t|H\right]\approx \\ \Delta\theta q_H^*\left(1-\mathbb{E}_{(\sigma,\mu)}\left[\delta^{\tilde{\mathbb{T}}}|H\right]\right)\approx\Delta\theta q_H^*\left(1-\frac{1-\hat{p}}{1-p_0}\frac{\xi}{1-\hat{p}}\right) \tag{10}$$

Hence, since $p_0 < \hat{p}$ and $\xi > 0$, (9) is greater than (10), implying the existence of a profitable deviation for values of δ close to one.

Finally, the last two results in Proposition 2 Part II are direct consequences of the result above. Intuitively, if the high type never quits the relationship, the best option for the firm is to implement the pooling allocation.

6 Rehiring

In the model analyzed so far, the worker's decision to reject all the contracts in the menu is an irreversible action that ends the relationship. In other words, the firm cannot rehire the worker after a period of unemployment. As we showed above, this impairs the firm's ability to screen the worker. Once the worker reveals his type, his continuation payoff must be equal to zero. The firm can afford to pay the reservation wage because the worker has no other alternative than ending the relationship.

This logic does not apply when rehiring is possible. In this case, the worker can credibly threaten the firm to reject offers that pay slightly above the reservation wage because he expects to obtain a large payoff in the rest of the relationship. As we will see below, with rehiring it is possible to sustain equilibrium outcomes in which the worker's payoff remains strictly positive even when his type is known to the other party. This, in turn, make it easier for the firm to screen the worker.

There are different ways to break the automatic link (present in the benchmark model) between the decision to reject all the contracts and the decision to end the relationship. One possibility is to assume that the relationship lasts forever and quitting is now allowed. Another possibility is to add to the benchmark model the option for the worker to reject all the contracts and remain in the relationship. In the rest of the section, we analyze these extensions of the model.

We start with the infinitely repeated game in which, in each period, the firm proposes a menu of contracts. The worker either accepts a contract in the menu or rejects all of them. Both parties obtain a payoff equal to zero in each period in which the worker rejects all the contracts in menu.

Consider the standard mechanism design problem with commitment. We say that the payoffs $(V_{F,H}, V_{F,L}, W_H, W_L)$ are *incentive-compatible* and *ex-post strictly individually rational* if there exists an incentive compatible direct mechanism $\{(x_H, q_H), (x_L, q_L)\}$, $(x_i, q_i) \in \mathbb{R}_{++} \times [0,1]$ for $i = H, L$, satisfying:

i) For $i = H, L$, the firm's payoff $V_{F,i}$ when the worker is of type i is strictly positive: $V_{F,i} := v(q_i) - x_i > 0$;

ii) For $i = H, L$, type i's payoff is strictly positive: $W_i := x_i - \theta_i q_i - \alpha > 0$.

The main result of this section is a folk theorem. We show that any profile of incentive-compatible and ex-post strictly individual rational payoffs can be achieved in the infinitely repeated game when the parties are sufficiently patient.

Proposition 3 *For every tuple $(V_{F,H}, V_{F,L}, W_H, W_L) \in \mathbb{R}^4_{++}$ of incentive-compatible and ex-post strictly individually rational payoffs there exists $\delta^\dagger \in (0,1)$ such that for every $\delta \geqslant \delta^\dagger$ there exists a perfect Bayesian equilibrium (of infinitely repeated game) that leads to such payoffs.*

The proof of Proposition 3 is in Appendix E. Fix a tuple $(V_{F,H}, V_{F,L}, W_H, W_L)$ of incentive-compatible and ex-post strictly individually rational payoffs and let $\{(x_H, q_H), (x_L, q_L)\}$ denote the corresponding direct mechanism. We construct an equilibrium which consists of two phases. The screening phase takes place in the first period when the firm offers the menu $\{(x_H, q_H), (x_L, q_L)\}$. Each type $i = H, L$ selects the menu (x_i, q_i) and the firm learns the worker's type. The post-screening phase with type $i = H, L$ starts in the second period and implements the contract (x_i, q_i) in every period.

In equilibrium, the firm never updates its belief in the post-screening phase. It is therefore necessary to show that in the game with *complete* information with type i, there exists an equilibrium that implements (x_i, q_i) in every period (when the parties are sufficiently patient). The following lemma establishes this important result.

Lemma 8 *Consider the infinitely repeated game with complete information in which the firm interacts with type $i = H, L$. Let (x_i, q_i) be a contract yielding the payoff $V_{F,i} =$*

$v(q_i) - x_i > 0$ *to the firm and the payoff* $W_i = x_i - \theta_i q_i - \alpha > 0$ *to the worker. There exists* $\delta^\dagger \in (0,1)$ *such that for every* $\delta \geqslant \delta^\dagger$ *there exists a subgame perfect equilibrium that leads to the payoffs* $(V_{F,i}, W_i)$.

Proof of Lemma 8.

Fix $\varepsilon \in (0, \min\{V_{F,i}, W_i\})$. Let $(\underline{x}_i, q_i^*)$, $\underline{x}_i = \alpha + \theta_i q_i^* + \frac{\varepsilon}{2}$, denote the efficient contract that yields the payoff $\frac{\varepsilon}{2}$ to the worker. Also, let $(\bar{x}_i, q_i^*)$, $\bar{x}_i = v(q_i^*) - \frac{\varepsilon}{2}$, denote the efficient contract that yields the payoff $\frac{\varepsilon}{2}$ to the firm.

Consider the following strategy profile, generated by a simple three states automaton.

State $(i,0)$: This is the initial state. The automaton prescribes that the firm offers the menu $\{(x_i, q_i)\}$ and the worker accepts the contract (x_i, q_i). The state remains $(i,0)$ unless there is a deviation by the firm, in which case the state changes to $(i,1)$ irrespective of the worker's decision. When the firm deviates and offers a menu different from $\{(x_i, q_i)\}$, the worker accepts the contract which maximizes his current payoff, provided that this is positive (here and in what follows, we require the worker to select the contract with the smallest index if there are multiple contracts yielding the highest current payoff). Finally, the worker rejects all the contracts if they all yield a negative payoff.

State $(i,1)$: The automaton prescribes that the firm offers the menu $\{(\bar{x}_i, q_i^*)\}$ and the worker accepts $(\bar{x}_i, q_i^*)$. If the firm offers $\{(\bar{x}_i, q_i^*)\}$, the state remains $(i,1)$ irrespective of the worker's decision. Suppose instead that the firm deviates and offers the menu $m \neq \{(\bar{x}_i, q_i^*)\}$. The worker rejects every contract (x,q) with $x < v(1) + \alpha$ and selects, among the remaining ones, the contract that yields the highest current payoff, provided that this is positive. If the worker accepts a contract (x,q) with $x < v(1) + \alpha$, the state changes to $(i,2)$. In all other cases, the state remains $(i,1)$.

State $(i,2)$: The automaton prescribes that the firm offers the menu $\{(\underline{x}_i, q_i^*)\}$ and the worker accepts $(\underline{x}_i, q_i^*)$. The state remains $(i,2)$ unless there is a deviation by the firm. In this case, the state changes to $(i,1)$ irrespective of the worker's decision. When the firm deviates and offers a menu different from $\{(\underline{x}_i, q_i^*)\}$, the worker accepts the contract which maximizes his current payoff, provided that this is positive.

This concludes the description of the automaton. We now verify that this strategy profile is a subgame perfect equilibrium for values of δ sufficiently large. The firm's equilibrium payoffs are $V_{F,i}$ in state $(i,0)$, $\frac{\varepsilon}{2}$ in state $(i,1)$, and $v(q_i^*) - \underline{x}_i$ in state $(i,2)$. The worker's equilibrium payoffs are W_i in state $(i,0)$, $\bar{x}_i - \theta_i q_i^* - \alpha$ in state $(i,1)$, and $\frac{\varepsilon}{2}$ in state $(i,2)$.

Thus, the firm obtains the largest payoff in state $(i,2)$ and the lowest payoff in state $(i,1)$, while the worker obtains the largest payoff in state $(i,1)$ and the lowest payoff in state $(i,2)$. In fact, notice that $V_{F,i}+W_i=\pi_i(q_i)$ and $\pi_i(q_i^*)\geqslant\pi_i(q_i)$ for every q_i. This and the fact that $\varepsilon\in(0,\min\{V_{F,i},W_i\})$ imply:

$$v(q_i^*)-\underline{x}_i>V_{F,i}>v(q_i^*)-\bar{x}_i=\frac{\varepsilon}{2}$$
$$\bar{x}_i-\theta_i q_i^*-\alpha>W_i>\underline{x}_i-\theta_i q_i^*-\alpha=\frac{\varepsilon}{2}$$

It is then straightforward to check that for δ sufficiently large one-shot deviations from the automaton described above are not profitable. ■

Recall that the mechanism $\{(x_H,q_H),(x_L,q_L)\}$ is incentive compatible. Thus, if the post-screening phase with type $i=H,L$ implements the contract (x_i,q_i) in every period, then in the screening phase both types of the worker have an incentive to reveal their types. The rest of the proof of Proposition 3 provides a complete description of the equilibrium strategies and beliefs and verifies that sequential rationality is satisfied at all histories.

Consider now a different extension of the benchmark model in which the worker has the option to reject all the contracts and remain in the relationship. It is easy to extend the result of Proposition 3 to this setup. To see this, notice that the strategy of rejecting all the contracts and quitting is weakly dominated by the strategy of rejecting all the contracts and remaining in the relationship. Once we remove the strategy of quitting, we are back to the infinitely repeated game analyzed above.

To sum up, our analysis shows that when the relationship can continue after the worker rejects all the contracts (either because the game is infinitely repeated or because the worker has the option to reject all the contracts and stay in the relationship), many screening opportunities are available to the firm, including separation with employment. This is in contrast with the benchmark model in which the relationship ends automatically after the worker rejects all the contracts. In this case, only the pooling and the firing allocations can be implemented when the parties are sufficiently patient.

7 Concluding Remarks

We studied a dynamic-contracting model with adverse selection and limited commitment. In our benchmark model, the relationship ends when the worker rejects all contracts from the firm's menu. We characterized the limit equilibrium outcome as the parties become arbitrarily patient. If the prior probability that the worker has a low cost is low, the firm offers a pooling contract in every period. In contrast, if this prior probability that the worker has a low cost is high, the firm fires the worker with a high cost at the beginning of the relationship.

In this paper, the worker's action is verifiable. In some situations the agent's effort leads to stochastic outcomes and monitoring is thus imperfect. This would add a moral hazard component to the screening problem. We leave this interesting extension for future research.

Appendix A

Proof of Lemma 2.

First, assume that $\mu(h^t) \geqslant p^C$ (recall that $p^C \in (0,1)$ denotes the critical value of the prior above which the menu $\{(\theta_L q_L^* + \alpha, q_L^*)\}$ is optimal when there is commitment). As explained in Section 3, in any PBE (σ, μ), after a history h^t with $\mu(h^t) \geqslant p^C$, the firm's menu in period $t, t+1, \ldots,$ contains the contract $(\theta_L q_L^* + \alpha, q_L^*)$ and this contract is accepted by the low type (the high type quits the relationship in period t).

Next, assume that $\mu(h^t) = 0$. Again, the firm's continuation payoff is bounded above by $\pi_H(q_H^*)$, and the firm can guarantee this payoff by offering the menu $\{(\theta_H q_H^* + \alpha, q_H^*)\}$ in every period. Therefore, in equilibrium, the firm's menu in period $t, t+1, \ldots,$ must contain the contract $(\theta_H q_H^* + \alpha, q_H^*)$. Clearly, the low type strictly prefers to accept this contract rather than quit the relationship.

Finally, consider the case $\mu(h^t) \in (0, p^C)$ and let m_t denote a menu offered by the firm at h^t. By contradiction, assume that the low type quits the relationship with positive probability. This immediately implies that the high type must accept one of the contracts in m_t with positive probability. In fact, if the high type quits with probability one, then the firm's payoff is strictly smaller than $\mu(h^t)\pi_L(q_L^*)$, contradicting Lemma 1.

Let $m_t^H \subseteq m_t$ denotes the set of contracts in m_t accepted by the high type with positive probability. We claim that there is a contract $(x_t^H, q_t^H) \in m_t^H$ such that

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}q_\tau | h^t, \left(x_t^H, q_t^H\right), H\right] > 0, \tag{11}$$

where the left hand side of the inequality denotes the expected discounted total quality delivered by the high type after he accepts the contract (x_t^H, q_t^H) ($\mathbb{T} \leqslant \infty$ denotes the random time at which the worker quits). In fact, if inequality (11) is violated for all the contracts in m_t^H, then we have

$$V_F(h^t;(\sigma,\mu)) < \mu(h^t)\,\pi_L(q_L^*) +$$
$$(1-\mu(h^t)) \sum_{(x_t,q_t)\in m_t^H} \sigma_t^H((x_t,q_t)\,|h^t,m_t)\,\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}(v(0)-\alpha)\,|h^t,(x_t,q_t),H\right] \leqslant$$
$$\mu(h^t)\,\pi_L(q_L^*),$$

where the first inequality follows from the fact that the low type quits with positive probability and from the fact that the high type's strategy must be individually rational (see inequality (12) below). The second inequality uses $v(0)-\alpha \leqslant 0$.

Finally, notice that if $\left(x_t^H, q_t^H\right) \in m_t^H$, then

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}(x_\tau-\theta_H q_\tau-\alpha)\,|h^t,\left(x_t^H,q_t^H\right),H\right] \geqslant 0, \tag{12}$$

where the left hand side denotes the high type's continuation payoff after he accepts the contract $\left(x_t^H, q_t^H\right)$. Inequalities (11) and (12) imply that at the history (h^t, m_t) the low type can guarantee a strictly positive payoff by accepting the contract $\left(x_t^H, q_t^H\right)$ and mimicking the high type's behavior in every period $t+1, t+2, \ldots$. This shows that the low type's decision to rejects all the contracts in m_t is not optimal. ■

Proof of Lemma 3.

By contradiction, suppose there exist a PBE (σ,μ) and a history (h^t, m_t) satisfying the three properties in Lemma 3. First, consider the history $(h^t, m_t, (x_H, q_H))$. The firm's belief will be equal to zero and, in equilibrium, the menu offered by the firm in period $t+1, t+2, \ldots,$ will contain the contract $(\theta_H q_H^* + \alpha, q_H^*)$. Furthermore, the high type will select this contract in every period. We conclude that following $(h^t, m_t, (x_H, q_H))$, the high type's continuation payoff (evaluated at the beginning of period $t+1$) will be equal to zero. Furthermore, if the low type deviates and accepts the contract (x_H, q_H), then his continuation payoff will be at least $\Delta\theta q_H^*$ (in fact, the low type can mimic the high type and accept the contract $(\theta_H q_H^* + \alpha, q_H^*)$ in period $t+1, t+2, \ldots$).

Consider now the history $(h^t, m_t, (x_L, q_L))$. The firm's belief will be equal to one and, in equilibrium, the low type will accept the contract $(\theta_L q_L^* + \alpha, q_L^*)$ in period $t+1, t+2, \ldots$.

We conclude that after the history $(h^t, m_t, (x_L, q_L))$ the equilibrium continuation payoff of both types (again, evaluated at the beginning of period $t+1$) is equal to zero.

