Published for SISSA by Springer

Received: May 25, 2020
Accepted: July 19, 2020
Published: August 13, 2020

Surface defects from fractional branes. Part II

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Abstract: A generic half-BPS surface defect of $\mathcal{N} = 4$ supersymmetric U(N) Yang-Mills theory is described by a partition of $N = n_1 + \ldots + n_M$ and a set of $4M$ continuous parameters. We show that such a defect can be realized by n_I stacks of fractional D3-branes in Type II B string theory on a $\mathbb{Z}_M$ orbifold background in which the brane world-volume is partially extended along the orbifold directions. In this set up we show that the $4M$ continuous parameters correspond to constant background values of certain twisted closed string scalars of the orbifold. These results extend and generalize what we have presented for the simple defects in a previous paper.

Keywords: D-branes, Extended Supersymmetry, Nonperturbative Effects, Duality in Gauge Field Theories

ArXiv ePrint: 2005.03701

https://doi.org/10.1007/JHEP08(2020)058

JHEP08(2020)058

Contents

1 Introduction

In this article, which is a direct extension of the companion paper [1], we discuss how to embed into string theory a generic Gukov-Witten (GW) surface defect [2, 3] of the $\mathcal{N} = 4$ supersymmetric U(N) Yang-Mills theory. We do this by analyzing the massless fields on the world-volume of fractional D3-branes in Type II B string theory on an orbifold background, following the proposal of Kanno-Tachikawa (KT) [4]. Here, we shall only provide a brief introduction and refer the reader to the introductory sections of the companion paper [1] for a more detailed account and for a discussion of the relation of our approach to others already present in the literature as well as for the relevant references.

We consider Type II B string theory on the following orbifold space-time

$$\mathbb{C}_{(1)} \times \frac{\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}}{\mathbb{Z}_M} \times \mathbb{C}_{(4)} \times \mathbb{C}_{(5)} \tag{1.1}$$

with constant vacuum expectation values turned on for particular twisted scalar fields in the Neveu-Schwarz/Neveu-Schwarz (NS/NS) and Ramond/Ramond (R/R) sectors. In this background we engineer a $4d$ gauge theory by introducing stacks of fractional D3-branes that extend along the first two complex planes $\mathbb{C}_{(1)}$ and $\mathbb{C}_{(2)}$. In this combined orbifold/D-brane set-up, which we refer to as the KT configuration [4], we compute the profile in configuration space of the massless open strings by means of open/closed world-sheet correlators, and show that these exactly reproduce the singular profiles that characterize the GW surface defect in the $\mathcal{N} = 4$ gauge theory [2]. In this way we are therefore able to provide an explicit identification of the continuous parameters of the GW solution with the vacuum expectation values of the twisted scalars.

In [1] we already worked out this identification for the simple surface defects that correspond to the $\mathbb{Z}_2$ orbifold. In this paper, we extend our analysis to the $\mathbb{Z}_M$ orbifolds for $M > 2$ which can describe the most general defect corresponding to the breaking of the U(N) gauge group to the Levi subgroup $\mathrm{U}(n_0) \times \ldots \times \mathrm{U}(n_{M-1})$ with $\sum_I n_I = N$.

While the basic conceptual issues in realizing such a surface defect using fractional D3-branes remain the same for all M, the main difference with respect to [1] lies in the treatment of the closed string background. For $M = 2$ the massless fields of the NS/NS and R/R twisted sectors correspond to degenerate ground states and their vertex operators are realized using spin fields [5]. This is no longer the case for $M > 2$ and, in fact, the massless fields of the NS/NS sector arise from excited states created by the oscillators of the fermionic string coordinates. Furthermore, pairs of twisted sectors are related by complex conjugation and this turns out to play an important role in the identification of the closed string background with the real parameters in the GW profiles.

In the $\mathbb{Z}_M$ orbifold, there are $(M - 1)$ twisted sectors. One could treat all of them at once using the bosonization formalism [5, 6], but in order to keep track of all the relative phases it would be necessary to introduce the so-called cocycle factors. Since dealing with these cocycle factors is quite involved, and since the relative phases are crucial to obtain the correct identification of the continuous parameters of the GW surface defects, we adopt an explicit fermionic approach and use the bosonization formalism only where

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no phase ambiguities arise. The advantage of this method is that the relative phases among the contributions from different sectors are easily tracked and fixed by the fermionic statistics. Moreover, in this fermionic approach we can describe the fractional D3-branes using boundary states (for a review see for example [7, 8]). Even though the KT brane configuration has not been explicitly considered so far from the boundary state point of view, we can exploit many of the results that already exist in the literature [9–12] and generalize them to the present case, in which the fractional D3-branes partially extend along the orbifold. The price we have to pay for using this fermionic approach is that we have to distinguish between the twisted sectors and treat separately those whose twist parameter is smaller or bigger than $\frac{1}{2}$.

The open string sector, instead, is similar to that of the $M = 2$ case. We recall that for the KT configuration, the fractional D3-branes have the same field content as the regular D3-branes, since in this case the orbifold does not project away any of the open string excitations, unlike the case when the branes are entirely transverse to the orbifolded space. Indeed, on the world-volume of the fractional D3-branes we find a gauge vector and three complex massless scalars plus their fermionic partners. However, the corresponding vertex operators are linear combinations that behave covariantly under the action of the orbifold. When $M > 2$, these combinations are slightly more involved than for $M = 2$ and are written in terms of generalized plane-waves.

Once the vertex operators for the massless open and closed string states are derived, the discussion proceeds along the same lines as in the $M = 2$ case, but with the important technical differences and peculiarities that we have just mentioned.

This paper is structured as follows. In section 2 we provide a detailed description of the twisted closed string spectrum of Type II B strings on the orbifold (1.1). We then proceed in section 3 to introduce the fractional D3-branes of the $\mathbb{Z}_M$ orbifold and study two different aspects. Firstly, from the closed string point of view we write the boundary states and use them to derive the reflection rules that relate the left- and right-moving modes of the twisted closed strings in all sectors. Secondly, we derive the $\mathbb{Z}_M$-invariant vertex operators describing the massless open string excitations that live on the D3-brane world-volume. This turns out to be non-trivial given that the D3-brane extends partially along the orbifolded space. In section 4 we calculate mixed correlators among the open and closed string excitations and use them to derive in section 5 the singular profiles of the gauge fields near the location of the surface defect. The field profiles we obtain precisely match those of the GW defect once the background values of the twisted closed string field are identified with the continuous parameters of the surface defect.

Our analysis provides an explicit realization of the monodromy defects of the $\mathcal{N} = 4$ super Yang-Mills theory using perturbative string theory methods. As we discuss in the concluding section 6 we believe that this stringy realization may prove to be useful in further investigations of surface defects and their properties, and it may even offer an alternative approach to the study of extended objects in ordinary field theory through their embedding into string theory.

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2 Twisted closed strings in the $\mathbb{Z}_M$ orbifold

We consider Type II B string theory on the orbifold (1.1). The i-th complex plane $\mathbb{C}_{(i)}$ is parametrized by

$$z_i = \frac{x_{2i-1} + \mathrm{i}\, x_{2i}}{\sqrt{2}} \qquad \text{and} \qquad \bar{z}_i = \frac{x_{2i-1} - \mathrm{i}\, x_{2i}}{\sqrt{2}} \tag{2.1}$$

where x_μ are the ten real coordinates of space-time. The orbifold group $\mathbb{Z}_M$ is generated by an element g such that $g^M = 1$, with the following action on z_2 and z_3:

$$g \;:\; (z_2\,,\, z_3) \longrightarrow (\omega\, z_2\,,\, \omega^{-1} z_3) \tag{2.2}$$

where

$$\omega = \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}}\;. \tag{2.3}$$

The action of g on $\bar{z}_2$ and $\bar{z}_3$ follows from complex conjugation. This breaks the $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ isometry group of $\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}$ to $\mathrm{SU}(2)_+$.

To describe the closed strings propagating on this orbifold, we use the complex notation and denote the bosonic string coordinates by $\big\{Z^i(z), \overline{Z}^i(z)\big\}$ for the left-movers and $\big\{\widetilde{Z^i}(\bar{z}), \widetilde{\overline{Z}^i}(\bar{z})\big\}$ for the right-movers, with z and $\bar{z}$ parametrizing the closed string world-sheet. Similarly, we denote the fermionic string coordinates by $\big\{\Psi^i(z), \overline{\Psi}^i(z)\big\}$ for the left-movers and $\big\{\widetilde{\Psi^i}(\bar{z}), \widetilde{\overline{\Psi}^i}(\bar{z})\big\}$ for the right-movers.

2.1 Twisted sectors

In the $\mathbb{Z}_M$ orbifold theory, there are $(M-1)$ twisted sectors labeled by the index $\widehat{a} = 1, \ldots, M-1$. If M is odd, we can divide the twisted sectors in two sets, each one containing $\frac{M-1}{2}$ elements. The sectors of the first set are labeled by $\widehat{a} = a = 1, \cdots, \frac{M-1}{2}$ and are characterized by a twist parameter

$$\nu_a = \frac{a}{M} < \frac{1}{2}\;. \tag{2.4}$$

The sectors of the second set have, instead, a twist parameter

$$1 - \nu_a = \frac{M-a}{M} > \frac{1}{2} \tag{2.5}$$

and are labeled by $\widehat{a} = (M-a)$. If M is even, in addition there is an extra sector with twist parameter $\frac{1}{2}$, which has to be treated separately. For most of the discussion we will assume that M is odd and briefly comment on the special case with twist $\frac{1}{2}$, occurring when M is even, only at the very end, since this case has already been discussed in detail in the companion paper [1].

In the sectors with label a and twist parameter as in (2.4), the left-movers of the bosonic and fermionic string coordinates satisfy the following monodromy properties on the world-sheet:

$$Z^2(\mathrm{e}^{2\pi\mathrm{i}}\, z) = \mathrm{e}^{2\pi\mathrm{i}\nu_a}\, Z^2(z)\,, \qquad\qquad Z^3(\mathrm{e}^{2\pi\mathrm{i}}\, z) = \mathrm{e}^{-2\pi\mathrm{i}\nu_a}\, Z^3(z)\,, \tag{2.6a}$$

$$\Psi^2(\mathrm{e}^{2\pi\mathrm{i}}\, z) = \pm\,\mathrm{e}^{2\pi\mathrm{i}\nu_a}\, \Psi^2(z)\,, \qquad\qquad \Psi^3(\mathrm{e}^{2\pi\mathrm{i}}\, z) = \pm\,\mathrm{e}^{-2\pi\mathrm{i}\nu_a}\, \Psi^3(z)\,, \tag{2.6b}$$

where the $+(-)$ sign refers to the NS (R) sector. The analogous relations for $\overline{Z}^i$ and $\overline{\Psi}^i$ can be obtained by complex conjugation. On the other hand, the right-movers satisfy the monodromy properties:

$$\widetilde{Z}^2(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \mathrm{e}^{-2\pi\mathrm{i}\nu_a}\,\widetilde{Z}^2(\bar{z})\,, \qquad \widetilde{Z}^3(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \mathrm{e}^{2\pi\mathrm{i}\nu_a}\,\widetilde{Z}^3(\bar{z})\,, \tag{2.7a}$$

$$\widetilde{\Psi}^2(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \pm\,\mathrm{e}^{-2\pi\mathrm{i}\nu_a}\,\widetilde{\Psi}^2(\bar{z})\,, \qquad \widetilde{\Psi}^3(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \pm\,\mathrm{e}^{2\pi\mathrm{i}\nu_a}\,\widetilde{\Psi}^3(\bar{z})\,. \tag{2.7b}$$

Again, the relations for $\widetilde{\overline{Z}}^i$ and $\widetilde{\overline{\Psi}}^i$ are obtained by complex conjugation.

