An Adaptive Refinement Scheme for Radial Basis Function Collocation

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Abstract. In this paper we present an adaptive refinement algorithm for solving elliptic partial differential equations via a radial basis function (RBF) collocation method. The adaptive scheme is based on the use of an error indicator, which is characterized by the comparison of two RBF collocation solutions evaluated on a coarser set and a finer one. This estimate allows us to detect the domain parts that need to be refined by adding points in the selected areas. Numerical results support our study and point out the effectiveness of our algorithm.

Keywords: Meshfree methods · Adaptive algorithms · Refinement techniques · elliptic PDEs.

1 Introduction

In this paper we present a new adaptive refinement scheme for solving elliptic partial differential equations (PDEs). Our adaptive algorithm is applied to a nonsymmetric radial basis function (RBF) collocation method, which was originally proposed by Kansa [5]. This approach has engendered a large number of works, mainly by scientists from several different areas of science and engineering (see e.g. [1,2,3,4,7] and references therein). Basically, the adaptive scheme we propose is based on the use of an error indicator characterized by the comparison of two approximate RBF collocation solutions, which are evaluated on a coarser set and a finer one. This estimate allows us to identify the domain parts that need to be refined by adding points in the selected areas. In our numerical experiments we show the efficacy of our refinement algorithm, which is tested by modeling some Poisson-type problems.

The paper is organized as follows. In Section 2 we review some basic information on Kansa's collocation method, which is applied to elliptic PDEs. Section 3 describes the adaptive refinement algorithm. In Section 4 we show some numerical results carried out to illustrate the performance of the adaptive scheme. Section 5 contains conclusions.

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2 Nonsymmetric RBF Collocation

Given a domain $\Omega \subset \mathbb{R}^d$, we consider a (time independent) elliptic PDE along with its boundary conditions

$$\begin{aligned} \mathcal{L}u(\boldsymbol{x}) &= f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ \mathcal{B}u(\boldsymbol{x}) &= g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega, \end{aligned} \tag{1}$$

where $\mathcal{L}$ is a linear elliptic partial differential operator and $\mathcal{B}$ is a linear boundary operator.

For Kansa's collocation method we choose to represent the approximate solution $\hat{u}$ by a RBF expansion analogous to that used in the field of RBF interpolation [3], i.e. $\hat{u}$ is expressed as a linear combination of basis functions

$$\hat{u}(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \phi_\varepsilon(||\boldsymbol{x} - \boldsymbol{z}_j||_2), \tag{2}$$

where c_j is an unknown real coefficient, $||\cdot||_2$ denotes the Euclidean norm, and $\phi_\varepsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is some RBF depending on a *shape parameter* $\varepsilon > 0$ such that

$$\phi_\varepsilon(||\boldsymbol{x} - \boldsymbol{z}||_2) = \phi(\varepsilon||\boldsymbol{x} - \boldsymbol{z}||_2), \qquad \forall\, \boldsymbol{x}, \boldsymbol{z} \in \Omega.$$

In Table 1 we list some examples of popular globally supported RBFs, which are commonly used for solving PDEs (see [3] for details).

RBF	$\phi_\varepsilon(r)$
Gaussian (GA)	$\mathrm{e}^{-\varepsilon^2 r^2}$
Inverse MultiQuadric (IMQ)	$(1+\varepsilon^2 r^2)^{-1/2}$
MultiQuadric (MQ)	$(1+\varepsilon^2 r^2)^{1/2}$

Table 1. Some examples of popular RBFs.

In (2) we can distinguish between the set $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of *collocation points* and the set $Z = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N\}$ of *centers*. Additionally, for the sake of convenience we split the set X into a subset X_I of interior points and a subset X_B of boundary points, so that $X = X_I \cup X_B$.

Matching the PDE and the boundary conditions in (1) at the collocation points X, we obtain a linear system of equations

$$\boldsymbol{\Phi c} = \boldsymbol{v},$$

where $\boldsymbol{c} = (c_1, \ldots, c_N)^T$ is the vector of coefficients, $\boldsymbol{v} = (v_1, \ldots, v_N)^T$ is the vector of entries

$$v_i = \begin{cases} f(\boldsymbol{x}_i), & \boldsymbol{x}_i \in X_I, \\ g(\boldsymbol{x}_i), & \boldsymbol{x}_i \in X_B. \end{cases}$$

and $\Phi \in \mathbb{R}^{N\times N}$ is the collocation matrix

$$\Phi = \begin{bmatrix} \Phi_{\mathcal{L}} \\ \Phi_{\mathcal{B}} \end{bmatrix}. \tag{3}$$