Clearly, in equilibrium, the worker's decision must be sequentially rational. Therefore, the contracts (x_H, q_H) and (x_L, q_L) must satisfy the following IC constraints:

$$x_H - \theta_H q_H - \alpha \geqslant x_L - \theta_H q_L - \alpha \tag{13}$$

and

$$(1-\delta)(x_L - \theta_L q_L - \alpha) \geqslant (1-\delta)(x_H - \theta_L q_H - \alpha) + \delta \Delta\theta q_H^*. \tag{14}$$

Combining the two constraints we obtain

$$\theta_H (q_L - q_H) \geqslant x_L - x_H \geqslant \theta_L (q_L - q_H) + \frac{\delta}{1-\delta} \Delta\theta q_H^*,$$

which implies

$$\Delta\theta \geqslant \Delta\theta (q_L - q_H) \geqslant \frac{\delta}{1-\delta} \Delta\theta q_H^*.$$

Clearly, the second inequality cannot be satisfied if $\delta > \hat{\delta}$. ■

Appendix B: Proof of Proposition 1

In this appendix we prove the existence of PBE for generic values of the parameters. In particular, we construct an equilibrium in which the high type's payoff is equal to zero. Although not Markovian, in our PBE the firm and the low type's equilibrium continuation payoffs depend on the firm's belief. We use the function $V : [0,1] \to \mathbb{R}_+$ to denote the firm's payoff (as function of its belief) and the correspondence $\Phi : [0,1] \rightrightarrows \mathbb{R}_+$ to denote the set of payoffs of the low type. To simplify the notation, we write $\Phi(p) = z$ for $\Phi(p) = \{z\}$. We also write $\min \Phi(p)$ $(\max \Phi(p))$ to denote the smallest (largest) element of $\Phi(p)$.

For $\delta > \hat{\delta}$ we define V and Φ as follows:

$$V(p) = \max\{\pi_H(q_H^*), p\pi_L(q_L^*)\}, \tag{15}$$

$$\Phi(p) = \begin{cases} \Delta\theta q_H^* & \text{for } p < \hat{p} \\ [0, \Delta\theta q_H^*] & \text{for } p = \hat{p} \\ 0 & \text{for } p > \hat{p} \end{cases} \tag{16}$$

where $\hat{p}$ satisfies $\pi_H(q_H^*) = \hat{p}\pi_L(q_L^*)$.

Finally, recall that if $\delta > \hat{\delta}$, then it is impossible to find two contracts (x_H, q_H) and (x_L, q_L) that satisfy the constraints (13) and (14).

Next, we show that for generic values of δ smaller than $\hat{\delta}$ there exists a pair (V, Φ) satisfying a number of properties.

Lemma 9 *Fix the parameters* $(\theta_H, \theta_L, \alpha, v(\cdot))$. *For all but at most two values of* δ *in* $\left(0, \hat{\delta}\right]$, *there exists a pair* (V, Φ) *satisfying the following conditions:*

i) V *is continuous and* Φ *is upper hemicontinuous;*

ii) there exists $\bar{p} \in (0,1)$ *such that* $V(p) = p\pi_L^*$ *and* $\Phi(p) = 0$ *for* $p > \bar{p}$;

iii) there exists $\underline{p} \in [0, \bar{p}]$ *such that* $V(p) = \pi_H^*$ *for* $p \leqslant \underline{p}$, $\Phi(p) = \Delta\theta q_H^*$ *for* $p < \underline{p}$, *and* $\Delta\theta q_H^* \in \Phi\left(\underline{p}\right)$;

iv) $V(p) = \tilde{V}(p)$ *for* $p \in \left[\underline{p}, \bar{p}\right]$, *and* $V(p) > \tilde{V}(p)$ *for* $p \in \left(0, \underline{p}\right) \cup (\bar{p}, 1)$, *where* $\tilde{V}(p)$ *is defined by*

$$\begin{aligned} \tilde{V}(p) = & \max_{(q_H, q_L) \in [0,1]^2, x \in \mathbb{R}, \tilde{p} \leqslant p} \tfrac{1-p}{1-\tilde{p}} \left[(1-\delta)\pi_H(q_H) + \delta V(\tilde{p})\right] + \\ & \qquad \tfrac{p-\tilde{p}}{1-\tilde{p}} \left[(1-\delta)(v(q_L) - x) + \delta\pi_L(q_L^*)\right] \\ \text{s.t.} \quad & x - \theta_H q_L - \alpha \leqslant 0 \\ & (1-\delta)(x - \theta_L q_L - \alpha) = (1-\delta)\Delta\theta q_H + \delta \min\Phi(\tilde{p}) \end{aligned} \tag{17}$$

v) If $\min\Phi(p) < \Delta\theta q_H^*$ *and* $v \in \left[\min\Phi(p), \Delta\theta q_H^*\right]$, *there exists* $p' \in [0, p]$ *such that* $v \in \Phi(p')$.

Proof of Lemma 9.

We develop an iterative procedure which will deliver the pair (V, Φ) with the desired properties.

Step 1

First, we allow the firm to propose a menu which separates the two types (with employment). Specifically, for every belief p we consider the following optimization problem:

$$\begin{aligned} V^1(p) := & \max_{(q_H, q_L) \in [0,1]^2, x \in \mathbb{R}} (1-p)\left[(1-\delta)\pi_H(q_H) + \delta\pi_H(q_H^*)\right] + \\ & \qquad p\left[(1-\delta)(v(q_L) - x) + \delta\pi_L(q_L^*)\right] \\ \text{s.t.} \quad & x - \theta_H q_L - \alpha \leqslant 0 \\ & (1-\delta)(x - \theta_L q_L - \alpha) \geqslant (1-\delta)\Delta\theta q_H + \delta\Delta\theta q_H^* \end{aligned}$$

The firm offers the contracts $(\theta_H q_H + \alpha, q_H)$ to the high type and the contract (x, q_L) to the low type. Clearly, at the optimum the low type's IC constraint is binding. Thus, we can rewrite the problem as

$$\begin{aligned} V^1(p) = \max_{(q_H, q_L) \in [0,1]^2, x \in \mathbb{R}} & (1-p)\left[(1-\delta)\pi_H(q_H) + \delta\pi_H(q_H^*)\right] + \\ & p\left[(1-\delta)\pi_L(q_L) + \delta\pi_L(q_L^*) - (1-\delta)\Delta\theta q_H - \delta\Delta\theta q_H^*\right] \end{aligned} \tag{18}$$

$$\text{s.t.} \quad q_H - q_L + \tfrac{\delta}{1-\delta} q_H^* \leqslant 0 \tag{19}$$

We let $(q_H^1(p), q_L^1(p))$ denote the solution to the above problem. It follows from the concavity of the functions π_H and π_L that $q_H^1(p)$ is uniquely defined for $p \in [0,1)$, and $q_L^1(p)$ is uniquely defined for $p \in (0,1]$. Furthermore $q_H^1(\cdot)$ and $q_L^1(\cdot)$ are upper hemicontinuous (theorem of the maximum), and $V^1(\cdot)$ is continuous (again, theorem of the maximum) and convex (notice that the pairs (q_H, q_L) satisfying constraint (19) do not vary with p). Finally, it immediate to check that for any p, $q_H^1(p) \leqslant q_H^*$, and $q_L^1(p) \geqslant q_L^*$, and that $q_H^1(\cdot)$ is decreasing in p.

We now distinguish among different cases.

Case 1.1. For every $p \in [0,1]$,

$$V^1(p) \leqslant \max\{\pi_H(q_H^*), p\pi_L(q_L^*)\}.$$

In this case, we let V and Φ be defined as in equations (15) and (16), respectively.

Case 1.2. There exists $p \in (0,1)$ such that

$$V^1(p) > \max\{\pi_H(q_H^*), p\pi_L(q_L^*)\}. \tag{20}$$

Notice that

$$\frac{\partial V^1(p)}{\partial p} = (1-\delta)\pi_L\left(q_L^1(p)\right) + \delta\pi_L(q_L^*) - (1-\delta)\Delta\theta q_H^1(p) - \delta\Delta\theta q_H^* - (1-\delta)\pi_H\left(q_H^1(p)\right) - \delta\pi_H(q_H^*)$$

If $V^1(p) > \pi_H(q_H^*)$ it must be that

$$(1-\delta)\pi_L\left(q_L^1(p)\right) + \delta\pi_L(q_L^*) - (1-\delta)\Delta\theta q_H - \delta\Delta\theta q_H^* > \pi_H(q_H^*)$$

and, therefore, $\partial V^1(p)/\partial p$ must be strictly positive at any point p which satisfies inequality (20).

Also, recall that V^1 is convex and $V^1(p) \leqslant p\pi_L(q_L^*)$ for every $p \geqslant p^C$. We conclude that the set of beliefs for which inequality (20) holds is an interval $\left(\underline{p}_1, \bar{p}_1\right)$, with $\underline{p}_1 \in [0, \hat{p})$ and $\bar{p}_1 \in \left(\hat{p}, p^C\right]$.

Case 1.2.1. $\underline{p}_1 = 0$.

In this case, $q_H^1(0) = q_H^*$. We point out that the case $\underline{p}_1 = 0$ can arise only if $\delta \leqslant 1 - q_H^*$ (if $\delta > 1 - q_H^*$ it is impossible to find q_L such that the pair (q_H^*, q_L) satisfies constraint (19)).

We claim that for generic values of δ, if $\underline{p}_1 = 0$, then

$$\partial_+ V^1(0) = \lim_{p \downarrow 0} (1-\delta)\pi_L\left(q_L^1(p)\right) + \delta\pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*)$$

is strictly positive.

First, for $\delta \leqslant 1 - q_H^*/q_L^*$, $q_L^1(p) = q_L^*$ for every $p > 0$, and thus

$$\partial_+ V^1(0) = \pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*) = \pi_L(q_L^*) - \pi_L(q_H^*) > 0.$$

Suppose now that $\delta \in (1 - q_H^*/q_L^*, 1 - q_H^*]$ and $q_H^1(0) = q_H^*$. Then for each δ, there exists ε such that

$$q_L^1(p) = q_H^1(p) + \frac{\delta}{1-\delta} q_H^*$$

for every $p \in [0, \varepsilon]$. Therefore, we have

$$\partial_+ V^1(0) = (1-\delta)\pi_L\left(\frac{q_H^*}{1-\delta}\right) + \delta\pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*).$$

Notice that the function $g(\cdot)$ defined by

$$g(\delta) = (1-\delta)\pi_L\left(\frac{q_H^*}{1-\delta}\right) + \delta\pi_L(q_L^*) - \Delta\theta q_H^* - \pi_H(q_H^*)$$

is strictly concave and, therefore, there can be at most two distinct values of δ for which $g(\delta)$ is equal to zero. This shows that generically, if $\underline{p}_1 = 0$, then $\partial_+ V^1(p) > 0$. In what follows, we say that the value of δ is generic if $g(\delta) \neq 0$.

When $\underline{p}_1 = 0$ we define $V(\cdot; 1)$ and $\Phi(\cdot; 1)$ as follows:

$$V(p;1) = \begin{cases} V^1(p) & \text{for } p \leqslant \bar{p}_1 \\ p\pi_L(q_L^*) & \text{for } p > \bar{p}_1 \end{cases}$$

$$\Phi(p;1) = \begin{cases} (1-\delta)\Delta\theta q_H^1(p) + \delta\Delta\theta q_H^* & \text{for } p < \bar{p}_1 \\ [0, (1-\delta)\Delta\theta q_H^1(\bar{p}_1) + \delta\Delta\theta q_H^*] & \text{for } p = \bar{p}_1 \\ 0 & \text{for } p > \bar{p}_1 \end{cases}$$

Case 1.2.2. $\underline{p}_1 > 0$.

We claim that for every δ we have $\partial_+ V^1\left(\underline{p}_1\right) > 0$. Notice that $V^1(\cdot)$ cannot be constant and equal to $\pi_H(q_H^*)$ in the interval $\left[0, \underline{p}_1\right)$. In fact, if $V^1(0) = \pi_H(q_H^*)$, then we have $q_H^1(0) = q_H^*$. This and the firm's optimality condition imply that $q_H^1(p)$ is strictly decreasing in p in a neighborhood of zero, which, in turn, implies the strict convexity of $V^1(\cdot)$ near zero. Therefore, we conclude that either $V^1(0) < \pi_H(q_H^*)$ or $V^1(0) = \pi_H(q_H^*)$ and $V^1(\cdot)$ is strictly convex in a neighborhood of zero. In either case, $V^1(\cdot)$ achieves a minimum at $p_\dagger \in \left[0, \underline{p}_1\right)$ and $V^1(p_\dagger) < \pi_H(q_H^*) = V^1\left(\underline{p}_1\right)$. This and the convexity of $V^1(\cdot)$ imply $\partial_+ V^1\left(\underline{p}_1\right) > 0$.

In this case ($\underline{p}_1 > 0$), we define $V(\cdot;1)$ and $\Phi(\cdot;1)$ as follows:

$$V(p;1) = \begin{cases} \pi_H(q_H^*) & \text{for } p \leqslant \underline{p}_1 \\ V^1(p) & \text{for } p \in \left(\underline{p}_1, \bar{p}_1\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_1 \end{cases}$$

$$\Phi(p;1) = \begin{cases} \Delta\theta q_H^* & p < \underline{p}_1 \\ \left[(1-\delta)\Delta\theta q_H^1\left(\underline{p}_1\right) + \delta\Delta\theta q_H^*, \Delta\theta q_H^*\right] & p = \underline{p}_1 \\ (1-\delta)\Delta\theta q_H^1(p) + \delta\Delta\theta q_H^* & p \in \left(\underline{p}_1, \bar{p}_1\right) \\ \left[0, (1-\delta)\Delta\theta q_H^1(\bar{p}_1) + \delta\Delta\theta q_H^*\right] & p = \bar{p}_1 \\ 0 & p > \bar{p}_1 \end{cases}$$

Step 2

We now consider the case of probabilistic separation. That is, the firm offers two contracts. The high type chooses the first contract, while the low type randomizes between the two contracts.

For every $p \geqslant \underline{p}_1$, we consider the following optimization problem

$$V^2(p) := \max_{(q_H,q_L)\in[0,1]^2, x\in\mathbb{R}, \tilde{p}\in\left[\underline{p}_1, \min\{p,\bar{p}_1\}\right]} \tfrac{1-p}{1-\tilde{p}}\left[(1-\delta)\pi_H(q_H) + \delta V(\tilde{p};1)\right] + \tfrac{p-\tilde{p}}{1-\tilde{p}}\left[(1-\delta)(v(q_L) - x) + \delta\pi_L(q_L^*)\right]$$

$$\text{s.t.} \quad x - \theta_H q_L - \alpha \leqslant 0$$

$$(1-\delta)(x - \theta_L q_L - \alpha) \geqslant (1-\delta)\Delta\theta q_H + \delta\min\Phi(\tilde{p};1)$$

The second constraint must bind and we can rewrite the problems as

$$\begin{gathered}V^2(p) = \max_{(q_H, q_L) \in [0,1]^2, \tilde{p} \in \left[\underline{p}_1, \min\{p, \bar{p}_1\}\right]} \tfrac{1-p}{1-\tilde{p}} \left[(1-\delta)\pi_H(q_H) + \delta V(\tilde{p};1)\right] + \\ \tfrac{p-\tilde{p}}{1-\tilde{p}} \left[(1-\delta)\pi_L(q_L) + \delta \pi_L(q_L^*) - (1-\delta)\Delta\theta q_H - \delta \min \Phi(\tilde{p};1)\right]\end{gathered} \tag{21}$$

$$\text{s.t.} \quad (q_H - q_L)\Delta\theta + \tfrac{\delta}{1-\delta} \min \Phi(\tilde{p};1) \leqslant 0.$$

If $V^2(p) \leqslant V(p;1)$ for every $p \in [0,1]$, then we set $V(\cdot)$ equal to $V(\cdot;1)$, and $\Phi(\cdot)$ equal to $\Phi(\cdot;1)$. On the other hand, if $V^2(p) > V(p;1)$ for some p, we distinguish among different cases.

Case 2.1. $\underline{p}_1 = 0$

First, we assume that $\underline{p}_1 = 0$ and consider the generic values of δ for which $\partial_+ V(0;1) > 0$. We show that when the belief is sufficiently small the firm does not benefit from an additional possibility of screening the worker.

Claim 4 *Assume that* $\underline{p}_1 = 0$. *There exists* $\varepsilon > 0$ *such that* $V^2(p) = V(p;1)$ *for every* $p \in [0, \varepsilon]$.

Proof of Claim 4.

