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DETERMINISTIC AND STOCHASTIC CAUCHY PROBLEMS FOR A CLASS OF WEAKLY HYPERBOLIC OPERATORS ON $\mathbb{R}^n$

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Abstract. We study a class of hyperbolic Cauchy problems, associated with linear operators and systems with polynomially bounded coefficients, variable multiplicities and involutive characteristics, globally defined on $\mathbb{R}^n$. We prove well-posedness in Sobolev-Kato spaces, with loss of smoothness and decay at infinity. We also obtain results about propagation of singularities, in terms of wave-front sets describing the evolution of both smoothness and decay singularities of temperate distributions. Moreover, we can prove the existence of random-field solutions for the associated stochastic Cauchy problems. To these aims, we first discuss algebraic properties for iterated integrals of suitable parameter-dependent families of Fourier integral operators, associated with the characteristic roots, which are involved in the construction of the fundamental solution. In particular, we show that, also for this operator class, the involutiveness of the characteristics implies commutative properties for such expressions.

1. Introduction

In the present paper we focus on the Cauchy problem

$$\begin{cases} Lu(t) = f(t), & t \in (0,T] \\ D_t^k u(0) = g_k, & k = 0, \dots, m-1, \end{cases} \tag{1.1}$$

where $T > 0$ and $L = L(t, D_t; x, D_x)$ is a linear partial differential operator of the form

$$L(t, D_t; x, D_x) = D_t^m + \sum_{j=1}^{m} \sum_{|\alpha| \leqslant j} c_{j\alpha}(t; x) D_x^\alpha D_t^{m-j}, \tag{1.2}$$

$D = -i\partial$, with $(t,x)-$smoothly depending coefficients $c_{j\alpha}$, possibly admitting a polynomial growth, namely,

$$|\partial_t^k \partial_x^\beta c_{j\alpha}(t;x)| \leqslant C_{k\alpha} \langle x \rangle^{j-|\beta|}, \qquad \alpha, \beta \in \mathbb{Z}_+^n,\ |\alpha| \leqslant j,\ j = 1, \dots, m, \tag{1.3}$$

for $\langle x \rangle := (1 + |x|^2)^{1/2}$, some $C_{k\alpha} > 0$ and all $t \in [0,T]$, $x \in \mathbb{R}^n$. The hypothesis of smoothness with respect to t is assumed only for the sake of simplicity, since here we will not deal with questions concerning low regularity in time for the coefficients of the Cauchy problem.

The assumption (1.3) suggests to set the problem within the framework of the so-called SG calculus (see [10, 30]), defined through symbols satisfying global estimates on $\mathbb{R}^n \times \mathbb{R}^n$. Explicitly, given $(m, \mu) \in \mathbb{R}^2$, the class of SG symbols $S^{m,\mu}$ of order (m, μ) consists of all symbols $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{m-|\alpha|} \langle \xi \rangle^{\mu-|\beta|} \tag{1.4}$$

for all $x, \xi \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{Z}_+^n$, and some $C_{\alpha\beta} > 0$, see Section 2 below for some basic properties of the corresponding calculus. We are interested in existence and

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uniqueness of the solution to (1.1) in suitable weighted Sobolev-type spaces of functions or distributions, according to the regularity of the Cauchy data and of the operator L. More precisely, it is well-estabilished (see, e.g., [28] for PDEs with uniformly bounded coefficients, and [10] for PDEs in the SG framework) that, to have existence of a unique solution to (1.1) in Sobolev-type spaces, a hyperbolicity assumption is needed. The operator L is said to be *hyperbolic* if its principal symbol

$$L_m(t,\tau;x,\xi) := \tau^m + \sum_{j=1}^{m} \sum_{|\alpha|=j} c_{j\alpha}(t;x)\xi^\alpha \tau^{m-j} \tag{1.5}$$

factorizes as

$$L_m(t,\tau;x,\xi) = \prod_{j=1}^{m} (\tau - \tau_j(t;x,\xi)), \tag{1.6}$$

with real-valued and smooth roots τ_j, usually called *characteristic roots* of the operator L. A sufficient condition for existence of a unique solution is the *separation between the roots*, which reads either as

(H) $\quad |\tau_j(t;x,\xi) - \tau_k(t;x,\xi)| \geqslant C\langle\xi\rangle \quad$ (uniformly bounded coefficients) or,

(SGH) $\quad |\tau_j(t;x,\xi) - \tau_k(t;x,\xi)| \geqslant C\langle x\rangle\langle\xi\rangle \quad$ (SG type coefficients),

for a suitable $C > 0$ and every $(t;x,\xi) \in [0,T] \times \mathbb{R}^{2n}$. Conditions (H) or (SGH), respectively, are assumed to hold true either for every $j \neq k$ (*strict hyperbolicity*), or at least for every τ_j, τ_k belonging to different groups of coinciding roots (*weak hyperbolicity with roots of constant multiplicity*). The strict hyperbolicity condition (H) (respectively, (SGH)) allows to prove existence and uniqueness in Sobolev spaces (respectively, in Sobolev-Kato spaces) of a solution to (1.1) for every f, g_k, $k = 0, \dots, m-1$, in Sobolev spaces (respectivey, Sobolev-Kato spaces). A weak hyperbolicity condition with roots of constant multiplicity, together with a condition on the lower order terms of the operator L (a so-called *Levi condition*) allows to prove a similar result, and the phenomena of loss of derivatives and/or modification of the behavior as $|x| \to \infty$ with respect to the initial data appear. The first phenomenon has been observed for the first time in [8], the second one in [2, 12].

If there is no separation condition on the roots, then the only available results in the literature that we are aware of are those due to Morimoto [29] and Taniguchi [37], dealing with *involutive roots* in the uniformly bounded coefficients case. The condition of involutiveness, which is weaker than the separation condition (H), requires that, for all $j,k \in \mathbb{N}$, $t \in [0,T]$, $x,\xi \in \mathbb{R}^n$, the Poisson brackets

$$\{\tau - \tau_j, \tau - \tau_k\} := \partial_t\tau_j - \partial_t\tau_k + \tau'_{j,\xi} \cdot \tau'_{k,x} - \tau'_{j,x} \cdot \tau'_{k,\xi}$$

may be written as

$$\{\tau - \tau_j, \tau - \tau_k\} = b_{jk} \cdot (\tau_j - \tau_k) + d_{jk}, \tag{INV}$$

for suitable parameter-dependent, real-valued, uniformly bounded symbols b_{jk}, d_{jk}, $j,k \in \mathbb{N}$. Under the hypothesis (INV), adding a Levi condition, the authors reduced (1.1) to an equivalent first order system

$$\begin{cases} \mathbf{L}U(t) = F(t), \quad 0 \leqslant t \leqslant T, \qquad \mathbf{L} = D_t + \Lambda(t,x,D_x) + R(t,x,D_x), \\ U(0) = G, \end{cases} \tag{1.7}$$

with Λ a diagonal matrix having entries given by operators with symbol coinciding with the τ_j, $j = 1, \dots, m$, and R a (full) matrix of operators of order zero. They constructed the fundamental solution to (1.7), that is, a smooth family $\{E(t,s)\}_{0\leqslant s\leqslant t\leqslant T}$

of operators such that

$$\begin{cases} \mathbf{L}E(t,s) = 0 & 0 \leqslant s \leqslant t \leqslant T, \\ E(s,s) = I, & s \in [0,T), \end{cases}$$

so obtaining by Duhamel's formula the solution

$$U(t) = E(t,0)G + i\int_0^t E(t,s)F(s)ds$$

to the system and then the solution $u(t)$ to the corresponding higher order Cauchy problem (1.1) by the reduction procedure.

In the present paper we consider the Cauchy problem (1.1) in the SG setting, that is, under the growth condition (1.3), the hyperbolicity condition (1.6) and the involutiveness assumption (INV) for suitable parameter-dependent, real-valued symbols $b_{jk}, d_{jk} \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2n}))$, $j,k \in \mathbb{N}$. The involutiveness of the characteristic roots of the operator, or, equivalently, of the eigenvalues of the (diagonal) principal part of the corresponding first order system, is here the main assumption (see Assumption I in Section 3.3 for its precise statement).

We mainly work in the framework of Sobolev-Kato spaces $H^{r,\rho}$, see Definition (2.5), and we obtain several results concerning the Cauchy problem (1.1):

- we prove, in Theorem 5.6, the existence, for every $f \in C^\infty([0,T]; H^{r,\rho}(\mathbb{R}^n))$ and $g_k \in H^{r+m-1-k,\rho+m-1-k}(\mathbb{R}^n)$, $k = 0,\dots,m-1$, of a unique
$$u \in \bigcap_{k\in\mathbb{Z}^+} C^k([0,T']; H^{r-k,\rho-k}(\mathbb{R}^n)),$$
for a suitably small $0 < T' \leqslant T$, solution to (1.1); we also provide, in (5.10), an expression of u, in terms of SG Fourier integral operators (SG FIOs);
- we give a precise description, in Theorem 5.17, of a global wave-front set of the solution, under a (mild) additional condition on the operator L; namely, we prove that, when L is SG-classical, the set of (smoothness and decay) singularities of the solution of (1.1) with $f \equiv 0$ is included in unions of arcs of bicharacteristics, generated by the phase functions of the SG FIOs appearing in the expression (5.10), and emanating from (smoothness and decay) singularities of the Cauchy data g_k, $k = 0,\dots,m-1$;
- we deal, in Theorem 6.3, with a stochastic version of the Cauchy problem (1.1), namely, with the case when $f(t;x) = \gamma(t;x) + \sigma(t;x)\dot{\Xi}(t;x)$, and Ξ is a random noise of Gaussian type, white in time and with possible correlation in space; we give conditions on the noise Ξ, on the coefficients γ, σ, and on the data g_k, $k = 0,\dots,m-1$, such that there exists a so-called *random field solution* of (1.1), that is, a stochastic process which gives, for every fixed $(t,x) \in [0,T] \times \mathbb{R}^n$, a well-defined random variable (see Section 6 below for details).

All the results described above are achieved through the use of SG FIOs, which are recalled in Section 2.1. The *involutiveness assumption* is the key to prove that the fundamental solution of the first order system (1.7) may be written as a *finite sum* of (iterated integrals of smooth parameter-dependent families of) SG FIOs (modulo a regularizing term). This is stated in Theorem 4.1, a furher main result of this paper, and follows from the analysis performed in Section 3. The (rather technical) results in Sections 3 and 4 are indeed crucial to achieve our main results. Namely, in Section 3 we prove that, under the involutiveness assumption, multi-products of regular SG-phase functions given as solutions of eikonal equations satisfy a *commutative law* (Theorem 3.4). Similar to the classical situation, this property extends to the SG case the possibility to reduce the fundamental solution to a finite sum of terms (modulo a regularizing operator), see Section 4. This also implies commutativity properties of the *flows at infinity* generated by the corresponding (classical) SG-phase functions. Their relationship with the symplectic structures studied in [19] will be investigated elsewhere.

The paper is organized as follows. In Section 2 we recall basic elements of the SG calculus of pseudo-differential and Fourier integral operators. In Section 3 we prove a commutative property for multi-products of suitable families of regular SG-phase functions, determined by involutive families of SG-symbols. In Section 4 we discuss how such commutative property can be employed to obtain the fundamental solution for certain $(N \times N)$-dimensional first order linear systems with diagonal principal part and involutive characteristics. In Section 5 we study the Cauchy problem for SG-hyperbolic, involutive linear differential operators. Here two of the main theorems are proved: a well-posedness result (with loss of smoothness and decay) and a propagation of singularity result, for wave-front sets of global type, whose definition and basic properties are also recalled. Finally, in Section 6 we prove our final main theorem, namely, the existence of random field solutions for a stochastic version of the Cauchy problems studied in Section 5.

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2. Symbolic calculus for pseudo-differential and Fourier integral operators of SG type

In this section we recall some properties of SG operators, a class defined through symbols satisfying global estimates on $\mathbb{R}^n \times \mathbb{R}^n$. In particular, we state several fundamental results for SG FIOs. Further details can be found, e.g., in [3, 10, 23, 25, 36] and the references therein, from which we took most of the materials included in this section. Here and in what follows, $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$, where $A \lesssim B$ means that $A \leqslant c \cdot B$, for a suitable constant $c > 0$.

2.1. **Pseudo-differential operators and Fourier integral operators of SG type.**

For $a(x,\xi) \in S^{m,\mu}(\mathbb{R}^{2n})$, $m, \mu \in \mathbb{R}$, see (1.4), we define the semi-norms $|||a|||_l^{m,\mu}$ by

$$|||a|||_l^{m,\mu} = \max_{|\alpha+\beta|\leqslant l} \sup_{x,\xi\in\mathbb{R}^n} \langle x\rangle^{-m+|\beta|}\langle\xi\rangle^{-\mu+|\alpha|}|D_\xi^\alpha D_x^\beta a(x,\xi)|, \tag{2.1}$$

where $l \in \mathbb{Z}_+$. The quantities (2.1) define a Fréchet topology of $S^{m,\mu}(\mathbb{R}^{2n})$. Moreover, let

$$S^{\infty,\infty}(\mathbb{R}^{2n}) = \bigcup_{m,\mu\in\mathbb{R}} S^{m,\mu}(\mathbb{R}^{2n}), \qquad S^{-\infty,-\infty}(\mathbb{R}^{2n}) = \bigcap_{m,\mu\in\mathbb{R}} S^{m,\mu}(\mathbb{R}^{2n}).$$

The functions $a \in S^{m,\mu}(\mathbb{R}^{2n})$ can be $(\nu \times \nu)$-matrix-valued. In such case the estimate (1.4) must be valid for each entry of the matrix. The next technical lemma is useful when dealing with compositions of SG symbols.

Lemma 2.1. *Let $f \in S^{m,\mu}(\mathbb{R}^{2n})$, $m, \mu \in \mathbb{R}$, and g vector-valued in $\mathbb{R}^n$ such that $g \in S^{0,1}(\mathbb{R}^{2n}) \otimes \mathbb{R}^n$ and $\langle g(x,\xi)\rangle \asymp \langle\xi\rangle$. Then $f(x, g(x,\xi))$ belongs to $S^{m,\mu}(\mathbb{R}^{2n})$.*

The previous result can be found in [13], and can of course be extended to the other composition case, namely $h(x,\xi)$ vector valued in $\mathbb{R}^n$ such that it belongs to $S^{1,0}(\mathbb{R}^{2n}) \otimes \mathbb{R}^n$ and $\langle h(x,\xi)\rangle \asymp \langle x\rangle$, implying that $f(h(x,\xi),\xi)$ belongs to $S^{m,\mu}(\mathbb{R}^{2n})$.

We now recall definition and properties of the pseudo-differential operators $a(x,D) = \mathrm{Op}(a)$ where $a \in S^{m,\mu}$. Given $a \in S^{m,\mu}$, $\mathrm{Op}(a)$ is defined through the left-quantization (cf. Chapter XVIII in [24])

$$(\mathrm{Op}(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,d\xi, \quad u \in \mathcal{S}, \tag{2.2}$$

with $\widehat{u}$ the Fourier transform of $u \in \mathcal{S}$, given by $\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx$. The operators in (2.2) form a graded algebra with respect to composition, i.e.,

$$\mathrm{Op}(S^{m_1,\mu_1}) \circ \mathrm{Op}(S^{m_2,\mu_2}) \subseteq \mathrm{Op}(S^{m_1+m_2,\mu_1+\mu_2}).$$

The symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}$ of the composed operator $\mathrm{Op}(a)\circ\mathrm{Op}(b)$, where $a \in S^{m_1,\mu_1}$, $b \in S^{m_2,\mu_2}$, admits the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\, D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi), \tag{2.3}$$

which implies that the symbol c equals $a \cdot b$ modulo $S^{m_1+m_2-1,\mu_1+\mu_2-1}$.