For the sectors with label $(M-a)$ and twist parameter as in (2.5), similar monodromy relations hold for the world-sheet fields but with ν_a everywhere replaced by $(1-\nu_a)$.

2.2 Twisted NS sectors

We now turn to a discussion of the spectrum of massless string states in the various twisted sectors, focusing mainly on the fermionic fields in the complex directions 2 and 3. In the fermionic formalism, when the NS boundary conditions are imposed, we have to treat separately the sectors with twist parameter smaller than $\frac{1}{2}$ and those with twist parameter bigger than $\frac{1}{2}$.

2.2.1 Sectors with twist parameter $\nu_a < \frac{1}{2}$

In this case the monodromy properties (2.6b) and their complex conjugate lead to the following mode expansions for the left-moving fermionic fields (see for example [13] and references therein):

$$\begin{aligned}
\Psi^2(z) &= \sum_{r=1/2}^{\infty}\Big(\overline{\Psi}^2_{r-\nu_a}\,z^{-r+\nu_a-\frac{1}{2}} + \Psi^2_{-r-\nu_a}\,z^{r+\nu_a-\frac{1}{2}}\Big)\,,\\
\overline{\Psi}^2(z) &= \sum_{r=1/2}^{\infty}\Big(\Psi^2_{r+\nu_a}\,z^{-r-\nu_a-\frac{1}{2}} + \overline{\Psi}^2_{-r+\nu_a}\,z^{r-\nu_a-\frac{1}{2}}\Big)\,,
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\Psi^3(z) &= \sum_{r=1/2}^{\infty}\Big(\overline{\Psi}^3_{r+\nu_a}\,z^{-r-\nu_a-\frac{1}{2}} + \Psi^3_{-r+\nu_a}\,z^{r-\nu_a-\frac{1}{2}}\Big)\,,\\
\overline{\Psi}^3(z) &= \sum_{r=1/2}^{\infty}\Big(\Psi^3_{r-\nu_a}\,z^{-r+\nu_a-\frac{1}{2}} + \overline{\Psi}^3_{-r-\nu_a}\,z^{r+\nu_a-\frac{1}{2}}\Big)\,.
\end{aligned} \tag{2.9}$$

The oscillators $\Psi^2_{-r-\nu_a}$, $\overline{\Psi}^2_{-r+\nu_a}$, $\Psi^3_{-r+\nu_a}$ and $\overline{\Psi}^3_{-r-\nu_a}$ are creation modes acting on the twisted vacuum of the a-th sector which we denote by $|\Omega_a\rangle$. Such a state is defined by

$$|\Omega_a\rangle = \lim_{z\to 0}\,\sigma_a(z)\,s_a(z)\,|0\rangle \tag{2.10}$$

where $|0\rangle$ is the Fock vacuum and $\sigma_a(z)$ and $s_a(z)$ are, respectively, the bosonic and fermionic twist fields [14]. More precisely, these twist fields take the form

$$\sigma_a(z) = \sigma^2_{\nu_a}(z)\,\sigma^3_{1-\nu_a}(z) \quad \text{and} \quad s_a(z) = s^2_{\nu_a}(z)\,s^3_{-\nu_a}(z)\,, \tag{2.11}$$

where the superscripts refer to the complex directions where the twist takes place, and the subscripts indicate the twist parameters. The bosonic twist field $\sigma_a(z)$ is a conformal field of weight $\nu_a(1-\nu_a)$ while the fermionic twist field $s_a(w)$ is a conformal field of weight ν_a^2. Therefore, the total conformal weight of the operator associated to the twisted ground state is ν_a. This means that $|\Omega_a\rangle$ is massive with a mass m given by

$$m^2 = \nu_a - \frac{1}{2} < 0 \ . \tag{2.12}$$

This tachyonic state is removed by the GSO projection.

The first set of physical states one finds in the GSO projected spectrum are those obtained by acting with one fermionic creation mode with index $r = \frac{1}{2}$ on the twisted vacuum. In particular, the oscillators $\Psi^3_{-\frac{1}{2}+\nu_a}$ and $\overline{\Psi}^2_{-\frac{1}{2}+\nu_a}$ increase the energy by $(\frac{1}{2}-\nu_a)$ and thus, when acting on the twisted vacuum, they create two massless states.[1] The vertex operators corresponding to these massless excitations, in the (-1)-superghost picture and at zero momentum,[2] are:

$$\begin{aligned} \mathcal{V}^1_a(z) &= \sigma_a(z)\, {:}\Psi^3(w)\, s_a(z){:}\, \mathrm{e}^{-\phi(z)} \ , \\ \mathcal{V}^2_a(z) &= \sigma_a(z)\, {:}\overline{\Psi}^{\,2}(w)\, s_a(z){:}\, \mathrm{e}^{-\phi(z)} \ . \end{aligned} \tag{2.13}$$

Here $\phi(z)$ is the bosonic field appearing in the bosonization formulas of the superghosts [5] and, as usual, the symbol $: \, :$ denotes the normal ordering. The vertex operators (2.13) are conformal fields of weight 1 and we collectively denote them as $\mathcal{V}^\alpha_a(z)$ with $\alpha = 1, 2$. As explained in appendix A.1, they form a doublet transforming as a spinor of $\mathrm{SU}(2)_+$.

In the right-moving part, the monodromy properties (2.7b) lead to the following mode expansions for the fermionic fields

$$\begin{aligned} \widetilde{\Psi}^2(\bar z) &= \sum_{r=1/2}^{\infty} \Big(\widetilde{\overline{\Psi}}2_{r+\nu_a}\, \bar z^{-r-\nu_a-\frac{1}{2}} + \widetilde{\Psi}2_{-r+\nu_a}\, \bar z^{r-\nu_a-\frac{1}{2}} \Big) \ , \\ \widetilde{\overline{\Psi}}^2(\bar z) &= \sum_{r=1/2}^{\infty} \Big(\widetilde{\Psi}2_{r-\nu_a}\, \bar z^{-r+\nu_a-\frac{1}{2}} + \widetilde{\overline{\Psi}}2_{-r-\nu_a}\, \bar z^{r+\nu_a-\frac{1}{2}} \Big) \ , \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \widetilde{\Psi}^3(\bar z) &= \sum_{r=1/2}^{\infty} \Big(\widetilde{\overline{\Psi}}3_{r-\nu_a}\, \bar z^{-r+\nu_a-\frac{1}{2}} + \widetilde{\Psi}3_{-r-\nu_a}\, \bar z^{r+\nu_a-\frac{1}{2}} \Big) \ , \\ \widetilde{\overline{\Psi}}^3(\bar z) &= \sum_{r=1/2}^{\infty} \Big(\widetilde{\Psi}3_{r+\nu_a}\, \bar z^{-r-\nu_a-\frac{1}{2}} + \widetilde{\overline{\Psi}}3_{-r+\nu_a}\, \bar z^{r-\nu_a-\frac{1}{2}} \Big) \ . \end{aligned} \tag{2.15}$$

[1] The oscillators $\Psi^2_{-\frac{1}{2}-\nu_a}$ and $\overline{\Psi}^3_{-\frac{1}{2}-\nu_a}$, instead, carry an energy $(\frac{1}{2}+\nu_a)$ and, upon acting on the twisted vacuum, they create massive states with $m^2 = 2\nu_a$.

[2] The reason to write the vertex operators at zero momentum is because, as in [1], ultimately we will be interested in describing a constant twisted closed string background to account for the continuous parameters of the GW surface defects.

The oscillators $\widetilde{\Psi^2}_{-r+\nu_a}$, $\widetilde{\overline{\Psi}^2}_{-r-\nu_a}$, $\widetilde{\Psi^3}_{-r-\nu_a}$ and $\widetilde{\overline{\Psi}^3}_{-r+\nu_a}$ are creation modes acting on the twisted vacuum of the right sector which we denote by $|\widetilde{\Omega}_a\rangle$. This is defined by

$$|\widetilde{\Omega}_a\rangle = \lim_{\bar{z}\to 0} \widetilde{\sigma}_a(\bar{z})\, \widetilde{s}_a(\bar{z})\, |\widetilde{0}\rangle \tag{2.16}$$

where $|\widetilde{0}\rangle$ is the Fock vacuum of this sector and

$$\widetilde{\sigma}_a(\bar{z}) = \widetilde{\sigma}^2_{1-\nu_a}(\bar{z})\, \widetilde{\sigma}^3_{\nu_a}(\bar{z}) \quad \text{and} \quad \widetilde{s}_a(\bar{z}) \ = \widetilde{s}^2_{-\nu_a}(\bar{z})\, \widetilde{s}^3_{\nu_a}(\bar{z}) \ . \tag{2.17}$$

The bosonic twist field $\widetilde{\sigma}_a(\bar{z})$ is a conformal field of weight $(1-\nu_a)\nu_a$ while the fermionic twist field $\widetilde{s}_a(\bar{z})$ is a conformal field of weight ν_a^2, so that the total conformal weight of the operator associated to $|\widetilde{\Omega}_a\rangle$ is ν_a. The right-moving ground state is then tachyonic with a mass given by (2.12) and it is removed by the GSO projection.

The first set of physical states in the GSO projected spectrum are those created by a fermionic creation mode with index $r = \frac{1}{2}$. In particular those generated by the oscillators $\widetilde{\Psi^2}_{-\frac{1}{2}+\nu_a}$ and $\widetilde{\overline{\Psi}^3}_{-\frac{1}{2}+\nu_a}$ are massless since the energy carried by these modes exactly cancels that of the vacuum. Therefore, the vertex operators at zero momentum associated to these right-moving massless excitations in the (-1)-superghost picture are:

$$\begin{aligned} \widetilde{\mathcal{V}}^1_a(\bar{z}) &= -\widetilde{\sigma}_a(\bar{z}) :\widetilde{\Psi^2}(\bar{z})\, \widetilde{s}_a(\bar{z}) \!: \mathrm{e}^{-\widetilde{\phi}(\bar{z})} \,, \\ \widetilde{\mathcal{V}}^2_a(\bar{z}) &= \widetilde{\sigma}_a(\bar{z}) :\widetilde{\overline{\Psi}^3}(\bar{z})\, \widetilde{s}_a(\bar{z}) \!: \mathrm{e}^{-\widetilde{\phi}(\bar{z})} \ . \end{aligned} \tag{2.18}$$

These are conformal fields of weight 1 and we collectively denote them as $\widetilde{\mathcal{V}}^\beta_a(\bar{z})$ with $\beta = 1, 2$. We point out that the $-$ sign in the first line above is introduced because in this way the two operators form a doublet transforming in the spinor representation of $\mathrm{SU}(2)_+$, as explained in appendix A.1.

2.2.2 Sectors with twist parameter $(1 - \nu_a) > \frac{1}{2}$

Apart from a few subtleties, the conclusions obtained in the previous subsection for the twisted sectors with $\nu_a < \frac{1}{2}$, are valid also in the twisted sectors with $(1-\nu_a) > \frac{1}{2}$ provided one exchanges the role of the complex directions 2 and 3, and uses the sector label $(M-a)$. Thus, we can rather brief in our presentation.