For every p and $\tilde{p} \leqslant p$ define $V^2(p, \tilde{p})$ as follows:

$$\begin{gathered}V^2(p, \tilde{p}) = \max_{(q_H, q_L) \in [0,1]^2} \tfrac{1-p}{1-\tilde{p}} \left[(1-\delta)\pi_H(q_H) + \delta V(\tilde{p};1)\right] + \\ \tfrac{p-\tilde{p}}{1-\tilde{p}} \left[(1-\delta)\pi_L(q_L) + \delta \pi_L(q_L^*) - (1-\delta)\Delta\theta q_H - \delta \min \Phi(\tilde{p};1)\right]\end{gathered} \tag{22}$$

$$\text{s.t.} \quad (q_H - q_L)\Delta\theta + \tfrac{\delta}{1-\delta} \min \Phi(\tilde{p};1) \leqslant 0.$$

and notice that $V^2(p, 0) = V(p;1)$ (recall that $\Phi(p;1) = \Delta\theta q_H^*$).

We show that for p close to zero, the function $V^2(p, \cdot)$ is decreasing in $\tilde{p}$. This will prove our claim.

We let $q_H(p, \tilde{p})$ and $q_L(p, \tilde{p})$ denote the solution to the above problem and let $\gamma(p, \tilde{p})$ denote the Lagrangian multiplier. From the first order conditions with respect to q_L we have

$$\frac{p - \tilde{p}}{1 - \tilde{p}} (1-\delta) \frac{\partial \pi_L(q_L(p, \tilde{p}))}{q_L} = \gamma(p, \tilde{p}).$$

We apply the envelope theorem and obtain[18]

$$\frac{\partial V^2(p,\tilde{p})}{\partial \tilde{p}} = \frac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\,\pi_H(q_H(p,\tilde{p})) + \delta V(\tilde{p};1)\right] -$$

$$\frac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\,\pi_L(q_L(p,\tilde{p})) + \delta\pi_L(q_L^*) - (1-\delta)\,\Delta\theta q_H(\tilde{p}) - \delta \min\Phi(\tilde{p};1)\right] +$$

$$\left(\frac{1-p}{1-\tilde{p}}\right)\delta\frac{\partial V(\tilde{p};1)}{\partial \tilde{p}} - \left(\frac{p-\tilde{p}}{1-\tilde{p}}\right)\delta\frac{\partial \min\Phi(\tilde{p};1)}{\partial \tilde{p}} + \gamma(p,\tilde{p})\,\frac{\delta}{1-\delta}\frac{\partial \min\Phi(\tilde{p};1)}{\partial \tilde{p}} =$$

$$\frac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\,\pi_H(q_H(p,\tilde{p})) + \delta V(\tilde{p};1)\right] -$$

$$\frac{1-p}{(1-\tilde{p})^2}\left[(1-\delta)\,\pi_L(q_L(p,\tilde{p})) + \delta\pi_L(q_L^*) - (1-\delta)\,\Delta\theta q_H(p,\tilde{p}) - \delta \min\Phi(\tilde{p};1)\right] +$$

$$\left(\frac{1-p}{1-\tilde{p}}\right)\delta\frac{\partial V(\tilde{p};1)}{\partial \tilde{p}} - \left(\frac{p-\tilde{p}}{1-\tilde{p}}\right)\delta\frac{\partial \min\Phi(\tilde{p};1)}{\partial \tilde{p}} + \delta\frac{p-\tilde{p}}{1-\tilde{p}}\frac{\partial\pi_L(q_L(p,\tilde{p}))}{q_L}\frac{\partial \min\Phi(\tilde{p};1)}{\partial \tilde{p}}$$

Recall that we are considering the case in which $\underline{p}_1 = 0$. Therefore, as p converges to zero $\min_{\tilde{p}\leqslant p} q_H(p,\tilde{p})$ must converge to q_H^*. Also, as $\tilde{p}$ shrinks to zero, $V(\tilde{p};1)$ and $\min\Phi(\tilde{p};1)$ converge to $\pi_H(q_H^*)$ and $\Delta\theta q_H^*$, respectively, and the derivative of $\min\Phi(\tilde{p};1)$ (with respect to $\tilde{p}$) is bounded. Therefore, we have

$$\lim_{p\downarrow 0}\max_{\tilde{p}\leqslant p}\frac{\partial V^2(p,\tilde{p})}{\partial \tilde{p}} = \pi_H(q_H^*) - \left[(1-\delta)\,\pi_L(\max\left\{q_L^*,\frac{q_H^*}{1-\delta}\right\}) + \delta\pi_L(q_L^*) - \Delta\theta q_H^*\right] +$$

$$\delta\partial_+ V(0;1) = -(1-\delta)\,\partial_+ V(0;1) < 0$$

where the inequality follows from our genericity assumption.

We conclude that when $\underline{p}_1 = 0$ (and δ is generic), there exists $\varepsilon > 0$ such that for $p \leqslant \varepsilon$ the function $V^2(p,\cdot)$ is decreasing in the interval $[0,p]$. Thus, for $p\leqslant\varepsilon$, $V^2(p) = V^2(p,0) = V(p;1)$.

In general, the value of ε above depends on δ. However, is easy to see that there exists ε such that for any (generic) $\delta \leqslant 1 - q_H^*/q_L^*$ and for any $p\leqslant\varepsilon$, $V^2(p) = V^2(p,0) = V(p;1)$. ■

We define $\underline{p}_2 > 0$ as

$$\underline{p}_2 = \inf\left\{p : V^2(p) > V(p;1)\right\}$$

We now show that the function $V^2(\cdot)$ is convex. Clearly, the restriction of $V^2(\cdot)$ to the interval $\left[0,\underline{p}_2\right]$ is convex since, in this interval, $V^2(\cdot)$ is equal to $V(\cdot;1)$.

We now consider the interval $\left[\underline{p}_2, 1\right]$ and observe that there exists $\eta > 0$ such that

$$V^2(p) \geqslant V(p;1) > \pi_H(q_H^*) + \eta$$

[18] It is easy to see that the function $\min\Phi(\cdot;1)$ is differentiable in a neighborhood of zero.

for every $p \in \left[\underline{p}_2, 1\right]$. Thus, for $p \geqslant \underline{p}_2$ we have

$$V^2(p,p) \leqslant (1-\delta)\,\pi_H(q_H^*) + \delta V(p;1) < V(p;1) - (1-\delta)\,\eta.$$

This, together with the continuity of $V^2(p,\tilde{p})$ with respect to $\tilde{p}$, imply that for every $p' \geqslant \underline{p}_2$, there exists $\varepsilon > 0$ such that for any $p \in (p'-\varepsilon, p'+\varepsilon)$ the optimal value of $\tilde{p}$ (in the optimization problem (21)) is below $p'-\varepsilon$. This means that the restriction $V^2(\cdot)$ to the interval $(p'-\varepsilon, p'+\varepsilon)$ is the upper envelope of a fixed family of affine functions. Thus, the function $V^2(\cdot)$ is locally convex in $[0,1]$, and, therefore, convex.

It follows from the convexity of $V^2(\cdot)$ that there exists a point $\bar{p}_2 \in \left(\bar{p}_1, p^C\right]$ such that $V^2(\cdot) < p\pi_L(q_L^*)$ if $p < \bar{p}_2$, and $V^2(\cdot) > p\pi_L(q_L^*)$ if $p > \bar{p}_2$.

We conclude Step 2.1 by defining $V(\cdot;2)$ and $\Phi(\cdot;2)$ as follows:

$$V(p;2) = \begin{cases} V(p;1) & \text{for } p \leqslant \underline{p}_2 \\ V^2(p) & \text{for } p \in \left(\underline{p}_2, \bar{p}_2\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_2 \end{cases} \tag{23}$$

$$\Phi(p;2) = \begin{cases} \Phi(p;1) & p < \underline{p}_2 \\ Conv\left(\left\{(1-\delta)\,\Delta\theta q_H^2\left(\underline{p}_2\right) + \delta \min \Phi\left(\tilde{p}^2\left(\underline{p}_2\right);1\right)\right\} \cup \Phi\left(\underline{p}_2;1\right)\right) & p = \underline{p}_2 \\ (1-\delta)\,\Delta\theta q_H^2(p) + \delta \min \Phi\left(\tilde{p}^2(p);1\right) & p \in \left(\underline{p}_2, \bar{p}_2\right) \\ \left[0, (1-\delta)\,\Delta\theta q_H^2(\bar{p}_2) + \delta \min \Phi\left(\tilde{p}^2(p);1\right)\right] & p = \bar{p}_2 \\ 0 & p > \bar{p}_2 \end{cases} \tag{24}$$

where $q_H^2(p)$ and $\tilde{p}^2(p)$ denote the optimal values of q_H and $\tilde{p}$, respectively, in the optimization problem (21), and $Conv(\cdot)$ denotes the convex hull of a given set.

Case 2.2. $\underline{p}_1 > 0$.

We distinguish between two cases.

Case 2.2.1. There exists $\varepsilon > 0$ such that $V^2(p) = V(p;1)$ for every $p \in \left[\underline{p}_1, \underline{p}_1 + \varepsilon\right]$. We let $\underline{p}_2$ denote

$$\inf\left\{p : V^2(p) > V(p;1)\right\}$$

Similarly to the previous case, the function $V^2(\cdot)$ is convex and we let $\bar{p}_2 \in \left(\bar{p}_1, p^C\right]$

denote the point at which $V^2(\cdot)$ intersects the function $p\pi_L(q_L^*)$.

We define $V(\cdot;2)$ and $\Phi(\cdot;2)$ as in (23) and (24), respectively.

Case 2.2.2. For every $\varepsilon > 0$, there exists $p \in \left(\underline{p}_1, \underline{p}_1 + \varepsilon\right)$ such that $V^2(p) > V(p;1)$.

In this case we have $V^2\left(\underline{p}_1\right) = V\left(\underline{p}_1;1\right) = \pi_H(q_H^*)$, $q_H^2\left(\underline{p}_1\right) = q_H^*$ and

$$0 < \partial_+ V^1\left(\underline{p}_1\right) < \partial_+ V^2\left(\underline{p}_1\right) = \lim_{p \downarrow \underline{p}_1} \frac{\partial V^2(p)}{\partial p} =$$

$$\frac{1}{1-\underline{p}_1}\left[(1-\delta)\pi_L\left(q_L^2\left(\underline{p}_1\right)\right) + \delta\pi_L(q_L^*) - (1-\delta)\Delta\theta q_H^* - \delta\min\Phi\left(\underline{p}_1;1\right) - \pi_H(q_H^*)\right] \tag{25}$$

where $q_L^2(p)$ denotes the optimal value of q_L (given the belief p) in the optimization problem (21).

Recall the definition of $V^2(p,\tilde{p})$ in the optimization problem (21). It follows from inequality (25) that there exists $\varepsilon > 0$ such that $V^2\left(p,\underline{p}_1\right) > V(p;1)$ for every $p \in \left(\underline{p}_1, \underline{p}_1 + \varepsilon\right)$.

The function $V^2\left(p,\underline{p}_1\right)$ is convex is p. Thus, there exists $\bar{p}_2 \in \left(\bar{p}_1, p^C\right]$ at which the function $V^2\left(p,\underline{p}_1\right)$ and the function $p\pi_L(q_L^*)$ intersect. We define $V(\cdot;2)$ and $\Phi(\cdot;2)$ as follows:

$$V(p;2) = \begin{cases} V(p;1) & \text{for } p \leqslant \underline{p}_1 \\ V^2\left(p,\underline{p}_1\right) & \text{for } p \in \left(\underline{p}_1, \bar{p}_2\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_2 \end{cases}$$

$$\Phi(p;2) = \begin{cases} \Phi(p;1) & p < \underline{p}_1 \\ Conv\left(\left\{(1-\delta)\Delta\theta q_H\left(\underline{p}_1,\underline{p}_1\right) + \delta\min\Phi\left(\underline{p}_1;1\right)\right\} \cup \Phi\left(\underline{p}_1;1\right)\right) & p = \underline{p}_1 \\ (1-\delta)\Delta\theta q_H\left(p,\underline{p}_1\right) + \delta\min\Phi\left(\underline{p}_1;1\right) & p \in \left(\underline{p}_1, \bar{p}_2\right) \\ \left[0, (1-\delta)\Delta\theta q_H\left(\bar{p}_2,\underline{p}_1\right) + \delta\min\Phi(\bar{p}_1;1)\right] & p = \bar{p}_2 \\ 0 & p > \bar{p}_2 \end{cases}$$

Then for every $p \geqslant \underline{p}_1$, we consider the following optimization problem

$$V^3(p) = \max_{(q_H, q_L) \in [0,1]^2, \tilde{p} \in \left[\underline{p}_1, \min\{p, \bar{p}_2\}\right]} \tfrac{1-p}{1-\tilde{p}} \left[(1-\delta)\pi_H(q_H) + \delta V(\tilde{p}; 1)\right] +$$
$$\tfrac{p-\tilde{p}}{1-\tilde{p}} \left[(1-\delta)\pi_L(q_L) + \delta \pi_L(q_L^*) - (1-\delta)\Delta\theta q_H - \delta \min \Phi(\tilde{p}; 1)\right]$$

$$\text{s.t.} \quad (q_H - q_L)\Delta\theta + \tfrac{\delta}{1-\delta} \min \Phi(\tilde{p}; 1) \leqslant 0.$$

It is easy to show that there exists $\varepsilon > 0$ such that $V^3(p) = V(p; 2)$ for every $p \in \left[\underline{p}_1, \underline{p}_1 + \varepsilon\right]$ (the proof of this fact is similar to the proof of Claim 4 and we omit it).

If $V^3(p) \leqslant V(p; 2)$ for every p, then we set $V(\cdot)$ equal to $V(\cdot; 2)$, and $\Phi(\cdot)$ equal to $\Phi(\cdot; 2)$. Otherwise we define $\underline{p}_3 > \underline{p}_1$ as

$$\underline{p}_3 = \inf\left\{p : V^3(p) > V(p; 2)\right\},$$

and let $\bar{p}_3 > \underline{p}_3$ denote the point at which the function $V^2\left(p, \underline{p}_1\right)$ and the function $p\pi_L(q_L^*)$ intersect (it is easy to show that the function $V^3(\cdot)$ is convex).

Finally, we define the function $V(\cdot; 3)$, and $\Phi(\cdot; 3)$ as follows:

$$V(p; 3) = \begin{cases} V(p; 2) & \text{for } p \leqslant \underline{p}_3 \\ V^3(p) & \text{for } p \in \left(\underline{p}_3, \bar{p}_3\right) \\ p\pi_L(q_L^*) & \text{for } p \geqslant \bar{p}_2 \end{cases}$$

$$\Phi(p; 3) = \begin{cases} \Phi(p; 2) & p < \underline{p}_3 \\ Conv\left(\left\{(1-\delta)\Delta\theta q_H^3\left(\underline{p}_3\right) + \delta \min \Phi\left(\tilde{p}\left(\underline{p}_3\right); 2\right)\right\} \cup \Phi\left(\underline{p}_3; 2\right)\right) & p = \underline{p}_3 \\ (1-\delta)\Delta\theta q_H^3(p) + \delta \min \Phi\left(\tilde{p}^3(p); 2\right) & p \in \left(\underline{p}_3, \bar{p}_3\right) \\ \left[0, (1-\delta)\Delta\theta q_H^3(\bar{p}_3) + \delta \min \Phi\left(\tilde{p}^3(\bar{p}_3); 2\right)\right] & p = \bar{p}_3 \\ 0 & p > \bar{p}_3 \end{cases}$$

This concludes Step 2.

Step 3.

The analysis in Step 2 shows that there exists $\hat{k} = 2, 3$ such that $V\left(\cdot; \hat{k}\right)$ and $V\left(\cdot; \hat{k} - 1\right)$ coincides in the interval $\left[0,\ \underline{p}_{\hat{k}}\right]$, $\underline{p}_{\hat{k}} > 0$, and $V\left(\underline{p}_{\hat{k}}; \hat{k}\right)$ is strictly larger than $\pi_H(q_H^*)$.

We now proceed by induction. For any $k = \hat{k}, \hat{k}+1, \ldots,$ we take as given the pair $(V(\cdot;k), \Phi(\cdot;k))$ and construct the pair $(V(\cdot;k+1), \Phi(\cdot;k+1))$ using the same procedure described in Step 2 (see the optimization problem (21)).