By induction, it is then possible to deal with the composition (or multi-product) of $(M+1)$ SG pseudo-differential operators, where $M \geqslant 1$. The *multi-product*

$$Q_{M+1} = P_1 \cdots P_{M+1}$$

of the operators $P_j = \mathrm{Op}(p_j)$, with $p_j(x,\xi) \in S^{m_j,\mu_j}(\mathbb{R}^{2n})$, $m_j, \mu_j \in \mathbb{R}$, $j = 1,\dots,M+1$ is a SG pseudo-differential operator with symbol $q_{M+1}(x,\xi) \in S^{m,\mu}(\mathbb{R}^{2n})$, where $m = m_1 + \cdots + m_{M+1}$ and $\mu = \mu_1 + \cdots + \mu_{M+1}$. Moreover, the boundedness of $\oplus_{j=1}^{M+1} p_j$ in $\oplus_{j=1}^{M+1} S^{m_j,\mu_j}(\mathbb{R}^{2n})$ implies the boundedness of $q_{M+1}(x,\xi)$ in $S^{m,\mu}(\mathbb{R}^{2n})$.

The residual elements of the calculus are operators with symbols in

$$S^{-\infty,-\infty} = \bigcap_{(m,\mu)\in\mathbb{R}^2} S^{m,\mu} = \mathcal{S},$$

that is, those having kernel in $\mathcal{S}$, continuously mapping $\mathcal{S}'$ to $\mathcal{S}$.

An operator $A = \mathrm{Op}(a)$, is called *elliptic* (or $S^{m,\mu}$*-elliptic*) if $a \in S^{m,\mu}$ and there exists $R \geqslant 0$ such that

$$C\langle x\rangle^{m}\langle\xi\rangle^{\mu} \leqslant |a(x,\xi)|, \qquad |x| + |\xi| \geqslant R, \tag{2.4}$$

for some constant $C > 0$. An elliptic SG operator $A \in \mathrm{Op}(S^{m,\mu})$ admits a parametrix $P \in \mathrm{Op}(S^{-m,-\mu})$ such that $PA = I + K_1$, $AP = I + K_2$, for suitable $K_1, K_2 \in \mathrm{Op}(S^{-\infty,-\infty})$, where I denotes the identity operator.

It is a well-known fact that SG-operators give rise to linear continuous mappings from $\mathcal{S}$ to itself, extendable as linear continuous mappings from $\mathcal{S}'$ to itself. They also act continuously between the so-called Sobolev-Kato (or weighted Sobolev) spaces, that is from $H^{r,\varrho}(\mathbb{R}^n)$ to $H^{r-m,\varrho-\mu}(\mathbb{R}^n)$, where $H^{r,\varrho}(\mathbb{R}^n)$, $r,\varrho \in \mathbb{R}$, is defined as

$$H^{r,\varrho}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n)\colon \|u\|_{r,\varrho} = \|\langle.\rangle^{r}\langle D\rangle^{\varrho}u\|_{L^2} < \infty\right\}. \tag{2.5}$$

Next, we recall the class of Fourier integral operators we are interested in, together with some relevant properties. A SG FIO is defined, for $u \in \mathcal{S}$, as

$$u \mapsto (\mathrm{Op}_\varphi(a)u)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}a(x,\xi)\widehat{u}(\xi)\,d\xi, \tag{2.6}$$

where a is a symbol in $SG^{m,\mu}(\mathbb{R}^n)$ and φ is a SG phase function, that is a real valued symbol in $S^{1,1}(\mathbb{R}^{2n})$ such that, as $|(x,\xi)| \to \infty$, $\langle\varphi'_x(x,\xi)\rangle \asymp \langle\xi\rangle$ and $\langle\varphi'_\xi(x,\xi)\rangle \asymp \langle x\rangle$. In the sequel, we usually write "phase function" in place of "SG phase function".

It is well known, see [11], that given a phase function φ and two symbols $p \in S^{t,\tau}(\mathbb{R}^{2n})$, $a \in S^{m,\mu}(\mathbb{R}^{2n})$, we have the following composition rules:

$$\mathrm{Op}(p)\circ\mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c_1 + r_1), \qquad \mathrm{Op}_\varphi(a)\circ\mathrm{Op}(p) = \mathrm{Op}_\varphi(c_2 + r_2),$$

for some $c_1, c_2 \in S^{m+t,\mu+\tau}(\mathbb{R}^{2n})$, $r_1, r_2 \in S^{-\infty,-\infty}(\mathbb{R}^{2n})$. Moreover, given $a \in S^{m,\mu}(\mathbb{R}^{2n})$ and φ a regular phase function (see Definition 2.2 here below), $\mathrm{Op}_\varphi(a)$ maps continuously $H^{r,\rho}(\mathbb{R}^n)$ to $H^{r-m,\rho-\mu}(\mathbb{R}^n)$, for any $r, \rho, m, \mu \in \mathbb{R}$.

Definition 2.2 (Regular SG phase functions). *Let $\tau \in [0, 1)$ and $r > 0$. A SG phase function φ belongs to the class $\mathcal{P}_r(\tau)$ of (r, τ)-regular SG phase functions if it satisfies the following conditions:*

1: $|\det(\varphi''_{x\xi})(x, \xi)| \geqslant r$, *for any* $x, \xi \in \mathbb{R}^n$;

2: *the function* $J(x, \xi) := \varphi(x, \xi) - x \cdot \xi$ *is such that* $\displaystyle\sup_{\substack{x,\xi\in\mathbb{R}^n \\ |\alpha+\beta|\leqslant 2}} \frac{|D^\alpha_\xi D^\beta_x J(x, \xi)|}{\langle x\rangle^{1-|\beta|}\langle \xi\rangle^{1-|\alpha|}} \leqslant \tau.$

If only condition (1) holds, we write $\varphi \in \mathcal{P}_r$, and call it a regular SG phase function.

In spite of providing a result formally somehow similar to the analogous one involving pseudo-differential operators recalled above, the multi-product of $(M+1)$ SG FIOs with regular phase functions, $M \geqslant 1$, is a lot more difficult and technical issue. It has been studied in [3] (see [37] for the classical case), extending to the SG case the concept of multi-product of phase functions (see also Section 3 below).

2.2. Eikonal equations in SG classes.

To study evolution equations within the SG environment, we need first to introduce parameter-dependent symbols, where the variation of the parameters give rise to bounded sets in $S^{m,\mu}$.

Definition 2.3. *Let $\Omega \subseteq \mathbb{R}^N$. We write $f \in C^k\left(\Omega; S^{m,\mu}(\mathbb{R}^{2n})\right)$, with m, $\mu \in \mathbb{R}$ and $k \in \mathbb{Z}_+$ or $k = \infty$, if*

(i) $f = f(\omega; x, \xi)$, $\omega \in \Omega$, x, $\xi \in \mathbb{R}^n$;

(ii) *for any* $\omega \in \Omega$, $\partial^\alpha_\omega f(\omega) \in S^{m,\mu}(\mathbb{R}^{2n})$, *for all* $\alpha \in \mathbb{Z}^N_+$, $|\alpha| \leqslant k$;

(iii) $\{\partial^\alpha_\omega f(\omega)\}_{\omega\in\Omega}$ *is bounded in* $S^{m,\mu}(\mathbb{R}^{2n})$, *for all* $\alpha \in \mathbb{Z}^N_+$, $|\alpha| \leqslant k$.

The next technical Lemma is one of the key tools needed to prove the commutative law for multi-products of a class of regular phase functions studied in Section 3. Its proof can be found in [1].

Lemma 2.4. *Let $\Omega \subseteq \mathbb{R}^N$, $a \in C^k\left(\Omega; S^{m,\mu}(\mathbb{R}^{2n})\right)$ and $h \in C^k\left(\Omega; S^{0,0}(\mathbb{R}^{2n}) \otimes \mathbb{R}^N\right)$ such that $k \in \mathbb{Z}_+$ or $k = \infty$. Assume also that, for any $\omega \in \Omega$, $x, \xi \in \mathbb{R}^n$, the function $h(\omega; x, \xi)$ takes values in Ω. Then, setting $b(\omega) = a(h(\omega))$, that is, $b(\omega; x, \xi) = a(h(\omega; x, \xi); x, \xi)$, we find $b \in C^k\left(\Omega; S^{m,\mu}(\mathbb{R}^{2n})\right)$.*

Given a real-valued parameter-dependent symbol $a \in C([0, T]; S^{1,1}(\mathbb{R}^{2n}))$, consider its associated *eikonal equation*

$$\begin{cases} \partial_t\varphi(t, s; x, \xi) = a(t; x, \varphi'_x(t, s; x, \xi)), & t \in [0, T'], \\ \varphi(s, s; x, \xi) = x \cdot \xi, & s \in [0, T'], \end{cases} \tag{2.7}$$

with $0 < T' \leqslant T$. By an extension of the theory developed in [12], it is possible to prove that the following Proposition 2.5 holds true (see below).

Proposition 2.5. *Let $a \in C([0, T]; S^{1,1}(\mathbb{R}^{2n}))$ be real-valued. Then, for a small enough $T' \in (0, T]$, equation (2.7) admits a unique solution $\varphi \in C^1([0, T']^2; S^{1,1}(\mathbb{R}^{2n}))$. Moreover, for every $h \geqslant 0$ there exists $c_h \geqslant 1$ and $T_h \in [0, T']$ such that $\varphi(t, s; x, \xi) \in \mathcal{P}_r(c_h|t - s|)$, for all $0 \leqslant s \leqslant t \leqslant T_h$.*

In the sequel we will sometimes write $\varphi_{ts}(x, \xi) := \varphi(t, s; x, \xi)$, for a solution φ of (2.7).

We now focus on the Hamilton-Jacobi system corresponding to the real-valued Hamiltonian $a \in C([0, T]; S^{1,1}(\mathbb{R}^{2n}))$, namely,

$$\begin{cases} \partial_t q(t, s; y, \eta) &= -a'_\xi\left(t; q(t, s; y, \eta), p(t, s; y, \eta)\right), \\ \partial_t p(t, s; y, \eta) &= a'_x(t; q(t, s; y, \eta), p(t, s; y, \eta)), \end{cases} \tag{2.8}$$

where $t, s \in [0, T]$, $T > 0$, and the Cauchy data are

$$\begin{cases} q(s,s;y,\eta) = y, \\ p(s,s;y,\eta) = \eta. \end{cases} \tag{2.9}$$

The next Proposition 2.6 describes the properties of the solution of (2.8). Here we mainly refer to known results from [10, Ch. 6] and [12], and their extensions studied in [1] (see also [25]).

Proposition 2.6. *The solution (q, p) of the Hamilton-Jacobi system (2.8) with $a \in C([0,T]; S^{1,1}(\mathbb{R}^{2n}))$ real-valued, and the Cauchy data (2.9), is such that:*

(1) q belongs to $C^1([0,T]^2; S^{1,0}(\mathbb{R}^{2n}))$, and $\langle q(t,s;y,\eta)\rangle \asymp \langle y \rangle$;
p belongs to $C^1([0,T]^2; S^{0,1}(\mathbb{R}^{2n}))$, and $\langle p(t,s;y,\eta)\rangle \asymp \langle \eta \rangle$;

(2) for a suffcently small $T' \in (0, T]$ and a fixed t such that $0 \leqslant s, t \leqslant T'$, $s \neq t$,

$$\begin{cases} (q(t,s;y,\eta) - y)/(t-s) & \text{is bounded in} \quad S^{1,0}(\mathbb{R}^{2n}), \\ (p(t,s;y,\eta) - \eta)/(t-s) & \text{is bounded in} \quad S^{0,1}(\mathbb{R}^{2n}), \end{cases}$$

and

$$\begin{cases} q(t,s;y,\eta), q(t,s;y,\eta) - y \in C^1(I(T'); S^{1,0}(\mathbb{R}^{2n})), \\ p(t,s;y,\eta), p(t,s;y,\eta) - \eta \in C^1(I(T'); S^{0,1}(\mathbb{R}^{2n})), \end{cases}$$

where, for $T > 0$, $I(T) = \{(t,s) \colon 0 \leqslant t, s \leqslant T\}$.

(3) if, additionally, $a \in C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2n}))$, then, $q(t,s;y,\eta) - y \in C^\infty(I(T'); S^{1,0}(\mathbb{R}^{2n}))$ and $p(t,s;y,\eta) - \eta \in C^\infty(I(T'); S^{0,1}(\mathbb{R}^{2n}))$;

(4) there exists $T_1 \in (0, T']$ such that the mapping $x = q(t,s;y,\eta) : \mathbb{R}^n_y \ni y \longmapsto x \in \mathbb{R}^n_x$, with (t,s,η) understood as parameter, has the inverse function $y = \overline{q}(t,s;x,\eta)$ for any $(t,s) \in I(T_1)$ and any $\eta \in \mathbb{R}^n$; it also holds $\overline{q} \in C^1(I(T_1); S^{1,0}(\mathbb{R}^{2n}))$ and $\langle \overline{q}(t,s;x,\eta)\rangle \asymp \langle x \rangle$;

(5) let $T_1 \in (0, T']$, $\epsilon_1 \in (0, 1]$ be constants such that on $I(T_1)$ we have

$$\left\| \frac{\partial q}{\partial y} - I_n \right\| \leqslant 1 - \epsilon_1;$$

then, the inverse function $\overline{q}$ from the previous point satisfies

$$\begin{cases} \overline{q}(t,s;x,\eta) - x \in C^1(I(T_1); S^{1,0}(\mathbb{R}^{2n})), \\ (\overline{q}(t,s;x,\eta) - x)/(t-s) \text{ is bounded in } S^{1,0}(\mathbb{R}^{2n}), \text{ whenever } 0 \leqslant s, t \leqslant T_1; s \neq t; \end{cases}$$

(6) if, additionally, $a \in C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2n}))$, we also have $\overline{q}, \overline{q}(t,s;x,\eta) - x \in C^\infty(I(T_1); S^{1,0}(\mathbb{R}^{2n}))$.

The next Proposition 2.7 explains how the solution of (2.9) is related with the solution of (2.7) in the SG context. Its proof follows by slight modifications of the classical arguments given, e.g., in [25], and relies on Proposition 2.6 (see [1, 12] for details).

Proposition 2.7. *Let $a \in C([0,T]; S^{1,1}(\mathbb{R}^{2n}))$ be real-valued, $q(t,s;y,\eta)$, $p(t,s;y,\eta)$ and $\overline{q}(t,s;x,\xi)$ the parameter-dependent symbols constructed in the previous Proposition 2.6. Define $u(t,s;y,\eta)$ by*

$$u(t,s;y,\eta) = y \cdot \eta + \int_s^t \Big(a(\tau; q(\tau,s;y,\eta), p(\tau,s;y,\eta)) - a'_\xi(\tau; q(\tau,s;y,\eta), p(\tau,s;y,\eta)) \cdot p(\tau,s;y,\eta) \Big) \, d\tau, \tag{2.10}$$

and set

$$\varphi(t,s;x,\xi) = u(t,s;\overline{q}(t,s;x,\xi),\xi). \tag{2.11}$$

Then, $\varphi(t,s;x,\xi)$ is a solution of the eikonal equation (2.7) and satisfies

$$\varphi'_\xi(t,s;x,\xi) = \overline{q}(t,s;x,\xi), \tag{2.12}$$

$$\varphi'_x(t,s;x,\xi) = p(t,s;\overline{q}(t,s;x,\xi),\xi), \tag{2.13}$$

$$\partial_s\varphi(t,s;x,\xi) = -a(s;\varphi'_\xi(t,s;x,\xi),\xi), \tag{2.14}$$

$$\langle\varphi'_x(t,s;x,\xi)\rangle \asymp \langle\xi\rangle \text{ and } \langle\varphi'_\xi(t,s;x,\xi)\rangle \asymp \langle x\rangle\,. \tag{2.15}$$

Moreover, for any $l \geqslant 0$ there exists a constant $c_l \geqslant 1$ and $T_l \in (0,T_1]$ such that $c_lT_l < 1$, $\varphi(t,s;x,\xi)$ belongs to $\mathcal{P}_r(c_l|t-s|)$ and $\{J(t,s)/|t-s|\}$ is bounded in $S^{1,1}(\mathbb{R}^{2n})$ for $0 \leqslant t,s \leqslant T_l \leqslant T_1$, $t \neq s$, where $J(t,s;x,\xi) = \varphi(t,s;x,\xi) - x\cdot\xi$. Finally, if, additionally, $a \in C^\infty([0,T];S^{1,1}(\mathbb{R}^{2n}))$, we find $\varphi, J \in C^\infty(I(T_1);S^{1,1}(\mathbb{R}^{2n}))$.