In the left-moving part, the fermionic creation modes are the oscillators $\Psi^2_{-r+\nu_a}$, $\overline{\Psi}^2_{-r-\nu_a}$, $\Psi^3_{-r-\nu_a}$ and $\overline{\Psi}^3_{-r+\nu_a}$ where r is a positive half-integer. They act on the twisted vacuum $|\Omega_{M-a}\rangle$ which is defined by

$$|\Omega_{M-a}\rangle = \lim_{z\to 0} \sigma_{M-a}(z)\, s_{M-a}(z)\, |0\rangle \tag{2.19}$$

with

$$\sigma_{M-a}(z) = \sigma^2_{1-\nu_a}(z)\, \sigma^3_{\nu_a}(z) \,, \quad \text{and} \quad s_{M-a}(z) = s^2_{-\nu_a}(z)\, s^3_{\nu_a}(z) \ . \tag{2.20}$$

The ground state $|\Omega_{M-a}\rangle$ is tachyonic with a mass given by (2.12) and is removed by the GSO projection. At the first excited level, instead, we find two massless states created by the oscillators $\Psi^2_{-\frac{1}{2}+\nu_a}$ and $\overline{\Psi}^3_{-\frac{1}{2}+\nu_a}$, which correspond to the following vertex operators at zero momentum:

$$\begin{aligned}\mathcal{V}^1_{M-a}(z) &= -\sigma_{M-a}(z) :\Psi^2(z)\, s_{M-a}(z):\mathrm{e}^{-\phi(z)}\,,\\ \mathcal{V}^2_{M-a}(z) &= \sigma_{M-a}(z) :\overline{\Psi}^3(z)\, s_{M-a}(z):\mathrm{e}^{-\phi(z)}\;.\end{aligned} \tag{2.21}$$

These are conformal fields of weight 1 which we collectively denote as $\mathcal{V}^\alpha_{M-a}(z)$ with $\alpha = 1,2$. Again the $-$ sign in the first line is inserted so that these two operators transform as a doublet in the spinor representation of $\mathrm{SU}(2)_+$ (see appendix A.1).

Finally, in the right-moving part the oscillators $\widetilde{\Psi}2_{-r-\nu_a}$, $\widetilde{\overline{\Psi}}2_{-r+\nu_a}$, $\widetilde{\Psi}3_{-r+\nu_a}$ and $\widetilde{\overline{\Psi}}3_{-r-\nu_a}$ where r is a positive half-integer, are creation modes. They act on the twisted vacuum defined by

$$|\widetilde{\Omega}_{M-a}\rangle = \lim_{\bar{z}\to 0} \widetilde{\sigma}_{M-a}(\bar{z})\, \widetilde{s}_{M-a}(\bar{z})\,|0\rangle \tag{2.22}$$

where

$$\widetilde{\sigma}_{M-a}(\bar{z}) = \widetilde{\sigma}^2_{\nu_a}(\bar{z})\,\widetilde{\sigma}^3_{1-\nu_a}(\bar{z})\,,\quad \text{and}\quad \widetilde{s}_{M-a}(\bar{z}) = \widetilde{s}^2_{\nu_a}(\bar{z})\,\widetilde{s}^3_{-\nu_a}(\bar{z})\;. \tag{2.23}$$

As before this vacuum state is tachyonic and removed by the GSO projection. On the other hand, the states created by $\widetilde{\overline{\Psi}}2_{-\frac{1}{2}+\nu_a}$ and $\widetilde{\Psi}3_{-\frac{1}{2}+\nu_a}$ are massless and selected by the GSO projection. They correspond to the following vertex operators at zero momentum:

$$\begin{aligned}\widetilde{\mathcal{V}}^1_{M-a}(\bar{z}) &= \widetilde{\sigma}_{M-a}(\bar{z}) :\widetilde{\Psi}^3(\bar{z})\, \widetilde{s}_{M-a}(\bar{z}):\mathrm{e}^{-\widetilde{\phi}(\bar{z})}\,,\\ \widetilde{\mathcal{V}}^2_{M-a}(\bar{z}) &= \widetilde{\sigma}_{M-a}(\bar{z}) :\widetilde{\overline{\Psi}}^2(\bar{z})\, \widetilde{s}_{M-a}(\bar{z}):\mathrm{e}^{-\widetilde{\phi}(\bar{z})}\,,\end{aligned} \tag{2.24}$$

which are conformal fields of weight 1. We collectively denote these vertex operators as $\widetilde{\mathcal{V}}^\beta_{M-a}(\bar{z})$ with $\beta = 1,2$, since they transform as a doublet of $\mathrm{SU}(2)_+$ (see appendix A.1).

We summarize our results on the massless vertex operators of the twisted NS sectors in table 1 below.

2.2.3 Two-point functions in the twisted NS sectors

Given the explicit form of the vertex operators that we have derived, it is rather straightforward to compute their two-point functions. As a first step, we observe that there are no non-vanishing correlators between left (or right) operators of the same twisted sector, due to the presence of the bosonic twist fields; in fact for any complex direction j one has [14]

$$\big\langle \sigma^j_{\nu_a}(z_1)\,\sigma^j_{\nu_b}(z_2)\big\rangle = \frac{\delta_{\nu_b,1-\nu_a}}{(z_1-z_2)^{\nu_a(1-\nu_a)}}\,, \tag{2.25}$$

and similarly in the right sector. This implies that only the correlator $\langle\sigma_a(w_1)\,\sigma_{M-a}(w_2)\rangle$ is non vanishing. Therefore, only the two-point functions between vertex operators in sectors a and $(M-a)$ are non-zero. Another important point to consider is that these vertex operators inherit the fermionic statistics from the fermionic fields that are present in their definitions.

Vertex operator	State
$\mathcal{V}^1_a(z) = \sigma_a(z)\ :\Psi^3(z)\, s_a(z)\!: \mathrm{e}^{-\phi(z)}$	$\Psi^3_{-\frac{1}{2}+\nu_a}\,\vert\Omega_a\rangle_{(-1)}$
$\mathcal{V}^2_a(z) = \sigma_a(z)\ :\overline{\Psi}^2(z)\, s_a(z)\!: \mathrm{e}^{-\phi(z)}$	$\overline{\Psi}^2_{-\frac{1}{2}+\nu_a}\,\vert\Omega_a\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^1_a(\bar{z}) = -\widetilde{\sigma}_a(\bar{z})\ :\widetilde{\Psi^2}(\bar{z})\, \widetilde{s}_a(\bar{z})\!: \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$-\widetilde{\Psi^2}_{-\frac{1}{2}+\nu_a}\,\vert\widetilde{\Omega}_a\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^2_a(\bar{z}) = \widetilde{\sigma}_a(\bar{z})\ :\widetilde{\overline{\Psi}^3}(\bar{z})\, \widetilde{s}_a(\bar{z})\ : \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$\widetilde{\overline{\Psi}^3}_{-\frac{1}{2}+\nu_a}\,\vert\widetilde{\Omega}_a\rangle_{(-1)}$
$\mathcal{V}^1_{M-a}(z) = -\sigma_{M-a}(z)\ :\Psi^2(z)\, s_{M-a}(z)\!: \mathrm{e}^{-\phi(z)}$	$-\Psi^2_{-\frac{1}{2}+\nu_a}\,\vert\Omega_{M-a}\rangle_{(-1)}$
$\mathcal{V}^2_{M-a}(z) = \sigma_{M-a}(z)\ :\overline{\Psi}^{\,3}(z)\, s_{M-a}(z)\!: \mathrm{e}^{-\phi(z)}$	$\overline{\Psi}^3_{-\frac{1}{2}+\nu_a}\,\vert\Omega_{M-a}\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^1_{M-a}(\bar{z}) = \widetilde{\sigma}_{M-a}(\bar{z})\ :\widetilde{\Psi^3}(\bar{z})\, \widetilde{s}_{M-a}(\bar{z})\!: \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$\widetilde{\Psi^3}_{-\frac{1}{2}+\nu_a}\,\vert\widetilde{\Omega}_{M-a}\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^2_{M-a}(\bar{z}) = \widetilde{\sigma}_{M-a}(\bar{z})\ :\widetilde{\overline{\Psi}^2}(\bar{z})\, \widetilde{s}_{M-a}(\bar{z})\!: \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$\widetilde{\overline{\Psi}^2}_{-\frac{1}{2}+\nu_a}\,\vert\widetilde{\Omega}_{M-a}\rangle_{(-1)}$

Table 1. The vertex operators and the corresponding states in the left- and right-moving parts of the various twisted NS sectors. Here the label a takes values in the range $\left[1, \frac{M-1}{2}\right]$, and in the last column the subscript (-1) on the kets identifies the superghost picture.

Let us then compute the two-point function between $\mathcal{V}^1_a$ and $\mathcal{V}^2_{M-a}$. Using (2.25) and the basic conformal field theory correlators

$$\begin{aligned}\Big\langle :\Psi^3(z_1)\, s_a(z_1)\!: :\overline{\Psi}^3(z_2)\, s_{M-a}(z_2)\!:\Big\rangle &= \frac{1}{(z_1-z_2)^{1-2\nu_a(1-\nu_a)}}\,,\\ \Big\langle \mathrm{e}^{-\phi(z_1)}\, \mathrm{e}^{-\phi(z_2)}\Big\rangle &= \frac{1}{z_1-z_2}\,,\end{aligned} \tag{2.26}$$

we obtain

$$\Big\langle \mathcal{V}^1_a(z_1)\, \mathcal{V}^2_{M-a}(z_2)\Big\rangle = \frac{1}{(z_1-z_2)^2}\ . \tag{2.27}$$

In a similar way, using

$$\Big\langle :\overline{\Psi}^2(z_1)\, s_a(z_1)\!: :\Psi^2(z_2)\, s_{M-a}(z_2)\!:\Big\rangle = \frac{1}{(z_1-z_2)^{1-2\nu_a(1-\nu_a)}}\,, \tag{2.28}$$

and taking into account the explicit negative sign in $\mathcal{V}^1_{M-a}$, we get

$$\Big\langle \mathcal{V}^2_a(z_1)\, \mathcal{V}^1_{M-a}(z_2)\Big\rangle = \frac{-1}{(z_1-z_2)^2}\ . \tag{2.29}$$

Furthermore, the two-point functions between $\mathcal{V}^1_a$ and $\mathcal{V}^1_{M-a}$ and between $\mathcal{V}^2_a$ and $\mathcal{V}^2_{M-a}$ vanish since their fermionic charges do not match. Thus, altogether, we have

$$\Big\langle \mathcal{V}^\alpha_a(z_1)\, \mathcal{V}^\beta_{M-a}(z_2)\Big\rangle = \frac{(\epsilon^{-1})^{\alpha\beta}}{(z_1-z_2)^2} \tag{2.30}$$

where we have defined

$$\epsilon = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\ . \tag{2.31}$$

By taking into account the fermionic statistics of the vertex operators and the anti-symmetry of ϵ, we also find

$$\left\langle \mathcal{V}^{\alpha}_{M-a}(z_1)\, \mathcal{V}^{\beta}_{a}(z_2)\right\rangle = \frac{(\epsilon^{-1})^{\alpha\beta}}{(z_1-z_2)^2}\ . \tag{2.32}$$

Notice that (2.30) and (2.32) may be unified in a single formula by promoting the index a to the complete index $\widehat{a}$. This shows that despite the differences in the structure of the states and vertex operators in the fermionic formalism, all twisted sectors are actually treated on equal footing.