It is easy to show that for any k, the function $V(\cdot;k)$ is increasing and convex. Also, by construction, there exists $\hat{\eta} > 0$ such that for any k, and any $p \geqslant \underline{p}_k$ the following inequality holds:

$$V(p;k) > \pi_H(q_H^*) + \hat{\eta}.$$

We use this fact to show that the iterative procedure ends after finitely many rounds. Recall that p^C is the belief above which the unique optimal mechanism with commitment is to offer the menu $\{(\theta_L q_L^* + \alpha, q_L^*)\}$. Therefore, $\underline{p}_k \leqslant p^C$ for any k.

Claim 5 *For any* $k = \hat{k}, \hat{k}+1, \ldots,$

$$\underline{p}_{k+1} - \underline{p}_k > \frac{(1-\delta)\,\hat{\eta}\left(1-p^C\right)}{2\pi_L(q_L^*)}. \tag{26}$$

Proof of Claim 5.

Fix k and consider the optimization problem which defines the pair $(V(\cdot;k+1), \Phi(\cdot;k+1))$. Consider $p \geqslant \underline{p}_k$ and let $\tilde{p}^{k+1}(p)$ denote the optimal value of $\tilde{p}$.

Suppose that inequality (26) does not hold. Thus, there exists $p \in \left[\underline{p}_k, \underline{p}_k + \frac{(1-\delta)\hat{\eta}\left(1-p^C\right)}{\pi_L\left(q_L^*\right)}\right]$ such that $V^{k+1}(p) > V(p;k)$. Clearly, the last inequality holds only if $\tilde{p}^{k+1}(p) \geqslant \underline{p}_k$. However, this implies the following contradiction:

$$\begin{aligned} V^{k+1}(p) &\leqslant \tfrac{1-p}{1-\tilde{p}^{k+1}(p)}\left[(1-\delta)\pi_H(q_H^*) + \delta V\left(\tilde{p}^{k+1}(p);k\right)\right] + \tfrac{p-\tilde{p}^{k+1}(p)}{1-\tilde{p}^{k+1}(p)}\pi_L(q_L^*) \leqslant \\ &\tfrac{1-p}{1-\tilde{p}^{k+1}(p)}\left[(1-\delta)\pi_H(q_H^*) + \delta V(p;k)\right] + \tfrac{p-\tilde{p}^{k+1}(p)}{1-\tilde{p}^{k+1}(p)}\pi_L(q_L^*) \leqslant \\ &\tfrac{1-p}{1-\underline{p}_k}\left[(1-\delta)\pi_H(q_H^*) + \delta V(p;k)\right] + \tfrac{p-\underline{p}_k}{1-\underline{p}_k}\pi_L(q_L^*) \leqslant \\ &\left[(1-\delta)\pi_H(q_H^*) + \delta V(p;k)\right] + \tfrac{p-\underline{p}_k}{1-p^C}\pi_L(q_L^*) < \\ &V(p;k) - (1-\delta)\hat{\eta} + \tfrac{p-\underline{p}_k}{1-p^C}\pi_L(q_L^*) \leqslant V(p;k) \end{aligned}$$

where the second inequality follows from the monotonicity of $V(\cdot,k)$. This concludes the proof of Claim 5. ■

This shows that there exists an integer k^* for which the pairs $(V(\cdot;k^*), \Phi(\cdot;k^*))$ and $(V(\cdot;k^*+1), \Phi(\cdot;k^*+1))$ coincide on the entire unit interval. We set $(V(\cdot), \Phi(\cdot))$ equal to $(V(\cdot;k^*),\Phi(\cdot;k^*))$. By construction, $(V(\cdot),\Phi(\cdot))$ satisfies all the properties in Lemma 9

The number of iterations k^* necessary to get the fixed point $(V(\cdot),\Phi(\cdot))$ generally depends on the value of the discount factor. However, it is immediate to verify that there exists $\breve{k}$ such that for generic values of δ in $(0,1-q_H^*/q_L^*]$, the number of iterations necessary to get the fixed point $(V(\cdot),\Phi(\cdot))$ is bounded by $\breve{k}$. This is because there exists $\breve{\eta}>0$ such that for any generic value of $\delta\leqslant 1-q_H^*/q_L^*$, for any k, and any $p\geqslant \underline{p}_k$, we have $V(p;k)>\pi_H(q_H^*)+\breve{\eta}$ (this, in turn, follows from the convexity of the function $V(\cdot;k)$ and our discussion at the end of the proof of Claim 4. ■

We are now ready to conclude the proof of Proposition 1. First, we consider generic values of $\delta\leqslant\hat{\delta}$ and use the pair $(V(\cdot),\Phi(\cdot))$ defined in Lemma 9 to construct the equilibrium strategies. We assume that $\underline{p}<\bar{p}$ (the case $\underline{p}=\bar{p}=\hat{p}$ will be discussed below together with the case $\delta>\hat{\delta}$).

For every $p\in[0,1]$, we construct a set of menus $\mathbf{m}(p)$.

If $p<\underline{p}$, the set $\mathbf{m}(p)$ contains only the menu $m(p)=\{(\theta_H q_H^*+\alpha,q_H^*)\}$. If $p>\bar{p}$, $\mathbf{m}(p)$ contains only the menu $m(p)=\{(\theta_L q_L^*+\alpha,q_L^*)\}$.

Consider now $p\in[\underline{p},\bar{p})$, and let $(q_H(p),q_L(p),\tilde{p}(p))$ denote the solution to the optimization problem (17) such that

$$(1-\delta)\Delta\theta q_H(p)+\delta\min\Phi(\tilde{p}(p))=\min\Phi(p)$$

We let $m(p)$ denote the menus containing the contracts $(x_H(p),q_H(p))$ and $(x_L(p),q_L(p))$, where the payments $x_H(p)$ and $x_L(p)$ are given by:

$$x_H(p)=\theta_H q_H(p)+\alpha$$
$$x_L(p)=\theta_L q_L(p)+\alpha+\Delta\theta q_H(p)+\tfrac{\delta}{1-\delta}\min\Phi(\tilde{p}(p))$$

If $\max\Phi(p)=\min\Phi(p)$, then $\mathbf{m}(p)=\{m(p)\}$. If $\max\Phi(p)>\min\Phi(p)$ and $p\in(\underline{p},\bar{p})$, then we let $(q_H'(p),q_L'(p),\tilde{p}'(p))$ denote the solution to the optimization problem (17) such that

$$(1-\delta)\Delta\theta q_H'(p)+\delta\min\Phi(\tilde{p}'(p))=\max\Phi(p) \tag{27}$$

We also let $m'(p)$ denote the menus containing the contracts $(x_H'(p),q_H'(p))$ and

$(x'_L(p), q'_L(p))$, where the payments $x'_H(p)$ and $x'_L(p)$ are given by:

$$\begin{aligned} x'_H(p) &= \theta_H q'_H(p) + \alpha \\ x'_L(p) &= \theta_L q'_L(p) + \alpha + \Delta\theta q'_H(p) + \tfrac{\delta}{1-\delta}\min\Phi(\tilde{p}'(p)) \end{aligned} \tag{28}$$

In this case, we set $\mathbf{m}(p) = \{m(p), m'(p)\}$.

If $\max\Phi(\underline{p}) > \min\Phi(\underline{p})$, then we set $m'(p) = \{(\theta_H q^*_H + \alpha, q^*_H)\}$ and $\mathbf{m}(p) = \{m(p), m'(p)\}$.

Finally, we consider $\bar{p}$ and set $m(\bar{p}) = \{(\theta_L q^*_L + \alpha, q^*_L)\}$. If $\max\Phi(\bar{p}) = \min\Phi(\bar{p})$, then $\mathbf{m}(\bar{p}) = \{m(\bar{p})\}$. Otherwise, we set $m'(\bar{p}) = \{(x'_H(\bar{p}), q'_H(\bar{p})), (x'_L(\bar{p}), q'_L(\bar{p}))\}$ (see equations (27) and (28)) and $\mathbf{m}(\bar{p}) = \{m(\bar{p}), m'(\bar{p})\}$.

The equilibrium strategies are described in terms of the state which consists of a belief $p \in [0,1]$ and a continuation payoff $v \in \Phi(p)$. The initial state is $(p_0, \min\Phi(p_0))$, where p_0 is the prior.

Consider an arbitrary public history h^t and suppose the state is (p, v). In equilibrium, if $v = \min\Phi(p)$, then the firm offers the menu $m(p)$. On the other hand, if $v > \min\Phi(p)$, then the firm randomizes between the two menus in $\mathbf{m}(p)$ and proposes $m(p)$ with probability β defined by

$$\beta\min\Phi(p) + (1-\beta)\max\Phi(p) = v.$$

We now turn to the worker's strategy. Consider a public history h^t in which the firm's belief $\mu(h^t)$ is equal to p. Let $m = ((x_1, q_1), \ldots, (x_k, q_k))$ denote the menu offered by the firm, and for $i = H, L$ define

$$(\bar{x}_i, \bar{q}_i) := \arg\max_{i=1,\ldots,k} x_i - \theta_i q_i - \alpha$$

If $\bar{x}_L - \theta_L\bar{q}_L - \alpha < 0$, then both types reject all the contracts in the menu and quit the relationship. Furthermore, if the worker accepts a contract, then the firm's belief will be equal to one (in other words, the new state will be $(1, 0)$).

Suppose that $\bar{x}_L - \theta_L\bar{q}_L - \alpha \geqslant 0$ and $\bar{x}_H - \theta_H\bar{q}_H - \alpha < 0$. In this case, the low type picks the contract $(\bar{x}_L, \bar{q}_L)$, while the high type quits the relationship. Again, if the worker accepts a contract, the firm's belief will be equal to one.

We now turn to the case $\bar{x}_H - \theta_H\bar{q}_H - \alpha \geqslant 0$, and distinguish among three different possibilities. First, assume that the contracts $(\bar{x}_L, \bar{q}_L)$ and $(\bar{x}_H, \bar{q}_H)$ are such that

$$(1-\delta)(\bar{x}_L - \theta_L\bar{q}_L - \alpha) \leqslant (1-\delta)(\bar{x}_H - \theta_L\bar{q}_H - \alpha) + \delta\min\Phi(p)$$

In this case, both types accept the contract $(\bar{x}_H, \bar{q}_H)$, and the firm's belief remains unchanged. If the worker accepts any other contract, the firm's belief will jump to one.

Second, if

$$(1-\delta)(\bar{x}_L - \theta_L \bar{q}_L - \alpha) \geqslant (1-\delta)(\bar{x}_H - \theta_L \bar{q}_H - \alpha) + \delta \Delta\theta q_H^*,$$

then type $i = H, L$ accepts the contract $(\bar{x}_i, \bar{q}_i)$. The firm's belief will become zero if the worker accepts the contract $(\bar{x}_H, \bar{q}_H)$, and one if the worker accepts any other contract.

Finally, assume that

$$\frac{(1-\delta)\left[(\bar{x}_L - \theta_L \bar{q}_L) - (\bar{x}_H - \theta_L \bar{q}_H)\right]}{\delta} \in \left(\min \Phi(p), \Delta\theta q_H^*\right),$$

and thus

$$(1-\delta)(\bar{x}_L - \theta_L \bar{q}_L - \alpha) = (1-\delta)(\bar{x}_H - \theta_L \bar{q}_H - \alpha) + \delta\left[\tilde{\beta} \min \Phi(p') + \left(1-\tilde{\beta}\right) \max \Phi(p')\right]$$

for some $p' \leqslant p$ and some $\tilde{\beta} \in [0,1]$. In this case, the high type accepts the contract $(\bar{x}_H, \bar{q}_H)$, while the low type chooses the contract $(\bar{x}_H, \bar{q}_H)$ with probability $\frac{p'}{1-p'}\frac{1-p}{p}$, and the contract $(\bar{x}_L, \bar{q}_L)$ with probability $1 - \frac{p'}{1-p'}\frac{1-p}{p}$. Following the acceptance of the contract $(\bar{x}_H, \bar{q}_H)$ the new state will be $\left(p', \tilde{\beta} \min \Phi(p') + \left(1-\tilde{\beta}\right) \max \Phi(p')\right)$. If the worker accepts the contract $(\bar{x}_L, \bar{q}_L)$ or any other contract, the firm's belief will be equal to one.

It is easy to check that the above strategy profile, together with the firm's belief, constitute a PBE. The sequential rationality of the firm's strategy follows from the construction of the pair (V, Φ). In equilibrium, the high type behaves myopically and maximizes his period-t payoff at any history h^t. This behavior is indeed optimal since the high type's continuation payoff (computed at the beginning of period $t+1$) is equal to zero after any public history. Finally, notice that when the low type randomizes, all the contracts in the strategy's support yield the same expected payoff (and this is greater than the payoff of any other contract).

We now briefly turn to the case $\delta > \hat{\delta}$ and the case $\underline{p} = \bar{p} = \hat{p}$ (when $\delta \leqslant \hat{\delta}$). Recall the definitions of V and Φ in equations (15) and (16), respectively. For every belief p, we define the set of menus $\mathbf{m}(p)$ as follows. If $p < \hat{p}$, the set $\mathbf{m}(p)$ contains only the menu $m(p) = \{(\theta_H q_H^* + \alpha, q_H^*)\}$. If $p > \hat{p}$, $\mathbf{m}(p)$ contains only the menu $m(p) = \{(\theta_L q_L^* + \alpha, q_L^*)\}$. Finally, the set $\mathbf{m}(\hat{p})$ contains both the menu $m(\hat{p}) = \{(\theta_L q_L^* + \alpha, q_L^*)\}$ and the menu $m'(\hat{p}) = \{(\theta_H q_H^* + \alpha, q_H^*)\}$. The equilibrium strategies and beliefs are defined similarly to the case $\underline{p} < \bar{p}$ above and we omit the details. ■

Appendix C

Proof of Lemma 4.

First, notice that $W_H(h^t) = 0$ implies that almost all menus offered at h^t yield a continuation payoff of zero to the high type. Let $m_t = \left(\left(x_t^1, q_t^1\right), \ldots, \left(x_t^k, q_t^k\right)\right)$ be a menu offered at h^t. For $i = H, L$, let m_t^i denote the set of contracts in m_t accepted with positive probability by the type i. If there exists a contract $\left(x_t^j, q_t^j\right)$ in $m_t^L \setminus m_t^H$, then $\tilde{\mathbb{T}}$ coincides with t (in fact, the firm's belief jumps to one if the worker accepts the contract $\left(x_t^j, q_t^j\right)$) and there is nothing to prove. Therefore, assume that $m_t^L \subseteq m_t^H$ and recall that m_t^L is non-empty.

For every contract $\left(x_t^j, q_t^j\right)$ in m_t^H, we let $h_j^{t+1} = \left(h^t, m_t, \left(x_t^j, q_t^j\right)\right)$ denote the history in which the worker accepts the contract $\left(x_t^j, q_t^j\right)$ in period t, and recall that $W_H\left(h_j^{t+1}\right) \geqslant 0$ represents the high-worker's payoff at h_j^{t+1}. For every contract $\left(x_t^j, q_t^j\right) \in m_t^H$ we replace the payment x_t^j with the payment

$$\tilde{x}_t^j = x_t^j + \frac{\delta}{1-\delta} W_H\left(h_j^{t+1}\right)$$

which, clearly, implies $\tilde{x}_t^j = \theta_H q_t^j + \alpha$.

To keep the parties' payoffs unchanged, we also modify the payments in period $t+1$. In particular, for every $\left(x_t^j, q_t^j\right) \in m_t^H$ consider the history h_j^{t+1}. Let m_{t+1} denote a menu offered at h_j^{t+1} with positive probability. We subtract $\frac{1}{1-\delta} W_H\left(h_j^{t+1}, m_{t+1}\right)$, the high type's continuation payoff at the moment that the menu m_{t+1} is offered, from the payment of every contract in m_{t+1}. Notice that this yields to the high type a continuation payoff (evaluated at the beginning of period $t+1$) equal to zero.