2.3. **Classical symbols of SG type.** In the last part of Section 5 we will focus on the subclass of symbols and operators which are SG-classical, that is, $a \in S^{m,\mu}_{\mathrm{cl}} \subset S^{m,\mu}$. In this subsection we recall their definition, using material coming from [7] (see, e.g., [23] for additional details and proofs). We fix a cut-off function $\omega \in C_0^\infty(\mathbb{R}^n)$ with $\omega \equiv 1$ on the ball of radius $1/2$ centred at the origin.

Definition 2.8. *(i) A symbol $a(x,\xi)$ belongs to the class $S^{m,\mu}_{\mathrm{cl}(\xi)}(\mathbb{R}^{2n})$ if there exist functions $a_{\cdot,\mu-j}(x,\xi)$, $j = 0,1,\dots$, homogeneous of degree $\mu-j$ with respect to the variable ξ, smooth with respect to the variable x, such that,*

$$a(x,\xi) - \sum_{j=0}^{M-1}(1-\omega(\xi))\,a_{\cdot,\mu-j}(x,\xi) \in S^{m,\mu-M}(\mathbb{R}^{2n}), \quad M = 1,2,\dots$$

(ii) A symbol a belongs to the class $S^{m,\mu}_{\mathrm{cl}(x)}(\mathbb{R}^{2n})$ if $a\circ R \in S^{\mu,m}_{\mathrm{cl}(\xi)}(\mathbb{R}^{2n})$, with $R(x,\xi) = (\xi,x)$, that is, $a(x,\xi)$ has an asymptotic expansion into terms homogeneous in x.

Definition 2.9. *A symbol a is called SG-classical, and we write $a \in S^{m,\mu}_{\mathrm{cl}(x,\xi)}(\mathbb{R}^{2n}) = S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2n})$, if the following two conditions hold true:*

(i) there exist functions $a_{\cdot,\mu-j}(x,\xi)$, homogeneous of degree $\mu-j$ with respect to ξ and smooth in x, such that $(1-\omega(\xi))a_{\cdot,\mu-j}(x,\xi) \in S^{m,\mu-j}_{\mathrm{cl}(x)}(\mathbb{R}^{2n})$ and

$$a(x,\xi) - \sum_{j=0}^{M-1}(1-\omega(\xi))\,a_{\cdot,\mu-j}(x,\xi) \in S^{m,\mu-M}_{\mathrm{cl}(x)}(\mathbb{R}^{2n}), \quad M = 1,2,\dots;$$

(ii) there exist functions $a_{m-k,\cdot}(x,\xi)$, homogeneous of degree $m-k$ with respect to the x and smooth in ξ, such that $(1-\omega(x))a_{m-k,\cdot}(x,\xi) \in S^{m-k,\mu}_{\mathrm{cl}(\xi)}(\mathbb{R}^{2n})$ and

$$a(x,\xi) - \sum_{k=0}^{M-1}(1-\omega(x))\,a_{m-k,\cdot}(x,\xi) \in S^{m-M,\mu}_{\mathrm{cl}(\xi)}(\mathbb{R}^{2n}), \quad M = 1,2,\dots$$

We set $L^{m,\mu}_{\mathrm{cl}(x,\xi)}(\mathbb{R}^{2n}) = L^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2n}) = \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2n}))$.

The next results is especially useful when dealing with SG-classical symbols.

Theorem 2.10. *Let $a_k \in S^{m-k,\mu-k}_{\mathrm{cl}}(\mathbb{R}^{2n})$, $k = 0,1,\dots$, be a sequence of SG-classical symbols and $a \sim \sum_{k=0}^{\infty} a_k$ its asymptotic sum in the general SG-calculus. Then, $a \in S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^{2n})$.*

The algebra property of SG-operators and Theorem 2.10 imply that the composition of two SG-classical operators is still classical.

Remark 2.11. *The results in the previous Subsections 2.1 and 2.2 have classical counterparts. Namely, when all the involved starting elements are SG-classical, the resulting objects (multi-products, amplitudes, phase functions, etc.) are SG-classical as well.*

3. Commutative law for multi-products of regular SG-phase functions

In this section we prove a commutative law for the multi-products of regular SG phase functions. Through this result, we further expand the theory of SG FIOs, which we will be able to apply to obtain the solution of Cauchy problems for weakly hyperbolic linear differential operators, with polynomially bounded coefficients, variable multiplicities and involutive characteristics.

More precisely, we focus on the $\sharp$-product of regular SG phase functions obtained as solutions to eikonal equations. Namely, let $[\varphi_j(t,s)](x,\xi) = \varphi_j(t,s;x,\xi)$ be the phase functions defined by the eikonal equations (2.7), with φ_j in place of φ and a_j in place of a, where $a_j \in C([0,T]; S^{1,1})$, a_j real-valued, $j \in \mathbb{N}$. Moreover, let $I_{\varphi_j}(t,s) = I_{\varphi_j(t,s)}$ be the SG FIO with phase function $\varphi_j(t,s)$ and symbol identically equal to 1.

Assume that $\{a_j\}_{j\in\mathbb{N}}$ is bounded in $C([0,T]; S^{1,1})$. Then, by the above Proposition 2.7, there exists a constant c, independent of j, such that $\varphi_j(t,s) \in \mathcal{P}_r(c|t-s|)$, $j \in \mathbb{N}$. We make a choice of $T_1 > 0$, once and for all, such that $cT_1 \leqslant \tau_0$ for a constant $\tau_0 < 1/4$. Moreover, for convenience below, we define, for $M \in \mathbb{Z}_+$,

$$
\begin{aligned}
&\mathbf{t}_{M+1} = (t_0, \dots, t_{M+1}) \in \Delta(T_1),\\
&\mathbf{t}_{M+1,j}(\tau) = (t_0, \dots, t_{j-1}, \tau, t_{j+1}, \dots, t_{M+1}),\ 0 \leqslant j \leqslant M+1,
\end{aligned}
\tag{3.1}
$$

where $\Delta(T_1) = \Delta_{M+1}(T_1) = \{(t_0, \dots, t_{M+1})\colon 0 \leqslant t_{M+1} \leqslant t_M \leqslant \dots \leqslant t_0 \leqslant T_1\}$.

3.1. **Parameter-dependent multi-products of regular SG phase functions.**

Let $M \geqslant 1$ be a fixed integer. We have the following well defined multi-product

$$
\phi(\mathbf{t}_{M+1}; x, \xi) = [\varphi_1(t_0,t_1) \sharp \varphi_2(t_1,t_2) \sharp \dots \sharp \varphi_{M+1}(t_M, t_{M+1})](x,\xi),
\tag{3.2}
$$

where we will often set $t_0 = t$, $t_{M+1} = s$, for $\mathbf{t}_{M+1} \in \Delta(T_1)$ from (3.1). Explicitly, ϕ is defined by means of the critical points $(Y,N) = (Y,N)(\mathbf{t}_{M+1}; x, \xi)$, obtained, when $M \geqslant 2$, as solutions of the system

$$
\begin{cases}
Y_1(\mathbf{t}_{M+1}; x,\xi) = \varphi'_{1,\xi}(t_0, t_1; x, N_1(\mathbf{t}_{M+1}; x, \xi))\\
Y_j(\mathbf{t}_{M+1}; x,\xi) = \varphi'_{j,\xi}(t_{j-1}, t_j; Y_{j-1}(\mathbf{t}_{M+1}; x, \xi), N_j(\mathbf{t}_{M+1}; x, \xi)), & j = 2, \dots, M\\
N_j(\mathbf{t}_{M+1}; x,\xi) = \varphi'_{j+1,x}(t_j, t_{j+1}; Y_j(\mathbf{t}_{M+1}; x, \xi), N_{j+1}(\mathbf{t}_{M+1}; x, \xi)), & j = 1, \dots, M-1\\
N_M(\mathbf{t}_{M+1}; x,\xi) = \varphi'_{M+1,x}(t_M, t_{M+1}; Y_M(\mathbf{t}_{M+1}; x, \xi), \xi),
\end{cases}
\tag{3.3}
$$

namely,

$$
\begin{aligned}
\phi(\mathbf{t}_{M+1}; x, \xi) := \sum_{j=1}^{M} \big[&\varphi_j(t_{j-1}, t_j; Y_{j-1}(\mathbf{t}_{M+1}; x, \xi), N_j(\mathbf{t}_{M+1}; x, \xi))\\
&- Y_j(\mathbf{t}_{M+1}; x, \xi) \cdot N_j(\mathbf{t}_{M+1}; x, \xi)\big] + \varphi_{M+1}(t_M, t_{M+1}; Y_M(\mathbf{t}_{M+1}; x, \xi), \xi).
\end{aligned}
\tag{3.4}
$$

We give below some properties of the multi-product ϕ (see [3] for the detailed construction of the multi-product of regular SG phase functions which do not depend on parameters; the proof of Proposition 3.1 here below can be found in [1]: it follows by the results in [3], together with a careful use of Lemmas 2.1 and 2.4).

Let ϕ be the multi-product (3.4), with real-valued $a_j \in C([0,T]; S^{1,1}(\mathbb{R}^{2n}))$, $j = 1, \dots, M+1$. For any φ_j solution to the eikonal equation associated with the Hamiltonian a_j, we have

$$
\begin{cases}
\varphi_i(t,s) \sharp \varphi_j(s,s) &= \varphi_i(t,s)\\
\varphi_i(s,s) \sharp \varphi_j(t,s) &= \varphi_j(t,s),
\end{cases}
\tag{3.5}
$$

for all $i, j = 1, \dots, M+1$.

Proposition 3.1. *Let $\{a_j\}_{j\in\mathbb{N}}$ be a family of parameter-dependent, real-valued symbols, bounded in $C^\infty([0,T];S^{1,1}(\mathbb{R}^{2n}))$, and (Υ,N) be the solution of (3.3). Then, for $\gamma_k\in\mathbb{Z}_+$, $k=0,1\dots,M+1$, the following properties hold true.*

i: *for some $T_1\in(0,T]$ as in Proposition 2.7,*

$$\begin{cases}\Upsilon_j \text{ belongs to } C^\infty\left(\Delta(T_1);S^{1,0}(\mathbb{R}^{2n})\right),\\ N_j \text{ belongs to } C^\infty\left(\Delta(T_1);S^{0,1}(\mathbb{R}^{2n})\right).\end{cases}$$

ii: *For $J_{M+1}(\mathbf{t}_{M+1};x,\xi)=\phi(\mathbf{t}_{M+1};x,\xi)-x\cdot\xi$, we have*

$$\{\partial_{t_0}^{\gamma_0}\dots\partial_{t_{M+1}}^{\gamma_{M+1}}J_{M+1}\} \text{ is bounded in } S^{1,1}(\mathbb{R}^{2n}).$$

3.2. **A useful quasilinear auxiliary PDE.** Consider the following quasi-linear partial differential equation

$$(3.6)\qquad \begin{cases}\partial_{t_{j-1}}\Upsilon(\mathbf{t}_{M+1})-H(\Upsilon(\mathbf{t}_{M+1}),\mathbf{t}_{M+1})\cdot\Upsilon'_x(\mathbf{t}_{M+1})-G(\mathbf{t}_{M+1,j}(\Upsilon(\mathbf{t}_{M+1})))=0,\\ \Upsilon|_{t_{j-1}=t_j}=t_{j+1}\end{cases}$$

where, for $s\in\mathbb{R}$, $\mathbf{t}_{M+1,j}(s)$ is defined in (3.1), $\Upsilon(\mathbf{t}_{M+1})=\Upsilon(\mathbf{t}_{M+1};x,\xi)$ satisfies $\Upsilon\in C^\infty(\Delta(T_1);S^{0,0})$, and $H(\tau,\mathbf{t}_{M+1})=L(\tau,\mathbf{t}_{M+1};x,\xi)$ is a vector-valued family of symbols of order $(1,0)$ such that $H\in C^\infty([t_{j+1},t_{j-1}]\times\Delta(T_1);S^{1,0}\otimes\mathbb{R}^n)$. We also assume that $G(\mathbf{t}_{M+1,j}(\tau))=G(\mathbf{t}_{M+1,j}(\tau);x,\xi)$ with $G\in C^\infty(\Delta(T_1);S^{0,0})$, and satisfies $G(\mathbf{t}_{M+1};x,\xi)>0$, $G(\mathbf{t}_{M+1};x,\xi)|_{t_j=t_{j-1}}\equiv 1$, for any $\mathbf{t}_{M+1}\in\Delta(T_1)$, $(x,\xi)\in\mathbb{R}^{2n}$.

The following Lemma 3.2 (cf. [37]) is a key step to prove our first main technical tool, Theorem 3.4. In fact, it gives the solution of the characteristics system

$$(3.7)\qquad \begin{cases}\partial_{t_{j-1}}R(\mathbf{t}_{M+1})=-H(K(\mathbf{t}_{M+1}),\mathbf{t}_{M+1};R(\mathbf{t}_{M+1}),\xi)\\ \partial_{t_{j-1}}K(\mathbf{t}_{M+1})=\quad G(\mathbf{t}_{M+1,j}(K(\mathbf{t}_{M+1}));R(\mathbf{t}_{M+1}),\xi)\\ R|_{t_{j-1}=t_j}=y,\ K|_{t_{j-1}=t_j}=t_{j+1},\end{cases}$$

which then provides, by standard arguments, the solution to the quasi-linear equation (3.6). The latter, in turn, is useful to simplify the computations in the proof of Theorem 3.4. In view of this, we give a sketch of the proof of Lemma 3.2.

Lemma 3.2. *There exists a constant $T_2\in(0,T_1]$ such that (3.7) admits a unique solution $(R,K)=(R,K)(\mathbf{t}_{M+1};y,\xi)\in C^\infty(\Delta(T_2);(S^{1,0}(\mathbb{R}^{2n})\otimes\mathbb{R}^n)\times S^{0,0}(\mathbb{R}^{2n}))$, $t_{j-1}\in[t_j,T_2]$, which satisfies, for any $\mathbf{t}_{M+1}\in\Delta(T_2)$, $(y,\xi)\in\mathbb{R}^{2n}$,*

$$(3.8)\qquad \left\|\frac{\partial R}{\partial y}(\mathbf{t}_{M+1};y,\xi)-I\right\|\leqslant C(t_{j-1}-t_{j+1}),$$

for a suitable constant $C>0$ independent of M, and

$$(3.9)\qquad \begin{cases}t_{j+1}\leqslant K(\mathbf{t}_{M+1};y,\xi)\leqslant t_{j-1}\\ K_{|t_{j-1}=t_j}=t_{j+1}.\end{cases}$$

Proof. First, we notice that, as a consequence of Lemmas 2.1 and 2.4, the compositions in the right-hand side of (3.7) are well defined, and produce symbols of order $(1,0)$ and $(0,0)$, respectively, provided that $(R,K)\in C^\infty(\Delta(T_2);(S^{1,0}\otimes\mathbb{R}^n)\times S^{0,0})$, $\langle R(\mathbf{t}_{M+1};y,\xi)\rangle\asymp\langle y\rangle$, and $K(\mathbf{t}_{M+1};y,\xi)\in[t_{j+1},t_{j-1}]$ for any $\mathbf{t}_{M+1}\in\Delta(T_2)$, $y,\xi\in\mathbb{R}^n$, $T_2\in(0,T_1]$ sufficiently small.