Similarly, in the right-moving sector, we obtain

$$\left\langle \widetilde{\mathcal{V}}^{\alpha}_{M-a}(\bar z_1)\, \widetilde{\mathcal{V}}^{\beta}_{a}(\bar z_2)\right\rangle = \left\langle \widetilde{\mathcal{V}}^{\alpha}_{a}(\bar z_1)\, \widetilde{\mathcal{V}}^{\beta}_{M-a}(\bar z_2)\right\rangle = \frac{(\epsilon^{-1})^{\alpha\beta}}{(\bar z_1-\bar z_2)^2}\ . \tag{2.33}$$

From these two-point functions it is possible to infer the conjugate vertex operators as follows:

$$\begin{aligned} \big(\mathcal{V}_{M-a}(z)\big)^{\dagger}_{\alpha} &= \mathcal{V}^{\beta}_{a}(z)\,\epsilon_{\beta\alpha}\ , & \big(\mathcal{V}_{a}(z)\big)^{\dagger}_{\alpha} &= \mathcal{V}^{\beta}_{M-a}(z)\,\epsilon_{\beta\alpha}\,, \\ \big(\widetilde{\mathcal{V}}_{a}(z)\big)^{\dagger}_{\alpha} &= \widetilde{V}^{\beta}_{M-a}(z)\,\epsilon_{\beta\alpha}\ , & \big(\widetilde{\mathcal{V}}_{M-a}(z)\big)^{\dagger}_{\alpha} &= \widetilde{V}^{\beta}_{a}(z)\,\epsilon_{\beta\alpha}\ . \end{aligned} \tag{2.34}$$

2.3 The massless NS/NS vertex operators

The massless closed string excitations in the twisted NS/NS sectors are obtained by combining the left- and right-moving massless states that we have obtained in the previous subsection. In the sectors with twist parameter $\nu_a < \frac{1}{2}$, they are then described by the following vertex operators at zero momentum

$$b^{(a)}_{\alpha\beta}\ \mathcal{V}^{\alpha}_{a}(z)\,\widetilde{\mathcal{V}}^{\beta}_{a}(\bar z) \tag{2.35}$$

where $b^{(a)}_{\alpha\beta}$ are four constant complex fields.

Similarly, in the sectors with twist parameter $(1-\nu_a) > \frac{1}{2}$, the massless closed string excitations are described by the vertex operators at zero momentum

$$b^{(M-a)}_{\alpha\beta}\ \mathcal{V}^{\alpha}_{M-a}(z)\,\widetilde{\mathcal{V}}^{\beta}_{M-a}(\bar z) \tag{2.36}$$

where again $b^{(M-a)}_{\alpha\beta}$ are four constant complex fields.

The constants $b^{(a)}$ and $b^{(M-a)}$ can be considered as a background in which the string theory on the orbifold is defined. Given the structure of the vertex operators there are non-trivial relations among them. In particular, using (2.34) one finds that

$$\Big(b^{(a)}_{\alpha\beta}\,\mathcal{V}^{\alpha}_{a}(z)\,\widetilde{\mathcal{V}}^{\beta}_{a}(\bar z)\Big)^{\dagger} = b^{(M-a)}_{\alpha\beta}\ \mathcal{V}^{\alpha}_{M-a}(z)\,\widetilde{\mathcal{V}}^{\beta}_{M-a}(\bar z) \tag{2.37}$$

where

$$b^{(M-a)}_{11} = -b^{(a)\,\star}_{22}\,,\quad b^{(M-a)}_{12} = b^{(a)\,\star}_{21}\,,\quad b^{(M-a)}_{21} = b^{(a)\,\star}_{12}\,,\quad b^{(M-a)}_{22} = -b^{(a)\,\star}_{11}\,, \tag{2.38}$$

or, equivalently in matrix notation,

$$b^{(M-a)} = \epsilon\, b^{(a)\,\star}\,\epsilon\ . \tag{2.39}$$

These relations, which also appear in [9], show that if one turns on background values for the closed string fields in the twisted sector a, one also turns on background values for the fields in the twisted sector $(M-a)$ and *viceversa*, in such a way that the total background configuration is real.

2.4 Twisted R sectors

The $\mathbb{Z}_M$ orbifold (1.1) breaks the isometry of the ten-dimensional space as follows:

$$\mathrm{SO}(10) \longrightarrow \mathrm{SO}(6) \times \mathrm{SO}(2) \times \mathrm{SO}(2)\,, \tag{2.40}$$

where SO(6) acts on the first, fourth and fifth complex directions, which are not affected by the orbifold action. Correspondingly, the untwisted vacuum of the R sector which carries the 32-dimensional spinor representation of SO(10) decomposes into eight massless spinors of SO(6). Four of these are chiral and four anti-chiral. We denote the four chiral vacuum states by

$$\left|A, \pm\frac{1}{2}, \pm\frac{1}{2}\right\rangle \tag{2.41}$$

where $A \in \mathbf{4}$ labels the four different components of the chiral spinor representation of SO(6) and the four pairs of $\pm\frac{1}{2}$ denote the spinor weights along the second and third complex directions where the orbifold acts. Similarly, the four anti-chiral vacuum states are denoted by

$$\left|\dot{A}, \pm\frac{1}{2}, \pm\frac{1}{2}\right\rangle \tag{2.42}$$

where $\dot{A} \in \bar{\mathbf{4}}$ spans the four-dimensional anti-chiral spinor representation of SO(6).

In the twisted R sectors, not all such chiral and anti-chiral states remain massless. Indeed, the fermionic twist fields change the spinor weights in the orbifolded directions, so that conformal dimensions and the GSO parities of the corresponding vertex operators are modified. In the following we present a brief description of the spectrum in the various twisted R sectors, focusing on the massless excitations.

2.4.1 Sectors with twist parameter $\nu_a < \frac{1}{2}$

In these sectors the left-moving bosonic and fermionic twist fields σ_a and s_a are given in (2.11). When we act with s_a on the states (2.41) and (2.42), the charges in the directions 2 and 3 become

$$\varepsilon_2 = \pm\frac{1}{2} + \nu_a \quad \text{and} \quad \varepsilon_3 = \pm\frac{1}{2} - \nu_a \tag{2.43}$$

depending on their initial values. Because of this, not all choices of signs lead to massless configurations. In fact, the mass vanishes only if

$$\varepsilon_2^2 = \varepsilon_3^2 = \left(\frac{1}{2} - \nu_a\right)^2 . \tag{2.44}$$

Combining this with (2.43), we see that the only solution is

$$\varepsilon_2 = -\varepsilon_3 = -\frac{1}{2} + \nu_a\,, \tag{2.45}$$

so that, instead of $s_a(z)$, we can consider the effective fermionic twist

$$r_a(z) = s^2_{\nu_a-\frac{1}{2}}(z)\, s^3_{-\nu_a+\frac{1}{2}}(z) \tag{2.46}$$

which is a conformal field of weight $\left(\frac{1}{4} - \nu_a(1-\nu_a)\right)$.

In the R sector, there are two fundamental superghost pictures that one considers: the $(-\frac{1}{2})$- and the $(-\frac{3}{2})$-pictures [5]. Enforcing the GSO projection, in the $(-\frac{1}{2})$-picture one selects the chiral spinor of SO(6), while in the $(-\frac{3}{2})$-picture one selects the anti-chiral one. In this way, in fact, the sum of the spinor weights minus the superghost-charge is always an even integer. Thus we are led to introduce the following two vertex operators at zero momentum

$$\mathcal{V}^A_a(z) = \sigma_a(z)\, r_a(z)\, S^A(z) \mathrm{e}^{-\frac{1}{2}\phi(z)}\,, \tag{2.47a}$$

$$\mathcal{V}^{\dot{A}}_a(z) = \sigma_a(z)\, r_a(z)\, S^{\dot{A}}(z) \mathrm{e}^{-\frac{3}{2}\phi(z)}\,, \tag{2.47b}$$

where S^A and $S^{\dot{A}}$ are, respectively, the chiral and anti-chiral spin-fields of SO(6) [5, 6]. Both vertex operators are conformal fields of weight 1 and define the following massless twisted vacuum states:

$$\begin{aligned} |A_a\rangle_{(-\frac{1}{2})} &= \lim_{z\to 0} \mathcal{V}^A_a(z)\, |0\rangle\,, \\ |\dot{A}_a\rangle_{(-\frac{3}{2})} &= \lim_{z\to 0} \mathcal{V}^{\dot{A}}_a(z)\, |0\rangle\ . \end{aligned} \tag{2.48}$$

As far as the right-moving part is concerned, the bosonic and fermionic twist fields are given in (2.17). Therefore, we can repeat the previous analysis by simply replacing everywhere ν_a with $(1-\nu_a)$. In this way we find the following two physical vertex operators of weight 1:

$$\widetilde{\mathcal{V}}^A_a(\bar{z}) = \widetilde{\sigma}_a(\bar{z})\, \widetilde{r}_a(\bar{z})\, \widetilde{S}^A(\bar{z})\, \mathrm{e}^{-\frac{1}{2}\widetilde{\phi}(\bar{z})}\,, \tag{2.49a}$$

$$\widetilde{\mathcal{V}}^{\dot{A}}_a(\bar{z}) = \widetilde{\sigma}_a(\bar{z})\, \widetilde{r}_a(\bar{z})\, \widetilde{S}^{\dot{A}}(\bar{z})\, \mathrm{e}^{-\frac{3}{2}\widetilde{\phi}(\bar{z})}\,, \tag{2.49b}$$

where the effective fermionic twist is given by

$$\widetilde{r}_a(\bar{z}) = \widetilde{s}^2_{-\nu_a+\frac{1}{2}}(\bar{z})\, \widetilde{s}^3_{\nu_a-\frac{1}{2}}(\bar{z})\ . \tag{2.50}$$

The massless states corresponding to these vertex operators are

$$\begin{aligned} |\widetilde{A}_a\rangle_{(-\frac{1}{2})} &= \lim_{\bar{z}\to 0} \widetilde{\mathcal{V}}^A_a(\bar{z})\, |0\rangle\,, \\ |\widetilde{\dot{A}}_a\rangle_{(-\frac{3}{2})} &= \lim_{\bar{z}\to 0} \widetilde{\mathcal{V}}^{\dot{A}}_a(\bar{z})\, |0\rangle\ . \end{aligned} \tag{2.51}$$

2.4.2 Sectors with twist parameter $(1 - \nu_a) > \frac{1}{2}$

These sectors can be described in the same manner as before by simply exchanging the roles of the complex directions 2 and 3, and using $(M - a)$ as twist label. Thus, we merely

Vertex operator	State
$\mathcal{V}^{A}_{a}(z)=\sigma_a(z)\,r_a(z)\,S^{A}(z)\mathrm{e}^{-\frac{1}{2}\phi(z)}$	$\vert A_a\rangle_{(-\frac{1}{2})}$
$\mathcal{V}^{\dot{A}}_{a}(z)=\sigma_a(z)\,r_a(z)\,S^{\dot{A}}(z)\mathrm{e}^{-\frac{3}{2}\phi(z)}$	$\vert \dot{A}_a\rangle_{(-\frac{3}{2})}$
$\widetilde{\mathcal{V}}^{A}_{a}(\bar{z})=\widetilde{\sigma}_a(\bar{z})\,\widetilde{r}_a(\bar{z})\,\widetilde{S}^{A}(\bar{z})\,\mathrm{e}^{-\frac{1}{2}\widetilde{\phi}(\bar{z})}$	$\vert \widetilde{A}_a\rangle_{(-\frac{1}{2})}$
$\widetilde{\mathcal{V}}^{\dot{A}}_{a}(\bar{z})=\widetilde{\sigma}_a(\bar{z})\,\widetilde{r}_a(\bar{z})\,\widetilde{S}^{\dot{A}}(\bar{z})\,\mathrm{e}^{-\frac{3}{2}\widetilde{\phi}(\bar{z})}$	$\vert \widetilde{\dot{A}}_a\rangle_{(-\frac{3}{2})}$
$\mathcal{V}^{A}_{M-a}(z)=\sigma_{M-a}(z)\,r_{M-a}(z)\,S^{A}(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}$	$\vert A_{M-a}\rangle_{(-\frac{1}{2})}$
$\mathcal{V}^{\dot{A}}_{M-a}(z)=\sigma_{M-a}(z)\,r_{M-a}(z)\,S^{\dot{A}}(z)\,\mathrm{e}^{-\frac{3}{2}\phi(z)}$	$\vert \dot{A}_{M-a}\rangle_{(-\frac{3}{2})}$
$\widetilde{\mathcal{V}}^{A}_{M-a}(\bar{z})=\widetilde{\sigma}_{M-a}(\bar{z})\,\widetilde{r}_{M-a}(\bar{z})\,\widetilde{S}^{A}(\bar{z})\,\mathrm{e}^{-\frac{1}{2}\widetilde{\phi}(\bar{z})}$	$\vert \widetilde{A}_{M-a}\rangle_{(-\frac{1}{2})}$
$\widetilde{\mathcal{V}}^{\dot{A}}_{M-a}(\bar{z})=\widetilde{\sigma}_{M-a}(\bar{z})\,\widetilde{r}_{M-a}(\bar{z})\,\widetilde{S}^{\dot{A}}(\bar{z})\,\mathrm{e}^{-\frac{3}{2}\widetilde{\phi}(\bar{z})}$	$\vert \widetilde{\dot{A}}_{M-a}\rangle_{(-\frac{3}{2})}$