We recursively apply the procedure outlined above to periods $\tau = t+1, \ldots, \tilde{\mathbb{T}}$ (i.e., we increase the payments in period τ and, at the same time, decrease the payments in period $\tau + 1$). By construction, every contract (x_τ, q_τ) accepted with positive probability by the high type in period $\tau = t, \ldots, \tilde{\mathbb{T}} - 1$ is replaced with the contract $(\theta_H q_\tau + \alpha, q_\tau)$, while the payments in every menu offered in period $\tilde{\mathbb{T}}$ are uniformly decreased by the high type's continuation payoff. Finally, in every period $\tau = t, \ldots, \tilde{\mathbb{T}} - 1$, the set of contracts accepted by the low type is contained in the set of contracts accepted by the high type (this follows from the definition of $\tilde{\mathbb{T}}$). It is therefore immediate to check that all the results stated in the lemma hold. ■

Proof of Lemma 5.

Fix a PBE (σ, μ) and a history h^t $(\mu(h^t) < p)$ at which the firm offers a menu m_t with the properties described in the statement of the lemma. First, notice that if the high type rejects all the contracts in m_t with probability one (i.e., the high type quits), then the high type's length of the relationship, the firm's payoff (conditional on the high type), and the low type's payoff are all equal to zero (if the low type's payoff is strictly positive, the firm's payoff would fall below $\mu(h^t)\pi_L(q_L^*)$).

Consider now the case in which the high type accepts a contract in m_t, say $\left(x_t^H, q_t^H\right)$, with positive probability. We let h_H^{t+1} denote the history $\left(h^t, m_t, \left(x_t^H, q_t^H\right)\right)$. We also let h_L^{t+1} denote the history $\left(h^t, m_t, \left(x_t^L, q_t^L\right)\right)$.

The fact that type $i = H, L$ accepts with positive probability the contract (x_t^i, q_t^i) implies

$$(1-\delta)\left(x_t^H - \theta_H q_t^H - \alpha\right) + \delta W_H\left(h_H^{t+1}\right) \geqslant (1-\delta)\left(x_t^L - \theta_H q_t^L - \alpha\right) + \delta W_H\left(h_L^{t+1}\right)$$
$$(1-\delta)\left(x_t^L - \theta_L q_t^L - \alpha\right) + \delta W_L\left(h_L^{t+1}\right) \geqslant (1-\delta)\left(x_t^H - \theta_L q_t^H - \alpha\right) + \delta W_L\left(h_H^{t+1}\right)$$

We add the two incentive compatibility constraints and obtain

$$(1-\delta)\Delta\theta\left(q_t^L - q_t^H\right) + \delta\left(W_L\left(h_L^{t+1}\right) - W_H\left(h_L^{t+1}\right)\right) \geqslant \delta\left(W_L\left(h_H^{t+1}\right) - W_H\left(h_H^{t+1}\right)\right)$$

Recall that $\mu\left(h_L^{t+1}\right) \geqslant p$ and that $[p, 1]$ is a firing region. Therefore, there exist $\bar{K}$ and $\bar{\delta} < 1$ such that $W_L\left(h_L^{t+1}\right) \leqslant \bar{K}(1-\delta)$ for $\delta > \bar{\delta}$. Of course, $W_H\left(h_L^{t+1}\right) \geqslant 0$. This, together with the above inequality, implies:

$$(1-\delta)\Delta\theta\left(q_t^L - q_t^H\right) + \bar{K}(1-\delta) \geqslant \delta\left(W_L\left(h_H^{t+1}\right) - W_H\left(h_H^{t+1}\right)\right) \tag{29}$$

We now let $\mathcal{D}_H\left(h_H^{t+1}\right) := \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t+1}^{\mathbb{T}-1}\delta^{\tau-t-1}|h_H^{t+1}, H\right]$ denote the expected discounted time, computed at h_H^{t+1}, until the high type quits. Our next goal is to provide an upper bound to $\mathcal{D}_H\left(h_H^{t+1}\right)$. Thus, without loss, assume that $\mathcal{D}_H\left(h_H^{t+1}\right)$ is strictly positive. We let

$$\mathcal{Q}_{t+1} = \frac{\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t+1}^{\mathbb{T}-1}\delta^{\tau-t-1}q_\tau|h_H^{t+1}, H\right]}{\mathcal{D}_H\left(h_H^{t+1}\right)}$$

the expected discounted total quality provided by the high type at the history h_H^{t+1}.

Using Jensen's inequality (recall that the function $\pi(\cdot)$ is concave), we can bound the firm's continuation payoff (conditional on type H) as follows:

$$V_F(h_H^{t+1};H) \leqslant \mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum\nolimits_{\tau=t+1}^{\mathbb{T}-1}\delta^{\tau-t-1}\pi\left(q_\tau\right)|h_H^{t+1},H\right] \leqslant$$
$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum\nolimits_{\tau=t+1}^{\mathbb{T}-1}\delta^{\tau-t-1}\pi\left(\mathcal{Q}_{t+1}\right)|h_H^{t+1},H\right] = \mathcal{D}_H\left(h_H^{t+1}\right)\pi\left(\mathcal{Q}_{t+1}\right)$$

Let $\breve{q}_H \in (0, q_H^*)$ be such that $\pi_H\left(\breve{q}_H\right) = 0$ and notice that $\pi_H\left(q\right) < 0$ for every $q < \breve{q}_H$.[19] This implies that $\mathcal{Q}_{t+1} \geqslant \breve{q}_H$. In fact, if the last inequality is violated, then $V_F(h_H^{t+1};H)$ is strictly negative, and $V_F(h_H^{t+1})$ is strictly less than $\mu\left(h_H^{t+1}\right)\pi_L\left(q_L^*\right)$, contradicting Lemma 1. Notice that one strategy available to the low type is to imitate the high type's behavior (in every period). Therefore, we conclude that

$$W_L\left(h_H^{t+1}\right) - W_H\left(h_H^{t+1}\right) \geqslant \Delta\theta\mathcal{D}_H\left(h_H^{t+1}\right)\mathcal{Q}_{t+1} \geqslant \Delta\theta\mathcal{D}_H\left(h_H^{t+1}\right)\breve{q}_H$$

Combining the inequality above with inequality (29) we obtain

$$\mathcal{D}_H\left(h_H^{t+1}\right) \leqslant \frac{(1-\delta)\left(q_t^L - q_t^H\right)}{\delta\breve{q}_H} + \frac{\bar{K}\left(1-\delta\right)}{\delta\Delta\theta\breve{q}_H} \leqslant \frac{(1-\delta)}{\delta\breve{q}_H}\left(1+\frac{\bar{K}}{\Delta\theta}\right).$$

Hence, for $\delta > \frac{1}{2}$ we have

$$\mathcal{D}_H\left(h_H^{t+1}\right) \leqslant \frac{2\left(1-\delta\right)}{\breve{q}_H}\left(1+\frac{\bar{K}}{\Delta\theta}\right)$$

This, in turn, implies that (for $\delta > \frac{1}{2}$)

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum\nolimits_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}|h^t,m_t,H\right] \leqslant (1-\delta) + \mathcal{D}_H\left(h_H^{t+1}\right) \leqslant$$
$$\left(1+\tfrac{2}{\breve{q}_H}+\tfrac{2\bar{K}}{\Delta\theta\breve{q}_H}\right)(1-\delta) := \tilde{K}\left(1-\delta\right)$$

and establishes part i).

To verify property ii), notice that the inequality above implies that the firm's continuation payoff $V_F(h^t,m_t;(\sigma,\mu),H)$ is bounded above by $v\left(1\right)\tilde{K}\left(1-\delta\right).$

Finally, we turn to property iii). The analysis above implies that

$$V_F(h^t,m_t) \leqslant \mu\left(h^t\right)\left[\pi_L\left(q_L^*\right) - W_L\left(h^t,m_t\right)\right] + v\left(1\right)\tilde{K}\left(1-\delta\right) \tag{30}$$

[19] This part of the proof uses the assumption $\alpha > 0$ to bound $\breve{q}_H$ away from zero. In Appendix D, we provide a different argument which does not require $\breve{q}_H > 0$.

Let δ' be such that

$$v(1)\tilde{K}(1-\delta') = \frac{\pi_H(q_H^*)}{4}$$

and notice that for $\delta \geqslant \tilde{\delta} = \max\left\{\delta', \bar{\delta}, \frac{1}{2}\right\}$ and $\mu(h^t) \leqslant \frac{\pi_H(q_H^*)}{2\pi_L(q_L^*)}$

$$\mu\left(h^t\right)\pi_L(q_L^*) + v(1)\tilde{K}(1-\delta) \leqslant \frac{3}{4}\pi_H(q_H^*)$$

It follows that if the firm offers the menu m_t at the history h^t and $\delta \geqslant \tilde{\delta}$, then $\mu(h^t) > \frac{\pi_H(q_H^*)}{2\pi_L(q_L^*)}$. Finally, recall that $V_F(h^t, m_t)$ is bounded below by $\mu(h^t)\pi_L(q_L^*)$. This and inequality (30) imply

$$W_L\left(h^t, m_t\right) \leqslant \frac{v(1)}{\mu(h^t)}\tilde{K}(1-\delta) \leqslant \frac{2\pi_L(q_L^*)v(1)\tilde{K}}{\pi_H(q_H^*)}(1-\delta)$$

This shows that there exists $K > 0$ satisfying the three properties in Lemma 5. ■

Proof of Claim 3.

First, assume that $\tilde{q}_0 \leqslant \frac{\check{q}_H}{2}$ and notice that $\pi_H(\tilde{q}_0) < \pi_H\left(\frac{\check{q}_H}{2}\right) < 0$ (recall that $\check{q}_H \in (0, q_H^*)$ satisfies $\pi_H(\check{q}_H) = 0$). Also, notice that $\hat{p} < p_0 < p$. If inequality (5) is satisfied, then we have

$$\begin{aligned} 0 \leqslant \left(1 - \tfrac{p_0}{p}\right)\Upsilon_0\pi_H(\tilde{q}_0) + \tfrac{p_0}{p}\left[\Upsilon_p\pi_H(\tilde{q}_p) + (1-\Upsilon_p)p\pi_L(q_L^*)\right] + K(1-\delta) - p_0\pi_L(q_L^*) \leqslant \\ \left(1 - \tfrac{p_0}{p}\right)\Upsilon_0\pi_H\left(\tfrac{\check{q}_H}{2}\right) + \tfrac{p_0}{p}\Upsilon_p\left[\pi_H(q_H^*) - p\pi_L(q_L^*)\right] + K(1-\delta) \leqslant \\ \left(1 - \tfrac{p_0}{p}\right)\Upsilon_0\pi_H\left(\tfrac{\check{q}_H}{2}\right) + K(1-\delta) \end{aligned}$$

Putting together this and $p_0 < p^C$ we obtain

$$\left(\frac{1}{1-p_0}\right)\left(1 - \frac{p_0}{p}\right)\Upsilon_0 \leqslant -\frac{K(1-\delta)}{(1-p^C)\pi_H\left(\frac{\check{q}_H}{2}\right)} \tag{31}$$

Similarly, we obtain

$$\begin{aligned} 0 \leqslant \left(1 - \tfrac{p_0}{p}\right)\Upsilon_0\pi_H(\tilde{q}_0) + \tfrac{p_0}{p}\left[\Upsilon_p\pi_H(\tilde{q}_p) + (1-\Upsilon_p)p\pi_L(q_L^*)\right] + K(1-\delta) - p_0\pi_L(q_L^*) \leqslant \\ \left(1 - \tfrac{p_0}{p}\right)\Upsilon_0\pi_H(\tilde{q}_0) + \tfrac{p_0}{p}\Upsilon_p\left[\pi_H(q_H^*) - p\pi_L(q_L^*)\right] + K(1-\delta) \leqslant \\ \tfrac{p_0}{p}\Upsilon_p\left[\pi_H(q_H^*) - p\pi_L(q_L^*)\right] + K(1-\delta) \leqslant \\ \left(1 - \tfrac{f(p)}{2p}\right)\Upsilon_p\left[\pi_H(q_H^*) - p\pi_L(q_L^*)\right] + K(1-\delta) \end{aligned}$$

where the last inequality follows from $p_0 \in \left[p - \frac{f(p)}{2}, p\right]$ and $\pi_H(q_H^*) - p\pi_L(q_L^*) < 0$.

Hence, we have:

$$\left(\frac{1-p}{1-p_0}\right)\left(\frac{p_0}{p}\right)\Upsilon_p \leqslant \Upsilon_p \leqslant \frac{K(1-\delta)}{\left(1-\frac{f(p)}{2p}\right)\left[p\pi_L(q_L^*) - \pi_H(q_H^*)\right]} \tag{32}$$

For the case $\tilde{q}_0 \leqslant \frac{\check{q}_H}{2}$, inequalities (31) and (32) imply the result.

We now move to the case $\tilde{q}_0 > \frac{\check{q}_H}{2}$. It follows from the concavity of $\pi_H(\cdot)$ that $\pi_H(\tilde{q}_0) \leqslant \pi_H(0) + \pi'_H(0)\tilde{q}_0 \leqslant \pi'_H(0)\tilde{q}_0$. Also, $\pi_H(\tilde{q}_p) < \pi_H(q_H^*)$ for any $\tilde{q}_p \neq q_H^*$. Thus, we have

$$\begin{aligned}\left(1-\tfrac{p_0}{p}\right)\Upsilon_0\pi_H(\tilde{q}_0) + \tfrac{p_0}{p}\left[\Upsilon_p\pi_H(\tilde{q}_p) + (1-\Upsilon_p)p\pi_L(q_L^*)\right] + K(1-\delta) \leqslant \\ \left(1-\tfrac{p_0}{p}\right)\Upsilon_0\pi'_H(0)\tilde{q}_0 + \tfrac{p_0}{p}\left[\Upsilon_p\pi_H(q_H^*) + (1-\Upsilon_p)p\pi_L(q_L^*)\right] + K(1-\delta)\end{aligned} \tag{33}$$

Suppose that inequality (6) holds. Clearly, the inequality continues to hold if we replace $\tilde{q}_p$ with one. This allows us to conclude that $\tilde{q}_0$, Υ_0, and Υ_p must satisfy

$$\Upsilon_0\tilde{q}_0 \leqslant \Upsilon_p + \frac{2K(1-\delta)}{\Delta\theta\left(\frac{1}{1-p_0}\right)\left(1-\frac{p_0}{p}\right)} \tag{34}$$

Combining inequalities (33) and (34) we obtain (recall that $p_0 > \hat{p}$):

$$\begin{aligned}\left(1-\tfrac{p_0}{p}\right)\Upsilon_0\pi_H(\tilde{q}_0) + \tfrac{p_0}{p}\left[\Upsilon_p\pi_H(\tilde{q}_p) + (1-\Upsilon_p)p\pi_L(q_L^*)\right] + K(1-\delta) \leqslant \\ \left(1-\tfrac{p_0}{p}\right)\pi'_H(0)\Upsilon_p + \tfrac{p_0}{p}\left[\Upsilon_p\pi_H(q_H^*) + (1-\Upsilon_p)p\pi_L(q_L^*)\right] + K\left(1+\tfrac{2\pi'_H(0)(1-\hat{p})}{\Delta\theta}\right)(1-\delta)\end{aligned}$$

We define $K_1 := \left(1+\frac{2\pi'_H(0)(1-\hat{p})}{\Delta\theta}\right)$. It follows from inequality (5) and the inequality above that

$$p_0\pi_L(q_L^*) \leqslant \left(1-\frac{p_0}{p}\right)\pi'_H(0)\Upsilon_p + \frac{p_0}{p}\left[\Upsilon_p\pi_H(q_H^*) + (1-\Upsilon_p)p\pi_L(q_L^*)\right] + K_1(1-\delta)$$

which leads to:

$$\begin{aligned}0 \leqslant \Upsilon_p\left[\left(1-\tfrac{p_0}{p}\right)\pi'_H(0) + \tfrac{p_0}{p}\left[\pi_H(q_H^*) - p\pi_L(q_L^*)\right]\right] + K_1(1-\delta) \leqslant \\ \Upsilon_p\left[\left(1-\tfrac{p-\frac{f(p)}{2}}{p}\right)\pi'_H(0) + \tfrac{p-\frac{f(p)}{2}}{p}\left[\pi_H(q_H^*) - p\pi_L(q_L^*)\right]\right] + K_1(1-\delta)\end{aligned} \tag{35}$$

The second inequality holds because the expression $\left(1-\frac{p_0}{p}\right)\pi_H'(0)+\frac{p_0}{p}\left[\pi_H(q_H^*)-p\pi_L(q_L^*)\right]$ is affine in p_0 and is negative for $p_0=p$ and equal to zero for $p_0=p-f(p)$ (see the definition of the function $f(\cdot)$ in equation (2)). Also, recall that $p_0\in\left[p-\frac{f(p)}{2},p\right]$.