We focus only on the variables $(t_{j-1},t_j,t_{j+1};y,\xi)$, since all the others here play the role of (fixed) parameters, on which the solution clearly depends smoothly. We then omit them in the next computations. We will also write, to shorten some of the formulae, $(R,K)(s)=(R,K)(\mathbf{t}_{M+1,j-1}(s);y,\xi)$, $s\in[t_j,T_2]$, $T_2\in(0,T_1]$ sufficiently small, $\mathbf{t}_{M+1}\in\Delta(T_2)$, $(y,\xi)\in\mathbb{R}^{2n}$.

We rewrite (3.7) in integral form, namely

$$
\begin{cases} R(s) = y - \displaystyle\int_{t_j}^{s} H(K(\sigma);\sigma,t_j,t_{j+1};R(\sigma),\xi)\,d\sigma \\ K(s) = t_{j+1} + \displaystyle\int_{t_j}^{s} G(\sigma,K(\sigma),t_{j+1};R(\sigma),\xi)\,d\sigma, \end{cases} \tag{3.10}
$$

$s \in [t_j, T_2]$, $\mathbf{t}_{M+1} \in \Delta(T_2)$, $(y,\xi) \in \mathbb{R}^{2n}$, and solve (3.10) by the customary Picard method of successive approximations. That is, we define the sequences

$$
\begin{cases} R_{l+1}(s) = y - \displaystyle\int_{t_j}^{s} H(K_l(\sigma);\sigma,t_j,t_{j+1};R_l(\sigma),\xi)\,d\sigma \\ K_{l+1}(s) = t_{j+1} + \displaystyle\int_{t_j}^{s} G(\sigma,K_l(\sigma),t_{j+1};R_l(\sigma),\xi)\,d\sigma, \end{cases} \tag{3.11}
$$

for $l = 1,2,\dots$, $s \in [t_j, T_2]$, $\mathbf{t}_{M+1} \in \Delta(T_2)$, $(y,\xi) \in \mathbb{R}^{2n}$, with $R_0(s) = y$, $K_0(s) = s - t_j + t_{j+1}$. Arguing as in [37, proof of Lemma 3.10], see [1] for detailed proof in the SG case, we can show that $\{R_l\}_l$ is bounded in $C^\infty(\Delta(T_2); S^{1,0} \otimes \mathbb{R}^n)$, while $\{K_l\}_l$ is bounded in $C^\infty(\Delta(T_2); S^{0,0})$. It also holds $\langle R_l(s)\rangle \asymp \langle y\rangle$, uniformly with respect to $s \in [t_j, T_2]$ and $l \in \mathbb{N}$, with T_2 sufficiently small. Moreover, for any $\alpha,\beta \in \mathbb{Z}_+$,

(3.12)
$$
\sup_{(y,\xi)\in\mathbb{R}^{2n}} \left| \partial_y^\alpha \partial_\xi^\beta \left(K_{l+1}(\mathbf{t}_{M+1,j-1}(t_{j-1}); y,\xi) - K_l(\mathbf{t}_{M+1,j-1}(t_{j-1}); y,\xi) \right) \langle y\rangle^{|\alpha|} \langle \xi\rangle^{|\beta|} \right| \leqslant C_{\alpha\beta} \frac{(t_{j-1} - t_j)^{N+1}}{(N+1)!},
$$

with $C_{\alpha\beta}$ independent of j, N, and, similarly,

(3.13)
$$
\sup_{(y,\xi)\in\mathbb{R}^{2n}} \left| \partial_y^\alpha \partial_\xi^\beta \left(R_{l+1}(\mathbf{t}_{M+1,j-1}(t_{j-1}); y,\xi) - R_l(\mathbf{t}_{M+1,j-1}(t_{j-1}); y,\xi) \right) \langle y\rangle^{-1+|\alpha|} \langle \xi\rangle^{|\beta|} \right| \leqslant \widetilde{C}_{\alpha\beta} \frac{(t_{j-1} - t_j)^{N+1}}{(N+1)!},
$$

where $\widetilde{C}_{\alpha\beta}$ is independent of j, N. This follows by induction, via a (standard) Taylor expansion technique, and direct computations. The main new aspect here, compared with the original argument in [37], concerns the necessity of obtaining global estimates also with respect to $y \in \mathbb{R}^n$. This (rather challenging) complication is overcome by carefully employing Lemmas 2.1 and 2.4, jointly with the known, classical techniques.

Writing l in place of N in the right-hand side of (3.12) and (3.13), it easily follows that (R_l, K_l) converges, for $l \to +\infty$, to a unique fixed point (R,K), which satisfies the stated symbol estimates. Since, by induction, the properties (3.8) and (3.9) hold true for (R_l, K_l) in place of (R,K), $l \geqslant 0$, uniformly with respect to M, j, l, they also hold true for the limit (R,K). □

The next Corollary 3.3 is a standard result in the theory of Cauchy problems for quasilinear PDEs of the form (3.6). Its proof is based on the hypotheses on L and H, and the properties of the solution of (3.7).

Corollary 3.3. *Under the same hypotheses of Lemma 3.2, denoting by $\bar{R}(\mathbf{t}_{M+1}; x,\xi)$ the solution of the equation $R(\mathbf{t}_{M+1}; y,\xi) = x$ with $\mathbf{t}_{M+1} \in \Delta(T_2), x,\xi \in \mathbb{R}^n$, the function*

$$
\Upsilon(\mathbf{t}_{M+1}; x,\xi) = K(\mathbf{t}_{M+1}; \bar{R}(\mathbf{t}_{M+1}; x,\xi),\xi)
$$

solves the Cauchy problem (3.6) *for* $x, \xi \in \mathbb{R}^n$, $\mathbf{t}_{M+1} \in \Delta(T_2)$, *for a sufficiently small* $T_2 \in (0, T_1]$.

3.3. **Commutative law for multi-products of SG phase functions given by solutions of eikonal equations.** Let $\{a_j\}_{j\in\mathbb{N}}$ be a bounded family of parameter-dependent, real-valued symbols in $C^\infty([0,T]; S^{1,1})$ and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be the corresponding family of regular phase functions in $\mathcal{P}_r(c|t-s|)$, obtained as solutions to the eikonal equations associated with a_j, $j \in \mathbb{N}$. In the aforementioned multi-product (3.2), we commute φ_j and φ_{j+1}, defining a new multi-product ϕ_j, namely

$$\begin{aligned}\phi_j(\mathbf{t}_{M+1}; x, \xi) = &(\varphi_1(t_0, t_1)\sharp\varphi_2(t_1, t_2)\sharp \ldots \sharp\varphi_{j-1}(t_{j-2}, t_{j-1}))\,\sharp \\ &\sharp\,(\varphi_{j+1}(t_{j-1}, t_j)\sharp\varphi_j(t_j, t_{j+1}))\,\sharp\,(\varphi_{j+2}(t_{j+1}, t_{j+2})\sharp \ldots \sharp\varphi_{M+1}(t_M, t_{M+1}))(x, \xi),\end{aligned} \tag{3.14}$$

where $\mathbf{t}_{M+1} = (t_0, t_1, \ldots, \ldots t_{M+1}) \in \Delta(T_1)$.

Assumption I (Involutiveness of symbol families)**.** *Given the family of parameter-dependent, real-valued symbols* $\{a_j\}_{j\in\mathbb{N}} \subset C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2n}))$, *there exist families of parameter-dependent, real-valued symbols* $\{b_{jk}\}_{j,k\in\mathbb{N}}$ *and* $\{d_{jk}\}_{j,k\in\mathbb{N}}$, *such that* $b_{jk}, d_{jk} \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2n}))$, $j, k \in \mathbb{N}$, *and the Poisson brackets*

$$\begin{aligned}\{\tau - a_j(t;x,\xi), \tau - a_k(t;x,\xi)\} := &\,\partial_t a_j(t;x,\xi) - \partial_t a_k(t;x,\xi) \\ &+ a'_{j,\xi}(t;x,\xi)\cdot a'_{k,x}(t;x,\xi) - a'_{j,x}(t;x,\xi)\cdot a'_{k,\xi}(t;x,\xi)\end{aligned}$$

satisfy

$$\{\tau - a_j(t;x,\xi), \tau - a_k(t;x,\xi)\} = b_{jk}(t;x,\xi)\cdot(a_j - a_k)(t;x,\xi) + d_{jk}(t;x,\xi), \tag{3.15}$$

for all $j, k \in \mathbb{N}$, $t \in [0,T]$, $x, \xi \in \mathbb{R}^n$.

We can now state our first technical, but crucial, result, Theorem 3.4. It provides a commutative law for multi-products of regular phase functions obtained by solving eikonal equations. The latter are related to the characteristic roots of the largest class, to our best knowledge, of linear hyperbolic equations treated in the literature by means of these microlocal techniques, to produce an expression of the solution with finitely many terms (modulo a regular remainder).

Theorem 3.4. *Let* $\{a_j\}_{j\in\mathbb{N}}$ *be a family of parameter-dependent, real-valued symbols, bounded in* $C^\infty\big([0,T]; S^{1,1}(\mathbb{R}^{2n})\big)$, *and let* $\varphi_j \in \mathcal{P}_r(c|t-s|)$, *for some* $c > 0$, *be the phase functions obtained as solutions to* (2.7) *with* a_j *in place of* a, $j \in \mathbb{N}$. *Consider* $\phi(\mathbf{t}_{M+1})$ *and* $\phi_j(\mathbf{t}_{M+1})$ *defined in* (3.2) *and* (3.14), *respectively, for any* $M \geqslant 2$ *and* $j \leqslant M$. *Then, Assumption I implies that there exists* $T' \in (0,T]$, *independent of* M, *such that we can find a symbol family* $Z_j \in C^\infty(\Delta(T'); S^{0,0}(\mathbb{R}^{2n}))$ *satisfying, for all* $\mathbf{t}_{M+1} \in \Delta(T')$, $x, \xi \in \mathbb{R}^n$,

$$\begin{aligned}&t_{j+1} \leqslant Z_j(\mathbf{t}_{M+1}; x, \xi) \leqslant t_{j-1}, \\ &Z_j|_{t_j = t_{j-1}} = t_{j+1}, \text{ and } Z_j|_{t_j = t_{j+1}} = t_{j-1}, \\ &-2 \leqslant \partial_{t_j} Z_j(\mathbf{t}_{M+1}; x, \xi) \leqslant 0.\end{aligned} \tag{3.16}$$

Moreover, we have

$$\phi_j(\mathbf{t}_{M+1}; x, \xi) = \phi(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1}; x, \xi)); x, \xi) + \Psi_j(\mathbf{t}_{M+1}; x, \xi), \mathbf{t}_{M+1} \in \Delta(T'), x, \xi \in \mathbb{R}^n,$$

where $\Psi_j \in C^\infty(\Delta(T'); S^{0,0}(\mathbb{R}^{2n}))$ *also satisfies* $\Psi_j \equiv 0$ *if* $d_j = d_{j,j+1} \equiv 0$, *with the symbols* $d_{j,j+1}$ *appearing in* (3.15) *for* $k = j+1$.

The argument originally given in [37], to prove the analog of Theorem 3.4 in the classical case, extends to the SG setting, in view of Lemmas 2.1 and 2.4 above, and the preparation in Lemma 3.2 and Corollary 3.3. To avoid departing too much from our main focus (applications to SG-hyperbolic Cauchy problems), here we just give a sketch of the proof of Theorem 3.4 (see [1] for more details).

Proof of Theorem 3.4. We set

$$\Psi_j(\mathbf{t}_{M+1};x,\xi) = \phi_j(\mathbf{t}_{M+1};x,\xi) - \phi(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi), \tag{3.17}$$

and look for a symbol $Z_j = Z_j(\mathbf{t}_{M+1};x,\xi)$ satisfying (3.16), such that we also have $\Psi_j \in C^\infty(\Delta(T');S^{0,0})$, $T' \in (0,T]$. Since the definition of Ψ_j is formally the same given in [37], by the same computations performed there, it turns out that Ψ_j must fulfill

$$\partial_{t_{j-1}}\Psi_j = H_j(Z_j)\cdot\Psi'_{j,x} - F_j(Z_j) - (\partial_{t_j}\phi)(Z_j)\Big(\partial_{t_{j-1}}Z_j - H_j(Z_j)\cdot Z'_{j,x} - G_j(Z_j)\Big), \tag{3.18}$$

where we omitted everywhere the dependence on $(\mathbf{t}_{M+1};x,\xi)$, $(\partial_{t_j}\phi)(Z_j)$ stands for $(\partial_{t_j}\phi)(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi)$, and

$$F_j(Z_j) = F_j(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi), G_j(Z_j) = G_j(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi).$$

This first step of the proof is achieved by employing the properties of the solutions of the Hamiltonian systems associated with the a_k, $k \in \mathbb{N}$, recalled in Proposition 2.6. In view of the composition Lemmas 2.1 and 2.4, all the performed computations produce symbols which remain in the SG classes. In particular, it turns out that $F_j, G_j \in C^\infty(\Delta(T_1);S^{0,0})$ and $H_j \in C^\infty([t_{j+1},t_{j-1}]\times\Delta(T_1);S^{1,0}\otimes\mathbb{R}^n)$, for a suitable $T_1 \in (0,T]$.

Now, in order to simplify (3.18), we choose Z_j as solution to the quasilinear Cauchy problem

$$\begin{cases} \partial_{t_{j-1}}Z_j &= H_j(Z_j)\cdot Z'_{j,x} + G_j(Z_j) \\ Z|_{t_j=t_{j-1}} &= t_{j+1}. \end{cases} \tag{3.19}$$

It turns out that (3.19) is a quasilinear Cauchy problem of the type considered in Subsection 3.2. In view of Lemma 3.2, we can solve (3.19) through its characteristic system (3.7), with H_j in place of H and G_j in place of G, choosing a sufficiently small parameter interval $[0,T']$, $T' \in (0,T_1]$. Indeed, by Corollary 3.3, defining

$$Z_j(\mathbf{t}_{M+1};x,\xi) = K(\mathbf{t}_{M+1};\bar{R}(\mathbf{t}_{M+1};x,\xi),\xi), \quad \mathbf{t}_{M+1} \in \Delta(T'), x,\xi \in \mathbb{R}^n,$$

gives a solution of (3.19) with all the desired properties.