The two blocks in (3) are defined as

$$(\Phi_{\mathcal{L}})_{ij} = \mathcal{L}\phi_\varepsilon(||\boldsymbol{x}_i - \boldsymbol{z}_j||_2), \qquad \boldsymbol{x}_i \in X_I,\ \ \boldsymbol{z}_j \in Z,$$
$$(\Phi_{\mathcal{B}})_{ij} = \mathcal{B}\phi_\varepsilon(||\boldsymbol{x}_i - \boldsymbol{z}_j||_2), \qquad \boldsymbol{x}_i \in X_B,\ \ \boldsymbol{z}_j \in Z,$$

Since the collocation matrix (3) may be singular for certain configurations of the centers $\boldsymbol{z}_j$, it follows that the nonsymmetric collocation method cannot be well-posed for arbitrary center locations. However, it is possible to find sufficient conditions on the centers so that invertibility of Kansa's matrix is ensured. For a more detailed analysis of Kansa's collocation method and some variations thereof derived from applications, see e.g. [3,6] and references therein.

3 Adaptive Refinement Algorithm

In this section we present the adaptive algorithm proposed to solve time independent PDE problems by Kansa's approach.

`Step 1`. We define two sets, $X_{N_1^{(0)}}$ and $X_{N_2^{(0)}}$, of collocation points and two sets, $Z_{N_1^{(0)}}$ and $Z_{N_2^{(0)}}$ of centers. Each couple of sets has size $N_1^{(0)}$ and $N_2^{(0)}$, respectively, with $N_1^{(0)} < N_2^{(0)}$ and the symbol $^{(0)}$ identifying the initial iteration. We then split the related sets as follows:

- $X_{N_1{}^{(0)}} = X_{I,N_1{}^{(0)}} \cup X_{B,N_1{}^{(0)}}$ and $X_{N_2{}^{(0)}} = X_{I,N_2{}^{(0)}} \cup X_{B,N_2{}^{(0)}}$ are sets of interior and boundary collocation points, respectively;
- $Z_{N_1{}^{(0)}} = Z_{I,N_1{}^{(0)}} \cup Z^A_{B,N_1{}^{(0)}}$ and $Z_{N_2{}^{(0)}} = Z_{I,N_2{}^{(0)}} \cup Z^A_{B,N_2{}^{(0)}}$ are sets of interior and additional boundary centers, respectively.

Here we assume that $X_{I,N_i{}^{(0)}} = Z_{I,N_i{}^{(0)}}$, with $i = 1, 2$, while the set $Z^A_{B,N_i{}^{(0)}}$ of centers is taken outside the domain Ω as suggested in [3]. However, we note that it is also possible to consider only a set of data as collocation points and centers.

`Step 2`. For $k = 0, 1, \ldots$, we iteratively find two collocation solutions of the form (2), called $\hat{u}_{N_1{}^{(k)}}$ and $\hat{u}_{N_2{}^{(k)}}$, which are respectively computed on $N_1^{(k)}$ and $N_2^{(k)}$ collocation points and centers.

`Step 3`. We compare the two approximate RBF solutions by evaluating error on the (coarser) set containing $N_1^{(k)}$ points, i.e.

$$|\hat{u}_{N_2{}^{(k)}}(\boldsymbol{x}_i) - \hat{u}_{N_1{}^{(k)}}(\boldsymbol{x}_i)|, \qquad \boldsymbol{x}_i \in X_{N_1{}^{(k)}}.$$

Observe that here we assume that the solution computed on $N_2^{(k)}$ discretization points gives more accurate results than the ones obtained with only $N_1^{(k)}$ points.

`Step 4`. After fixing a tolerance tol, we determine all points $\boldsymbol{x}_i \in X_{N_1{}^{(k)}}$ such that

$$|\hat{u}_{N_2{}^{(k)}}(\boldsymbol{x}_i) - \hat{u}_{N_1{}^{(k)}}(\boldsymbol{x}_i)| > tol. \tag{4}$$

`Step 5`. In order to refine the distribution of discretization points, we compute the *separation distance*

$$q_{X_{N_1{}^{(k)}}} = \frac{1}{2} \min_{i \neq j} ||\boldsymbol{x}_i - \boldsymbol{x}_j||_2, \qquad \boldsymbol{x}_i \in X_{N_1{}^{(k)}}. \tag{5}$$

`Step 6`. For $k = 0, 1, \ldots$ we update the two sets $X_{N_1{}^{(k+1)}}$ and $X_{N_2{}^{(k+1)}}$ of collocation points (and accordingly the corresponding sets $Z_{N_1{}^{(k+1)}}$ and $Z_{N_2{}^{(k+1)}}$ of centers) as follows. For each point $\boldsymbol{x}_i \in X_{N_1{}^{(k)}}$, such that the condition (4) is satisfied, we add to $\boldsymbol{x}_i$:

- four points (the blue circles depicted in the left frame of Figure 1), thus creating the set $X_{N_1{}^{(k+1)}}$;
- eight points (the red squares shown in the right frame of Figure 1), thus generating the set $X_{N_2{}^{(k+1)}}$.

