Structuralism and Mathematical Practice in Felix Klein's Work on Non-Euclidean Geometry

Published version:

DOI:10.1093/philmat/nkaa029

Structuralism and Mathematical Practice in Felix Klein's Work on Non-Euclidean Geometry

Francesca Biagioli

Department of Philosophy and Educational Sciences, University of Turin

ORCiD: 0000-0003-1842-690X

francesca.biagioli@unito.it

November 2, 2020

Abstract. It is well known that Felix Klein took a decisive step in investigating the invariants of transformation groups. However, less attention has been given to Klein's considerations on the epistemological implications of his work on geometry. This paper proposes an interpretation of Klein's view as a form of mathematical structuralism, according to which the study of mathematical structures provides the basis for a better understanding of how mathematical research and practice develop.

Contents

1 Introduction

Felix Klein's Erlangen Program [1872] occupies an important place in the history of mathematical structuralism. Not only did Klein provide a geometrical interpretation of the theory of algebraic invariants, but in doing so he has made important steps towards the modern understanding of invariants of transformation groups. The same notions features prominently in more recent versions of mathematical structuralism (*e.g.*, [Schiemer 2014]), as well as in the study of the modernist transformation of mathematics from the science of number and quantities to the study of abstract structures (see [Gray 2008]). This transformation entails a form of mathematical structuralism at least in the broad sense, as the view that mathematics is concerned with the investigation of structures independently of the nature of individual objects making up those structures. However, contemporary versions of mathematical structuralism disagree on what these structures are and what their role in mathematical practice is. So, the questions arise: What is Klein's notion of structure? What is the corresponding notion of mathematical abstraction? How are these notions reflected in Klein's methodology?

Perhaps even more puzzling, various foundational projects have been presented as generalizations of Klein's group-theoretical view. This includes set and model-theoretic approaches as well as category theory.[1] While all of these approaches are supposed to account for mathematical practice, they offer different conceptual frameworks for the understanding of structure. So, the questions arises whether Klein's work actually fits in the one or the other of these frameworks, or finally if Klein's approach might reveal some interrelations between different foundational programs.

In order to shed some light on these intricate problems, this paper takes into consideration the development of Klein's thought from his early works on non-Euclidean geometries to his epistemological writings from the 1890s. Most of these writings appeared as course materials and public lectures, which circulated widely in the 1890s but are less known today. Klein's Erlangen Program has been considered very influential by twentieth-century mathematicians. Historical studies, however, have emphasized that essential requirements for the implementation of the Erlangen Program were supplied in the works of other mathematicians, especially Lie's *Theorie der Transformationsgruppen*, which appeared in three volumes in 1888, 1890 and 1893 (see [Hawkins 1984], [Rowe 1988]). Therefore, Klein decided to give a wider circulation to the Erlangen Program with a second edition in *Mathematische Annalen* [1893] and elaborated on his earlier ideas in a number of contributions from the same period. As Klein reported in his collected mathematical papers, it is during this period that he started to work on the physical as well as philosophical implications of his classification of geometries. He wrote: "In the same period [the 1890s] I was able to cultivate my interests in mechanics and mathematical physics, as I have been intending to do since the beginning of my studies in mathematics. The first physical investigations in the theory of relativity emerged a few years later, and rapidly attracted general attention. I suddenly recognized that my classification of 1872 included even these investigations and provided the simplest way to clarify the newest physical (or even philosophical) ideas from a mathematical viewpoint" [Klein 1921, p.413].

I will argue in the following that, although Klein did not use the term "structure" or an equivalent term (at least to my knowledge), an *in rebus* notion of structure is implied in his characterization of abstraction as a process that leads from empirical notions to math-

[1] Mautner [1946] and Tarski in a lecture of 1966, which was published in 1986, deemed the idea of logic as invariant theory an extension of Klein's Erlangen Program. Eilenberg and Mac Lane [1945] is the main reference for Klein's influence on the development of category theory.

ematical notions such as groups and number fields. These are some of the typical examples of abstract objects in the current sense of structures that can be instantiated by a variety of mathematical domains. In Klein's account, abstraction includes abstracting away from empirical data, which are essentially approximate, and formulating exact definitions of mathematical notions. I will point out that Klein's definitions amount to understanding mathematical notions as instantiations of structures. I will argue, furthermore, that the implications of Klein's methodology amount to advocating a form of mathematical structuralism, according to which the study of mathematical structures provides the basis for a better understanding of mathematical practice (including the use of different approaches to mathematical disciplines, as well as the use of mathematics in physics).

Section 2 deals with the methodology of Klein's work on non-Euclidean geometry in 1871–1874, with a special focus on the case studies that informed his account of abstraction. This includes the foundation of projective geometry and the question of how this geometry is related to ordinary metrical geometry, as well as the use of transfer principles over transformation groups. This section will conclude with an attempt to spell out the notion of structure that is implied in Klein's understanding of "invariant theory" as deeply connected to projective geometry. Section 3 will consider how Klein elaborated on his philosophical ideas in 1889–1897. It will be argued that Klein especially relied on the way in which Dedekind introduced the supposition of the continuity of space in *Stetigkeit und irrationale Zahlen* [1872] to articulate his own account of the abstraction at work in the introduction of geometrical axioms. After discussing Klein's account, I will draw the relevant comparisons with mathematical structuralism. I will conclude by pointing out that Klein's account of abstraction allowed him to address the applicability of non-Euclidean geometries in physics, and that Klein's argument was taken up in the philosophical debate by Ernst Cassirer.

2 Klein's Work on Geometry as Structuralist Methodology

This section will discuss the methodology at work in Klein's classification of geometries of 1871, as well as in the plan of a group-theoretical investigation of a broader variety of geometries presented by him in the Erlangen Program. Section 2.1 will address how Klein accounted for the interplay of the so-called "synthetic" and "analytic" approaches (in particular Christian von Staudt's, on the one hand, and Arthur Cayley's, on the other) in the foundation of projective geometry. Klein presented his classification of geometries as the outcome of an attempt to integrate these two approaches by building on the work of mathematicians such as Julius Plücker, Alfred Clebsch and Otto Hesse. Section 2.2 will examine how Klein, in annotations to the later editions of the Erlangen Program, rephrased his view by saying that projective geometry offered a vantage point for his understanding of geometry as the study of the invariants of transformation groups. I will propose an interpretation of Klein's considerations as outlining a structuralist methodology, according to which the notion of structure is gained from the comparison of the various mathematical domains instantiating it.