With such choice of Z_j, (3.18) is then reduced to the linear, non-homogeneous PDE

$$\partial_{t_{j-1}}\Psi_j = H_j(Z_j)\cdot\Psi'_{j,x} - F_j(Z_j), \tag{3.20}$$

with the initial condition

$$\Psi_j|_{t_{j-1}=t_j} = 0. \tag{3.21}$$

Notice that (3.21) holds true since we have $Z_j|_{t_j=t_{j-1}} = t_{j+1}$, that, together with (3.5), gives

$$\begin{aligned}
&\Psi_j(\mathbf{t}_{M+1};x,\xi)|_{t_{j-1}=t_j} = \Psi_j(\mathbf{t}_{M+1,j-1}(t_j);x,\xi)\\
&= \phi_j(\mathbf{t}_{M+1,j-1}(t_j);x,\xi) - \phi(t_0,\dots,t_{j-2},t_j,Z_j(\mathbf{t}_{M+1};x,\xi)|_{t_j=t_{j-1}},t_{j+1},\dots,t_{M+1};x,\xi)\\
&\qquad = \big[\varphi_1(t_0,t_1)\sharp\dots\sharp\varphi_{j-1}(t_{j-2},t_j)\sharp \underbrace{(\varphi_{j+1}(t_j,t_j)\sharp\varphi_j(t_j,t_{j+1}))}_{=\varphi_j(t_j,t_{j+1})}\sharp\\
&\qquad\qquad \sharp\varphi_{j+2}(t_{j+1},t_{j+2})\sharp\dots\varphi_{M+1}(t_M,t_{M+1})\big]\,(x,\xi)\\
&\qquad - \big[\varphi_1(t_0,t_1)\sharp\dots\sharp\varphi_{j-1}(t_{j-2},t_j)\sharp \underbrace{(\varphi_j(t_j,t_{j+1})\sharp\varphi_{j+1}(t_{j+1},t_{j+1}))}_{=\varphi_j(t_j,t_{j+1})}\sharp\\
&\qquad\qquad\qquad \sharp\varphi_{j+2}(t_{j+1},t_{j+2})\sharp\dots\sharp\varphi_{M+1}(t_M,t_{M+1})\big]\,(x,\xi) = 0.
\end{aligned}$$

Then, the method of characteristics, applied to the linear, non-homogeneous PDE (3.20), shows that we can write Ψ_j in the form

$$\Psi_j(\mathbf{t}_{M+1};x,\xi) = \int_{t_j}^{t_{j-1}} \widetilde{F}_j(\mathbf{t}_{M+1,j-1}(\tau);\theta(\tau;\tilde{\theta}(\tau;x,\xi),\xi),\xi)d\tau, \tag{3.22}$$

where

$$\widetilde{F}_j(\mathbf{t}_{M+1};x,\xi) = -F_j(\mathbf{t}_{M+1,j}(Z_j(\mathbf{t}_{M+1};x,\xi));x,\xi),$$
$$\theta(\tau;y,\xi) = \theta(\mathbf{t}_{M+1,j}(\tau);y,\xi), \quad \tilde{\theta}(\tau;x,\xi) = \tilde{\theta}(\mathbf{t}_{M+1,j-1}(\tau);x,\xi),$$

for suitable vector-valued functions $\theta, \tilde{\theta}$. By arguments similar to those in [37], both θ and $\tilde{\theta}$ turn out to be elements of $C^\infty(\Delta(T');S^{1,0}\otimes\mathbb{R}^n)$, satisfying

$$\langle\theta(\mathbf{t}_{M+1,j}(\tau);y,\xi)\rangle \asymp \langle y\rangle\,, \quad \langle\tilde{\theta}(\mathbf{t}_{M+1,j-1}(\tau);x,\xi)\rangle \asymp \langle x\rangle,$$

with constants independent of $\mathbf{t}_{M+1}\in\Delta(T')$, $x,\xi\in\mathbb{R}^n$. Such result, together with the properties of Z_j and another application of Lemma 2.4, allows to conclude that $\widetilde{F}_j\in C^\infty(\Delta(T');S^{0,0})$, which in turn implies $\Psi_j\in C^\infty(\Delta(T');S^{0,0})$. That $\Psi_j\equiv 0$ when $d_j\equiv 0$ follows exactly as in [37], since, also for the SG case, $d_j\equiv 0 \Rightarrow \widetilde{F}_j\equiv 0$. ◻

4. Cauchy problems for weakly SG-hyperbolic first order systems

In the present section we deal with the Cauchy problem

$$\begin{cases} \mathbf{L}U(t,s) = F(t), & (t,s)\in\Delta_T,\\ U(s,s) = G, & s\in[0,T), \end{cases} \tag{4.1}$$

on the simplex $\Delta_T := \{(t,s)|\ 0\leqslant s\leqslant t\leqslant T\}$, where

$$\mathbf{L}(t,D_t;x,D_x) = D_t + \Lambda(t;x,D_x) + R(t;x,D_x), \tag{4.2}$$

Λ is a $(N\times N)$-dimensional, diagonal operator matrix, whose entries $\lambda_j(t;x,D_x)$, $j=1,\dots,N$, are pseudo-differential operators with real-valued, parameter-dependent symbols $\lambda_j(t;x,\xi)\in C^\infty([0,T];S^{1,1})$, R is a parameter-dependent, $(N\times N)$-dimensional operator matrix of pseudo-differential operators with symbols in $C^\infty([0,T];S^{0,0})$, $F\in C^\infty([0,T],H^{r,\varrho}\otimes\mathbb{R}^N)$, $G\in H^{r,\varrho}\otimes\mathbb{R}^N$, $r,\varrho\in\mathbb{R}$.

The system (4.2) is then of hyperbolic type, since the principal symbol part $\mathrm{diag}(\lambda_j(t;x,\xi))_{j=1,\dots,N}$ of the coefficient matrix is diagonal and real-valued. Then, its fundamental solution $E(t,s)$ exists (see [10]), and can be obtained as an infinte sum of matrices of Fourier integral operators (see [25, 37] and Section 5 of [3] for the SG case). Indeed, for T' small enough, it is possible to express $\{E(t,s)\}$ in the form

$$E(t,s) = I_\varphi(t,s) + \int_s^t I_\varphi(t,\theta) \sum_{\nu=1}^{\infty} W_\nu(\theta,s)\, d\theta, \tag{4.3}$$

where $I_\varphi(t,s)$ is the operator matrix defined by

$$I_\varphi(t,s) = \begin{pmatrix} I_{\varphi_1}(t,s) & & 0 \\ & \ddots & \\ 0 & & I_{\varphi_N}(t,s) \end{pmatrix}$$

$I_{\varphi_j} := Op_{\varphi_j}(1)$, $1 \leqslant j \leqslant N$, and the phase functions $\varphi_j = \varphi_j(t,s;x,\xi)$, $1 \leqslant j \leqslant N$, defined on $\Delta_{T'} \times \mathbb{R}^{2n}$, are solutions to the eikonal equations (2.7) with $-\lambda_j$ in place of a.

Here we are going to show that if (4.1) is of involutive type, then its fundamental solution $E(t,s)$ can be reduced to a finite sum expression, modulo a smoothing remainder, in the same spirit of [25, 37], by applying the results from Section 3 above.

We define the sequence of $(N \times N)$-dimensional matrices of SG FIOs $\{W_\nu(t,s); (t,s) \in \Delta(T')\}_{\nu\in\mathbb{N}}$ recursively as

$$W_{\nu+1}(t,s;x,D_x) = \int_s^t W_1(t,\theta;x,D_x) W_\nu(\theta,s;x,D_x) d\theta, \tag{4.4}$$

starting with W_1 defined as

$$\mathbf{L} I_\varphi(t,s) = i W_1(t,s). \tag{4.5}$$

We also set

$$w_j(t,s;x,\xi) = \sigma(W_j(t,s;x,D_x)), \quad j = 1,\dots,\nu+1,\dots. \tag{4.6}$$

the (matrix-valued) symbol of W_j. Notice that, since the phase functions φ_j are solutions of eikonal equations (2.7) associated with the Hamiltonians $-\lambda_j$, we have for $j = 1,\dots,N$ the relation

$$D_t \mathrm{Op}_{\varphi_j(t,s)}(1) + \mathrm{Op}(\lambda_j(t))\mathrm{Op}_{\varphi_j(t,s)}(1) = \mathrm{Op}_{\varphi_j(t,s)}(b_{0,j}(t,s)), \quad b_{0,j}(t,s) \in S^{0,0}(\mathbb{R}^{2n}).$$

Then,

$$W_1(t,s) := -i \left(\begin{pmatrix} B_{0,1}(t,s) & & 0 \\ & \ddots & \\ 0 & & B_{0,N}(t,s) \end{pmatrix} + R(t) I_\varphi(t,s) \right), \tag{4.7}$$

with $B_{0,j}(t,s) = \mathrm{Op}_{\varphi_j(t,s)}(b_{0,j}(t,s))$ and $b_{0,j}(t,s) \in S^{0,0}$, $j = 1,\dots,N$.

By (4.7) and algebraic properties of FIOs and PDOs, one can rewrite (4.5) as

$$L I_\varphi(t,s) = \sum_{j=1}^{N} \widetilde{W}_{\varphi_j}(t,s), \tag{4.8}$$

where $\widetilde{W}_{\varphi_j}(t,s)$ are $(N \times N)$-dimensional matrices, with entries given by Fourier integral operators with parameter-dependent phase function φ_j and symbol in $S^{0,0}$, $1 \leqslant j \leqslant N$. Thus, if we set $M_\nu = \{\mu = (N_1,\dots,N_\nu) : N_k = 1,\dots,N,\ k = 1,\dots,\nu\}$ for

$\nu \geqslant 2$, the operator matrix $W_\nu(t,s)$ can be written in the form of iterated integrals, namely

$$\int_s^t \int_s^{\theta_1} \dots \int_s^{\theta_{\nu-2}} \sum_{\mu \in M_\nu} W^{(\mu)}(t, \theta_1, \dots, \theta_{\nu-1}, s) d\theta_{\nu-1} \dots d\theta_1, \tag{4.9}$$

where

$$W^{(\mu)}(t, \theta_1, \dots, \theta_{\nu-1}, s) = W_{\varphi_{N_1}}(t, \theta_1) W_{\varphi_{N_2}}(\theta_1, \theta_2) \dots W_{\varphi_{N_\nu}}(\theta_{\nu-1}, s)$$

is the product of ν matrices of SG FIOs with regular phase functions φ_{N_j} and symbols $\sigma(W_{\varphi_{N_j}}(\theta_{j-1}, \theta_j)) = -i\sigma(\widetilde{W}_{\varphi_{N_j}}(\theta_{j-1}, \theta_j)) \in S^{0,0}$. By the algebra properties of SG FIOs (cf. [3]), $W^{(\mu)}(t, \theta_1, \dots, \theta_{\nu-1}, s)$ is a matrix of SG FIOs with phase function $\phi^{(\mu)} = \varphi_{N_1} \sharp \dots \sharp \varphi_{N_\nu}$ and parameter-dependent symbol $\omega^{(\mu)}(t, \theta_1, \dots, \theta_{\nu-1}, s)$ of order $(0,0)$. Consequently, we can write

$$\begin{aligned} E(t,s) = I_\varphi(t,s) + \int_s^t I_\varphi(t,\theta) \Bigg\{ \sum_{j=1}^N W_{\varphi_j}(\theta, s) \\ + \sum_{\nu=2}^\infty \sum_{\mu \in M_\nu} \int_s^\theta \int_s^{\theta_1} \dots \int_s^{\theta_{\nu-2}} W^{(\mu)}(\theta, \theta_1, \dots, \theta_{\nu-1}, s) d\theta_{\nu-1} \dots d\theta_1 \Bigg\} d\theta. \end{aligned} \tag{4.10}$$

Given the commutative properties of the product of the Fourier integral operators appearing in the expression of $E(t,s)$ under Assumption I, which follow from the results proved in Section 3, our main result for SG-involutive systems, the next Theorem 4.1, can be proved by an argument analogous to the one illustrated in [37] (the details of the proof in the SG symbol classes can be found in [1]).

Theorem 4.1. *Let (4.1) be an involutive SG-hyperbolic system, that is, Assumption I is fulfilled by the family* $\{\lambda_j\}_{j=1}^N$. *Then, the fundamental solution (4.10) can be reduced, modulo smoothing terms, to*

$$\begin{aligned} E(t,s) = & I_\varphi(t,s) + \sum_{j=1}^N W^\dagger_{\varphi_j}(t,s) \\ & + \sum_{j=2}^N \sum_{\mu \in M_j^\dagger} \int_s^t \int_s^{\theta_1} \dots \int_s^{\theta_{j-2}} W^{(\mu^\dagger)}(t, \theta_1, \dots, \theta_{j-1}, s) d\theta_{j-1} \dots d\theta_1, \end{aligned} \tag{4.11}$$

where the symbol of $W^\dagger_{\varphi_j}(t,s)$ *is* $\int_s^t w_j(\theta, s)\, d\theta$, *with* w_j *in (4.6),* $\mu^\dagger = (N_1, \dots, N_j) \in M_j^\dagger := \{N_1 < \dots < N_j : N_k = 1, \dots, N, k = 1, \dots, j\}$, *and* $W^{(\mu^\dagger)}(t, \theta_1, \dots, \theta_{j-1}, s)$ *is a* $(N \times N)$*-dimensional matrix of SG Fourier integral operators with regular phase function* $\phi^{(\mu^\dagger)} = \varphi_{N_1} \sharp \dots \sharp \varphi_{N_j}$ *and matrix-valued, parameter-dependent symbol* $\omega^{(\mu^\dagger)}(t, \theta_1, \dots, \theta_{j-1}, s) \in S^{0,0}(\mathbb{R}^{2n})$.

Remark 4.2. *Theorem 4.1 can clearly be applied to the case of a* $N \times N$ *system such that* Λ *is diagonal and its symbol entries* λ_j, $j = 1, \dots, N$, *coincide with the (repeated) elements of a family of real-valued, parameter-dependent symbols* $\{\tau_j\}_{j=1}^m$, $1 < m < N$, *satisfying Assumption I. In such situation, working initially "block by block" of coinciding elements, and then performing the reduction of (4.10) to (4.11), through further applications of the* *commutative properties proved above, we see that (4.11) can be further reduced to*

$$E(t,s) = I_\varphi(t,s) + \sum_{j=1}^{m} W^\dagger_{\varphi_j}(t,s) + \sum_{j=2}^{m} \sum_{\mu\in M^\dagger_j} \int_s^t \int_s^{\theta_1} \dots \int_s^{\theta_{j-2}} W^{(\mu^\dagger)}(t,\theta_1,\dots,\theta_{j-1},s)d\theta_{j-1}\dots d\theta_1, \tag{4.12}$$

with $\mu^\dagger = (m_1,\dots,m_j) \in M^\dagger_j := \{m_1 < \dots < m_j : m_k = 1,\dots,m,\ k = 1,\dots,j\}$, and $(N\times N)$-dimensional matrices of SG Fourier integral operators with phase function $\phi^{(\mu^\dagger)} = \varphi_{m_1}\sharp\dots\sharp\varphi_{m_j}$ and matrix-valued, parameter-dependent symbol as above.

The following result about existence and uniqueness of a solution $U(t,s)$ to the Cauchy problem (4.1) is a SG variant of the classical Duhamel formula, see [3, 10, 12].

Proposition 4.3. *For $F \in C^\infty([0,T];H^{r,\varrho}(\mathbb{R}^n)\otimes\mathbb{R}^N)$ and $G \in H^{r,\varrho}(\mathbb{R}^n)\otimes\mathbb{R}^N$, the solution $U(t,s)$ of the Cauchy problem (4.1), under the SG-hyperbolicity assumptions explained above, exists uniquely for $(t,s) \in \Delta_{T'}$, $T' \in (0,T]$ suitably small, it belongs to the class $\bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'};H^{r-k,\varrho-k}(\mathbb{R}^n)\otimes\mathbb{R}^N)$, and is given by*

$$U(t,s) = E(t,s)G + i\int_s^t E(t,\sigma)F(\sigma)d\sigma, \quad (t,s)\in\Delta_{T'}, s\in[0,T').$$

5. Cauchy problems for weakly SG-hyperbolic linear operators

Here we employ the results from the previous section to the study of Cauchy problems associated with linear hyperbolic differential operators of SG type. After obtaining the fundamental solution, we study the propagation of singularities in the case of SG-classical coefficients. We recall here just the basic definition, see [10, 11, 12, 13, 18] for more details.

Definition 5.1. *An operator L of the form (1.2) is called (SG-)hyperbolic if its principal symbol L_m in (1.5) satisfies (1.6) with real-valued, smooth roots $\tau_j \in C^\infty([0,T], S^{1,1}(\mathbb{R}^n))$, $j = 1,\dots,m$. The roots τ_j are usually called characteristics. More precisely, L is called* (weakly) SG-hyperbolic with involutive roots (or SG-involutive)*, if L_m satisfies (1.6) with real-valued characteristic roots such that the family $\{\tau_j\}_{j=1}^m$ satisfies Assumption I.*

5.1. Fundamental solution for linear SG-involutive operators of order $m \in \mathbb{N}$.

Given an SG-involutive operator L in the sense of Definition 5.1, it is possible to translate the Cauchy problem

$$\begin{cases} Lu(t,s) = f(t), & (t,s)\in\Delta_T, \\ D_t^k u(s,s) = g_k, & k = 0,\dots,m-1, s\in[0,T), \end{cases} \tag{5.1}$$

into a Cauchy problem for an involutive system (4.1) with suitable initial conditions, under a suitable factorization condition, see below.