Table 2. The vertex operators and the corresponding states in the left- and right-moving parts of the twisted R sectors.

present the physical GSO projected massless vertex operators at zero momentum. In the left-moving part they are

$$\mathcal{V}^{A}_{M-a}(z)=\sigma_{M-a}(z)\,r_{M-a}(z)\,S^{A}(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}\,, \tag{2.52a}$$

$$\mathcal{V}^{\dot{A}}_{M-a}(z)=\sigma_{M-a}(z)\,r_{M-a}(z)\,S^{\dot{A}}(z)\,\mathrm{e}^{-\frac{3}{2}\phi(z)}\,, \tag{2.52b}$$

with

$$r_{M-a}(z)=s^{2}_{-\nu_a+\frac{1}{2}}(z)\,s^{3}_{\nu_a-\frac{1}{2}}(z)\,. \tag{2.53}$$

In the right-moving part, instead, they are

$$\widetilde{\mathcal{V}}^{A}_{M-a}(\bar{z})=\widetilde{\sigma}_{M-a}(\bar{z})\,\widetilde{r}_{M-a}(\bar{z})\,\widetilde{S}^{A}(\bar{z})\,\mathrm{e}^{-\frac{1}{2}\widetilde{\phi}(\bar{z})}\,, \tag{2.54a}$$

$$\widetilde{\mathcal{V}}^{\dot{A}}_{M-a}(\bar{z})=\widetilde{\sigma}_{M-a}(\bar{z})\,\widetilde{r}_{M-a}(\bar{z})\,\widetilde{S}^{\dot{A}}(\bar{z})\,\mathrm{e}^{-\frac{3}{2}\widetilde{\phi}(\bar{z})}\,. \tag{2.54b}$$

where

$$\widetilde{r}_{M-a}(\bar{z})=\widetilde{s}^{2}_{\nu_a-\frac{1}{2}}(\bar{z})\,\widetilde{s}^{3}_{-\nu_a+\frac{1}{2}}(\bar{z})\,. \tag{2.55}$$

When acting on the Fock vacuum these vertex operators create the twisted ground states which have the same expressions as in (2.48) and (2.51) with the obvious changes in notation.

We summarize our findings in table 2 below.

2.4.3 Two-point functions in the twisted R sectors

As we have seen in the twisted NS sectors, the only non-vanishing two-point functions necessarily involve the left-moving (or right-moving) vertex operators in complementary sectors a and $(M-a)$, because of the two-point functions (2.25). Of course, the same is

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true in the twisted R sectors. Furthermore, in order to soak up the background charge in the superghost sector, only the overlaps between states in the $(-\frac{1}{2})$- and $(-\frac{3}{2})$-pictures, or *viceversa*, are non-zero. Taking this into account and using standard results from conformal field theory, we find

$$\begin{aligned}\left\langle \mathcal{V}_a^A(z_1)\, \mathcal{V}_{M-a}^{\dot{B}}(z_2)\right\rangle &= \left\langle \mathcal{V}_{M-a}^A(z_1)\, \mathcal{V}_a^{\dot{B}}(z_2)\right\rangle = \frac{(C^{-1})^{A\dot{B}}}{(z_1-z_2)^2}\,,\\ \left\langle \mathcal{V}_a^{\dot{A}}(z_1)\, \mathcal{V}_{M-a}^{B}(z_2)\right\rangle &= \left\langle \mathcal{V}_{M-a}^{\dot{A}}(z_1)\, \mathcal{V}_a^{B}(z_2)\right\rangle = \frac{(C^{-1})^{\dot{A}B}}{(z_1-z_2)^2}\,,\end{aligned} \tag{2.56}$$

where C is the charge conjugation matrix of SO(6) (see appendix A.2). Of course, analogous correlators hold for the right-moving vertex operators.

From the first line of (2.56), we read the following conjugation rules

$$\begin{aligned}\big(\mathcal{V}_{M-a}(z)\big)^\dagger_{\dot{B}} &= \mathcal{V}_a^A(z)\, C_{A\dot{B}}\,,\\ \big(\mathcal{V}_{a}(z)\big)^\dagger_{\dot{B}} &= \mathcal{V}_{M-a}^A(z)\, C_{A\dot{B}}\,,\end{aligned} \tag{2.57}$$

while from the second line we obtain the same relations with dotted and undotted indices exchanged. The same formulas apply also for the right-moving vertices.

2.5 The massless R/R vertex operators

The massless closed string excitations in the twisted R/R sectors are obtained by combining left and right movers. We shall work with the asymmetric superghost pictures $(-\frac{1}{2},-\frac{3}{2})$ or $(-\frac{3}{2},-\frac{1}{2})$, so that the corresponding closed string fields are R/R potentials. In the twisted sector labeled by a we choose the $(-\frac{1}{2},-\frac{3}{2})$-picture and write the following massless vertex operators at zero momentum:[3]

$$\mathcal{C}^{(a)}_{A\dot{B}}\, \mathcal{V}_a^A(z)\ \widetilde{\mathcal{V}}_a^{\dot{B}}(\bar{z}) \tag{2.58}$$

where $\mathcal{C}^{(a)}_{A\dot{B}}$ are sixteen constant complex fields. These constants can be considered as a background in which the orbifold closed string theory is defined.

In the twisted sector labeled by $(M-a)$ we choose, instead, the other asymmetric superghost picture, namely the $(-\frac{3}{2},-\frac{1}{2})$-picture, and consider the following massless vertex operators

$$\mathcal{C}^{(M-a)}_{\dot{A}B}\, \mathcal{V}_{M-a}^{\dot{A}}(z)\ \widetilde{\mathcal{V}}_{M-a}^{B}(\bar{z}) \tag{2.59}$$

where $\mathcal{C}^{(M-a)}_{A\dot{B}}$ are other sixteen constant complex fields contributing to the background in which the closed string propagates.

Notice that in writing the vertex operators (2.58) and (2.59) for the twisted R/R potentials, we have correlated the choice of picture numbers with the twisted sector. Of

[3]In [15] it is shown that the complete BRST invariant vertex operators in the asymmetric superghost pictures are an infinite sum of terms characterized by the number of superghost zero modes. For our purposes, however, only the first (and simplest) terms in these sums is relevant since all the others decouple and thus can be discarded.

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course we could have made different choices, but they would lead to the same results. In fact, it is well-known that in a BRST invariant framework like ours, the way in which the superghost pictures are distributed is completely arbitrary, provided one satisfies the global constraints due to the presence of a background charge, and that the physical results do not depend on this choice. However, our picture assignment is particularly convenient because it immediately implies that the R/R potentials in the a-th twisted sector are naturally related to those in the sector $(M-a)$ by complex conjugation, exactly as it happens in the twisted NS/NS sectors. Indeed, we have

$$\Big(\mathcal{C}^{(a)}_{A\dot{B}}\,\mathcal{V}^{A}_{a}(z)\,\widetilde{\mathcal{V}}^{\dot{B}}_{a}(\bar{z})\Big)^{\dagger}=\mathcal{C}^{(M-a)}_{\dot{A}B}\,\mathcal{V}^{\dot{A}}_{M-a}(z)\,\widetilde{\mathcal{V}}^{B}_{M-a}(\bar{z})\,, \tag{2.60}$$

where, in matrix notation,

$$\mathcal{C}^{(M-a)}=C\,\mathcal{C}^{(a)\star}\,C\,, \tag{2.61}$$

which is the strict analogue of (2.39) holding in the NS/NS sectors. We therefore see that by turning on a R/R background potential value in the twisted sector a, one also turns on a background R/R potential in the twisted sector $(M-a)$ and *viceversa*, in such a way that the total configuration is real.

3 Fractional D3-branes in the $\mathbb{Z}_M$ orbifold

We now turn to discuss the open strings in the $\mathbb{Z}_M$ orbifold with the aim of analyzing surface defects in $4d$ gauge theories engineered on stacks of (fractional) D3-branes. As is well-known, a D-brane introduces a boundary on the string world-sheet where non-trivial relations between the left and the right movers of the closed strings take place. We will investigate these relations using the boundary state formalism (for a review, see for example [7, 8]) and then will analyze the massless open string spectrum on the brane world-volume. Since our ultimate goal is to recover a string theory description of the surface defects in a $4d$ gauge theory, we place the (fractional) D3-branes in such a way that they are partially extended along the orbifold as originally proposed in [4]. More precisely, we take the D3-brane world-volume to be $\mathbb{C}_{(1)}\times\mathbb{C}_{(2)}$ in such a way that the orbifold action breaks the $4d$ Poincaré symmetry leaving unbroken the one in the first complex direction along which the surface defect is extended.

3.1 Boundary states and reflection rules

In the $\mathbb{Z}_M$ orbifold there are M different types of fractional D-branes, labeled by an index $I=0,1,\ldots,M-1$, corresponding to the M irreducible representations of $\mathbb{Z}_M$. A fractional D3-brane of type I can be described by a boundary state which contains an untwisted component $|U\rangle$, which is the same for all types of branes, and a twisted component $|T;I\rangle$, which depends on the type of brane considered:

$$|\mathrm{D3};I\rangle=\mathcal{N}\,|U\rangle+\mathcal{N}'\,|T;I\rangle \tag{3.1}$$

where $\mathcal{N}$ and $\mathcal{N}'$ are appropriate normalization factors related to the brane tensions (whose explicit expression is not relevant for our purposes). This schematic structure holds of

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course both in the NS/NS and in the R/R sectors, which we now discuss in turn, focusing on the fermionic twisted components.

3.1.1 NS/NS sector

The twisted component of the boundary state for a fractional D3-brane of type I is a sum of $(M-1)$ terms which refer to the $(M-1)$ twisted sectors of the closed strings on the orbifold and whose coefficients have to be chosen in a specific way in order to have a consistent description of the D-brane. By this we mean that the cylinder amplitude between two such boundary states, once translated into the open string channel, must correctly reproduce the $\mathbb{Z}_M$-invariant one-loop annulus amplitude. In [10] a thorough analysis of this issue was carried out in general, using the Cardy condition for the construction of consistent boundary states in rational conformal field theories [16]. Borrowing these results and adapting them to our case, we can write the twisted component of the boundary state for a D3-brane of type I in the NS/NS sector and its conjugate as follows:

$$
\begin{aligned}
|T;I|\rangle_{\mathrm{NS}} &= \sum_{\widehat{a}=1}^{M-1} \sin\left(\frac{\pi\widehat{a}}{M}\right)\omega^{I\,\widehat{a}}\,|\,\widehat{a}\,\rangle\!\rangle_{\mathrm{NS}}\,, \\
{}_{\mathrm{NS}}\langle T;I| &= \sum_{\widehat{a}=1}^{M-1} \sin\left(\frac{\pi\widehat{a}}{M}\right)\omega^{-I\,\widehat{a}}\,{}_{\mathrm{NS}}\langle\!\langle\,\widehat{a}\,|\;.
\end{aligned}
\tag{3.2}
$$

Here, the sum runs over all twisted sectors, ω is the M-th root of unity as in (2.3) and $|\,\widehat{a}\,\rangle\!\rangle_{\mathrm{NS}}$ is the GSO projected Ishibashi state for the twisted sector $\widehat{a}$. These Ishibashi states enforce the appropriate gluing conditions between the left-moving and right-moving modes. For our purposes it is not necessary to write the complete expression of these Ishibashi states, but it is enough to write the terms which may have a non-zero overlap with the massless states of the closed string twisted sectors discussed in section 2.