2.1 Klein on Projective and non-Euclidean Geometry

Klein's comparative approach had its roots in a variety of mathematical traditions, which Klein was able to combine in an unprecedented way.[2] Klein's first job as Julius Plücker's assistant at the University of Bonn from 1866 to 1868 was to set up and carry out demonstrations accompanying Plücker's lectures in experimental physics. Klein also assisted

[2] I rely in the following especially on Rowe [1992] and Gray [2008].

Plücker with his mathematical research.

When Plücker died, in 1868, Alfred Clebsch in Göttingen appointed Klein to complete the posthumous edition of Plücker's work on line geometry. This position enabled Klein to move into the circle of algebraic geometers inspired by Clebsch.

Klein began to engage with projective geometry while studying at the University of Berlin, in 1869–1870. This is quite surprising, given the fact that Berlin was considered to be the main School in Germany for the "arithmetization" of mathematics.[3] In fact the University of Berlin had a tradition of teaching what came to be known as projective geometry dating back to the appointment of Jacob Steiner in 1832. Weierstrass took over Steiner's classes on projective geometry in the 1860s. These were taught synthetically.[4]

Klein [1921, p.50] reported that he was introduced to projective geometry and non-Euclidean geometry by the Austrian mathematician Otto Stolz. It was then, Klein reported, that he came up with the idea to apply Arthur Cayley's projective metric to non-Euclidean cases. In other words, Klein provided a projective model of non-Euclidean geometry. Klein first presented his model in Weierstrass's seminar in 1870. Subsequently, Klein sketched the model in [1871a] and presented it in detail in a series of papers [Klein 1871b, 1873, 1874].

Klein emphasized the vantage point of these works in integrating "synthetic" and "analytic" approaches. According to Klein's definitions, "synthetic geometry is that which studies figures as such, without the addition of formulas, whereas analytic geometry makes a consistent use of formulas, particularly of the coordinate system x-y" [Klein, 1909, p.111]. Klein went on to point out that there exists only a different gradation between these two approaches. Plücker's analytic geometry, for example, used both geometric constructions and coordinate equations. Klein explained that: "in mathematics, as everywhere else, people tend to form parties, and so there arose schools of pure synthetists and schools of pure analysts, who valued above all the absolute 'purity of method,' and therefore where more one-sided than the nature of the subject would have required" [*ibid.*, p. 112].[5] Klein sought to overcome such one-sidedness by showing that synthetic proofs rest on tacit assumptions that require an explicitly analytical formulation; *vice versa*, the algebraic formulation of geometric relations can be translated into idealized constructions in some cases. The examples under consideration will provide some further elucidation.

The first implementation of a synthetic approach is found in Christian von Staudt's geometry of position [1847, 1856–1860]. Before von Staudt, it was common practice to define the fundamental invariants of this geometry, that is harmonic ratios, as cross-ratios of the coordinates with value -1. Von Staudt broke with this tradition by using a geometrical construction of the fourth harmonic to three collinear points. He used this construction to provide an autonomous foundation of the geometry of position in terms of relations of incidence and order alone. Von Staudt [1856–1860, vol.2] used the properties of harmonic progressions and involutions to introduce projective coordinates and to develop a generalized calculus of segments. Cross-ratios correspond to the projective coordinates, although there is no mention of the notion of cross-ratio in von Staudt's work. The idea behind this approach is that analytic geometry presupposes the geometry of position [see Nabonnand, 2008, p.230].

Cayley's projective metric exemplified what was meant to be a "purely analytic approach" in the above-mentioned tradition. Cayley called a pair of imaginary points the

[3] I will turn back to how Klein distanced himself from this tradition in Section 3.

[4] I am thankful to an anonymous referee for pointing out this information.

[5] On these meaning of "analysis" and "synthesis" as two different research styles in nineteenth-century geometry, see Epple [1997].

"absolute." Any pair of real points can be taken with respect to the absolute. Cayley defined distance as the cross-ratio of a 4-tuple formed by two such points and the absolute. Cayley [1859] used a projective metric based on his definition of distance as an auxiliary assumption to show that the formulas of ordinary metrical geometry can be derived from projective geometry.

Klein presented his work on non-Euclidean geometry as a complement to von Staudt's approach, on the one hand, and to Cayley's, on the other. Regarding von Staudt's approach, Klein was the first to notice a gap in von Staudt's considerations. The fundamental theorem of projective geometry states that it suffices for two lines to have 3 points in common, A, B and C, in order for them to have all points in common. Von Staudt's proof follows, by *reductio ad absurdum*, from the assumption that two such lines are not identical. This amounts to saying that there exists a segment whose limiting points M, N are common to the two lines while the inner points are not. Under this assumption, at least one of the points in common, A, B and C lies outside the segment MN. However, this implies that its harmonic conjugate with respect to M and N lies inside the segment. The contradiction follows by the definition of the harmonic conjugate, which is uniquely determined with respect to 3 given points. However, von Staudt also presupposed a property of the line, which is now known as connectedness and which was first defined by Dedekind (who called it continuity), in 1872. According to Dedekind's definition: "If all points of the line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes" [Dedekind, 1872, p.11]. Klein called Dedekind's continuity "the analytical content of von Staudt's considerations" [Klein, 1873, p.132]. Klein wrote: "Such a content can be summarized by demanding that projective space be represented by a numerical three-fold extended manifold. Besides, this is an assumption that lies at the foundation of any speculation about space" [ibid.].[6]

After Klein's remark, Jacob Lüroth and Hieronymus Zeuthen, independently of one another, gave one rigorous proof of the fundamental theorem of projective geometry and communicated their proof to Klein. A reproduction of the proof based on Dedekind's continuity is found in Klein [1874].

Regarding Cayley's projective metric, Klein gave a geometrical interpretation of Cayley's definition of distance by fixing a two-dimensional "fundamental surface" in the projective plane. Every two points in the projective plane define a straight line, and every such line intersects the fundamental surface in two other points. The distance between the former points is given by the logarithm of the cross-ratio they form with the points of the fundamental surface, multiplied by a constant. This definition enabled Klein to obtain different metrical geometries by taking into account different possible types of fundamental surfaces: (i) An imaginary second-order surface; (ii) The inner points of a real, non-degenerate surface of second order; (iii) The circle at infinity (*i.e.*, Cayley's absolute). Klein classified geometries into elliptic, hyperbolic and parabolic, according to each of these cases. Whereas parabolic geometry corresponds to Euclidean geometry, elliptic and hyperbolic geometries provide the interpretation of the non-Euclidean metrics that became known as "the projective model of non-Euclidean geometry" [see, *e.g.*, Gowers, 2008, p.94].