We write $\Theta_j = \mathrm{Op}(\tau_j)$, and also set, for convenience below, $\Gamma_j = D_t - \Theta_j$, $j = 1,\dots,m$. Moreover, with the permutations M_k of k elements of the set $\{1,\dots,m\}$ and their sorted counterparts $M^\dagger_k$, $1 \leqslant k \leqslant m$ from Section 4, we introduce the notation

$$M_0 = \{\varnothing\}, \quad M = \bigcup_{k=0}^{m-1} M_k, \quad M^\dagger = \bigcup_{k=1}^{m} M^\dagger_k.$$

For $\alpha \in M_k$, $0 \leqslant k \leqslant m$, we define $\mathbf{card}(\alpha) = k$ and

$$\Gamma_\varnothing = I, \qquad \Gamma_\alpha = \Gamma_{\alpha_1} \dots \Gamma_{\alpha_k},\ \alpha = (\alpha_1, \dots, \alpha_k) \in M_k, k \geqslant 1.$$

The proof of the following Lemma 5.2 can be found in [13]. Analogous results are used in [25] and [29].

Lemma 5.2. *When $\{\lambda_j\}$ is an involutive system, for all $\alpha \in M_m$ we have*

$$\Gamma_\alpha = \Gamma_1 \dots \Gamma_m + \sum_{\beta \in M} \mathrm{Op}(q^\alpha_\beta(t))\Gamma_\beta, \tag{5.2}$$

where $q^\alpha_\beta \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2n}))$.

A systemization and well-posedness (with loss of decay and regularity) theorem can be stated for the Cauchy problem (5.1) under a suitable condition for the operator L. This result is due, in its original local form, to Morimoto [29] and it has been extended to the SG case in [12], where the proof of the next result, based on Lemma 5.2, can be found.

Proposition 5.3. *Assume the SG-hyperbolic operator L to be of the form*

$$L = \Gamma_1 \cdots \Gamma_m + \sum_{\alpha \in M^\dagger} \mathrm{Op}(p_\alpha(t))\Gamma_\alpha \bmod \mathrm{Op}(C^\infty([0,T]; S^{-\infty,-\infty}(\mathbb{R}^{2n}))), \tag{5.3}$$

with $p_\alpha \in C^\infty([0,T]; S^{0,0}(\mathbb{R}^{2n}))$. Moreover, assume that the family of its characteristic roots $\{\tau_j\}_{j=1}^m$ sastisfies Assumption I. Then, the Cauchy problem (5.1) for L is equivalent to a Cauchy problem for a suitable first order system with diagonal principal part, of the form (4.1), where U, F and G are N-dimensional vector, K a $(N \times N)$-dimensional matrix, with N given by (5.4). U is defined in (5.5), (5.6), and (5.7). Namely,

$$N = \sum_{j=0}^{m-1} \frac{m!}{(m-j)!}, \tag{5.4}$$

$$U = {}^t\left(u_\varnothing \equiv u, u_{(1)}, \dots, u_{(m)}, u_{(1,2)}, u_{(2,1)}, \dots, u_\alpha, \dots\right), \tag{5.5}$$

with $\alpha \in M$, and

- *for $\alpha \in M_k$, $0 \leqslant k \leqslant m-2$ and $j = \max\{1, \dots, m\} - \alpha$, we set*

$$\Gamma_j u_\alpha = u_{\alpha_j} \tag{5.6}$$

with $\alpha_j = (j, \alpha_1, \dots, \alpha_k) \in M_{k+1}$;

- *for $\alpha \in M_{m-1}$ and $j \notin \{\alpha\}$, we set*

$$\Gamma_j u_\alpha = f - \sum_{\beta \in M^\dagger} \mathrm{Op}(p_\beta(t))u_\beta + \sum_{\beta \in M} \mathrm{Op}(q^{\alpha_j}_\beta(t))u_\beta, \tag{5.7}$$

with $\alpha_j = (j, \alpha_1, \dots, \alpha_k) \in M_m$ and the symbols p_β, q^α_β from (5.2) and (5.3).

Remark 5.4. *We call the SG-hyperbolic operators L satisfying the factorization condition (5.3) "operators of Levi type".*

Remark 5.5. *Since, for $\alpha \in M_k$, $k \geqslant 1$, we have*

$$\Gamma_\alpha = D_t^k + \sum_{j=0}^{k-1} \mathrm{Op}(\Upsilon^j_\alpha(s))D^j_t,\ \Upsilon^j_\alpha \in C^\infty([0,T]; S^{k-j,k-j}(\mathbb{R}^{2n})),$$

the initial conditions G for U can be expressed as

$$\begin{cases} G_\varnothing(s,s) = g_0, \\ G_\alpha(s,s) = g_{\mathbf{card}(\alpha)} + \displaystyle\sum_{j=0}^{\mathbf{card}(\alpha)-1} \mathrm{Op}(\Upsilon^j_\alpha(s))g_j, \quad \alpha \in M, \mathbf{card}(\alpha) > 0. \end{cases} \tag{5.8}$$

Notice that, in view of the continuity properties of the SG pseudo-differential operators and of the orders of the Υ^i_α, (5.8) *implies*

$$G_\alpha \in H^{m-1-\mathbf{card}(\alpha),\mu-1-\mathbf{card}(\alpha)}(\mathbb{R}^n), \quad \alpha \in M. \tag{5.9}$$

The next Theorem 5.6 is our first main result, namely, a well-posedness result, with decay and regularity loss, for the Cauchy problem (5.1). It is a consequence of Proposition 5.3 in combination with the analysis of first order systems in Section 4.

Theorem 5.6. *Let the operator L in* (5.1) *be SG-involutive, of the form considered in Proposition* 5.3. *Let* $f \in C^\infty([0,T]; H^{r,\varrho}(\mathbb{R}^n))$ *and* $g_k \in H^{r+m-1-k,\varrho+m-1-k}(\mathbb{R}^n)$, $k = 0,\dots,m-1$. *Then, for a suitable* $T' \in (0,T]$, *the Cauchy problem* (5.1) *admits a unique solution* $u(t,s)$, *belonging to* $\bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'}; H^{r-k,\varrho-k}(\mathbb{R}^n))$, *given, modulo elements in* $C^\infty(\Delta_{T'}; \mathcal{S}(\mathbb{R}^n))$, *by*

$$u(t,s) = \sum_{\alpha\in M} W_\alpha(t,s) G_\alpha + \sum_{\alpha\in M_{m-1}} \int_s^t W_\alpha(t,\sigma) f(\sigma)\, d\sigma, \quad (t,s)\in\Delta_{T'}, s\in[0,T'), \tag{5.10}$$

for suitable linear combinations of parameter-dependent families of (iterated integrals of) regular SG Fourier integral operators $W_\alpha(t,s)$, $\alpha \in M$, $(t,s)\in\Delta_{T'}$, *with phase functions and matrix-valued symbols determined through the characteristic roots of* L.

Proof. By the procedure explained in Proposition 5.3 and Remark 5.5, we can switch from the Cauchy problem (5.1) to an equivalent Cauchy problem (4.1), with $u \equiv U_\varnothing$. The uniqueness of the solution is then a consequence of known results about symmetric SG-hyperbolic systems, see [10], of which (4.1) is a special case.

The fundamental solution of (4.1) is given by (4.12), in view of Theorem 4.1 and Remark 4.2. It is a matrix-valued, parameter-dependent operator family $E(t,s) = (E_{\mu\mu'})_{\mu,\mu'\in M}(t,s)$, whose elements $E_{\mu\mu'}(t,s)$, $\mu,\mu'\in M$, are, modulo elements with kernels in $C^\infty(\Delta_{T'};\mathcal{S})$, linear combinations of parameter-dependent families of (iterated integrals of) regular SG FIOs, with phase functions of the type

$$\begin{aligned} \phi^{(\mu^\dagger)} &= \varphi_{m_1}, & \mu^\dagger &= (m_1) \in M^\dagger_1, \\ \phi^{(\mu^\dagger)} &= \varphi_{m_1}\sharp\dots\sharp\varphi_{m_j}, & \mu^\dagger &= (m_1,\dots,m_j)\in M^\dagger_j, j\geqslant 2, \end{aligned}$$

φ_k solution of the eikonal equation associated with the characteristic root τ_k of L, $k = 1,\dots,m$, and parameter-dependent, matrix-valued symbols of the type

$$\omega^{(\mu^\dagger)}(t,\theta_1,\dots,\theta_{j-1},s)\in S^{0,0}, \quad \mu\in M^\dagger_j,$$

$j = 1,\dots,m$. Then, the component $U_\varnothing \equiv u$ of the solution U of (4.1) has the form (5.10), with $W_\alpha = E_{\varnothing\alpha}$, taking into account (5.7) and (5.8).

We observe that the k-th order t-derivatives of the operators W_α, $\alpha\in M$, map continuously $H^{r,\varrho}$ to $H^{r-k,\varrho-k}$, $k\in\mathbb{Z}_+$, in view of the algebraic properties of FIOs and of the fact that, of course,

$$\partial_t[\mathrm{Op}_{\phi^{(\mu^\dagger)}(t,s)}(w^{(\mu^\dagger)}(t,s))] = \mathrm{Op}_{\phi^{(\mu^\dagger)}(t,s)}(i(\partial_t\phi^{(\mu^\dagger)})(t,s)\cdot w^{(\mu^\dagger)}(t,s) + \partial_t w^{(\mu^\dagger)}(t,s)),$$

obtaining a symbol of orders 1-unit higher in both components at any t-derivative step. This fact, together with the hypothesis on f, implies that the second sum in (5.10) belongs to $\bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'}; H^{r-k,\varrho-k}(\mathbb{R}^n))$.

The same is true for the elements of the first sum. In fact, recalling the embedding among the Sobolev-Kato spaces and (5.9), since $\alpha\in M \Rightarrow 0 \leqslant \mathbf{card}(\alpha) \leqslant m-1$, we find

$$\begin{aligned} W_\alpha(t,s)G_\alpha \in & \bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'}; H^{r+m-1-\mathbf{card}(\alpha)-k,\varrho+m-1-\mathbf{card}(\alpha)-k}) \\ & \hookrightarrow \bigcap_{k\in\mathbb{Z}_+} C^k(\Delta_{T'}; H^{r-k,\varrho-k}), \quad \alpha \in M, \end{aligned}$$

and this concludes the proof. □

5.2. **Propagation of singularities for classical SG-involutive operators.** Theorem 5.6, together with the propagation results proved in [16], implies our second main result, Theorem 5.17 below, about the global wave-front set of the solution of the Cauchy problem (5.1), in the case of a classical SG-involutive operator L of Levi type. We first recall the necessary definitions, adapting some material that appeared in [15, 16, 17].

Definition 5.7. *Let $\mathcal{B}$ be a topological vector space of distributions on $\mathbb{R}^n$ such that*

$$\mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbb{R}^n)$$

with continuous embeddings. Then $\mathcal{B}$ is called SG-admissible when $\mathrm{Op}_t(a)$ maps $\mathcal{B}$ continuously into itself, for every $a \in S^{0,0}(\mathbb{R}^{2n})$. If $\mathcal{B}$ and C are SG-admissible, then the pair $(\mathcal{B}, C)$ is called SG-ordered (with respect to $(m,\mu) \in \mathbb{R}^2$), when the mappings

$$\mathrm{Op}_t(a) : \mathcal{B} \to C \quad \text{and} \quad \mathrm{Op}_t(b) : C \to \mathcal{B}$$

are continuous for every $a \in S^{m,\mu}(\mathbb{R}^{2n})$ and $b \in S^{-m,-\mu}(\mathbb{R}^{2n})$.

Remark 5.8. *$\mathcal{S}(\mathbb{R}^n)$, $H^{r,\varrho}(\mathbb{R}^n)$, $r,\varrho \in \mathbb{R}$, and $\mathcal{S}'(\mathbb{R}^n)$ are SG-admissible. $(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$, $(H^{r,\varrho}(\mathbb{R}^n), H^{r-m,\varrho-\mu}(\mathbb{R}^n))$, $r,\varrho \in \mathbb{R}$, $(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ are SG-ordered (with respect to any $(m,\mu) \in \mathbb{R}^2$).*

Definition 5.9. *Let $\varphi \in \mathcal{P}_r$ be a regular phase function, $\mathcal{B}$ and C be SG-admissible and $\Omega \subseteq \mathbb{R}^n$ be open. Then the pair $(\mathcal{B}, C)$ is called weakly-I SG-ordered (with respect to (m,μ,φ,Ω)), when the mapping*

$$\mathrm{Op}_\varphi(a) : \mathcal{B} \to C$$

is continuous for every $a \in S^{m,\mu}(\mathbb{R}^{2n})$ with support such that the projection on the ξ-axis does not intersect $\mathbb{R}^n \backslash \Omega$.

Remark 5.10. *$(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$, $(H^{r,\varrho}(\mathbb{R}^n), H^{r-m,\varrho-\mu}(\mathbb{R}^n))$, $r,\varrho \in \mathbb{R}$, $(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ are weakly-I SG-ordered pairs (with respect to any $(m,\mu) \in \mathbb{R}^2$, $\varphi \in \mathcal{P}_r$, and $\Omega = \varnothing$).*

Now we recall the definition given in [15] of global wave-front sets for temperate distributions with respect to Banach or Fréchet spaces and state some of their properties. First of all, we recall the definitions of set of characteristic points that we use in this setting.

We need to deal with the situations where (2.4) holds only in certain (conic-shaped) subset of $\mathbb{R}^n \times \mathbb{R}^n$. Here we let Ω_m, $m = 1,2,3$, be the sets

$$\begin{gathered} \Omega_1 = \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}), \qquad \Omega_2 = (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^n, \\ \Omega_3 = (\mathbb{R}^n \backslash \{0\}) \times (\mathbb{R}^n \backslash \{0\}), \end{gathered} \tag{5.11}$$

Definition 5.11. *Let Ω_k, $k = 1,2,3$ be as in (5.11), and let $a \in S^{m,\mu}(\mathbb{R}^{2n})$.*

(1) a is called locally or type-1 invertible *with respect to m,μ at the point $(x_0,\xi_0) \in \Omega_1$, if there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and a positive constant R such that (2.4) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geqslant R$.*

(2) *a is called* Fourier-locally *or* type-2 invertible *with respect to m, μ at the point $(x_0, \xi_0) \in \Omega_2$, if there exist an open conical neighbourhood Γ of x_0, a neighbourhood X of ξ_0 and a positive constant R such that (2.4) holds for $x \in \Gamma$, $|x| \geqslant R$ and $\xi \in X$.*

(3) *a is called* oscillating *or* type-3 invertible *with respect to m, μ at the point $(x_0, \xi_0) \in \Omega_3$, if there exist open conical neighbourhoods Γ_1 of x_0 and Γ_2 of ξ_0, and a positive constant R such that (2.4) holds for $x \in \Gamma_1$, $|x| \geqslant R$, $\xi \in \Gamma_2$ and $|\xi| \geqslant R$.*

If $k \in \{1, 2, 3\}$ and a is not *type-k invertible with respect to m, μ at $(x_0, \xi_0) \in \Omega_k$, then (x_0, ξ_0) is called* type-k characteristic *for a with respect to m, μ. The set of type-k characteristic points for a with respect to m, μ is denoted by $\mathrm{Char}^k_{m,\mu}(a)$.*

The (global) set of characteristic points *(the characteristic set), for a symbol $a \in S^{m,\mu}(\mathbb{R}^{2n})$ with respect to m, μ is defined as*

$$\mathrm{Char}(a) = \mathrm{Char}_{m,\mu}(a) = \mathrm{Char}^1_{m,\mu}(a) \bigcup \mathrm{Char}^2_{m,\mu}(a) \bigcup \mathrm{Char}^3_{m,\mu}(a).$$

In the next Definition 5.12 we introduce different classes of cutoff functions (see also Definition 1.9 in [14]).