Let us suppose again that M is odd. If $\widehat{a}=a\in[1,\frac{M-1}{2}]$, we have

$$
|a\rangle\!\rangle_{\mathrm{NS}} = \Big(\mathrm{i}\,\overline{\Psi}{}^{2}_{-\frac{1}{2}+\nu_a}\,\widetilde{\overline{\Psi}}{}^{2}_{-\frac{1}{2}+\nu_a} - \mathrm{i}\,\Psi^{3}_{-\frac{1}{2}+\nu_a}\,\widetilde{\Psi}{}^{3}_{-\frac{1}{2}+\nu_a}\Big)|\Omega_a\rangle_{(-1)}\;|\widetilde{\Omega}_a\rangle_{(-1)} + \cdots \tag{3.3}
$$

where the ellipses stand for terms involving a higher number of oscillators or massive fermionic modes. The relative minus sign in the brackets of (3.3) is due to the fact that the complex direction 3 is transverse to the D3-brane while the complex direction 2 is longitudinal. If $\widehat{a}=(M-a)$, instead, we have

$$
|M-a\rangle\!\rangle_{\mathrm{NS}} = \Big(\mathrm{i}\,\Psi^{2}_{-\frac{1}{2}+\nu_a}\,\widetilde{\Psi}{}^{2}_{-\frac{1}{2}+\nu_a} - \mathrm{i}\,\overline{\Psi}{}^{3}_{-\frac{1}{2}+\nu_a}\,\widetilde{\overline{\Psi}}{}^{3}_{-\frac{1}{2}+\nu_a}\Big)|\Omega_{M-a}\rangle_{(-1)}\;|\widetilde{\Omega}_{M-a}\rangle_{(-1)} + \cdots\,. \tag{3.4}
$$

The corresponding Ishibashi bra states are

$$
\begin{aligned}
{}_{\mathrm{NS}}\langle\!\langle a| &= {}_{(-1)}\langle\widetilde{\Omega}_a|\;{}_{(-1)}\langle\Omega_a|\Big(-\mathrm{i}\,\widetilde{\Psi}{}^{2}_{\frac{1}{2}-\nu_a}\,\overline{\Psi}{}^{2}_{\frac{1}{2}-\nu_a} + \mathrm{i}\,\widetilde{\overline{\Psi}}{}^{3}_{\frac{1}{2}-\nu_a}\,\Psi^{3}_{\frac{1}{2}-\nu_a}\Big) + \cdots \\
{}_{\mathrm{NS}}\langle\!\langle M-a| &= {}_{(-1)}\langle\widetilde{\Omega}_{M-a}|\;{}_{(-1)}\langle\Omega_{M-a}|\Big(-\mathrm{i}\,\widetilde{\overline{\Psi}}{}^{2}_{\frac{1}{2}-\nu_a}\,\Psi^{2}_{\frac{1}{2}-\nu_a} + \mathrm{i}\,\widetilde{\Psi}{}^{3}_{\frac{1}{2}-\nu_a}\,\overline{\Psi}{}^{3}_{\frac{1}{2}-\nu_a}\Big) + \cdots
\end{aligned}
\tag{3.5}
$$

where the conjugate vacuum states are normalized in such a way that

$$
{}_{(-1)}\langle\Omega_a|\Omega_a\rangle_{(-1)} = 1 \quad\text{and}\quad {}_{(-1)}\langle\Omega_{M-a}|\Omega_{M-a}\rangle_{(-1)} = 1\,, \tag{3.6}
$$

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and similarly for the right-moving sectors.

In the presence of fractional D3-branes, the left and right moving parts of a twisted closed string have non-trivial correlation functions since the closed string world-sheet has a boundary. In the boundary state formalism, this boundary is the unit circle on which the Ishibashi states enforce an identification between the left and the right movers of the closed strings. In particular for the massless vertex operators of the twisted NS sector with label a and twist parameter ν_a described in section 2.2.1, given any two points w and $\bar{w}$ inside the unit disk $\mathbb{D}$ corresponding to a D3-brane of type I, we have

$$\big\langle \mathcal{V}_a^\alpha(w)\,\widetilde{\mathcal{V}}_a^\beta(\bar{w})\big\rangle_I \;\equiv\; {}_{\mathrm{NS}}\big\langle T;I\big|\,\mathcal{V}_a^\alpha(w)\,\widetilde{\mathcal{V}}_a^\beta(\bar{w})|0\rangle|\widetilde{0}\big\rangle = \frac{M_{I,a}^{\alpha\beta}}{(1-w\bar{w})^2}\,, \tag{3.7}$$

where the last step is a consequence of the conformal invariance which fixes the form of the two-point function of conformal fields of weight 1 on $\mathbb{D}$. The constant in the numerator can be obtained from the overlap between the twisted boundary state and the states created by the vertex operators $\mathcal{V}_a^\alpha$ and $\widetilde{\mathcal{V}}_a^\beta$. For example, fixing $\alpha = 1$ and $\beta = 2$ and referring to the explicit expressions in table 1, we have

$$\begin{aligned}
M_{I,a}^{12} &= \lim_{w\to 0}\,\lim_{\bar{w}\to 0}\,{}_{\mathrm{NS}}\langle T;I|\,\mathcal{V}_a^1(w)\,\widetilde{\mathcal{V}}_a^2(\bar{w})|0\rangle|\widetilde{0}\rangle\\
&= {}_{\mathrm{NS}}\langle T;I|\,\Psi^3{}_{-\frac{1}{2}+\nu_a}\,\widetilde{\overline{\Psi}}{}^3{}_{-\frac{1}{2}+\nu_a}\,|\Omega_a\rangle_{(-1)}\,|\widetilde{\Omega}_a\rangle_{(-1)}\\
&= \sin(\pi\nu_a)\,\omega^{-Ia}\,{}_{\mathrm{NS}}\langle\!\langle a|\,\Psi^3{}_{-\frac{1}{2}+\nu_a}\,\widetilde{\overline{\Psi}}{}^3{}_{-\frac{1}{2}+\nu_a}\,|\Omega_a\rangle_{(-1)}\,|\widetilde{\Omega}_a\rangle_{(-1)}\\
&= \mathrm{i}\,\sin(\pi\nu_a)\,\omega^{-Ia}\;.
\end{aligned} \tag{3.8}$$

Proceeding in a similar way, we find that $M_{I,a}^{21}$ is identical to (3.8), while $M_{I,a}^{11} = M_{I,a}^{22} = 0$, since in these cases the fermionic oscillators are unbalanced. We can thus summarize this result by rewriting (3.7) as

$$\big\langle \mathcal{V}_a^\alpha(w)\,\widetilde{\mathcal{V}}_a^\beta(\bar{w})\big\rangle_I \;=\; \frac{\mathrm{i}\,\sin(\pi\nu_a)\,\omega^{-Ia}\,(\tau_1)^{\alpha\beta}}{(1-w\bar{w})^2} \tag{3.9}$$

where τ_1 is the first Pauli matrix.

We now map this disk two-point function onto the complex plane by using the Cayley transformation

$$w = \frac{z-\mathrm{i}}{z+\mathrm{i}}\,, \tag{3.10}$$

obtaining

$$\big\langle \mathcal{V}_a^\alpha(z)\,\widetilde{\mathcal{V}}_a^\beta(\bar{z})\big\rangle_I \;=\; \big\langle \mathcal{V}_a^\alpha(w)\,\widetilde{\mathcal{V}}_a^\beta(\bar{w})\big\rangle_I\,\frac{dw}{dz}\,\frac{d\bar{w}}{d\bar{z}} \;=\; \frac{-\mathrm{i}\,\sin(\pi\nu_a)\,\omega^{-Ia}\,(\tau^1)^{\alpha\beta}}{(z-\bar{z})^2}\;. \tag{3.11}$$

Thus, using the doubling trick, we are led to introduce the following reflection rule for right moving vertex operators:

$$\widetilde{\mathcal{V}}_a^\beta(\bar{z}) \longrightarrow (R_{I,a})^\beta{}_\gamma\,\mathcal{V}_{M-a}^\gamma(\bar{z})\,, \tag{3.12}$$

so that

$$\left\langle \mathcal{V}_a^{\alpha}(z)\,\widetilde{\mathcal{V}}_a^{\beta}(\bar z)\right\rangle_I \longrightarrow (R_{I,a})^{\beta}{}_{\gamma}\left\langle \mathcal{V}_a^{\alpha}(z)\,\mathcal{V}_{M-a}^{\gamma}(\bar z)\right\rangle = (R_{I,a})^{\beta}{}_{\gamma}\,\frac{(\epsilon^{-1})^{\alpha\gamma}}{(z-\bar z)^2} \tag{3.13}$$

where, in the last step, we used (2.30). Comparing with (3.11) we find that the reflection matrix $R_{I,a}$ is given by

$$R_{I,a} = \mathrm{i}\,\sin(\pi\nu_a)\,\omega^{-Ia}\,\tau_3 \tag{3.14}$$

where τ_3 is the third Pauli matrix. Repeating the same calculations in the twisted sector labeled by $(M-a)$, we get

$$R_{I,M-a} = \mathrm{i}\,\sin(\pi\nu_a)\,\omega^{Ia}\,\tau_3\;. \tag{3.15}$$

Notice that even though the oscillator structure of the boundary states in the sectors a and $(M-a)$ is different, in the end the reflection matrices (3.14) and (3.15) have the same form and can be simultaneously written as

$$R_{I,\widehat{a}} = \mathrm{i}\,\sin\left(\frac{\pi\widehat{a}}{M}\right)\omega^{-I\,\widehat{a}}\,\tau_3 \tag{3.16}$$

with $\widehat{a} = 1,\ldots,M-1$.