To sum up, it was in integrating the opposed approaches of Cayley and von Staudt that

[6] It is important to notice that Klein referred to the foundation of projective geometry in the manner of von Staudt, that is, independently of the metrical notions of elementary geometry. As Klein pointed out, this requires Dedekind's continuity. Dedekind pointed out to Lipschitz in a letter from 1876 that, on the contrary, the continuity of space is not a requirement for Euclidean geometry [1932, p.479]. On how Dedekind's remark foreshadowed Hilbert's foundation of elementary geometry, see [Ferreirós 1999, Ch.4]. I turn back to Klein's interpretation of Dedekind's continuity in Section 3.

Klein presented his first classification of geometries. While attaching a (generalized) spatial meaning to analytic expressions such as Cayley's definition of distance, Klein supplemented von Staudt's geometry of position with an analytic representation of space that applies to a variety of geometries. Using the terminology of mathematical structuralism, one can say that Klein's way of proceeding offered a paradigmatic example of a structure (of a projective space) that can be instantiated in different ways (by the three cases of metrics). The following section considers how Klein generalized his approach further in the Erlangen Program.

2.2 From Projective Geometry to the Group-Theoretical View

Klein's "Vergleichende Betrachtungen über neuere geometrische Forschungen" first appeared as a pamphlet distributed during his inaugural address as newly appointed Professor at the University of Erlangen in 1872. It was translated into Italian (1890), French (1891) and English (1893), as "A Comparative Review of Recent Researches in Geometry." Klein published a second revised edition in 1893 and included a third commented edition in Klein [1921]. This pamphlet is now best known as the Erlangen Program (from now on EP).

The main mathematical result of Klein's EP was the proof that elliptic, hyperbolic and parabolic geometries are equivalent to the three cases of manifolds of constant curvature; where the curvature can be more, less than or equal 0, respectively.[8] The first sketch of the proof is found in Klein [1872, pp.43–45] and is presented in more detail in Klein [1873]. Klein's proof consists in showing that the properties of such manifolds are uniquely determined as invariants of the transformations of metrical geometry, insofar as these transformations form a group.[9] According to Klein's definition, transformations form a group iff: (i) the product of any two transformations of the group also belongs to the group; (ii) for every transformation of the group, there exists in the group an inverse transformation.[10] In modern axiomatizations, (i) and (ii) correspond to closure under the group operation and to the existence, for every such operation, of an inverse operation in the group. A complete formulation of the group axioms, including (iii) associativity and (iv) the existence of an identity element, was first presented by Walther von Dyck in 1882–83, and it was known to Klein by 1893. Klein referred to von Dyck's work for a detailed presentation of group theory in Klein [1884, p.6]. Even though Klein himself did not state (iii) and (iv) in the quite informal presentation of the EP, he was correct in claiming that the transformations under consideration form a group, insofar as transformation groups fulfil all of the group axioms.

Klein then developed a general procedure for comparing geometries based on Hesse's principle of transfer.[11]Hesse set up a one-to-one correspondence between the points in the plane and pairs of points belonging to a line. Under the assumption of this correspondence, his principle states that, for every theorem proved in plane geometry, there is a second

[8]These manifolds had been studied by Beltrami [1869].

[9]"Transformation" indicates a one-to-one mapping of space onto itself. Translations and rotations are typical examples of transformations of ordinary geometry, which leave invariant parallelism, lengths and the measure of angles. By contrast, projections or collineations leave invariant only such relations as, of points, to lie on the same line etc.

[10]Klein specified the second condition in 1893. Both conditions are necessary for the characterization of transformation groups. However, Klein did not revise the rest of the proof. In 1872, he adopted Camille Jordan's definition of 1870, which referred to finite groups of permutations. In that case, the closure of a set of elements relative to a fundamental operation (the first of the said conditions) is a sufficient condition for the set to form a group. Subsequently, Lie drew attention to the fact that the existence of an inverse operation is required in the case of infinite groups (see [Hawkins 1984]).

[11]For a detailed discussion of Hesse's principle and its generalizations in late nineteenth-century mathematics, see Hawkins [1988].

theorem obtained by transfer in line geometry and *vice versa*. Hesse [1866, p.15] also mentioned that it is possible to extend the principle, so that space geometry or even a geometry with more than three dimensions can be derived from line geometry. Hesse's remark refers to the fact that the lines in space form a four-dimensional manifold. The fundamental objects in line geometry are line complexes and congruences of lines defined by equations in line coordinates. Klein himself as well as Lie had worked on a transfer principle connecting line geometry with metric geometry in four variables in the period that immediately preceded the first version of the EP (see Rowe [1989], Schiemer [2020]).

Klein generalized Hesse's principle further in the EP by establishing the equivalence of geometries that have the same underlying group: Suppose that a manifold A has been investigated with reference to a group B, and by any transformation A is converted into A', then B becomes B' and the B'–based treatment of A' can be derived from the B–based treatment of A [Klein 1893a, p.72]. The principle showed that: "As long as our geometrical investigations are based on one and the same transformation group, the geometric content remains unvaried" or, as Klein also puts it, "the essential thing is the transformation group" [Klein 1893a, pp.73–74].

These claims seem to presuppose an abstract notion of structure, insofar as Klein abstracts away from any content other than the relative invariants of transformation groups. The geometric content is determined by the group. The study of, say, the Euclidean group determines what counts as the same type of figure in Euclidean geometry. In general, different geometrical figures can share the same invariant properties relative to the group. Consider, for example, two circles of different diameters. In Euclidean geometry, these figures belong to the same type and differ from other types of figures such as squares or ellipses. Considering the group of collineations, however, there is only a difference between curved and rectilinear figures. This means that, with reference to the group of collineations, circles, ellipses, hyperboles and parabolas belong to the same type of figure. As Schiemer [2020] has suggested, Klein's understanding of geometric content as determined by the group can be spelled out by saying that different geometric figures can share the same abstract form determined by invariant properties. Schiemer expresses such an assumption by introducing an abstraction principle stating that the types of two figures in a manifold are identical in case that they are congruent relative to the transformation group. This interpretation allows him to emphasize a connection between the Kleinian approach and *in rebus* structuralism. According to *in rebus* structuralism, mathematical theories describe abstract structures; however, such structures cannot be considered independently of their representations. This dependence relation can be thought of in terms of an act of abstraction from the particular mathematical systems to their shared structures.