From inequality (35) we obtain:

$$\Upsilon_p \leqslant \frac{K_1}{\left(p-\frac{f(p)}{2}\right)\pi_L(q_L^*)-\left(\frac{f(p)}{2p}\right)\pi_H'(0)-\left(1-\frac{f(p)}{2p}\right)\pi_H(q_H^*)}(1-\delta) := K_2(1-\delta)$$

and, thus,

$$\left(\frac{1-p}{1-p_0}\right)\left(\frac{p_0}{p}\right)\Upsilon_p \leqslant \Upsilon_p \leqslant K_2(1-\delta) \tag{36}$$

Finally, using (34) and (36) and $p_0<p^C$ we have:

$$\left(\frac{1}{1-p_0}\right)\left(1-\frac{p_0}{p}\right)\Upsilon_0\tilde{q}_0 \leqslant \left(\frac{1}{1-p^C}\right)\left(1-\frac{p_0}{p}\right)\Upsilon_p+\frac{2K(1-\delta)}{\Delta\theta} \leqslant$$
$$\left(\frac{1}{1-p^C}\right)\Upsilon_p+\frac{2K(1-\delta)}{\Delta\theta} \leqslant \left(\frac{1}{1-p^C}\right)K_2(1-\delta)+\frac{2K(1-\delta)}{\Delta\theta}$$

Recall that $\tilde{q}_0>\frac{\check{q}_H}{2}$. It follows from the last inequality that

$$\left(\frac{1}{1-p_0}\right)\left(1-\frac{p_0}{p}\right)\Upsilon_0 \leqslant \left(\frac{2}{\check{q}_H}\right)\left[\frac{K_2}{1-p^C}+\frac{2K}{\Delta\theta}\right](1-\delta)$$

which coupled with (36) implies the result. ■

Proof of Lemma 7.

Fix a PBE (σ,μ) and a history h^t as described in the statement of the lemma. The firm's continuation payoff after offering the menu m_t is equal to

$$V_F(h^t,m_t)=\left(1-\mu\left(h^t\right)\right)V_F(h^t,m_t;H)+\mu\left(h^t\right)V_F(h^t,m_t;L)$$

Recall from Lemma 6 that $[\hat{p}+\varepsilon,1]$ is a firing region. Therefore, it follows from Lemma 5 that there exist $\bar{K}$ and $\bar{\delta}>1-\frac{\varepsilon\pi_L\left(q_L^*\right)}{2\bar{K}}$ such that $\delta>\bar{\delta}$ implies

$$V_F(h^t,m_t;H) \leqslant \bar{K}(1-\delta)$$

Then it follows from the last inequality that

$$V_F(h^t,m_t) \leqslant (\hat{p}-\varepsilon)\pi_L(q_L^*)+\bar{K}(1-\delta) < \left(\hat{p}-\frac{\varepsilon}{2}\right)\pi_L(q_L^*) < \pi_H(q_H^*)$$

contradicting Lemma 1. ■

Proof of Part II of Proposition 2.

We start with the proof of the first property. By contradiction, suppose that there exists a sequence $\{\delta_n, (\sigma_n, \mu_n)\}_{n=1}^{\infty}$ such that δ_n converges to one, (σ_n, μ_n) is a PBE of the game with discount factor equal to δ_n, and

$$\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)}\left[\delta_n^{\mathbb{T}}\right] = \xi > 0 \tag{37}$$

Without loss of generality, we assume that $W_H\left(h^0; (\sigma_n, \mu_n)\right) = 0$ for every n.

Let $\underline{k} := \left\lfloor \frac{2}{1-\hat{p}} \right\rfloor$, and for $k = \underline{k}, \underline{k}+1, \ldots$, let $\tilde{\mathbb{T}}_k \leqslant \infty$ be the random time that stops the play at the first history $\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}\right)$ at which the menu $m_{\tilde{\mathbb{T}}}$ contains a contract $(x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})$ accepted with positive probability and for which $\mu\left(h^{\tilde{\mathbb{T}}}, m_{\tilde{\mathbb{T}}}, (x_{\tilde{\mathbb{T}}}, q_{\tilde{\mathbb{T}}})\right) \geqslant \hat{p} + \frac{1}{k}$. As usual, we set $\tilde{\mathbb{T}}_k = \infty$ if the event does not occur in finite time.

It follows from Lemma 5 that for every $k \geqslant \underline{k}$ there exist $n_k^1 \in \mathbf{N}$ and K_k^1 such that for every $n \geqslant n_k^1$ the PBE (σ_n, μ_n) satisfies the following property. If the firm offers a menu with a contract that is accepted with positive probability and leads to a belief weakly larger than $\hat{p} + \frac{1}{k}$, then the expected discounted time until the high type quits the relationship is bounded above by $K_k^1 (1 - \delta_n)$. Thus, for $n \geqslant n_k^1$ we have:

$$\left|\mathbb{E}_{(\sigma_n,\mu_n)}\left[\delta_n^{\mathbb{T}}\right] - \mathbb{E}_{(\sigma_n,\mu_n)}\left[\mathbb{I}_{\left\{\tilde{\mathbb{T}}_k<\infty\right\}}\left(1 - \mu_n\left(h^{\tilde{\mathbb{T}}_k}\right)\right)\delta_n^{\tilde{\mathbb{T}}_k}\right]\right| \leqslant K_k^1 (1 - \delta_n) \tag{38}$$

Next recall that $\left[\hat{p} + \frac{1}{k}\right]$ is a firing region and Lemma 5 (property ii) provides an upper bound to the firm's continuation payoff when it offers a menu with a contract the leads to a firing region. Therefore, for every $k \geqslant \underline{k}$ there exist $n_k^2 \in \mathbf{N}$ and K_k^2 such that for every $n \geqslant n_k^2$ the firm's equilibrium payoff is bounded as follows:

$$V_F(h^0; (\sigma_n, \mu_n)) \leqslant \mathbb{E}_{(\sigma_n,\mu_n)}\left[\mathbb{I}_{\left\{\tilde{\mathbb{T}}_k<\infty\right\}}\left[(1-\delta_n)\sum_{t=0}^{\tilde{\mathbb{T}}_k-1}\delta_n^t \pi_H(q_t) + \delta_n^{\tilde{\mathbb{T}}_k}\mu_n\left(h^{\tilde{\mathbb{T}}_k}\right)\pi_L(q_L^*)\right] + \right.$$
$$\left. \mathbb{I}_{\left\{\tilde{\mathbb{T}}_k=\infty\right\}}(1-\delta_n)\sum_{t=0}^{\tilde{\mathbb{T}}_k-1}\delta_n^t \pi_H(q_t)\right] + K_k^2 (1 - \delta_n)$$

Notice that when $\tilde{\mathbb{T}}_k < \infty$, the belief $\mu_n\left(h^{\tilde{\mathbb{T}}_k}\right)$ is, by definition, smaller than $\hat{p}+\frac{1}{k}$, and, therefore, we have:

$$\mu_n\left(h^{\tilde{\mathbb{T}}_k}\right)\pi_L\left(q_L^*\right) < \left(\hat{p}+\frac{1}{k}\right)\pi_L\left(q_L^*\right) = \pi_H\left(q_H^*\right)+\frac{1}{k}\pi_L\left(q_L^*\right)$$

Combining the last two inequalities, for every $n \geqslant n_k^2$ we obtain:

$$V_F(h^0;(\sigma_n,\mu_n)) \leqslant \mathbb{E}_{(\sigma_n,\mu_n)}\left[\mathbb{I}_{\left\{\tilde{\mathbb{T}}_k<\infty\right\}}\left[(1-\delta_n)\sum_{t=0}^{\tilde{\mathbb{T}}_k-1}\delta_n^t\pi_H\left(q_t\right)+\delta_n^{\tilde{\mathbb{T}}_k}\pi_H\left(q_H^*\right)\right]+\right.$$

$$\left.\mathbb{I}_{\left\{\tilde{\mathbb{T}}_k=\infty\right\}}(1-\delta_n)\sum_{t=0}^{\tilde{\mathbb{T}}_k-1}\delta_n^t\pi_H\left(q_t\right)\right]+K_k^2\left(1-\delta_n\right)+\tfrac{1}{k}\pi_L\left(q_L^*\right)=$$

$$\pi_H\left(q_H^*\right)-\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{t=0}^{\tilde{\mathbb{T}}_k-1}\delta_n^t\left[\pi_H\left(q_H^*\right)-\pi_H\left(q_t\right)\right]\right]+K_k^2\left(1-\delta_n\right)+\tfrac{1}{k}\pi_L\left(q_L^*\right)$$

This and the fact that the firm's payoff is bounded below by $\pi_H\left(q_H^*\right)$ lead to the following result. For every $k \geqslant \underline{k}$,

$$\limsup_{n\to\infty}\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{t=0}^{\tilde{\mathbb{T}}_k-1}\delta_n^t\left[\pi_H\left(q_H^*\right)-\pi_H\left(q_t\right)\right]\right]\leqslant\frac{1}{k}\pi_L\left(q_L^*\right) \tag{39}$$

Inequality (39) implies that for every $\eta > 0$ there exists $k_\eta \in \mathbf{N}$ such that for every $k \geqslant k_\eta$ there exists $\hat{n}_k \in \mathbf{N}$ such that for $n \geqslant \hat{n}_k$ we have

$$\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{t=0}^{\tilde{\mathbb{T}}_k-1}\delta_n^t\left|q_H^*-q_t\right|\right]\leqslant\eta \tag{40}$$

Furthermore, k_η and $\hat{n}_k$ are such that for every $k \geqslant k_\eta$ and every $n \geqslant \hat{n}_k$

$$\frac{\xi}{1-\hat{p}}-\eta\leqslant\mathbb{E}_{(\sigma_n,\mu_n)}\left[\mathbb{I}_{\left\{\tilde{\mathbb{T}}_k<\infty\right\}}\delta_n^{\tilde{\mathbb{T}}_k}\right]\leqslant\frac{\xi}{1-\hat{p}}+\eta$$

The above result is a consequence of equality (37), inequality (38), and Lemma 7.

Fix $\varepsilon \in \left(0, \frac{\Delta\theta q_H^*\xi}{4(1-\hat{p})\left(1+\Delta\theta q_H^*\right)}\left(\frac{\hat{p}}{p_0}-\frac{1-\hat{p}}{1-p_0}\right)\right)$. Recall that for every k, the interval $\left[\hat{p}+\frac{1}{k},1\right]$ is a firing region and Lemma 5 (property iii) provides an upper bound to the low type's continuation payoff when the firm's menu contains a contract that leads to a firing region. Finally, recall that if a history h^t is reached with probability $\Pr(h^t)$ under (σ_n, μ_n), then that history is reached with probability $\frac{\mu_n(h^t)}{p_0}\Pr(h^t)$ if the worker behaves according to σ_n^L, and with probability $\frac{(1-\mu_n(h^t))}{(1-p_0)}\Pr(h^t)$ if the worker behaves according to σ_n^H.

Putting together these observations and the last three inequalities we conclude that there exist $\tilde{k}$ and $\tilde{n}$ such that for every $n \geqslant \tilde{n}$ the low type obtains a payoff of at most

$$\Delta\theta q_H^*\left(1-\mathbb{E}_{(\sigma_n,\mu_n)}\left[\delta_n^{\tilde{\mathbb{T}}_{\tilde{k}}}|\sigma_n^L\right]\right)+\varepsilon \leqslant \Delta\theta q_H^*\left(1-\frac{\hat{p}}{p_0}\frac{\xi}{1-\hat{p}}+\varepsilon\right)+\varepsilon$$

when he behaves according to σ_n^L, and a payoff of at least

$$\Delta\theta q_H^*\left(1-\mathbb{E}_{(\sigma_n,\mu_n)}\left[\delta_n^{\tilde{\mathbb{T}}_{\tilde{k}}}|\sigma_n^H\right]\right)-\varepsilon \geqslant \Delta\theta q_H^*\left(1-\frac{1-\hat{p}}{1-p_0}\frac{\xi}{1-\hat{p}}-\varepsilon\right)-\varepsilon$$

when he behaves according to σ_n^H.

Notice that

$$\Delta\theta q_H^*\left(1-\tfrac{1-\hat{p}}{1-p_0}\tfrac{\xi}{1-\hat{p}}-\varepsilon\right)-\varepsilon-\Delta\theta q_H^*\left(1-\tfrac{\hat{p}}{p_0}\tfrac{\xi}{1-\hat{p}}+\varepsilon\right)-\varepsilon =$$
$$\Delta\theta q_H^*\tfrac{\xi}{1-\hat{p}}\left(\tfrac{\hat{p}}{p_0}-\tfrac{1-\hat{p}}{1-p_0}\right)-2\varepsilon\left(1+\Delta\theta q_H^*\right)>0$$

which implies that for n sufficiently large the low type has an incentive to deviate and follow σ_n^H instead of the equilibrium strategy σ_n^L.

Notice that property ii) in Part II of Proposition 2 follows directly from part i) and inequality (40).

Finally, we turn to part iii). Assume, towards a contradiction, that there exists a sequence $\{\delta_n, (\sigma_n, \mu_n)\}$ such that δ_n converges to one and

$$\lim_{n\to\infty}\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{t=0}^{\mathbb{T}-1}\delta_n^t\left(x_t-\theta_H q_H^*-\alpha\right)|i\right]=\tilde{\xi}>0$$

for some $i \in \{H, L\}$. Using part i) and part ii) it is immediate to conclude that

$$\limsup_{n\to\infty} V_F(h^0;(\sigma_n,\mu_n)) \leqslant \pi_H\left(q_H^*\right)-\min\left\{p_0, 1-p_0\right\}\tilde{\xi}<\pi_H\left(q_H^*\right)$$

which leads to a contradiction and concludes the proof. ■

Appendix D: The case $\alpha = 0$

In this appendix, we illustrate the changes needed in the proofs above to accommodate the case that $\alpha = 0$. First, we slightly modify the notion of firing region and replace Definition 2 with the following definition.

Definition 3 *The interval* $[p, 1]$ *is a firing region if there exists a function* $\varrho : (0,1] \to \mathbb{R}_{++}$ *with* $\lim_{\delta\to 1} \varrho(\delta) = 0$ *satisfying the following property. Let* (σ, μ) *be an arbitrary PBE of the game with discount factor* δ *and consider a history* h^t *at which* $\mu(h^t) \geqslant p$. *Then we have:*

i) $\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}|h^t, H\right]$, *the expected discounted time until the high type quits the relationship, is bounded by* $\varrho(\delta)$.

ii) $V_F(h^t; (\sigma,\mu), H)$, *the firm's continuation payoff at the history* h^t *conditional on type* H, *is bounded by* $\varrho(\delta)$;

iii) $W_L(h^t; (\sigma,\mu))$, *the low type's continuation payoff at the history* h^t, *is bounded by* $\varrho(\delta)$.

Notice that the interval $\left[p^C, 1\right]$ is a firing region. Lemma 5 should be replaced with Lemma 10.