3.1.2 R/R sector

The above analysis can be easily extended to the R/R sector where, in analogy with (3.2), the twisted components of the boundary state are given by

$$\begin{aligned}
|T;I|\rangle_{\mathrm{R}} &= \sum_{\widehat{a}=1}^{M-1}\sin\left(\frac{\pi\widehat{a}}{M}\right)\omega^{I\,\widehat{a}}\,|\,\widehat{a}\,\rangle\!\rangle_{\mathrm{R}}\,,\\
{}_{\mathrm{R}}\langle T;I| &= \sum_{\widehat{a}=1}^{M-1}\sin\left(\frac{\pi\widehat{a}}{M}\right)\omega^{-I\,\widehat{a}}\;{}_{\mathrm{R}}\langle\!\langle\,\widehat{a}\,|\;.
\end{aligned} \tag{3.17}$$

In writing the expressions for the GSO-projected Ishibashi states $|\,\widehat{a}\,\rangle\!\rangle_{\mathrm{R}}$ and their conjugates, we adopt the same picture assignments discussed in section 2.4: the $(-\frac{1}{2},-\frac{3}{2})$-picture for the twisted sectors labeled by $\widehat{a} = a \in [1,\frac{M-1}{2}]$, and the $(-\frac{3}{2},-\frac{1}{2})$-picture for the sectors with $\widehat{a} = (M-a)$. Apart from this, the structure of these states is similar to that of the twisted boundary states for D3-branes in the $\mathbb{Z}_2$ orbifold obtained in [12] from the factorization of the one-loop open string partition function, and already used in our companion paper [1]. In particular, for $\widehat{a} = a$ we have

$$|a\rangle\!\rangle_{\mathrm{R}} = \big(C\Gamma_1\Gamma_2\big)_{A\dot B}\,|A_a\rangle_{(-\frac{1}{2})}|\widetilde{\dot B}_a\rangle_{(-\frac{3}{2})} + \ldots \tag{3.18}$$

where the ellipses stand for contributions from massive fermionic modes, the vacuum states have been defined in (2.48) and (2.51), and Γ_1 and Γ_2 are the SO(6) Dirac matrices along the first two real longitudinal directions of the D3-branes. Likewise, when $\widehat{a} = (M-a)$ we have

$$|M-a\rangle\!\rangle_{\mathrm{R}} = \big(C\Gamma_1\Gamma_2\big)_{\dot A B}\,|\dot A_{M-a}\rangle_{(-\frac{3}{2})}|\widetilde{B}_{M-a}\rangle_{(-\frac{1}{2})} + \ldots\;. \tag{3.19}$$

The corresponding Ishibashi conjugate states are

$$
\begin{aligned}
{}_{\mathrm{R}}\langle\!\langle a| &= {}_{(-\frac{3}{2})}\langle\widetilde{\dot{A}}_a|\;{}_{(-\frac{1}{2})}\langle B_a|\left(\Gamma_2\Gamma_1 C^{-1}\right)^{\dot{A}B} + \dots \\
{}_{\mathrm{R}}\langle\!\langle M-a| &= {}_{(-\frac{1}{2})}\langle\widetilde{A}_{M-a}|\;{}_{(-\frac{3}{2})}\langle\dot{B}_{M-a}|\left(\Gamma_2\Gamma_1 C^{-1}\right)^{A\dot{B}} + \dots
\end{aligned}
\tag{3.20}
$$

where the bra vacuum states are defined such that

$$
{}_{(-\frac{1}{2})}\langle B_a|A_a\rangle_{(-\frac{1}{2})} = \delta^A_B \quad \text{and} \quad {}_{(-\frac{3}{2})}\langle\widetilde{\dot{B}}_{M-a}|\widetilde{\dot{A}}_{M-a}\rangle_{(-\frac{3}{2})} = \delta^{\dot{A}}_{\dot{B}} \tag{3.21}
$$

with analogous relations for the right-moving vacua.[4]

We can now repeat the same steps followed in the NS sector to prove that the boundary state enforces an identification between left-moving and right-moving vertex operators in the twisted R sector a according to

$$
\widetilde{\mathcal{V}}_a^{\dot{B}}(\bar{z}) \longrightarrow (R_{I,a})^{\dot{B}}_{\;\dot{C}}\,\mathcal{V}^{\dot{C}}_{M-a}(\bar{z})\,, \tag{3.22}
$$

where the reflection matrix is the anti-chiral/anti-chiral block of

$$
R_{I,a} = \sin(\pi\nu_a)\,\omega^{-Ia}\,\Gamma_1\Gamma_2\ . \tag{3.23}
$$

Similarly, in the twisted R sector labeled by $(M-a)$ the reflection matrix is the chiral/chiral block of

$$
R_{I,M-a} = \sin(\pi\nu_a)\,\omega^{Ia}\,\Gamma_1\Gamma_2\ . \tag{3.24}
$$

We can combine the last two formulas into

$$
R_{I,\widehat{a}} = \sin\left(\frac{\pi\widehat{a}}{M}\right)\omega^{-I\widehat{a}}\,\Gamma_1\Gamma_2 \tag{3.25}
$$

with the understanding that one has to take the lower-right and upper-left blocks for $\widehat{a} = a$ and $\widehat{a} = (M-a)$, respectively, as a consequence of the picture assignments.

3.2 Massless open string spectrum

We now analyze the spectrum of massless open strings that live on a configuration made of stacks of n_I fractional D3-branes of type I for $I = 0, \dots, M-1$, that engineer a theory with gauge group $\mathrm{U}(n_0) \times \dots \times \mathrm{U}(n_{M-1})$. We will restrict ourselves to listing the fields in the adjoint representation of $\mathrm{U}(n_I)$ as these will be the only fields that are sourced by the background values given to the twisted closed string scalars. We tailor our notations and conventions to be as close as possible to those in [1].

In the familiar case of D3-branes in flat space, in the (0)-superghost picture the bosonic massless open string states are represented by vertex operators of the form[5]

$$
\left(\mathrm{i}\,\partial Z^i + \kappa\cdot\Psi\,\Psi^i\right)\mathrm{e}^{\mathrm{i}\,\kappa\cdot Z}\ . \tag{3.26}
$$

[4] We remark that in (3.21) the superghost charges of the bra and ket states exactly soak up the background charge anomaly. For example the superghost charge of ${}_{(-\frac{1}{2})}\langle B_a|$ is $-\frac{3}{2}$, and that of $|A_a\rangle_{(-\frac{1}{2})}$ is $-\frac{1}{2}$.

[5] Here and in the following we always assume the operators to be normal ordered, unless this causes ambiguities.

where

$$\kappa_i = \frac{k_{2i-1} + \mathrm{i}\, k_{2i}}{\sqrt{2}} \quad \text{and} \quad \bar{\kappa}_i = \frac{k_{2i-1} + \mathrm{i}\, k_{2i}}{\sqrt{2}}\,, \tag{3.27}$$

with k_μ being the real momentum along the direction x^μ of the D3-brane world-volume. In addition we denote the complex direction 1 by the symbol $\parallel$ and the complex direction 2 by the symbol $\perp$, since these directions are, respectively, longitudinal and perpendicular to the surface defect realized by the D3-brane configuration on the orbifold. We also introduce the following convenient notation

$$\begin{aligned} \kappa_\parallel \!\cdot\! Z_\parallel &= \kappa_1\, \overline{Z}^1 + \bar{\kappa}_1\, Z^1\,, \qquad & \kappa_\perp \!\cdot\! Z_\perp &= \kappa_2\, \overline{Z}^2 + \bar{\kappa}_2\, Z^2\,, \\ \kappa_\parallel \!\cdot\! \Psi_\parallel &= \kappa_1\, \overline{\Psi}^1 + \bar{\kappa}_1\, \Psi^1\,, \qquad & \kappa_\perp \!\cdot\! \Psi_\perp &= \kappa_2\, \overline{\Psi}^2 + \bar{\kappa}_2\, \Psi^2\,, \end{aligned} \tag{3.28}$$

so that

$$\kappa \cdot Z = \kappa_\parallel \!\cdot\! Z_\parallel + \kappa_\perp \!\cdot\! Z_\perp \tag{3.29}$$

and similarly for $\kappa \cdot \Psi$. Clearly, the parallel terms $\kappa_\parallel \!\cdot\! Z_\parallel$ and $\kappa_\parallel \!\cdot\! \Psi_\parallel$ are invariant under the orbifold group $\mathbb{Z}_M$, but the perpendicular terms are not, since

$$g: \quad \begin{cases} \kappa_\perp \!\cdot\! Z_\perp \quad \longrightarrow \quad g[\kappa_\perp \!\cdot\! Z_\perp] = \omega^{-1}\kappa_2\, \overline{Z}^2 + \omega\, \bar{\kappa}_2\, Z^2\,, \\ \kappa_\perp \!\cdot\! \Psi_\perp \quad \longrightarrow \quad g[\kappa_\perp \!\cdot\! \Psi_\perp] = \omega^{-1}\kappa_2\, \overline{\Psi}^2 + \omega\, \bar{\kappa}_2\, \Psi^2\,. \end{cases} \tag{3.30}$$

This in particular implies that in order to write the open string vertex operators for the fractional D3-branes one cannot use the plane waves $\mathrm{e}^{\mathrm{i}\,\kappa_\perp \cdot Z_\perp}$ but instead decomposes these into functions that transform in the irreducible representations of $\mathbb{Z}_M$. These functions, which we denote by $\mathcal{E}_I$ with $I = 0, \ldots, M-1$, are simply obtained by summing the plane waves $\mathrm{e}^{\mathrm{i}\,\kappa_\perp \cdot Z_\perp}$ over the orbits of the group with coefficients chosen such that the combination transforms covariantly under the group action. So we are led to define:

$$\mathcal{E}_I = \frac{1}{M}\sum_{J=0}^{M-1} \omega^{-IJ}\, g^J\Big[\mathrm{e}^{\mathrm{i}\,\kappa_\perp \cdot Z_\perp}\Big] = \frac{1}{M}\sum_{J=0}^{M-1} \omega^{-IJ}\, \mathrm{e}^{\mathrm{i}\,(\omega^{-J}\kappa_2\, \overline{Z}^2 + \omega^J \bar{\kappa}_2\, Z^2)}\,. \tag{3.31}$$

One can easily check that

$$g\big[\mathcal{E}_I\big] = \frac{1}{M}\sum_{J=0}^{M-1} \omega^{-IJ}\, \mathrm{e}^{\mathrm{i}\,(\omega^{-J-1}\kappa_2\, \overline{Z}^2 + \omega^{J+1} \bar{\kappa}_2\, Z^2)} = \omega^I\, \mathcal{E}_I\,, \tag{3.32}$$

which shows that $\mathcal{E}_I$ transforms in the I-th irreducible representation of $\mathbb{Z}_M$. For $M = 2$ and $\omega = -1$, the functions $\mathcal{E}_I$ are simply

$$\mathcal{E}_0 = \cos(\kappa_\perp \!\cdot\! Z_\perp) \quad \text{and} \quad \mathcal{E}_1 = \mathrm{i}\, \sin(\kappa_\perp \!\cdot\! Z_\perp)\,, \tag{3.33}$$

which are exactly the two combinations used in the case of the $\mathbb{Z}_2$ orbifold in [1].

In a similar way, we have to break up the operators multiplying the plane wave in (3.26) into various pieces with definite charge I under the orbifold action and form invariant combinations with $\mathcal{E}_{M-I}$. In the orbifold theory, only such combinations represent vertex operators describing physical fields on the world-volume of the fractional D3-brane.

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Applying these considerations, we see that the gauge field A_1 along the parallel directions is described by the following vertex operator in the (0)-superghost picture:

$$\mathcal{V}_{A_1} = \Big[\big(\mathrm{i}\,\partial Z^1 + \kappa_\parallel\!\cdot\!\Psi_\parallel\,\Psi^1\big)\,\mathcal{E}_0 + \kappa_2\,\overline{\Psi}^2\,\Psi^1\,\mathcal{E}_1 + \bar{\kappa}_2\,\Psi^2\,\Psi^1\,\mathcal{E}_{M-1}\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ . \tag{3.34}$$

Each term in square brackets is invariant under $\mathbb{Z}_M$. For instance, the terms $\partial\overline{Z}^1$ and $\kappa_\parallel\!\cdot\!\Psi_\parallel\,\overline{\Psi}^1$, which are $\mathbb{Z}_M$ invariant, are multiplied with the invariant function $\mathcal{E}_0$. Similarly the term $\kappa_2\,\overline{\Psi}^2\,\Psi^1$, which gets a factor ω^{-1} under the orbifold action, is multiplied by $\mathcal{E}_1$ to make a $\mathbb{Z}_M$-invariant combination. Likewise, it is easy to see that the third term in (3.34) is also $\mathbb{Z}_M$ invariant. The vertex operator for the complex conjugate field component $\overline{A}_1$ is obtained by simply replacing ∂Z^1 and Ψ^1 with $\partial\overline{Z}^1$ and $\overline{\Psi}^1$.