Applying the categories of *in rebus* structuralism to describe Klein's methodology, one can say that: The study of abstract structures in Klein's work is instrumental to the development of concrete mathematical and scientific practices. For example, Klein's [1873] claim that Dedekind's continuity is a presupposition for any speculation about space can be interpreted in the following sense: Instead of presupposing the particular assumption of Euclidean space, the study of the projective properties of figures leads to the discovery of a more fundamental structure, which agrees locally with the system of the real numbers. The fundamental structure for the projective properties of figures can be specified in terms of Euclidean or non-Euclidean geometries. Klein added in a note to the 1893 edition of the EP that projective geometry, as it emerged with Clebsch and with Klein himself, had overcome the opposition between synthetic and analytic methods. Here, he characterized this opposition in philosophical terms by saying that the former method relies more on intuition. As a *prima facie* characterization, it will be helpful to recall the fact that, in nineteenth-century philosophy, "intuition" indicated an immediate mode of cognition, which is directed towards singular objects. However, most philosophers, in the wake of the Kantian tradition, held that knowledge is always mediated by different types of concepts which categorize the objects of intuition. Klein used "intuition" as an umbrella term for different types of evidence, from the use of diagrams in synthetic geometry to empirical data and experiments. Klein's notion of intuition reflects, nonetheless, the philosophical usage, insofar as he identified the main characteristics of intuitions as being inexact and restricted to a particular view. Therefore, Klein demanded that: "A mathematical subject matter ought not to be regarded as exhausted, until it has become conceptually evident [*begrifflich evident*]" [Klein 1893a, pp.93–94]. Klein did not spell out what he meant by "conceptually evident" here or use it elsewhere, at least to my knowledge. My suggestion in the next section is that this terminology relates to a notion of concept that Klein in his philosophical writings traced back to Dedekind and interpreted in his own way.

To conclude this section, it is noteworthy that Klein pointed even more clearly to the importance of projective geometry in the development of the group-theoretical view of geometry in a footnote to the 1921 edition of the Erlangen Program [Klein 1921, p.464]. Klein clarified in this footnote that what he meant by "invariant theory" does not relate to the question of finding the rational integer invariants of any given binary form as addressed by Klein and his former teacher Clebsch in the same period, but to the following passage from Klein's second paper on non-Euclidean geometry of 1872–1873 (also included as n. XVIII in the first volume of Klein [1921]):

> The simplest group of transformations appears to be the group of all the linear transformations, which are here understood as such that replace the initial variable with linear fractional functions of the same with a common denominator. It is this way of treating the manifold, which I will call *projective*, which is used in the *new algebra* [...]. The name "*invariant theory*," which is associated with the new algebra, characterizes well what is essential to any way of treating a manifold according to the view advocated here; it is always a matter of discovering the invariant relations under a given extent of variations. [Klein 1873, p.122]

As Klein himself made clear in addition to the above quote, various ways of treating a manifold are equally possible. Ordinary metrical geometry offers one such way. Another way to treat a manifold is provided by "*analysis situs*," which is now known as topology.

My suggestion is that, in Klein's eyes, the projective way of treating a manifold offers the first and paradigmatic illustration of a process of abstraction from the investigation of the connections between distinct domains (*i.e.*, invariant relations) to the definition of abstract concepts or structures that have different mathematical systems falling under them. Another example of such a concept is the concept of group. It is well known that the defining characteristics of transformation groups had been formulated first by Évariste Galois and Camille Jordan. Both Galois and Jordan had been dealing with permutation groups. As Wussing [1984, p.178] pointed out: "The progressive evolution of the Erlangen Program brought a change in the manner of picturing a group and in its definition, and contributed to the formulation of the concept of 'group' as a group of transformations." It is noteworthy, however, that Klein himself constantly relied on the projective approach rather than starting with abstract groups. I take this as an indication of Klein's "bottom up" approach to mathematical abstraction, so to speak, that is, the fact that the notion of structure in his methodology is abstracted in a stepwise fashion from the comparison of different instantiations of the same structure.

3 Klein and Mathematical Structuralism

This section offers a discussion of Klein's account of mathematical abstraction starting from a comparison with Dedekind's. Dedekind developed what is now considered to be one of the first versions of mathematical structuralism, according to which numbers are defined as places within an ordered system. In particular, I will refer to how Sieg and Schlimm have reconstructed the development of Dedekind's view from the way in which he established a connection between the rational numbers and the points of a straight line in 1872 to his definition of natural numbers as *abstracta* of a simply infinite system in 1888. Although it was only in 1888 that Dedekind presented a worked-out view of numbers as ordinals, Sieg and Schlimm have pointed out how the connection between numbers and points led Dedekind to the recognition of one paradigmatic case of an abstract concept of ordered system that has a variety of instantiations. The crucial step towards the characterization of such a system in axiomatic terms is the formulation of the laws that characterize the ordering relation, in particular the axiom of continuity, regardless the supposed nature of the things making up the system.

Section 3.1 will point out that Klein referred to Dedekind's work on irrational numbers on several occasions [Klein 1874, 1890, 1898, 1893b], often considering Dedekind's axiom in connection with other formulations by Cantor and Weierstrass. Section 3.2 will focus on how Klein in his lectures on non-Euclidean geometry from 1889-1890 adopted a conception of continuity that he dated back, more specifically, to Dedekind [1872] to introduce the epistemological ideas underlying his classification of geometries. In particular, I will consider how Klein extended Dedekind's connection between numbers and points to the case of a segment of a projective straight line. Klein started from the consideration of this ordered system to account for a process of abstraction that has its roots in spatial intuitions and culminates in the formulation of axiomatic definitions for a variety of topological forms of space. I will rely on this account to substantiate the interpretation of Klein's bottom-up approach to mathematical structures that has been proposed in Section 2.

3.1 Klein and Structural Axiomatics

As mentioned in the previous section, Klein maintained that Dedekind's continuity lies at the foundations of any speculation about space. Klein's consideration is reminiscent of Dedekind's view that, lacking any knowledge about the continuity or discontinuity of space, the assumption of the continuity of the line "is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line" [Dedekind 1872, p.12]. In Dedekind's view, we do not know whether space is a continuous manifold or not, but even under the supposition that space is discrete, it would be possible to construe it as continuous by filling the gaps in thought.

As Sieg and Schlimm [2014] have pointed out, the above passage suggests that Dedekind used the term "axiom" to denote true statements about definite mathematical objects: The axiom of continuity is true of the geometric line; however, it is impossible to adduce a proof of its correctness. While such a perspective on axioms is a traditional one, Sieg and Schlimm suggest that Dedekind's practice reveals an understanding of axioms (without using the word) that is much closer to the way in which Hilbert and most modern mathematicians use the word, namely, as characteristic conditions of abstract concepts.Dedekind developed his approach further in [1888] by taking such characteristic conditions as starting points for rigorous, although non-formal proofs.[12]Sieg has pointed out Dedekind's influence on Hilbert

[12]On the development of Dedekind's analysis of number, see Sieg and Schlimm [2005].

by showing that Hilbert's axiomatic system for the real numbers was presented not in the contemporary formal-logical style, but in an algebraic way. Hilbert assumed that a system exists whose elements satisfy some axiomatic conditions; where that assumption requires consistency proofs. Sieg [2014, p.135] called such an approach "structural" axiomatics and contrasted it with the formal axiomatics developed later by Hilbert himself as well as Bernays. In Sieg's interpretation, Hilbert thought of axiom systems in a structural way, insofar as he focused on the question whether such a system can be thought (*i.e.*, whether the axioms do not lead to contradiction), without addressing the question concerning the nature of the things making up the system.