In both cases the new points are given by properly either adding or subtracting the value of (5) to the components of $\boldsymbol{x}_i$. Furthermore, we remark that in the illustrative example of Figure 1 the point $\boldsymbol{x}_i$ is marked by a black cross, while the new sets are such that $X_{N_1{}^{(k)}} \subset X_{N_2{}^{(k)}}$, for $k = 1, 2, \ldots$.

`Step 7`. The iterative process stops when having no points anymore which fulfill the condition (4), giving the set $X_{N_2{}^{(k^*)}}$ back. Note that k^* is here used to denote the last algorithm iteration.

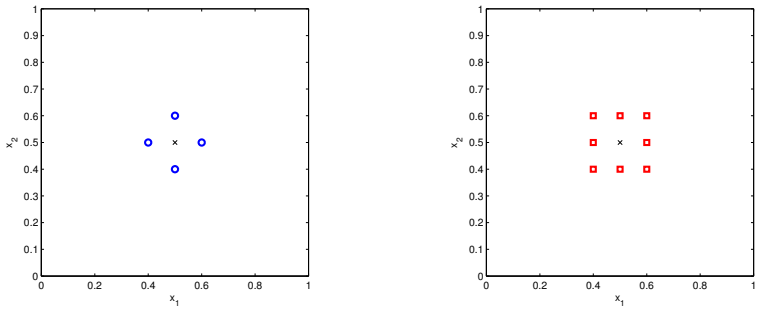

Fig. 1. Illustrative example of refinement for sets $X_{N_1{}^{(k)}}$ (left) and $X_{N_2{}^{(k)}}$ (right) in the adaptive algorithm.

4 Numerical Results

In this section we summarize the results derived from application of our adaptive refinement algorithm, which is implemented in MATLAB environment. All the results are carried out on a laptop with an Intel(R) Core(TM) i7-6500U CPU 2.50 GHz processor and 8GB RAM.

In the following we restrict our attention on solving some elliptic PDE problems via the nonsymmetric RBF collocation method. In particular, in (1) we consider a few Poisson-type problems, taking the Laplace operator $\mathcal{L} = -\Delta$ and assuming Dirichlet boundary conditions. Hence, the PDE problem in (1) can be defined as follows:

$$\begin{aligned} -\Delta u(\boldsymbol{x}) &= f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) &= g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \end{aligned} \tag{6}$$

Then, we focus on two test problems of the form (6) defined on the domain $\Omega = [0,1]^2$. The exact solutions of such Poisson problems are

$$\begin{aligned} \text{P1}: \quad & u_1(x_1, x_2) = \sin(x_1 + 2x_2^2) - \sin(2x_1^2 + (x_2 - 0.5)^2), \\ \text{P2}: \quad & u_2(x_1, x_2) = \frac{1}{2} x_2 \left[\cos(4x_1^2 + x_2^2 - 1)\right]^4 + \frac{1}{4} x_1. \end{aligned}$$

A graphical representation of these analytic solutions is shown in Figure 2.

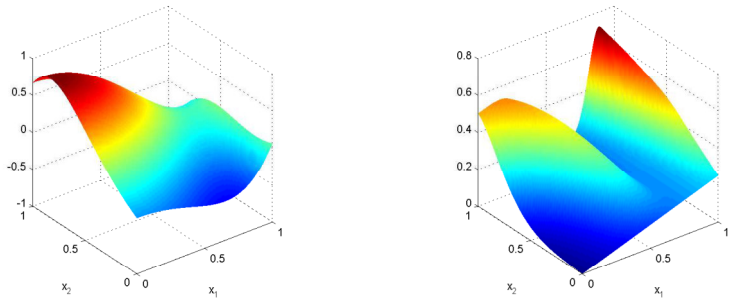

Fig. 2. Graphs of exact solutions u_1 (left) and u_2 (right) of Poisson problems.

In our numerical tests we analyze the performance of the adaptive refinement strategy applied to Kansa's collocation method by using globally supported RBFs such as MQ, IMQ and GA (see Table 1). We remark that the use of compactly supported RBFs is also possible and effective but our tests showed that the most accurate results were obtained with quite large supports. So the use of compactly supported functions does not provide particular benefits w.r.t. globally supported RBFs. For this reason and for the sake of brevity, we do not consider this case in the present paper.