Definition 5.12. *Let $X \subseteq \mathbb{R}^n$ be open, $\Gamma \subseteq \mathbb{R}^n \backslash \{0\}$ be an open cone, $x_0 \in X$ and $\xi_0 \in \Gamma$.*

(1) *A smooth function φ on $\mathbb{R}^n$ is called a cutoff (function) with respect to x_0 and X, if $0 \leqslant \varphi \leqslant 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of x_0. The set of cutoffs with respect to x_0 and X is denoted by $\mathscr{C}_{x_0}(X)$ or $\mathscr{C}_{x_0}$.*

(2) *A smooth function ψ on $\mathbb{R}^n$ is called a directional cutoff (function) with respect to ξ_0 and Γ, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of ξ_0 such that the following is true:*
   - *$0 \leqslant \psi \leqslant 1$ and $\mathrm{supp}\,(\psi) \subseteq \Gamma$;*
   - *$\psi(t\xi) = \psi(\xi)$ when $t \geqslant 1$ and $|\xi| \geqslant R$;*
   - *$\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geqslant R$.*

   The set of directional cutoffs with respect to ξ_0 and Γ is denoted by $\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ or $\mathscr{C}^{\mathrm{dir}}_{\xi_0}$.

Remark 5.13. *Let $X \subseteq \mathbb{R}^n$ be open and $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbb{R}^n \backslash 0$ be open cones. Then the following is true.*

(1) *if $x_0 \in X$, $\xi_0 \in \Gamma$, $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, then $c_1 = \varphi \otimes \psi$ belongs to $S^{0,0}(\mathbb{R}^{2n})$, and is type-1 invertible at (x_0, ξ_0);*

(2) *if $x_0 \in \Gamma$, $\xi_0 \in X$, $\psi \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma)$ and $\varphi \in \mathscr{C}_{\xi_0}(X)$, then $c_2 = \varphi \otimes \psi$ belongs to $S^{0,0}(\mathbb{R}^{2n})$, and is type-2 invertible at (x_0, ξ_0);*

(3) *if $x_0 \in \Gamma_1$, $\xi_0 \in \Gamma_2$, $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ and $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$, then $c_3 = \psi_1 \otimes \psi_2$ belongs to $S^{0,0}(\mathbb{R}^{2n})$, and is type-3 invertible at (x_0, ξ_0).*

We can now introduce the wave-front sets.

Definition 5.14. *Let $k \in \{1, 2, 3\}$, Ω_k be as in (5.11), and let $\mathcal{B}$ be a Banach or Fréchet space such that $\mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{B} \subset \mathcal{S}'(\mathbb{R}^n)$.*

(1) *The point $(x_0, \xi_0) \in \Omega_k$ is called* type-k regular *for f with respect to $\mathcal{B}$, if*

$$\mathrm{Op}(c_k) f \in \mathcal{B}, \tag{5.12}$$

   for some c_k in Remark 5.13, $k = 1, 2, 3$. The set of all type-k regular points for f with respect to $\mathcal{B}$, is denoted by $\Theta^k_{\mathcal{B}}(f)$.

(2) *The* type-k wave-front set *of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to $\mathcal{B}$ is the complement of $\Theta^k_{\mathcal{B}}(f)$ in Ω_k, and is denoted by $\mathrm{WF}^k_{\mathcal{B}}(f)$;*

(3) *The* global wave-front set $\mathrm{WF}_{\mathcal{B}}(f) \subseteq (\mathbb{R}^n \times \mathbb{R}^n) \backslash 0$ *is the set*

$$\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}^1_{\mathcal{B}}(f) \bigcup \mathrm{WF}^2_{\mathcal{B}}(f) \bigcup \mathrm{WF}^3_{\mathcal{B}}(f).$$

The sets $\mathrm{WF}^1_{\mathcal{B}}(f)$, $\mathrm{WF}^2_{\mathcal{B}}(f)$ and $\mathrm{WF}^3_{\mathcal{B}}(f)$ in Definition 5.14, are also called the *local*, *Fourier-local* and *oscillating* wave-front set of f with respect to $\mathcal{B}$.

Remark 5.15. *In the special case when $\mathcal{B} = H^{r,\varrho}(\mathbb{R}^n)$, $r,\varrho \in \mathbb{R}$, we write $\mathrm{WF}^k_{r,\varrho}(f)$, $k = 1,2,3$. In this situation, $\mathrm{WF}_{r,\varrho}(f) \equiv \mathrm{WF}^1_{r,\varrho}(f) \bigcup \mathrm{WF}^2_{r,\varrho}(f) \bigcup \mathrm{WF}^3_{r,\varrho}(f)$ coincides with the scattering wave front set of $f \in \mathcal{S}'(\mathbb{R}^n)$ introduced by Melrose [27]. In the case when $\mathcal{B} = \mathcal{S}(\mathbb{R}^n)$, $\mathrm{WF}_{\mathcal{B}}(f)$ coincides with the $\mathcal{S}$-wave-front set considered in [17] (see also [34]).*

The next result describes the relation between "regularity with respect to $\mathcal{B}$" of temperate distributions and global wave-front sets.

Proposition 5.16. *Let $\mathcal{B}$ be SG-admissible, and let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then*

$$f \in \mathcal{B} \quad \Longleftrightarrow \quad \mathrm{WF}_{\mathcal{B}}(f) = \emptyset.$$

Theorem 5.17 here below extends the analogous result in [16] to the more general case of a classical, SG-hyperbolic involutive operator L of Levi type, and the one in [37] to the global wave-front sets introduced above. It is a consequence of Theorem 5.6 and of Theorem 5.17 in [16].

Theorem 5.17. *Let L in (5.1) be a classical, SG-hyperbolic, involutive operator of Levi type, that is, of the type considered in Proposition 5.3 with SG-classical coefficients, of the form (5.3). Let $g_k \in \mathcal{S}'(\mathbb{R}^n)$, $k = 0,\dots,m-1$, and assume that the m-tuple $(\mathcal{B}_0,\dots,\mathcal{B}_{m-1})$ consists of SG-admissible spaces. Also assume that the SG-admissible space C is such that $(\mathcal{B}_k, C)$, $k = 0,\dots,m-1$, are weakly-I SG-ordered pairs with respect to*

$$(k-j, k-j, \phi^{(\alpha)}, \emptyset), \quad k = 0,\dots,m-1, j = 0,\dots,k, \alpha \in M.$$

Then, for the solution $u(t,s)$ of the Cauchy problem (5.1) with $f \equiv 0$, $(t,s) \in \Delta_{T'}$, $s \in [0,T')$, we find

$$\mathrm{WF}^k_C(u(t,s)) \subseteq \bigcup_{j=1}^{m} \bigcup_{\alpha\in M_j^\dagger} \bigcup_{\substack{\mathbf{t}_j\in\Delta_j(T')\\ t_0=t, t_j=s}} \bigcup_{\ell=0}^{m-1} (\Phi_\alpha(\mathbf{t}_j)(\mathrm{WF}^k_{\mathcal{B}_\ell}(g_\ell)))^{\mathrm{con}_k}, \quad k = 1,2,3, \tag{5.13}$$

where V^{con_k} for $V \subseteq \Omega_k$, is the smallest k-conical subset of Ω_k which includes V, $k \in \{1,2,3\}$ and $\Phi_\alpha(\mathbf{t}_j)$ is the canonical transformation of $T^\star\mathbb{R}^n$ into itself generated by the parameter-dependent SG-classical phase functions $\phi^{(\alpha)}(\mathbf{t}_j) \in \mathcal{P}_r$, $\alpha \in M_j^\dagger$, $\mathbf{t}_j \in \Delta_j(T')$, $t_0 = t$, $t_j = s$, $j = 1,\dots,m$, appearing in (5.10).

Proof. We prove (5.13) only for the case $k = 3$, since the arguments for the cases $k = 1$ and $k = 2$ are analogous. Let

$$(x_0,\xi_0) \in \bigcap_{j=1}^{m} \bigcap_{\alpha\in M_j^\dagger} \bigcap_{\substack{\mathbf{t}_j\in\Delta_j(T')\\ t_0=t, t_j=s}} \bigcap_{\ell=0}^{m-1} [\Omega_3 \backslash (\Phi_\alpha(\mathbf{t}_j)(\mathrm{WF}^3_{\mathcal{B}_\ell}(g_\ell)))^{\mathrm{con}_3}].$$

Then, by (5.10) and [16, Theorem 5.17], in view of (5.8) and the hypotheses on C, $\mathcal{B}_k$, $k = 0,\dots,m-1$, there exists $c_3 \in S^{0,0}$, type-3 invertible at (x_0,ξ_0), such that $\mathrm{Op}(c_3)(W_\alpha(t,s)G_\alpha) \in C$, $\alpha \in M_j^\dagger$, $j = 1,\dots,m$.

In fact, for the terms such that $\alpha \in M_1^\dagger$, this follows by a direct application of Theorem 5.17 in [16]. For the terms with $\alpha \in M_j^\dagger$, $j = 2,\dots,m$, we first observe that we can bring $\mathrm{Op}(c_3)$ within the iterated integrals analogous to those appearing in (4.11). Then, in view of the hypothesis on (x_0,ξ_0), again by Theorem 5.17 in [16] and the properties of the operators W_α in (5.10), the integrand belongs to C for any $\mathbf{t}_j$, $j = 2,\dots,m$, in the integration domain. In view of the smooth dependence on $\mathbf{t}_j$, the iterated integral belongs to $\mathcal{C}$ as well. Since the right-hand side of (5.10) is a finite sum, modulo a term in $\mathcal{S}$, we conclude that $\mathrm{Op}(c_3)(u(t,s)) \in \mathcal{C}$, which implies $(x_0, \xi_0) \notin \mathrm{WF}^3_{\mathcal{C}}(u(t,s))$ and proves the claim. □

Remark 5.18. *(1) We recall that the canonical transformation generated by an arbitrary regular phase function $\varphi \in \mathcal{P}_r$ is defined by the relations*

$$(x,\xi) = \Phi_\varphi(y,\eta) \quad \Longleftrightarrow \quad \begin{cases} y = \varphi'_\xi(x,\eta) = \varphi'_\eta(x,\eta), \\ \xi = \varphi'_x(x,\eta). \end{cases}$$

(2) Assume that the hypotheses of Theorem 5.17 hold true. Then $\mathrm{WF}^k_{\mathcal{C}}(u(t,s))$, $(t,s) \in \Delta_{T'}$, $k = 1,2,3$, consists of unions of arcs of bicharacteristics, generated by the phase functions appearing in (5.10) and emanating from points belonging to $\mathrm{WF}^m_{\mathcal{B}_k}(g_k)$, $k = 0,\dots,m-1$, cf. [17, 29, 37].

(3) The hypotheses on the spaces $\mathcal{B}_k$, $k = 0,\dots,m-1$, $\mathcal{C}$, are authomatically fulfilled for $\mathcal{B}_k = H^{r+m-1-k,\varrho+m-1-k}(\mathbb{R}^n)$, $\mathcal{C} = H^{r,\varrho}(\mathbb{R}^n)$, $r, \varrho \in \mathbb{R}$, $k = 0,\dots,m-1$. That is, the results in Theorem 5.17 and in point 2 above hold true, in particular, for the $\mathrm{WF}^k_{r,\varrho}(u(t,s))$ wave-front sets, $r, \varrho \in \mathbb{R}$, $k = 1,2,3$.

(4) A result similar to Theorem 5.17 holds true for the solution $U(t,s)$ of the system (4.1) when $F \equiv 0$ and Λ and R are matrices of SG-classical operators.

6. Stochastic Cauchy problems for weakly SG-hyperbolic linear operators

This section is devoted to the proof of the existence of random-field solutions of a stochastic PDE of the form

$$L(t, D_t; x, D_x)u(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x), \tag{6.1}$$

associated with the initial conditions $D^k_t u(0,x) = g_k(x)$, $k = 0,\dots,m-1$, for a SG-involutive operator L, where γ and σ are suitable real-valued functions, $\dot{\Xi}$ is a random noise, described below, and u is an unknown stochastic process called *solution* of the SPDE.

Since the sample paths of the solution u are, in general, not in the domain of the operator L, in view of the singularity of the random noise, we rewrite (6.1) in its corresponding integral (i.e., *weak*) form and look for *mild solutions of* (6.1), that is, stochastic processes $u(t,x)$ satisfying

(6.2)
$$u(t,x) = v_0(t,x) + \int_0^t \int_{\mathbb{R}^n} \Lambda(t,s,x,y)\gamma(s,y)dyds + \int_0^t \int_{\mathbb{R}^n} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)dyds,$$

where Λ is a suitable distribution, associated with the fundamental solution of the operator L, and in (6.2) we adopted the usual abuse of notation involving *distributional integrals*.

Based on the results of the previous Section 5, and on the analysis in [4], we can show that (6.2) has a meaning, and we call it the solution of (6.1) with the associated initial conditions.

6.1. Stochastic integration with respect to a martingale measure.

We recall here the definition of stochastic integral with respect to a martingale measure, using material coming from [4], to which we refer the reader for further details. Let us consider a distribution-valued Gaussian process $\{\Xi(\phi);\ \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)\}$ on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with mean zero and covariance functional given by

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^n} \big(\phi(t) * \tilde{\psi}(t)\big)(x)\,\Gamma(dx)dt, \tag{6.3}$$

where $\widetilde{\psi}(t,x) := \psi(t,-x)$, $*$ is the convolution operator and Γ is a nonnegative, nonnegative definite, tempered measure on $\mathbb{R}^n$. Then [35, Chapter VII, Théorème XVIII] implies that there exists a nonnegative tempered measure μ on $\mathbb{R}^n$ such that $\mathcal{F}\mu = \widehat{\mu} = \Gamma$. By Parseval's identity, the right-hand side of (6.3) can be rewritten as

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^n} [\mathcal{F}\phi(t)](\xi)\ \cdot \overline{[\mathcal{F}\psi(t)](\xi)}\,\mu(d\xi)dt.$$

The tempered measure Γ is usually called *correlation measure*. The tempered measure μ such that $\Gamma = \widehat{\mu}$ is usually called *spectral measure*.

In this section we consider the SPDE (6.1) and its mild solution (6.2): this is the way in which we understand (6.1); we provide conditions to show that each term on the right-hand side of (6.2) is meaningful.

In fact, we call *(mild) random-field solution to* (6.1) an $L^2(\Omega)$-family of random variables $u(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}^n$, jointly measurable, satisfying the stochastic integral equation (6.2).

To give a precise meaning to the stochastic integral in (6.2) we define

$$\int_0^t \int_{\mathbb{R}^n} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)dsdy := \int_0^t \int_{\mathbb{R}^n} \Lambda(t,s,x,y)\sigma(s,y)M(ds,dy), \tag{6.4}$$

where, on the right-hand side, we have a stochastic integral with respect to the martingale measure M related to Ξ. As explained in [22], by approximating indicator functions with C_0^∞-functions, the process Ξ can indeed be extended to a worthy martingale measure $M = (M_t(A);\ t\in\mathbb{R}_+, A\in\mathscr{B}_b(\mathbb{R}^n))$, where $\mathscr{B}_b(\mathbb{R}^n)$ denotes the bounded Borel subsets of $\mathbb{R}^n$. The natural filtration generated by this martingale measure will be denoted in the sequel by $(\mathscr{F}_t)_{t\geqslant 0}$. The stochastic integral with respect to the martingale measure M of stochastic processes f and g, indexed by $(t,x)\in[0,T]\times\mathbb{R}^n$ and satisfying suitable conditions, is constructed by steps (see [9, 21, 38]), starting from the class $\mathcal{E}$ of simple processes, and making use of the pre-inner product (defined for suitable f,g)

$$\begin{aligned}\langle f,g\rangle_0 &= \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}}^n \big(f(s) * \tilde{g}(s)\big)(x)\,\Gamma(dx)ds\Big] \\ &= \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}}^n [\mathcal{F}f(s)](\xi)\cdot\overline{[\mathcal{F}g(s)](\xi)}\,\mu(d\xi)ds\Big],\end{aligned} \tag{6.5}$$

with corresponding semi-norm $\|\cdot\|_0$, as follows.