Lemma 10 *Suppose that* $[p, 1]$ *is a firing region. There exists a function* $\zeta : (0,1] \to \mathbb{R}_{++}$ *with* $\lim_{\delta\to 1} \zeta(\delta) = 0$ *satisfying the following property. Let* (σ, μ) *be an arbitrary PBE of the game with discount factor* δ, *and consider a history* h^t *with* $\mu(h^t) < p$. *Suppose that at* h^t *the firm offers a menu* m_t *containing a contract* $\left(x_t^L, q_t^L\right)$ *accepted with positive probability and for which*

$$\mu\left(h^t, m_t, \left(x_t^L, q_t^L\right)\right) \geqslant p$$

Then we have:

i) $\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1}\delta^{\tau-t}|h^t, m_t, H\right]$, *the expected discounted time until the high type quits the relationship, is bounded by* $\zeta(\delta)$;

ii) $V_F(h^t, m_t; (\sigma,\mu), H)$, *the firm's continuation payoff at the history* (h^t, m_t) *conditional on type* H, *is bounded by* $\zeta(\delta)$;

iii) $W_L(h^t, m_t; (\sigma,\mu))$, *the low type's continuation payoff at the history* (h^t, m_t), *is bounded by* $\zeta(\delta)$.

Proof of Lemma 10.

Fix a PBE (σ, μ) and a history h^t $(\mu(h^t) < p)$ at which the firm offers a menu m_t with the properties described in the statement of the lemma. First, notice that if the high type rejects all the contracts in m_t with probability one (i.e., the high type quits), then the expected length of the relationship and the firm's payoff (conditional on the high type) are both equal to zero, while the low type's continuation payoff is bounded above by $\Delta\theta(1-\delta)$.

Consider now the case in which the high type accepts a contract in m_t, say $\left(x_t^H, q_t^H\right)$, with positive probability. We let h_H^{t+1} denote the history $\left(h^t, m_t, \left(x_t^H, q_t^H\right)\right)$. We also let h_L^{t+1} denote the history $\left(h^t, m_t, \left(x_t^L, q_t^L\right)\right)$.

The fact that type $i = H, L$ accepts with positive probability the contract (x_t^i, q_t^i) implies

$$(1-\delta)\left(x_t^H - \theta_H q_t^H - \alpha\right) + \delta W_H\left(h_H^{t+1}\right) \geqslant (1-\delta)\left(x_t^L - \theta_H q_t^L - \alpha\right) + \delta W_H\left(h_L^{t+1}\right)$$
$$(1-\delta)\left(x_t^L - \theta_L q_t^L - \alpha\right) + \delta W_L\left(h_L^{t+1}\right) \geqslant (1-\delta)\left(x_t^H - \theta_L q_t^H - \alpha\right) + \delta W_L\left(h_H^{t+1}\right)$$

We add the two incentive compatibility constraints and obtain

$$(1-\delta)\Delta\theta\left(q_t^L - q_t^H\right) + \delta\left(W_L\left(h_L^{t+1}\right) - W_H\left(h_L^{t+1}\right)\right) \geqslant \delta\left(W_L\left(h_H^{t+1}\right) - W_H\left(h_H^{t+1}\right)\right)$$

Recall that $\mu\left(h_L^{t+1}\right) \geqslant p$ and that $[p, 1]$ is a firing region. Therefore, there exists a function $\varrho(\cdot)$ such that $W_L\left(h_L^{t+1}\right) \leqslant \varrho(\delta)$. Of course, $W_H\left(h_L^{t+1}\right) \geqslant 0$. This, together with the above inequality, implies:

$$\frac{(1-\delta)\Delta\theta + \varrho(\delta)}{\delta} \geqslant \frac{(1-\delta)\Delta\theta\left(q_t^L - q_t^H\right) + \varrho(\delta)}{\delta} \geqslant \left(W_L\left(h_H^{t+1}\right) - W_H\left(h_H^{t+1}\right)\right)$$

Recall that a strategy available to the low type is to imitate, in every period, the high type. Therefore, we have

$$W_L\left(h_H^{t+1}\right) \geqslant W_H\left(h_H^{t+1}\right) + \Delta\theta\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t+1}^{\mathbb{T}-1}\delta^{\tau-t-1}q_\tau | h_H^{t+1}, H\right]$$

where $\mathbb{T}$ denotes the random time in which the worker quits the relationship.

Combining the last two inequalities, we obtain

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t+1}^{\mathbb{T}-1}\delta^{\tau-t-1}q_\tau | h_H^{t+1}, H\right] \leqslant \frac{(1-\delta)\Delta\theta + \varrho(\delta)}{\delta\Delta\theta} \tag{41}$$

We can now prove part i) of Lemma 10. Inequality (41) together with Claim 6 below show the existence of a function $\zeta : (0,1] \to \mathbb{R}_{++}$, with $\lim_{\delta\to 1} \zeta(\delta) = 0$, satisfying the condition in part i).

Claim 6 *Consider a sequence of discount factors $\{\delta_n\}_{n=1}^{\infty}$ converging to one. For each $n = 1, 2, \ldots,$ let (σ_n, μ_n) be a PBE of the game with discount factor δ_n, and let h_n^t be a history of the game.*[20] *If*

$$\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{\tau=t}^{\mathbb{T}-1} \delta_n^{\tau-t} q_\tau | h_n^t, H \right] = 0 \tag{42}$$

then

$$\lim_{n\to\infty} \mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{\tau=t}^{\mathbb{T}-1} \delta_n^{\tau-t} | h_n^t, H \right] = 0 \tag{43}$$

Proof of Claim 6.

Assume, towards a contradiction, that there exist $\varepsilon > 0$ and a sequence $\{\delta_n, (\sigma_n, \mu_n), h_n^t\}_{n=1}^{\infty}$ with $\{\delta_n\}$ converging to one, and for which equality (42) holds and

$$\mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{\tau=t}^{\mathbb{T}-1} \delta_n^{\tau-t} | h_n^t, H \right] \geqslant \varepsilon \tag{44}$$

for every n.

Fact 2 *We assume, without loss of generality, that $\mu_n(h_n^t) < p^C$ for any n.*

Recall that under any PBE (σ_n, μ_n), if $\mu_n(h_n^t) \geqslant p^C$, then the high type rejects all the contracts in the firm's menu with probability one.

Fact 3 *If $\mu_n(h_n^t) \leqslant \frac{\pi_H(q_H^*)}{2\pi_L(q_L^*)}$, then*

$$\mathbb{E}_{(\sigma_n,\mu_n)} \left[(1-\delta_n) \sum_{\tau=t}^{\mathbb{T}-1} \delta_n^{\tau-t} q_\tau | h_n^t, H \right] \geqslant \frac{\pi_H(q_H^*)}{2\pi'_H(0)}$$

[20] Notice that the length of the history h_n^t may vary with n. However, to ease the notation, we do not index the length t by n. This does not cause any confusion.

By contradiction, suppose that the inequality above is violated. It follows from the concavity of $\pi_H(\cdot)$ that

$$V_F(h_n^t;(\sigma_n,\mu_n)) \leqslant \mu_n\left(h_n^t\right)\pi_L\left(q_L^*\right) + \left(1-\mu_n\left(h_n^t\right)\right)\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{\tau=t}^{\mathbb{T}-1}\delta_n^{\tau-t}q_\tau|h_n^t,H\right]\pi_H'(0) <$$

$$\mu_n\left(h_n^t\right)\pi_L\left(q_L^*\right) + \left(1-\mu_n\left(h_n^t\right)\right)\frac{\pi_H\left(q_H^*\right)}{2} \leqslant \pi_H\left(q_H^*\right)$$

which contradicts Lemma 1.

Fact 4 *There exists $\bar{n}$ such that for $n \geqslant \bar{n}$, $\mu_n\left(h_n^t\right) \in \left(\frac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}, p^C\right)$.*

It follows immediately from the first two facts and equality (42).

Fact 5 *We have*

$$\lim_{n\to\infty}\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{\tau=t}^{\mathbb{T}-1}\delta_n^{\tau-t}\left|q_\tau-q_L^*\right||h_n^t,L\right]=0$$

Taking a subsequence if necessary, assume, towards a contradiction, that the limit above exists and is different from zero. Then it follows that

$$\limsup_{n\to\infty}\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{\tau=t}^{\mathbb{T}-1}\delta_n^{\tau-t}\left(v\left(q_\tau\right)-x_\tau\right)|h_n^t,L\right] < \pi_L\left(q_L^*\right)$$

Notice that equality (42) implies

$$\limsup_{n\to\infty}\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{\tau=t}^{\mathbb{T}-1}\delta_n^{\tau-t}\left(v\left(q_\tau\right)-x_\tau\right)|h_n^t,H\right] \leqslant 0$$

Putting together the last two inequalities, we obtain that for n sufficiently large

$$V_F(h_n^t;(\sigma_n,\mu_n)) < \mu_n\left(h_n^t\right)\pi_L\left(q_L^*\right)$$

which contradicts Lemma 1.

We can now conclude the proof of the claim. Let $T_n\left(\frac{\varepsilon}{2}\right)$ be the smallest positive integer such that $\delta_n^{T_n\left(\frac{\varepsilon}{2}\right)} \leqslant 1-\frac{\varepsilon}{4}$ and take $n^* \in \mathbb{N}$ such that $n \geqslant n^*$ implies $\delta_n^{T_n\left(\frac{\varepsilon}{2}\right)} > 1-\frac{\varepsilon}{2}$. Inequality (44) implies that for every $n \geqslant n^*$

$$\mathbb{P}_{(\sigma_n,\mu_n)}\left[\mathbb{T} > t+T_n\left(\frac{\varepsilon}{2}\right)|h_n^t,H\right] \geqslant \frac{\varepsilon}{4}$$

Also, given equality (42), we can take $n^{**} \geqslant n^{*}$ such that $n \geqslant n^{**}$ implies

$$\mathbb{P}_{(\sigma_n,\mu_n)}\left[\mathbb{T} > t + T_n\left(\frac{\varepsilon}{2}\right) \text{ and } (1-\delta_n)\sum_{\tau=t}^{t+T_n\left(\frac{\varepsilon}{2}\right)} \delta_n^{\tau-t} q_\tau < \frac{\varepsilon}{8} q_L^* \,|h_n^t, H\right] \geqslant \frac{\varepsilon}{8}$$

Finally, if follows from equality (42) and from Facts 3 and 4 that there exists $n^{***} \geqslant n^{**}$ such that for $n \geqslant n^{***}$ we have

$$\mathbb{P}_{(\sigma_n,\mu_n)}\left[\mathbb{T} > t + T_n\left(\tfrac{\varepsilon}{2}\right) \text{ and } (1-\delta_n)\sum_{\tau=t}^{t+T_n\left(\frac{\varepsilon}{2}\right)} \delta_n^{\tau-t} q_\tau < \tfrac{\varepsilon}{8} q_L^* \text{ and } \mu_n\left(h_n^{t+T_n\left(\frac{\varepsilon}{2}\right)}\right) \leqslant \tfrac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}|h_n^t, H\right] \geqslant \tfrac{\varepsilon}{16}$$

This means that for every $n \geqslant n^{***}$ there exists a subset of histories $h_n^{t+T_n\left(\frac{\varepsilon}{2}\right)}$ which, conditional on type H, are reached with probability of at least $\frac{\varepsilon}{16}$ and at which the firm's belief is at most $\frac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}$. It follows from Fact 3 that for every history $h_n^{t+T_n\left(\frac{\varepsilon}{2}\right)}$ with $\mu_n\left(h_n^{t+T_n\left(\frac{\varepsilon}{2}\right)}\right) \leqslant \frac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}$, we have

$$\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{\tau=t+T_n\left(\frac{\varepsilon}{2}\right)}^{\mathbb{T}-1} \delta_n^{\tau-t} q_\tau |h_n^{t+T_n\left(\frac{\varepsilon}{2}\right)}, H\right] \geqslant \frac{\pi_H\left(q_H^*\right)}{2\pi_H'\left(0\right)}$$

which, in turn, implies

$$\mathbb{E}_{(\sigma_n,\mu_n)}\left[(1-\delta_n)\sum_{\tau=t}^{\mathbb{T}-1} \delta_n^{\tau-t} q_\tau |h_n^t, H\right] \geqslant \frac{\varepsilon}{32}\frac{\pi_H\left(q_H^*\right)}{\pi_H'\left(0\right)}$$

for every $n \geqslant n^{***}$, contradicting equality (42). ■

To verify part ii) of Lemma 10 notice that

$$V_F(h^t, m_t; (\sigma,\mu), H) \leqslant v(1)\,\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{\tau=t}^{\mathbb{T}-1} \delta^{\tau-t} |h^t, m_t, H\right]$$

Therefore, it follows from part i) that there exists a function $\tilde{\zeta} : (0,1] \to \mathbb{R}_{++}$, with $\lim_{\delta\to 1} \tilde{\zeta}(\delta) = 0$ such that

$$V_F\left(h^t, m_t; (\sigma,\mu), H\right) \leqslant \tilde{\zeta}(\delta)$$

To establish part iii), notice that it follows from the above inequality that

$$\pi_H\left(q_H^*\right) \leqslant V_F\left(h^t, m_t; (\sigma, \mu), H\right) \leqslant \left(1-\mu\left(h^t\right)\right)\tilde{\zeta}(\delta)+\mu\left(h^t\right)\pi_L\left(q_L^*\right)$$

We take $\delta_1 < 1$ such that $\delta \geqslant \delta_1$ implies $\tilde{\zeta}(\delta) < \frac{\pi_H\left(q_H^*\right)\pi_L\left(q_L^*\right)}{2\pi_L\left(q_L^*\right)-\pi_H\left(q_H^*\right)}$. Therefore, for $\delta \geqslant \delta_1$ the last inequality implies $\mu\left(h^t\right) \geqslant \frac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}$.

Recall that the firm's payoff at h^t is bounded below by $\mu\left(h^t\right)\pi_L\left(q_L^*\right)$. Thus, we have

$$\mu\left(h^t\right)\pi_L\left(q_L^*\right) \leqslant V_F\left(h^t, m_t; (\sigma, \mu), H\right) \leqslant \left(1-\mu\left(h^t\right)\right)\tilde{\zeta}(\delta)+\mu\left(h^t\right)\left[\pi_L\left(q_L^*\right)-W_L\left(h^t, m_t; (\sigma, \mu), H\right)\right] \leqslant$$

$$\left(1-\frac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}\right)\tilde{\zeta}(\delta)+\mu\left(h^t\right)\left[\pi_L\left(q_L^*\right)-W_L\left(h^t, m_t; (\sigma, \mu), H\right)\right]$$

We conclude that for $\delta \geqslant \delta_1$

$$W_L\left(h^t, m_t; (\sigma, \mu), H\right) \leqslant \frac{1}{\mu\left(h^t\right)}\left(1-\frac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}\right)\tilde{\zeta}(\delta) \leqslant \left(\frac{2\pi_L\left(q_L^*\right)}{\pi_H\left(q_H^*\right)}\right)\left(1-\frac{\pi_H\left(q_H^*\right)}{2\pi_L\left(q_L^*\right)}\right)\tilde{\zeta}(\delta)$$

establishing part iii). ■

Next, we explain how to modify the proof of Claim 2. First, we replace the linear bound $\bar{K}_1(1-\delta)$ used in Claim 2 (see inequality (3)) with a bound $\rho(\delta)$ (satisfying $\lim_{\delta\to 1}\rho(\delta) = 0$). Proceeding exactly as in the proof of Claim 2, we conclude that the firm's payoff $V_F\left(h^0\right)$ is bounded above by

$$\left(1-\frac{p_0}{p}\right)\Upsilon(0)\,\pi_H\left(\tilde{q}_0\right)+\frac{p_0}{p}\left[\Upsilon(p)\,\pi_H\left(\tilde{q}_p\right)+\left(1-\Upsilon(p)\right)p\pi_L\left(q_L^*\right)\right]+\rho(\delta)\,, \qquad (45)$$

and the following incentive compatibility constraint must be satisfied.

$$\Upsilon(p)\,\tilde{q}_p\Delta\theta+\rho(\delta) \geqslant \left(\frac{1-p}{1-p_0}\right)\left(\frac{p_0}{p}\right)\Upsilon(p)\,\tilde{q}_p\Delta\theta+\left(\frac{1}{1-p_0}\right)\left(1-\frac{p_0}{p}\right)\Upsilon(0)\,\tilde{q}_0\Delta\theta. \quad (46)$$

It follows from the concavity of $\pi_H(\cdot)$ that $\pi_H\left(\tilde{q}_0\right) \leqslant \pi_H'(0)\,\tilde{q}_0$. Also, by replacing $\pi_H\left(\tilde{q}_p\right)$ with $\pi_H\left(q_H^*\right)$ in (45), and $\tilde{q}_p$ with one in (46) we conclude that

$$V_F\left(h^0\right) \leqslant \left(1-\frac{p_0}{p}\right)\pi_H'(0)\,\Upsilon(0)\,\tilde{q}_0+\frac{p_0}{p}\left[\Upsilon(p)\,\pi_H\left(q_H^*\right)+\left(1-\Upsilon(p)\right)p\pi_L\left(q_L^*\right)\right]+\rho(\delta) \quad (47)$$

and $\tilde{q}_0$, $\Upsilon(0)$, and $\Upsilon(p)$ must satisfy

$$\Upsilon(p) \geqslant \Upsilon(0)\,\tilde{q}_0 - \frac{\rho(\delta)}{\Delta\theta\left[1-\left(\frac{1-p}{1-p_0}\right)\left(\frac{p_0}{p}\right)\right]}$$

Recall that $\lim_{\delta\to 1}\rho(\delta) = 0$ and that $p_0 \geqslant p - \frac{f(p)}{2} > \hat{p}$. This implies that as δ goes to one, both $\Upsilon(0)\,\tilde{q}_0$ and must $\Upsilon(p)$ shrink to zero. In fact, if $\Upsilon(0)\,\tilde{q}_0$ remains bounded away from zero (as δ goes to one), then it follows from the last two inequalities that for δ sufficiently large $V_F(h^0)$ is strictly smaller than $p_0\pi_L(q_L^*)$, contradicting Lemma 1. Similarly, if $\Upsilon(0)\,\tilde{q}_0$ goes to zero and $\Upsilon(p)$ remains bounded away from zero, then it follows from inequality (47) that for δ sufficiently large $V_F(h^0)$ is strictly smaller than $p_0\pi_L(q_L^*)$.