In Sections I and II of Dedekind [1872], the characteristic conditions of continuity are inferred from the comparison of ordered systems of different kinds. He began by pointing out what he called an "analogy" between the laws that determine the order of rational numbers and the corresponding laws for the points of the line: The same laws apply to the "x is less than y" relation and to the relation "x is to the left of y." He then turned the analogy into a "real connection" by showing that there is an order-preserving mapping between the elements of the separate domains. The mapping also shows that there are some gaps in the system of rational numbers, namely, divisions of the system in two separate classes, for which there is not a rational number that produces the division. Dedekind went on to introduce the notion of "cut" to designate any such division. According to Dedekind, irrational numbers are "created" in correspondence with cuts that do not have a corresponding number in the original system.

From the standpoint of structural axiomatics, Dedekind's way of proceeding can be reformulated as follows.[13]A system O is an ordered system iff there is a relation R on O, such that: (1) R is transitive; (2) Between any two different elements u and v of O there are infinitely many elements z; either $R(u,z)$ and $R(z,v)$ or $R(v,z)$ and $R(z,u)$; (3) For every element x in O there is a partition of O into two infinite subsets O_1 and O_2, such that i) for all z in O_1 and for all u in O_2, $R(z,u)$, ii) for all z in O_1 different from x, $R(z,x)$, iii) for all u of O_2 different from x, $R(x,u)$. The element x itself may be taken either as the greatest element of O_1 or as the least element of O_2. For every such element x, the partition of (3) satisfying (i) – (iii) is necessarily unique; Dedekind called it the "cut determined by x" and denoted it (O_1, O_2). The system O is continuous iff the inverse of 3) also holds true: 4) For every cut (O_1, O_2) in O, there is either in O_1 or in O_2 an element x, which determines the cut and which can be either rational or irrational.

Sieg and Schlimm argue that Dedekind in later works modified his notion of "creation" by shifting the focus from the creation of objects (*i.e.*, the individual irrational numbers) to the creation of abstract concepts via structural definitions. Dedekind articulated an axiomatic extension of the number concept starting from the structure of a simply infinite system. He asserted in a famous passage: "If in the consideration of a simply infinite system N set in order by a transformation φ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers* [Dedekind, 1888, p. 68].[14]

[13]I rely in the following on Sieg and Schlimm [2014].

[14]The interpretation of this passage is controversial, because Dedekind stated also that "numbers are free creations of the human mind" and "a thing is completely determined by all that can be affirmed or thought concerning it" [1888, pp.31, 44]. According to Reck, Dedekind's system of natural numbers is "created" in the following sense: "It is identified as a new system of mathematical objects, one that is neither located in the physical, spatio-temporal world, nor coincides with any of the previously constructed set-theoretic simple infinities" [Reck 2003, p.400]. Without addressing this complicated issue here, it is

Sieg and Morris [2018] point out that there are two different notions of abstraction at work in Dedekind's structuralism. On the one hand, numbers are abstracted *from* simply infinite systems in the negative sense: No new objects are introduced, but familiar object fall under more abstract concepts. On the other hand, the abstraction *to* such concepts as simply infinite systems and continuous ordered systems imply a different form of abstraction, which, in Sieg's and Morris's reconstruction, goes back to Lotze. According to this conception, there are cases in which abstraction does not drop specific characteristics, but rather replaces some by more general ones and yields in this way more abstract concepts.

Klein's methodology appears *prima facie* rather distant from Dedekind's, insofar as their works involve very different subject matters. There is evidence that, nevertheless, Klein relied on Dedekind to define his own stance towards structural axiomatics. Klein wrote:

> Regarding the origin of axioms, I cannot say more than this: The abstraction that leads to them here as in other domains takes place unconsciously. That which is only given approximately in intuition or experiment, is what we formulate in an exact way, because otherwise we would not know how to make a start. – With that, I can clarify my stance towards the theory of irrationals. Certainly, the formation of irrational numbers was triggered by the fact that our spatial intuition is seemingly continuous. Since I do not attribute any precision to spatial intuition, however, I will not want the existence of irrationals to be derived from such an intuition. I think that the theory of irrationals should be justified or defined arithmetically, to be then brought into geometry by means of axioms, and hereby enable the rigor of distinctions (*Schärfe der Distinktionen*) that is required for the mathematical treatment. [Klein 1890, p.572]

Klein distanced himself from the traditional conception of axioms as expressing some facts that are immediately evident and cannot be proven. He considered an axiom to be "nothing else than the demand, by which one reads exact statements into the inexact intuition" [Klein 1890, p.571]. The reliance on stated axioms does not exclude the use of diagrams in order to anticipate the results of proofs; a rigorous proof requires, nonetheless, that one refers back to the axioms formulated as exact statements. Therefore, Klein in the above passage referred in particular to Dedekind's characterization of irrational numbers as exact notions triggered by the inexact intuition of the continuity of space.[15]

In order to clarify his position concerning the use of diagrams in synthetic geometry, Klein [1890] introduced a distinction between "concrete" and "abstract" intuition. The former is directly derived from observations and therefore essentially inexact; the latter is "refined" by the substitution of inexact and specific characteristics with exact and more general ones.[16] Klein's example is the following. Klein [1890] presented a solution to the Clifford-Klein problem of determining the class of all surfaces in elliptic, hyperbolic, and

noteworthy that Klein borrowed from Dedekind the notion of abstraction to "conceptual properties," which Klein also identified as "axiomatic definitions" [Klein 1897, p.588]. However, Klein avoided the talk of "creations," arguably because in the case of geometry, unlike in the case of number theory, it appears more straightforward to assume that concepts subsume objects, including those that are located in the physical world.

[15]Further examples of "inexact" and "exact notions" from Klein's lectures on non-Euclidean geometry are discussed in Section 3.2.

[16]Elsewhere Klein expresses this as a distinction between naïve and refined intuitions: "The naïve intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact" [Klein 1894, p.42].

parabolic space that are locally isometric to the Euclidean plane.[17] The solution of this problem also sheds light on the (abstract) intuition of space, which, like all intuitions, is always restricted to a spatial region. The assumptions of Euclidean geometry that appear to be more intuitive in what Klein called the concrete sense of intuition are replaced by the more general assumptions that characterize the projective plane. Such assumptions imply the more general concept of space-form, whose characteristics define a whole class of geometries.