The two starting sets defined in Section 3 consist of ${N_1}^{(0)} = 289$ and ${N_2}^{(0)} = 1089$ grid collocation points, while the tolerance in (4) is given by $tol = 10^{-4}$. In particular, in order to measure the quality of our results, we compute the Maximum Absolute Error (MAE), i.e.,

$$\text{MAE} = \max_{1 \le i \le N_{eval}} |u(\boldsymbol{y}_i) - \hat{u}(\boldsymbol{y}_i)|.$$

which is evaluated on a grid of $N_{eval} = 40 \times 40$ evaluation points. Moreover, in regards to the efficiency of the adaptive scheme, we report the CPU times computed in seconds.

In Tables 2–4 we present the results obtained, also indicating the final number N_{fin} of collocation points required to achieve the fixed tolerance. Further, as an example, for brevity in only one case for each test problem, we report the "refined grids" after applying iteratively the adaptive algorithm. More precisely, in Figure 3 we graphically represent the final distribution of points obtained after the last algorithm iteration by applying: the MQ-RBF with $\varepsilon = 4$ for the test problem P1 (left), and the IMQ-RBF with $\varepsilon = 3$ for the test problem P2 (right).

Test Problem	N_{fin}	MAE	CPU time
P1	856	8.28×10^{-5}	1.2
P2	798	2.66×10^{-4}	1.7

Table 2. MQ, $\varepsilon = 4$, $tol = 10^{-4}$.

Test Problem	N_{fin}	MAE	CPU time
P1	1001	3.10×10^{-5}	1.7
P2	808	1.10×10^{-4}	1.8

Table 3. IMQ, $\varepsilon = 3$, $tol = 10^{-4}$.

Test Problem	N_{fin}	MAE	CPU time
P1	898	3.49×10^{-4}	2.1
P2	843	2.08×10^{-4}	1.9

Table 4. GA, $\varepsilon = 9$, $tol = 10^{-4}$.

Analyzing the numerical results, we can observe as the adaptive algorithm allows us to increase the number of points in the regions where the solution is not

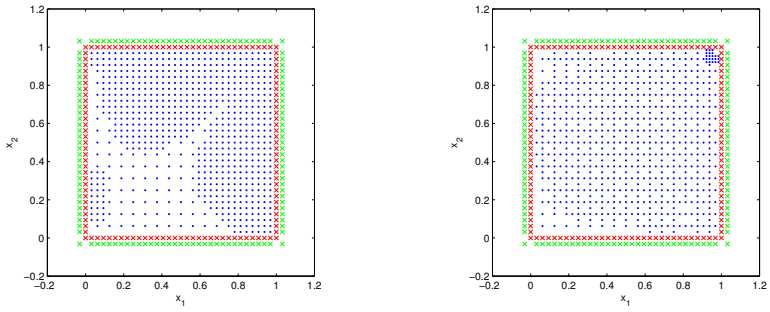

Fig. 3. Final distribution of points obtained after applying the refinement process with MQ, $\varepsilon = 4$, for problem P1 (left) and with IMQ, $\varepsilon = 3$, for problem P2.

accurate enough. From the tables we note as MQ and IMQ give more accurate results than GA. In fact, though the number of points required to satisfy the fixed tolerance is quite similar for all used RBFs, we can remark a greater instability of GA that needs a larger value of ε to work effectively. Finally, in terms of computational efficiency the algorithm converges in few seconds in each of the tests carried out.

5 Conclusions

In this work we presented an adaptive refinement algorithm to solve time independent elliptic PDEs. This refinement strategy is tested on a nonsymmetric RBF collocation scheme, known as Kansa's method. More precisely, here we proposed an adaptive approach based on a refinement technique, which consisted in comparing two collocation solutions computed on a coarser set of collocation points and a finer one. This process allowed us to detect the domain areas where it is necessary to adaptively add points, thus enhancing accuracy of the method. Numerical results supported this study by showing the algorithm performance on some Poisson-type problems.

As future work we are interested in investigating and possibly extending our adaptive schemes to hyperbolic and parabolic PDE problems. Moreover, we are currently working on the optimal selection of the RBF shape parameter in collocation schemes. However, this is out of the scopes of the present paper and it will be dealt with in forthcoming works.

Acknowledgments

The authors acknowledge support from the Department of Mathematics "Giuseppe Peano" of the University of Torino via Project 2018 "Algebra, geometry and numerical analysis" and Project 2019 "Mathematics for applications". Moreover,

this work was partially supported by INdAM – GNCS Project 2019 "Kernel-based approximation, multiresolution and subdivision methods and related applications". This research has been accomplished within RITA (Research ITalian network on Approximation).

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