Clearly, if both $\Upsilon(0)\,\tilde{q}_0$ and $\Upsilon(p)$ converge to zero as δ goes to one, then we have:

$$\lim_{\delta\to 1}\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t q_t \mid h^0, H\right] = 0$$

We then apply Claim 6 to conclude that

$$\lim_{\delta\to 1}\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t \mid h^0, H\right] = 0$$

This shows that there exists a function $\zeta(\cdot)$ with $\lim_{\delta\to 1}\zeta(\delta) = 0$ such that

$$\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t \mid h^0, H\right] \leqslant \zeta(\delta)$$

Next, notice that

$$V_F(h^0;H) \leqslant v(1)\,\mathbb{E}_{(\sigma,\mu)}\left[(1-\delta)\sum_{t=0}^{\mathbb{T}-1}\delta^t|h^0, H\right] \leqslant v(1)\,\zeta(\delta)$$

Also, the argument in the proof of Claim 2 shows that

$$W_L\left(h^0\right) \leqslant \frac{1-p_0}{p_0}V_F(h^0;H) < \frac{1-\hat{p}}{\hat{p}}V_F(h^0;H) \leq \left(\frac{1-\hat{p}}{\hat{p}}\right)v(1)\,\zeta(\delta),$$

delivering the desired result.

Finally, we remark that the proof of part II) of Theorem 2 works in the same way if one replaces the respective linear bounds $K(1-\delta)$ with functions $\zeta(\cdot)$ satisfying $\lim_{\delta\to 1}\zeta(\delta) = 0$.

Appendix E: Proof of Proposition 3

We now describe a strategy profile and a system of beliefs which yield the payoffs $(V_{F,H}, V_{F,L}, W_H, W_L)$ (we divide the description into different phases). Then we show that unilateral deviations are not profitable when the discount factor δ is sufficiently large.

Screening Phase: In the first period, the firm offers the menu $\{(x_H, q_H), (x_L, q_L)\}$ (recall that (x_i, q_i), $i = H, L$, is a contract yielding the payoff $V_{F,i}$ to the firm and the payoff W_i to type i). If both contracts are rejected, the firm does not update its belief and insists on the same menu until a contract (x_i, q_i), $i = H, L$, is accepted. In this case, the firm's belief assigns probability one to type i. Furthermore, the firm does not revise its belief in future periods and the continuation equilibrium consistent with the automaton described below starting at the state $(i, 0)$ follows.

Suppose that during the screening phase the firm deviates and offers a menu m different from $\{(x_H, q_H), (x_L, q_L)\}$. Let $(x^*(m), q^*(m)) \in m$ denote the optimal contract for the high type in m. Formally:

$$x^*(m) - \theta_H q^*(m) - \alpha \geqslant x_j - \theta_H q_j - \alpha$$

for all $(x_j, q_j) \in m$.[21]

If $x^*(m) < \alpha + \theta_H + v(1)$, every type of the worker rejects all the contracts and the screening phase continues in the next period with the firm insisting on the menu $\{(x_L, q_L), (x_H, q_H)\}$. If any contract $(x_k, q_k) \in m$ is selected, the firm's belief assigns probability one to the low type and the continuation equilibrium consistent with the automaton described below starting at the state $(L, 2)$ follows.

If $x^*(m) \geqslant \alpha + \theta_H + v(1)$, every type of the worker accepts the contract $(x^*(m), q^*(m))$ and the screening phase continues in the next period. If any other contract $(x_k, q_k) \in m$ is accepted or if all the contracts are rejected, the firm's belief assigns probability one to the low type and the continuation equilibrium consistent with the automaton described below starting at the state $(L, 2)$ follows.

Post-Screening Phase: According to the description above, a post-screening phase can be reached in a state $(i, r) \in \{H, L\} \times \{0, 1, 2\}$. The transition function among the states and the action prescription for the firm and for type $i = H, L$ in state (i, r) are the

[21] If there are several optimal contracts for type H, we select the contract with the smallest index.

same as the ones in the automaton for type i presented in Section 2. The action prescription for type $j \neq i$ in a state (i, r) are defined below.

Actions of type L in the state $(H, 0)$: If the firm offers the menu $\{(x_H, q_H)\}$, the low type accepts (x_H, q_H). If the firm deviates and offers a different menu, then type L accepts the contract that yields the largest current payoff, provided that this is positive (if it is negative, the worker rejects all the contracts).[22]

Actions of type L in the state $(H, 1)$: If the firm offers the menu $\{(\bar{x}_H, q_H^*)\}$, the low type accepts $(\bar{x}_H, q_H^*)$. Consider a deviation by the firm. The low type rejects all the contracts (x, q) with $x < v(1) + \alpha$. Among the remaining contracts, the low type selects the contract which yields the largest current payoff, provided that this is positive (if it is negative, the worker rejects all the contracts).

Actions of type L in the state $(H, 2)$: If the firm offers the menu $\{(\underline{x}_H, q_H^*)\}$, the low type accepts $(\underline{x}_H, q_H^*)$. If the firm deviates and offers a different menu, then type L accepts the contract that yields the largest current payoff, provided that this is positive

Actions of type H in the state $(L, 0)$: If the firm offers the menu $\{(x_L, q_L)\}$, the high type accepts (x_L, q_L) if and only if $x_L - \theta_H q_L - \alpha \geqslant 0$. If the firm deviates and offers a different menu, then type H accepts the contract that yields the largest current payoff, provided that this is positive (if it is negative, the worker rejects all the contracts).

Actions of type H in the state $(L, 1)$: We distinguish between two cases. First, assume that $\bar{x}_L - \theta_H q_L^* - \alpha > 0$. In this case, if the firm offers the menu $\{(\bar{x}_L, q_L^*)\}$, the high type accepts $(\bar{x}_L, q_L^*)$. Consider a deviation by the firm. The high type rejects all the contracts (x, q) with $x < v(1) + \alpha$. Among the remaining contracts, the high type selects the contract which yields the largest current payoff, provided that this is positive.

Suppose now that $\bar{x}_L - \theta_H q_L^* - \alpha \leqslant 0$. In this case, the high type selects the contract which yields the largest current payoff, provided that this is positive.

Actions of type H in the state $(L, 2)$: If the firm offers the menu $\{(\underline{x}_L, q_L^*)\}$, the high type accepts $(\underline{x}_L, q_L^*)$ provided that it yields a positive current payoff. If the firm deviates and offers a different menu, then type H accepts the contract that yields the largest current payoff, provided that this is positive.

Optimality of the Proposed Strategies. We now analyze the parties' incentives and show deviations are not profitable for δ sufficiently large. Let h^t be an arbitrary history

[22] As usual, the worker selects the contract with the smallest index among those who yield the largest current payoff.

in the screening phase. We let $V_F(S)$ denote the firm's continuation payoff at h^t (the payoff is computed before the firm offers the menu). Recall that the firm's belief at h^t is equal to the prior p_0. We also let $W_i(S)$, $i = H, L$, denote the continuation payoff of type i at h^t. We have:

$$V_F(S) = (1 - p_0) V_{F,H} + p_0 V_{F,L} \quad W_L(S) = W_L \quad W_H(S) = W_H$$

We now turn to the post-screening phase. For $i \in \{H, L\}$ and $r \in \{0, 1, 2\}$, let $V_F(i, r)$ and $W_i(i, r)$ denote the firm and type i's continuation payoff, respectively, in the state (i, r).[23] These payoffs are equal to:

$$\begin{array}{llll}
V_F(0,H) = V_{F,H} & V_F(0,L) = V_{F,L} & W_H(0,H) = W_H & W_L(0,L) = W_L \\
V_F(1,H) = \frac{\varepsilon}{2} & V_F(1,L) = \frac{\varepsilon}{2} & W_H(1,H) = \pi_H(q_H^*) - \frac{\varepsilon}{2} & W_L(1,L) = \pi_L(q_L^*) - \frac{\varepsilon}{2} \\
V_F(2,H) = \pi_H(q_H^*) - \frac{\varepsilon}{2} & V_F(2,L) = \pi_L(q_L^*) - \frac{\varepsilon}{2} & W_H(2,H) = \frac{\varepsilon}{2} & W_L(2,L) = \frac{\varepsilon}{2}
\end{array}$$

Next, we specify the continuation payoff of type $i = H, L$ in the state (j, r), $j \neq i$ and $r \in \{0, 1, 2\}$. We have:

$$\begin{array}{ll}
W_L(0,H) = W_H + \Delta\theta q_H & W_H(0,L) = \max\{W_L - \Delta\theta q_L, 0\} \\
W_L(1,H) = \pi_H(q_H^*) + \Delta\theta q_H^* - \frac{\varepsilon}{2} & W_H(1,L) = \max\left\{\pi_L(q_L^*) - \Delta\theta q_L^* - \frac{\varepsilon}{2}, 0\right\} \\
W_L(2,H) = \Delta\theta q_H^* + \frac{\varepsilon}{2} & W_H(2,L) = \max\left\{-\Delta\theta q_L^* + \frac{\varepsilon}{2}, 0\right\}
\end{array}$$

To show that unilateral deviations from the proposed strategy profile are not profitable, it is enough to verify that finitely many inequalities are satisfied. Given the payoffs above, it is immediate to check that for every inequality, there is a critical value of δ above which the inequality is satisfied. Since the number of inequalities is finite, we conclude that there exists $\delta^\dagger \in (0, 1)$ such that for $\delta \geqslant \delta^\dagger$ no unilateral deviation is profitable.

Belief Update. It is straightforward to check that, after each menu posted by the firm, the proposed system of beliefs satisfies Bayes's rule after each choice that is taken by the worker with positive probability.

We conclude that the strategy profile and the system of beliefs presented above constitute a PBE when $\delta \geqslant \delta^\dagger$. ■

[23] The action prescription for type i in the state (i, r) is specified in Section 2.

References

[1] Acemoglu, D., M. Golosov, and A. Tsyvinski (2010): "Dynamic Mirrlees Taxation under Political Economy Constraints," *Review of Economic Studies*, 77, 841-881.

[2] Aumann, R. J., and M. B. Maschler (1995): *Repeated Games with Incomplete Information*, The MIT Press.

[3] Ausubel, L. M., and R. J. Deneckere (1989): "Reputation in Bargaining and Durable Goods Monopoly," *Econometrica*, 57, 511-531.

[4] Beccuti, J., and M. Möller (2018): "Dynamic Adverse Selection with a Patient Seller," *Journal of Economic Theory*, 173, 95-117.

[5] Bhaskar, V. (2014): "The Ratchet Effect Re-examined: A Learning Perspective," mimeo.

[6] Bhaskar, V. and G. Mailath (2017): "The Curse of Long Horizons," mimeo.

[7] Carmichael, L., and B. MacLeod (2000): "Worker Cooperation and the Ratchet Effect," *Journal of Labor Economics*, 18, 1-19.

[8] Farhi, E., C. Sleet, I. Werning, and S. Yeltekin (2012): "Non-linear Capital Taxation Without Commitment," *Review of Economic Studies*, 79, 1469-1493.

[9] Freixas, X., R. Guesnerie, and J. Tirole (1985): "Planning under Incomplete Information and the Ratchet Effect," *The Review of Economic Studies,* 52, 173-191.

[10] Fudenberg, D., and J. Tirole (1991): "Perfect Bayesian Equilibrium and Sequential Equilibrium," *Journal of Economic Theory*, 53, 236-260.

[11] Gibbons, R. (1987): "Piece-Rate Incentive Schemes," *Journal of Labor Economics*, 5, 413-429.

[12] Halac, M. (2012): "Relational Contracts and the Value of Relationships," *American Economic Review*, 102, 750-779.

[13] Hart, O., and J. Tirole (1988): "Contract Renegotiation and Coasian Dynamics," *Review of Economic Studies*, 55, 509–540.

[14] Kamenica, E., and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590-2615.

[15] Kanemoto, Y., and B. MacLeod (1992): "The Ratchet Effect and the Market for Secondhand Workers," *Journal of Labor Economics*, 10, 85-98.

[16] Laffont, J.J., and J. Tirole (1987): "Comparative Statics of the Optimal Dynamic Incentive Contract," *European Economic Review*, 31, 901-926.

[17] Laffont, J.J., and J. Tirole (1988): "The Dynamics of Incentive Contracts," *Econometrica,* 56, 1153–1175.

[18] Laffont, J.J., and J. Tirole (1990): "Adverse Selection and Renegotiation in Procurement," *Review of Economic Studies,* 57, 597-625.

[19] Laffont, J.J., and J. Tirole (1993): *A Theory of Incentives in Procurement and Regulation*, The MIT Press.

[20] Litwack, J. M. (1991): "Discretionary Behaviour and Soviet Economic Reform," *Soviet Studies,* 43, 255-279.

[21] Maestri, L. (2017): "Dynamic Contracting under Adverse Selection and Renegotiation," *Journal of Economic Theory*, 171, 136-173.

[22] Pęski, M. (2008): "Repeated Games with Incomplete Information on One Side," *Theoretical Economics*, 3, 29-84.

[23] Roy, D. (1952): "Quota Restriction and Goldbricking in a Machine Shop," *American Journal of Sociology,* 57, 427-442.

[24] Scheuer, F., and A. Wolitzky (2016): "Capital Taxation under Political Constraints," *American Economic Review*, 106, 2304-2328.

[25] Skreta, V. (2006): "Sequentially Optimal Mechanisms," *Review of Economic Studies*, 73, 1085-1111.

[26] Skreta, V. (2015): "Optimal Auction Design under Non-Commitment," *Journal of Economic Theory*, 159, 854-890.

[27] Strulovici, B. (2017): “Contract Negotiation and the Coase Conjecture: A Strategic Foundation for Renegotiation-Proof Contracts,” *Econometrica*, 85, 585-616.