My suggestion is that Klein, similar to Dedekind, used a Lotzean notion of abstraction to characterize the formation of mathematical concepts. Furthermore, Klein pointed out that the exact and general characteristics that replace concrete intuitions in mathematical concept formation can be expressed axiomatically.

This might seem in contrast with the fact that Klein himself did not adopt an explicitly axiomatic approach in his original presentation of his geometrical work from the 1870s. It is noteworthy, however, that he outlined how to make explicit such an approach after the development of structural axiomatics by other mathematicians, for example, in his review of the third volume of Lie's *Theorie der Transformationsgruppen*. Klein delivered this review as a lecture for the Physical-Mathematical Society of the University of Kazan when Lie was awarded the first Lobachevsky prize. Klein wrote: "For any geometry of a manifold one can demand an axiomatic definition, *i.e.*, a definition that determines the geometry under consideration via conceptual properties without the use of explicit formulas (or better say, independently of the arbitrary coordinate assignment)" [Klein 1898, p.588]. Klein went on to explain that such properties define the concept of the group underlying geometry. In ordinary metrical geometry, the principles of congruence define the group of rigid transformations. Alternatively, Klein pointed out that one can start from the consideration the possible configurations of elements, which are equivalent in the sense of the group. The definition of the group in the case of projective geometry is given by the principles concerning incidence and order relations. Klein identified the underlying group of collineations as the more general concept of space that is presupposed in the investigation of connections between different geometries. He maintained that this way of proceeding sheds new light on the problem of determining the preconditions of measurement:

> When it comes to taking into consideration the topologically different forms of space for the determination of the geometry of actual space, we are faced not so much with an arbitrary but with an inner consequence. Our empirical measurement has an upper limit, which is given by the dimensions of the objects accessible to us or to our observation. What do we know about spatial relations in the infinitely large? From the start absolutely nothing. Therefore, we rely on the postulates that we formulate. I consider all of the different topological forms of space equally compatible with experience. The fact that we put first some of these forms of space in our theoretical considerations (*i.e.*, the original types, that is, the properly parabolic, hyperbolic, and elliptic geometries) and finally select parabolic geometry (*i.e.*, the usual Euclidean geometry), depends solely on the principle of economy. [Klein 1898, p.595][18]

[17] Klein learned about this class of surfaces in a lecture taught by William Kingdon Clifford in 1873. The problem is known as the "Clifford-Klein problem" or "the problem of the form of space " [see Torretti 1978, p.151].

[18] Here and in the rest of the lecture, Klein uses "postulates" and "axioms" as synonyms. My suggestion is that this terminology is justified by the definition of axioms discussed above: If axioms are "demands that read exact statements into inexact intuitions," they are also "postulates" in the etymological sense that they "demand" what such statements imply.

Summing up, it appears that in the 1890s Klein subscribed to the basic tenets of what we now call structural axiomatics. Firstly, Klein held that axioms provide definitions of abstract concepts – where abstraction replaces some particular and inexact notions with more general and exact characteristics. Secondly, specific mathematical domains are characterized as falling under such concepts by identifying their structural properties. This is exemplified by the way in which arithmetical operations are defined in the theory of irrational numbers, via operations with cuts. Another example is the way in which Klein defined geometrical properties as relative invariants of transformation groups. Thirdly, connections between distinct domains are explored via mappings.

Klein pointed out in his mathematical works that the reliance on stated axioms is required for the rigorization of analysis and of geometry. At the same time, especially in lecture courses and popular lectures for a wider audience, Klein introduced motives other than rigor in the discussion on axiomatic definitions. His focus in [1898], as well as in other lectures from the 1890s, was on the physical interpretation of the projective determination of measure rather than on rigorization. The difference of emphasis between Klein's more technical and more popular writings reflects the different scopes of these writings. However, there is evidence that he saw the views expressed in his popular writings as compatible, if not complementary to his technical works.

Klein offered a clear explanation of his approach in another popular lecture from 1895, on the so-called "arithmetization" of mathematics. Klein distanced himself from the project of arithmetizing mathematics in the sense of Weierstrass, Kummer and Kronecker in Berlin. This was the view that only the arithmetical and finitary way of proof is justified while additional assumptions can be avoided in analysis or may be allowed for the purposes of geometry and mechanics. In presenting himself as a representative of the Göttingen tradition of Gauss, Riemann and Dedekind, Klein agreed with the overall aim of arithmetization as a way to elaborate different branches of mathematics with the same rigor as the definition of numbers. As he put it, however, this positive side of arithmetization comes with a negative one: "On the negative side, I have to point out that mathematics goes beyond logical deduction. Besides the latter, intuition still retains its full and specific meaning" [Klein 1895, p.84]. According to Klein, intuitions play an important role in the practice of geometry, in applied mathematics and in mathematics teaching.

In sum, Klein did not call into question the advantages of arithmetization. However, he emphasized the epistemic value of intuition as an important complement to arithmetization when it comes to domains other than analysis. One might rephrase his position by saying that Klein at least gradually came to advocate an axiomatic approach, when it comes to contexts of justification. However, he defended some role for the different kinds of evidence that he summarized under the term "intuition," when it comes to contexts of discovery.[19]

My suggestion is that, when considered in more detail, Klein's writings show a more complicated picture, which is not captured by the justification/discovery dichotomy. Klein tends to associate different epistemic goals with the abstraction at work in mathematics. These goals include justification, but also understanding and discovery. We have seen that Klein in [1890] emphasized the advantage of axiomatics when it comes to justifying existential assumptions, *e.g.*, in the case of irrational numbers. We saw also that Klein in [1895] and [1898] advocated the use of axiomatic definitions in geometry as an indispensable presupposition for rigorous proofs. At the same time, the latter writings suggest that such definitions set higher standards of rigor than concrete intuitions. Therefore, Klein said that mathematicians turn concrete intuitions into abstract intuitions.

My suggestion is that abstract intuitions in Klein's sense have two important goals: On

[19]For this reading, see Schlimm [manuscript].

the one hand, they allowed him to account for mathematical practices in use in nineteenth-century geometry; on the other hand, they play a heuristic role in Klein's investigation of different possible physical interpretations of geometry. In order to further elucidate these two goals, the following section will focus on examples offered by Klein in his lectures on non-Euclidean geometry from 1899–1890. For reasons of space, I will leave the discussion of how discovery operates in Klein's work on the application of geometry in physics for another paper.