(1) For a *simple process* $g(t,x;\omega) = \sum_{j=1}^m 1_{(a_j,b_j]}(t)1_{A_j}(x)X_j(\omega)\in\mathcal{E}$ (with $m\in\mathbb{N}$, $0\leqslant a_j < b_j \leqslant T$, $A_j\in\mathscr{B}_b(\mathbb{R}^n)$, X_j bounded, $\mathscr{F}_{A_j}$-measurable random variable, $1\leqslant j\leqslant n$) the stochastic integral with respect to M is given by

$$(g\cdot M)_t := \sum_{j=1}^m \big(M_{t\wedge b_j}(A_j) - M_{t\wedge a_j}(A_j)\big)X_j,$$

where $x\wedge y := \min\{x,y\}$. One can show, by applying the definition, that for all $g\in\mathcal{E}$ the following fundamental isometry is valid:

$$\mathbb{E}\big[(g\cdot M)_t^2\big] = \|g\|_0^2. \tag{6.6}$$

(2) Since the pre-inner product (6.5) is well-defined on elements of $\mathcal{E}$, if now we define $\mathcal{P}_0$ as the completion of $\mathcal{E}$ with respect to $\langle\cdot,\cdot\rangle_0$, then, for all the elements g of the Hilbert space $\mathcal{P}_0$, we can construct the stochastic integral with respect to M as an $L^2(\Omega)$-limit of simple processes via the isometry (6.6). So, $\mathcal{P}_0$ turns out to be the space of all integrable processes (with respect to M). Moreover, by Lemma 2.2 in [32] we know that $\mathcal{P}_0 = L^2_p([0,T]\times\Omega,\mathcal{H})$, where here $L^2_p(\dots)$ stands for the predictable stochastic processes in $L^2(\dots)$ and $\mathcal{H}$ is the Hilbert space which is obtained by completing the Schwartz functions with respect to the inner product $\langle\cdot,\cdot\rangle_0$. Thus, $\mathcal{P}_0$ consists of predictable processes which may contain tempered distributions in the x-argument (whose Fourier transforms are functions, $\mathbb{P}$-almost surely).

Now, to give a meaning to the integral (6.4), we need to impose conditions on the distribution Λ and on the coefficient σ such that $\Lambda\sigma \in \mathcal{P}_0$. In [6], sufficient conditions for the existence of the integral on the right-hand side of (6.4) have been given, in the case that σ depends on the spatial argument y, assuming that the spatial Fourier transform of the function σ is a complex-valued measure with finite total variation. Namely, we assume that, for all $s \in [0,T]$,

$$|\mathcal{F}\sigma(\cdot,s)| = |\mathcal{F}\sigma(\cdot,s)|(\mathbb{R}^n) = \sup_{\pi}\sum_{A\in\pi}|\mathcal{F}\sigma(\cdot,s)|(A) < \infty,$$

where π is any partition on $\mathbb{R}^n$ into measurable sets A, and the supremum is taken over all such partitions. Let, in the sequel, $\nu_s := \mathcal{F}\sigma(\cdot,s)$, and let $|\nu_s|_{\mathrm{tv}}$ denote its total variation. We summarize all these conditions in the following theorem; for its proof, see [6, Theorem 2.6].

Theorem 6.1. *Let Δ_T be the simplex given by $0 \leqslant t \leqslant T$ and $0 \leqslant s \leqslant t$. Let, for $(t,s,x) \in \Delta_T \times \mathbb{R}^n$, $\Lambda(t,s,x)$ be a deterministic function with values in $\mathcal{S}'(\mathbb{R}^n)_\infty$, the space of rapidly decreasing temperate distributions, and let σ be a function in $L^2([0,T],C_b(\mathbb{R}^n))$. Assume that:*

(A1) the function $(t,s,x,\xi) \mapsto [\mathcal{F}\Lambda(t,s,x)](\xi)$ is measurable, the function $s \mapsto \mathcal{F}\sigma(s) = \nu_s \in L^2([0,T],\mathcal{M}_b(\mathbb{R}^n))$, and, moreover,

$$\int_0^T\left(\sup_{\eta\in\mathbb{R}^n}\int_{\mathbb{R}^n}|[\mathcal{F}\Lambda(t,s,x)](\xi+\eta)|^2\mu(d\xi)\right)|\nu_s|^2_{\mathrm{tv}}\,ds < \infty; \tag{6.7}$$

(A2) Λ and σ are as in (A1) and

$$\lim_{h\downarrow 0}\int_0^T\left(\sup_{\eta\in\mathbb{R}^n}\int_{\mathbb{R}^n}\sup_{r\in(s,s+h)}|[\mathcal{F}(\Lambda(t,s,x)-\Lambda(t,r,x))](\xi+\eta)|^2\mu(d\xi)\right)|\nu_s|^2_{\mathrm{tv}}\,ds = 0.$$

Then $\Lambda\sigma \in \mathcal{P}_0$, so the stochastic integral on the right-hand side of (6.4) is well-defined and

$$\mathbb{E}\big[((\Lambda(t,\cdot,x,*)\sigma(\cdot,*))\cdot M)_t^2\big] \leqslant \int_0^t\left(\sup_{\eta\in\mathbb{R}^n}\int_{\mathbb{R}^n}|[\mathcal{F}\Lambda(t,s,x)](\xi+\eta)|^2\mu(d\xi)\right)|\nu_s|^2_{\mathrm{tv}}\,ds.$$

The reason for the assumption that $\Lambda(t) \in \mathcal{S}'(\mathbb{R}^n)_\infty$ is that, in this case, the Fourier transform in the second spatial argument is a smooth function of slow growth and the convolution of such a distribution with any other distribution in $\mathcal{S}'(\mathbb{R}^n)$ is well-defined, see [35, Chapter VII, Section 5]. A necessary and sufficient condition for $T \in \mathcal{S}'(\mathbb{R}^n)_\infty$ is that each regularization of T with a C_0^∞-function is a Schwartz function. This is true in our application, due to next Proposition 6.2 and the fact that the Fourier transform is a bijection on the Schwartz functions.

Proposition 6.2. *Let $A = \mathrm{Op}_\varphi(a)$ be a SG Fourier integral operator, with symbol $a \in S^{m,\mu}(\mathbb{R}^{2n})$, $(m,\mu) \in \mathbb{R}^2$, and phase function φ. Let K_A denote its Schwartz kernel. Then,* *the Fourier transform with respect to the second argument of K_A, $\mathcal{F}_{\cdot\mapsto\eta}K_A(x,\cdot)$, is given by*

$$\mathcal{F}_{\cdot\mapsto\eta}K_A(x,\cdot) = e^{i\varphi(x,-\eta)}a(x,-\eta). \tag{6.8}$$

In the next subsection we will apply Theorem 6.1, Proposition 6.2 and the theory developed in the previous sections to prove the existence of random-field solutions for stochastic PDEs associated with a SG-involutive operator.

6.2. **Random field solutions of stochastic linear SG-involutive equations.** We conclude the paper with our third main result.

Theorem 6.3. *Let us consider the Cauchy problem*

$$\begin{cases} L(t,D_t;x,\partial_x)u(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x), & t\in[0,T], x\in\mathbb{R}^n, \\ D_t^k u(0,x) = g_k(x), & k=0,\dots,m-1, x\in\mathbb{R}^n, \end{cases} \tag{6.9}$$

for a SPDE associated with a SG-involutive operator L of the type (1.2), $m\in\mathbb{N}$, of Levi type, that is, satisfying (5.3). Assume also, for the initial conditions, that $g_k\in H^{z+m-k-1,\zeta+m-k-1}(\mathbb{R}^n)$, $0\leqslant k\leqslant m-1$, with $z\in\mathbb{R}$ and $\zeta>n/2$. Consider a Gaussian noise $\dot{\Xi}$ of the type described in Subsection 6.1, with associated spectral measure such that

$$\int_{\mathbb{R}^n}\mu(d\xi)<\infty. \tag{6.10}$$

Finally, assume that $\gamma\in C([0,T];H^{z,\zeta}(\mathbb{R}^n))$, $\sigma\in C([0,T],H^{0,\zeta}(\mathbb{R}^n))$, and $s\mapsto\mathcal{F}\sigma(s)=\nu_s\in L^2([0,T],\mathcal{M}_b(\mathbb{R}^n))$.

Then, for some $T'\in(0,T]$, there exists a random-field solution u of (6.9). Moreover, $\mathbb{E}[u]\in C([0,T'],H^{z,\zeta}(\mathbb{R}^n))$.

Proof. The operator L is of the form considered in Section 5, and by Theorem 5.6 we know that the solution to (6.9) is formally given by (5.10), that is

$$u(t) = \sum_{\alpha\in M}E_{\emptyset\alpha}(t,0)G_\alpha + \sum_{\alpha\in M_{m-1}}\int_0^t E_{\emptyset\alpha}(t,s)f(s)\,ds, \quad t\in[0,T'], \tag{6.11}$$

where $M=\bigcup_{k=0}^m M_k$ with M_k the permutations of k elements of the set $\{1,\dots,m\}$, and where $E_{\emptyset\alpha}(t,s)$, $\alpha\in M$, $(t,s)\in\Delta_{T'}$ are (modulo elements with kernels in $C^\infty(\Delta_{T'};\mathcal{S})$) suitable linear combinations of parameter-dependent families of iterated integrals of regular SG FIOs, with phase functions given by (sorted) sharp products of solutions to the eikonal equations associated with the characteristic roots of L, and matrix-valued symbols in $S^{(0,0)}$ determined again through the characteristic roots of L. Let us insert now $f(t,x)=\gamma(t,x)+\sigma(t,x)\dot{\Xi}(t,x)$ in (6.11), so that, formally,

(6.12)
$$\begin{aligned} u(t,x) &= v_0(t,x) + \int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\gamma(s,y)\,dyds + \int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)\,dyds \\ &= v_0(t,x)+v_1(t,x)+v_2(t,x), \end{aligned}$$

In (6.12), $\Lambda(t,s,x,y)$ is the kernel of the finite sum $\sum_{\alpha\in M_{m-1}}E_{\emptyset\alpha}(t,s)$, that is, Λ is (modulo an element of $C^\infty(\Delta_{T'};\mathcal{S})$) a linear combination of (iterated integrals of) kernels of parameter-dependent FIOs with symbols of order $(0,0)$. We have already observed (see the last lines of the proof of Theorem 5.6) that $v_0\in C([0,T'],H^{z,\zeta})$. Since, by assumption, $\zeta>\frac{n}{2}$, v_0 produces a function which is continuous and L^2 with respect to $x\in\mathbb{R}^n$ and $t\in[0,T']$, respectively. We have that

$$\forall(t,x)\in[0,T]\times\mathbb{R}^n,\ v_0(t,x) \text{ is finite.}$$

Since all the operators $E_{\emptyset\alpha}(t,s)$ in (6.11) are linear and continuous from $H^{z,\zeta}$ to itself, and $\gamma \in C([0,T'], H^{z,\zeta})$, the first integral in (6.12) certainly makes sense, and also $v_1 \in C([0,T'], H^{z,\zeta})$. This gives:

$$\forall (t,x) \in [0,T] \times \mathbb{R}^n,\ v_1(t,x) \text{ is finite.}$$

Let us now focus on the term v_2. We can rewrite it as

$$v_2(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)M(ds,dy), \tag{6.13}$$

where M is the martingale measure associated with the stochastic noise Ξ, as defined in Section 6.1. By Proposition 6.2 we then find

$$|[\mathcal{F}_{y\mapsto\eta}\Lambda(t,s,x,\cdot)](\eta)|^2 \leqslant C_{t,s}\langle x\rangle^0\langle\eta\rangle^0 = C_{t,s}, \tag{6.14}$$

where $C_{t,s}$ can be chosen to be continuous in s and t, since Λ differs by an element of $C^\infty(\Delta_{T'}, \mathcal{S})$ from the kernel of a linear combination of (iterated integrals of) kernels of (parameter-dependent) SG FIOs with symbol in $C(\Delta_{T'}, S^{0,0})$. Using (6.14), we get that condition (A1) in Theorem 6.1 is satisfied if

$$\int_0^t \left(\sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{R}^n} |[\mathcal{F}_{y\mapsto\eta}\Lambda(t,s,x,\cdot)](\eta+\xi)|^2 \mu(d\eta)\right) |\nu_s|^2_{\text{tv}}\, ds \lesssim \int_0^t |\nu_s|^2_{\text{tv}}\, ds \int_{\mathbb{R}^n} \mu(d\eta) < \infty.$$

In view of the assumptions on σ, we conclude that assumption $(A1)$ holds true as long as (6.10) does. To check the continuity condition $(A2)$ in Theorem 6.1, since Λ is regular with respect to s and t, it suffices to show that

$$\sup_{r\in(s,s+h)} |\mathcal{F}(\Lambda(t,s,x) - \Lambda(t,r,x))(\xi+\eta)|^2 \leqslant C^2_{t,s,h}, \tag{6.15}$$

with $C_{t,s,h} \to 0$ as $h \to 0$ and $C_{t,s,h} \leqslant C_{T'}$ for every $h \in [0, t-s]$, $(t,s) \in \Delta_{T'}$.

Indeed, if (6.15) holds, then (A2) holds via Lebesgue's Dominated Convergence Theorem, in view of assumption (6.10), the fact that $|\nu_s|^2_{\text{tv}} \in L^1[0,T]$, and $C_{t,s,h} \leqslant C_{T_0}$.

Then, it only remains to show that (6.15) holds true. But this follows from the fact that the function $s \mapsto \mathcal{F}\Lambda(t,s,\cdot)(*)$ is, by (6.14), uniformly continuous on $[0,t]$ with values in the Fréchet space $S^{0,0}(\mathbb{R}^{2n})$, endowed with the norm

$$||a-b|| = \sum_{\ell=0}^{\infty} \frac{1}{2^\ell} \frac{\|a-b\|^{0,0}_\ell}{1+\|a-b\|^{0,0}_\ell},$$

so its modulus of continuity,

$$\omega_{t,s}(h) = \sup_{r\in(s,s+h)} ||(\mathcal{F}\Lambda(t,s,\cdot)(*) - \mathcal{F}\Lambda(t,r,\cdot)(*))||$$

tends to 0 as $h \to 0$. For more details see [4]. The argument above shows that we can apply Theorem 6.1 to get that v_2 is well-defined as a stochastic integral with respect to the martingale measure canonically associated with Ξ.

Summing up, the random-field solution $u(t,x)$ in (6.12) makes sense: its deterministic part is well-defined for every $(t,x) \in [0,T]$ and its stochastic part makes sense as a stochastic integral with respect to a martingale measure.

The regularity claim $\mathbb{E}[u] \in C([0,T'], H^{z,\zeta}(\mathbb{R}^d))$ follows from the regularity properties of the $E_{\emptyset\alpha}$, of γ and of the Cauchy data, taking expectation on both sides of (6.12), and recalling the fact that the expected value $\mathbb{E}[v_2]$ of the stochastic integral is zero, being Ξ a Gaussian process with mean zero. It follows that the regularity of $\mathbb{E}[u]$ is the same as the one of the solution of the associated deterministic Cauchy problem.

□

Remark 6.4. *One could say that the random-field solution u of (6.9) found in Theorem 6.3 "is unique" in the following sense. First, when $\sigma \equiv 0$, it reduces to the unique solution of the deterministic Cauchy problem (5.1), with $f \equiv \gamma$ and $s = 0$. Moreover, by linearity, if u_1 and u_2 are two solutions of the linear Cauchy problem (6.9), $u = u_1 - u_2$ satisfies the deterministic equation $Lu = 0$ with trivial initial conditions, and such Cauchy problem admits in $\mathcal{S}'$ only the trivial solution. The latter follows immediately by the $\mathcal{S}'$ well-posedness (with loss of smoothness and decay) of the Cauchy problem for the homogeneous deterministic linear equation $Lu = 0$ proved in Theorem 5.6.*

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