3.2 Klein's Lectures on Non-Euclidean Geometry: 1889–1890

[20]

Klein in his lectures on non-Euclidean geometry introduced the problem of the arithmetization of geometry by starting from a general consideration on the foundations of analysis. Since the development of the calculus, there had been a debate about whether infinitesimal quantities exist as ultimate constituents of a continuum or the talk of infinitesimals is but an auxiliary assumption for calculating limits. According to what Klein, following Paul du Bois-Reymond [1882], called the "idealist" approach, infinitesimals exist as presuppositions of the calculus although neither infinite nor infinitesimal quantities are imaginable in the sense of concrete intuition. According to the opposed "empiricist" approach, in du Bois-Reymond's terminology, the representation of such abstract concepts as limits is derived indirectly by the use of geometrical constructions. Du Bois-Reymond's point was that both of the above views provide equally possible accounts of the foundations of analysis, the ultimate grounds for choosing one rather than the other being a matter of the mathematicians' inclination. By contrast, Klein proposed a sort of synthesis: Klein himself considered the geometrical properties of figures from the standpoint of the empiricist; as far as the proper mathematical consideration is concerned, he agreed with idealism in demanding "conceptual characterizations" because of their absolute rigor [1893b, p.312]. As pointed out in the previous section, these characterizations correspond to structural definitions in an explicitly axiomatic approach.

Klein's first example of the two approaches was about continuity. According to the received, empiricist view, the notion of continuity is immediately given in the intuition of the continuity of the points on a line and the assumption of irrational numbers is only a consequence of this intuition. In the modern, idealist view, which Klein drew back to Dedekind, the inexact intuition of how points are related is only the trigger for the formation of the exact concept of irrational [Klein 1893, p.310].

Klein went on to argue that the opposition between idealism and empiricism presents itself when it comes to the foundations of projective geometry. From the idealist view, the definition of continuity is presupposed as given in the theory of irrational numbers and applied to the projective line that features in von Staudt's proof of the fundamental theorem. For the rigorization of this proof by the use of Dedekind's continuity, Klein referred to his earlier work in collaboration with Lüroth-Zeuthen and to the subsequent presentation of the proof by Jean-Gaston Darboux. From the empiricist view, it appears more appropriate to base the proof on the consideration of geometrical relations. Klein referred, for example, to the proof of the fundamental theorem by Friedrich Schur, who avoided Dedekind's continuity by obtaining the notion of projectivity from that of perspectivity.

[20]Klein's lectures were transcribed by Friedrich Schilling and published in 1893. A second edition by Walter Rosemann appeared posthumously in 1928. I will refer to Klein's most detailed discussion of the fundamental theorem of projective geometry, which is found in the 1893 edition. Subsequently, I will draw the relevant comparisons with the 1928 edition.

Although Klein admitted that both proofs were correct, he still preferred the first one mainly for what appear to be epistemological reasons: Only the first proof shows that projective space is a concept that can have Euclidean and non-Euclidean metrics as special cases. The characterization of space based on a structural definition of continuity such as Dedekind's offers a very natural way not only to address internal questions about geometry but also to address empirical questions.[21] Klein had proved the relative consistency of non-Euclidean geometry with the projective interpretation. However, he did not limit himself to this result, but he also investigated the applications of his interpretation in measurement.[22] The projective determination of measure offered an example of the inner connection between the structural methodology at work in mathematical theories and their various applications. Klein wrote:

> I will say that everything that is mathematically well-founded, sooner or later will find a far-reaching significance beyond its original ground. [...] Firstly, the projective determination of measure obtained a great epistemological significance, in that it proved to be the simplest foundation of non-Euclidean geometries, which originated from the investigations about the independence of the axiom of parallels from the other axioms and which appeared particularly hard to grasp at first. [...] And now, in recent years, it turns out that the projective determination of measure provides a rational ground for the latest physical speculations. [Klein 1910, p.22]

The rigorization of geometry does not contradict concrete intuitions (in the sense elucidated in Section 3.1) but the particular assumptions that appeared to be evident in the received view. By contrast, Klein's suggestion is to start from the recognition that intuitions are essentially inexact to reconstruct all the steps towards the axioms. The introduction of axioms turns our initially concrete and inexact intuitions into abstract and exact ones, which in Klein's view are the same as concepts.

Klein's example in [1893] was the construction of a numerical scale on the projective line. Given two points, 0 and 1, on the line and a point at infinity, ∞, the construction of the fourth harmonic point 2 can be reiterated with the constructed points to generate 3, 4, and so on. The series corresponds to the integer numbers. Points corresponding to negative and fractional numbers can be constructed by projecting the same construction from an external point. The step that shows the continuity of the line is the postulate that such a construction has a limit in the point at infinity. This allows for the correlation of every point with a particular number, which can be rational or irrational. The inverse claim that every number corresponds to a point is incorrect, because the construction only considers a part of the line.[23] Alternatively, it is possible to start with an unlimited figure of first order, such as a line bundle. In this case the correspondence of the lines with the real numbers is one to one.

The above way of proceeding mirrors Dedekind's way to establish a real connection between rational numbers and the points of a line by starting from the consideration of an

[21] Klein [1873, p.114] referred for his stance also to the following passage from [Riemann 1876, pp.267–268] about the empirical investigation of metrical relations: "Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices."

[22] Klein's investigations include the abovementioned lecture from 1897 and a series of studies in the geometrical foundations of relativistic physics [see esp. Klein 1910]. A thorough account of these investigations is beyond the scope of the present paper.

[23] For a detailed description of this construction with illustrations, see Klein [1893b, pp.337–354].

analogy between the laws of the distinct domains (*i.e.*, numbers and geometrical points). The analogy is apparent in the reformulation of Dedekind's law of continuity given by Sieg and Schlimm [2014] and reproduced here in Section 3.1. Klein emphasized the fact that the two domains are distinct by considering the construction of a harmonic progression, on the one side, and the systems of numbers, on the other. He then showed that both kinds of domains form a continuous ordered system, which implies the existence of limiting elements. Not only did this procedure provide a convenient way to introduce projective coordinates, but, in Klein's view, it showed something important about spatial concepts: Thinking of space in analogy with the numerical manifold shed light on what can and cannot be assumed about an observable region of space, given the fact that the projective determination of measure can be specified in three different cases.

Klein gave a more thorough account of the different ways to assign projective coordinates in [1928], including via Hilbert's calculus of segments. Nevertheless, Klein highlighted again that the advantage of the above construction is that it introduces projective coordinates independently of Euclidean geometry. Klein wrote:

> We especially appreciate the fact that, in order to introduce the coordinates, we do not need the whole projective space and we can restrict every consideration to a suitably finite region of space, say this room. Such a restriction has a principled interest, mainly because it takes into account that, in the application of geometry to the space of empirical intuition, we are not warranted to make statements about spatial elements at any distance; in particular the discussion about whether parallels exist or not does not make any sense from the very beginning. [1928, p.154]

Summing up, both in the proof of the fundamental theorem of projective geometry and in the introduction of projective coordinates, Klein was faced with the alternative: Either to rely on geometrical relations and proceed without Dedekind's continuity, or to presuppose the theory of irrational numbers and proceed synthetically in the sense of von Staudt, namely, without metrical notions. While Klein admitted that both ways of proceeding can be adopted for different purposes, he favored the second approach when it comes to providing an epistemic justification of non-Euclidean geometries as belonging to projective geometry and as viable hypotheses in the application of geometry to the space of empirical intuition.

Regarding the philosophical background of Klein's argument, it is worth noticing that the late nineteenth-century philosophical tradition was much influenced by Kantian philosophy. Non-Euclidean geometry appeared as a challenge against the view that there must be some core of *a priori* knowledge grounded in spatial intuition, which accounts for the applicability of geometric concepts to physical objects. However, most of the neo-Kantians distanced themselves from Kant's theory of pure intuitions on philosophical grounds and proposed different arguments for the applicability of geometry.[24]

Cassirer, for example, saw the formation of group concepts as a way to address the applicability problem at stake with Klein's classification of geometries. This problem appeared more complicated with the recognition that Euclidean geometry is only one of the possible hypotheses. This shows that even presuppositions that once appeared necessary, such as the Euclidean notions of distance, congruence, and straight line, can be called into question by scientific developments. Cassirer maintained that, nevertheless, the reason why geometric notions are indispensable in physics becomes even clearer in a somewhat

[24] On the nineteenth-century discussion of Kant's Transcendental Aesthetics and its developments in Marburg neo-Kantianism, see [Biagioli 2016, Ch.2].

paradoxical way. The revolutionary aspects of the EP, according to Cassirer, lies in the fact that the only implicit assumption that lies at the foundation of geometry is "a system of possible transformations" rather than immutable geometrical properties [Cassirer 1910, p.91]. It is because the system can vary according to the initial assumptions, and not because we possess some kind of immediate knowledge about space, that geometrical transformations provide such suitable tools to represent spatial relations and account for different hypotheses that might or might not be confirmed empirically.[25]

4 Concluding Remarks

The striking difference between Cassirer's reception of Klein's work and the twentieth-century generalizations of the EP lies in the focus of the former on the epistemological implications of Klein's methodology. As mentioned before, this depends in part on the neo-Kantian background of late nineteenth-century philosophy. Nevertheless, Cassirer offered a philosophical account for the view that the structural definition of geometric concepts provides a rigorous foundation of projective geometry, but has also a heuristic aspect, when it comes to exploring the non-Euclidean hypotheses in geometry and physics.

Much more could be said about the comparison of Klein and contemporary variants of mathematical structuralism. Based on my previous reconstruction, I limit myself to point out that much of what Reck says about the structuralist aspects of Dedekind's work applies to Klein, insofar as the latter advocated some of the tenets of Dedekind's structuralism. To sum up, both Dedekind and Klein advocated what we now call a methodological structuralism. However, Dedekind's structuralism has further implications that differ from *ante rem* structuralism. According to Shapiro's [1997] *ante rem* structuralism, numbers are mere "places" or "positions" in patterned structures. By contrast, we saw that Dedekind distinguished numbers both from physical objects and from objects in other simple infinities. Dedekind's characterization of numbers implies a positive notion of abstraction, according to which numbers can be identified as a new system of mathematical objects that have all and only the characteristic conditions established in arithmetic.

I have shown that Klein relied on Dedekind with regard to the characterization of continuous ordered systems. Klein's view in this regard is equally distant from *ante rem* structuralism. Based on Klein's account of abstraction in the context of geometry, his conception of structure can be spelled out in terms *in rebus* structuralism (see Schiemer [2020]), as moving from the comparison of concrete mathematical systems. Klein gave an original interpretation of the idea that there is a positive side to mathematical abstraction by emphasizing that nineteenth-century mathematicians, such as von Staudt, Cayley, and Klein himself, in investigating a plurality of systems of geometrical axioms, also shaped the intuitions of their contemporaries about spatial concepts. These intuitions, in Klein's terminology, have become more and more abstract with the development of projective geometry and group theory.

Finally, I have discussed the fact that Klein departed from the Berlin School, as well as from more standard approaches to the philosophy of mathematics in the twentieth century, in emphasizing the role of intuitions in geometry and applied mathematics. A closer look at Klein's explanations reveals, however, that even what he called intuition relates to a form of mathematical structuralism. The need to take into account geometrical constructions and empirical evidence derives from the idea that the structural reasoning at work in arithmetic finds fruitful applications in geometry, and can be extended further in the formulation of scientific hypotheses. As van Fraassen emphasized, Klein took important steps

[25] For a discussion of Cassirer's argument, see [Biagioli 2018, 2020].

towards scientific structuralism with the study of the projective model of non-Euclidean geometry. This model showed that different congruence relations, corresponding in practice to different measurement standards, imposed on the same space give it a different geometry. Therefore, van Fraassen [2008, p.216] has called Klein's model an "initial locus" for the problem of coordinating abstract mathematical structures to empirical reality.

This paper focused on aspects of Klein's work that, in philosophical writings, are usually read in a set-theoretic framework (see *e.g.*, Torretti [1978], Wussing [1984], Suppes [2002]) but reveal Klein's own philosophical views, when considered in context. It would be no less important to extend the consideration to the reception of Klein in the development of category theory. As far as I can tell from my previous considerations, the kind of structuralism that would emerge from Klein's epistemological writings lends plausibility to Marquis's [2008] comparison with Eilenberg and Mac Lane [1945]. In this paper, they introduced categories as algebraic structures that include groups as special cases, and, just as groups, can have various interpretations (*e.g.*, geometrical, logical, combinatorial). They gave an abstract definition of categories, along the lines of the axiomatic definition of a group (see Section 2.2). Category in this sense works as an auxiliary notion in preparation for what Eilenberg and Mac Lane called functors and natural transformations. After discussing Eilenberg's and Mac Lane's claim that their theory generalizes Klein's group-theoretical view, Marquis argues that a deeper connection lies somewhere else, in Klein's comparative approach. The subsequent development of category theory as a foundational framework for mathematics made extensive use of comparison of constructions and of the isomorphisms occurring in different branches of mathematics to suggest new result by analogy.

My suggestion is that much of what Klein and Cassirer said about the epistemological implications of structural methods was motivated likewise by the desire for a uniform treatment of different mathematical disciplines and the conviction that this would contribute to the advancement of science.