In a similar way we can write the vertex operators for the gauge field A_2 in the directions transverse to the surface defect, which is

$$\mathcal{V}_{A_2} = \Big[\big(\mathrm{i}\,\partial Z^2 + \kappa_\parallel\!\cdot\!\Psi_\parallel\,\Psi^2\big)\,\mathcal{E}_{M-1} + \kappa_2\,\overline{\Psi}^2\,\Psi^2\,\mathcal{E}_0\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ . \tag{3.35}$$

The vertex operator for $\overline{A}_2$ can be obtained from the above expression by replacing ∂Z^2 and Ψ^2 with $\partial\overline{Z}^2$ and $\overline{\Psi}^2$, and $\mathcal{E}_{M-1}$ with $\mathcal{E}_1$.

Finally, let us consider the scalar fields. On the fractional D3-brane world-volume there are three complex scalars that together with the gauge vector provide the bosonic content of the $\mathcal{N}=4$ vector multiplet. When the orbifold acts partially along the world-volume as in our case, all three complex scalars remain in the spectrum. Denoting them by Φ and Φ_r with $r=4,5$, they and their complex conjugates are described by the following $\mathbb{Z}_M$-invariant vertices:

$$\begin{aligned}
\mathcal{V}_{\Phi} &= \Big[\big(\mathrm{i}\,\partial Z^3 + \kappa_\parallel\!\cdot\!\Psi_\parallel\,\Psi^3\big)\,\mathcal{E}_1 + \kappa_2\,\overline{\Psi}^2\,\Psi^3\,\mathcal{E}_2 + \bar{\kappa}_2\,\Psi^2\,\Psi^3\,\mathcal{E}_0\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ ,\\
\mathcal{V}_{\overline{\Phi}} &= \Big[\big(\mathrm{i}\,\partial \overline{Z}^3 + \kappa_\parallel\!\cdot\!\Psi_\parallel\,\overline{\Psi}^3\big)\,\mathcal{E}_{M-1} + \kappa_2\,\overline{\Psi}^2\,\overline{\Psi}^3\,\mathcal{E}_0 + \bar{\kappa}_2\,\Psi^2\,\overline{\Psi}^3\,\mathcal{E}_{M-2}\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ ,
\end{aligned} \tag{3.36}$$

and

$$\mathcal{V}_{\Phi_r} = \Big[\big(\mathrm{i}\,\partial Z^r + \kappa_\parallel\!\cdot\!\Psi_\parallel\,\Psi^r\big)\,\mathcal{E}_0 + \kappa_2\,\overline{\Psi}^2\,\Psi^r\,\mathcal{E}_1 + \bar{\kappa}_2\,\Psi^2\,\Psi^r\,\mathcal{E}_{M-1}\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ , \tag{3.37}$$

with $\mathcal{V}_{\overline{\Phi}_r}$ obtained by simply replacing Ψ^r with $\overline{\Psi}^r$.

All these vertex operators have conformal dimension 1 provided the corresponding fields are massless, *i.e.* if $\kappa\cdot\bar{\kappa} = \frac{1}{2}k^2 = 0$.

4 Open/closed correlators

In this section we study the mixed amplitudes between the twisted closed string fields discussed in section 2 and the massless open string fields introduced in the previous section by calculating open/closed disk correlators (see [17] for a review of scattering of strings off D-branes). An example of such a mixed amplitude is shown in figure 1, in which the closed string field is the NS/NS scalar $b^{(\widehat{a})}_{\alpha\beta}$ in the twisted sector $\widehat{a}$.
The open/closed string amplitudes we consider correspond to disk diagrams with a closed string vertex inserted in the interior and an open string vertex inserted on the boundary.

$$\left\langle \mathcal{V}_{\text{open}} \right\rangle_{b^{(\widehat{a})}_{\alpha\beta};I} \quad \equiv$$

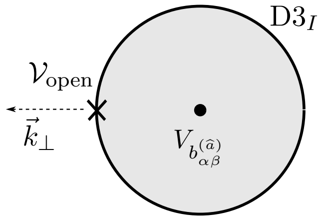

Figure 1. An example of a mixed open/closed string amplitude on a D3-brane of type I. The closed string vertex operator in the bulk represents the insertion of the twisted NS/NS scalar $b^{(\widehat{a})}_{\alpha\beta}$; the open string field on the boundary is a generic massless excitation on the D3-brane which can couple to $b^{(\widehat{a})}_{\alpha\beta}$. The result is a function of the open string momentum $\vec{k}_\perp$ along the two orbifolded directions of the D3-brane world-volume which are transverse to the surface defect.

These diagrams are generically non-vanishing due to the D3-brane boundary conditions that enforce an identification between the left and right movers of the closed strings.

We now explain how to compute these mixed amplitudes starting from the NS/NS twisted fields.

4.1 Correlators with NS/NS twisted fields

Let us consider the scalar $b^{(\widehat{a})}_{\alpha\beta}$ in the NS/NS twisted sector $\widehat{a}$. Its coupling with a massless open string excitation on a D3-brane of type I described by the vertex operator $\mathcal{V}_{\text{open}}$ is given by the following expression:

$$\left\langle \mathcal{V}_{\text{open}} \right\rangle_{b^{(\widehat{a})}_{\alpha\beta};I} = b^{(\widehat{a})}_{\alpha\beta} \int \frac{dz\, d\bar{z}\, dx}{dV_{\text{proj}}} \left\langle \mathcal{V}^{\alpha}_{\widehat{a}}(z) \widetilde{\mathcal{V}}^{\beta}_{\widehat{a}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \right\rangle_I , \tag{4.1}$$

where

$$dV_{\text{proj}} = \frac{dz\, d\bar{z}\, dx}{(z-\bar{z})(\bar{z}-x)(x-z)} \tag{4.2}$$

is the projective invariant volume element and the integrals are performed on the string world-sheet. In particular the closed string insertion points z and $\bar{z}$, are in the upper and lower half complex plane, respectively, while the open string insertion point x is on the real axis.

Since we are interested in the couplings with constant background fields $b^{(\widehat{a})}_{\alpha\beta}$, the left and right vertex operators in (4.1) are at zero momentum. The open string vertex, instead, has a non-vanishing momentum. Since the fractional brane is located at the orbifold fixed point $z_2 = 0$, translation invariance is broken in the complex direction 2. Therefore, the components κ_2 and $\bar{\kappa}_2$ of the open string momentum are arbitrary, while the components κ_1 and $\bar{\kappa}_1$ are set to zero by momentum conservation in the parallel directions and the final amplitude will be proportional to $\delta^{(2)}(\kappa_\parallel)$.

Using the reflection rule (3.16), the integrand of (4.1) can be rewritten as

$$\left\langle \mathcal{V}^{\alpha}_{\widehat{a}}(z) \widetilde{\mathcal{V}}^{\beta}_{\widehat{a}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \right\rangle_I = \mathrm{i}\,\sin\left(\frac{\pi\widehat{a}}{M}\right) \omega^{-I\,\widehat{a}}\, (\tau_3)^{\beta}{}_{\gamma} \left\langle \mathcal{V}^{\alpha}_{\widehat{a}}(z)\, \mathcal{V}^{\gamma}_{M-\widehat{a}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \right\rangle . \tag{4.3}$$

Thus, the calculation is reduced to the evaluation of a three-point function of vertex operators of conformal weight 1. The functional dependence on the word-sheet variables is fixed

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by conformal invariance and exactly cancels that of the projective invariant volume (4.2) so that in the end the result will be a constant that depends on the detailed structure of the vertex operators.

There are, however, some features that can be described in generality, and are independent of the specific components of $b^{(\widehat{a})}_{\alpha\beta}$ and of the particular open string vertices that are considered. When we write the three-point functions in (4.3) as products of correlators for each of the independent conformal fields, we easily recognize that the superghost contribution is always given by

$$\left\langle \mathrm{e}^{-\phi(z)}\, \mathrm{e}^{-\phi(\bar{z})} \right\rangle = \frac{1}{z-\bar{z}} \ . \tag{4.4}$$

It is perhaps less obvious but it turns out that also the contribution arising from the bosonic string coordinates is the same for all amplitudes. Indeed, the only non-vanishing correlator involving the bosonic coordinates along the parallel direction is

$$\left\langle \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}(x) \right\rangle = \delta^{(2)}(\kappa_\parallel)\,, \tag{4.5}$$

which enforces the anticipated momentum conservation for $\kappa_\parallel$, while the terms containing ∂Z^1 or $\partial \overline{Z}^1$ always vanish inside the correlators and thus they can be ignored. As far as the perpendicular direction is concerned, we have to take into account the presence of the bosonic twist fields and the fact that the plane waves appear in the combinations $\mathcal{E}_I$ defined in (3.31). Thus, one typically has to evaluate a correlator of the form

$$\left\langle \sigma_{\widehat{a}}(z)\,\sigma_{M-\widehat{a}}(\bar{z})\,\mathcal{E}_I(x) \right\rangle = \frac{1}{M}\sum_{J=0}^{M-1} \omega^{-IJ} \left\langle \sigma_{\widehat{a}}(z)\,\sigma_{M-\widehat{a}}(\bar{z})\,\mathrm{e}^{\mathrm{i}\left(\omega^{-J}\kappa_2\,\overline{Z}^2(x)+\omega^{J}\bar{\kappa}_2\,Z^2(x)\right)} \right\rangle \ . \tag{4.6}$$

For any value of J, the correlator in the sum is equal simply to $\langle \sigma_{\widehat{a}}(z)\,\sigma_{M-\widehat{a}}(\bar{z})\rangle$, so that

$$\left\langle \sigma_{\widehat{a}}(z)\,\sigma_{M-\widehat{a}}(\bar{z})\,\mathcal{E}_I(x) \right\rangle = \frac{1}{M}\left(\sum_{J=0}^{M-1} \omega^{-IJ}\right)\left\langle \sigma_{\widehat{a}}(z)\,\sigma_{M-\widehat{a}}(\bar{z}) \right\rangle = \delta_{I,0}\left\langle \sigma_{\widehat{a}}(z)\,\sigma_{M-\widehat{a}}(\bar{z}) \right\rangle \ . \tag{4.7}$$

This means that in the open string vertex operators we can just focus on the terms proportional to $\mathcal{E}_0$ and disregard the other terms, as they will not contribute. Furthermore, we can also neglect the terms involving ∂Z^2 or $\partial\overline{Z}^2$, since they always give a vanishing contribution inside the correlators. With this in mind, we can proceed to the explicit evaluation of the mixed amplitudes with the twisted NS/NS scalars.

4.1.1 Explicit computations

We start by considering the correlator (4.3) with $\widehat{a} = a \in [1, \frac{M-1}{2}]$ and $\alpha = 1$ and $\beta = 2$, corresponding to the twisted field $b^{(a)}_{12}$. Applying the above considerations, one realizes that this scalar does not couple to any open string field except A_2 and $\overline{A}_2$. Indeed, the terms of the vertex operators of A_1, Φ, Φ_r and their conjugates which contain $\mathcal{E}_0$ always contain other structures with unbalanced bosonic or fermionic fields, which therefore vanish inside the correlator. Let us then consider the coupling with A_2. In this case, inserting the explicit expressions of the vertex operators in (4.3), we have

$$\left\langle \mathcal{V}_a^1(z)\widetilde{\mathcal{V}}_a^2(\bar z)\,\mathcal{V}_{A_2}(x)\right\rangle_I = -\mathrm{i}\,\sin\pi\nu_a\,\omega^{-Ia}\left\langle \mathcal{V}_a^1(z)\,\mathcal{V}_{M-a}^2(\bar z)\,\mathcal{V}_{A_2}(x)\right\rangle \tag{4.8}$$

with

$$\begin{aligned}\left\langle \mathcal{V}_a^1(z)\,\mathcal{V}_{M-a}^2(\bar z)\,\mathcal{V}_{A_2}(x)\right\rangle = \kappa_2\left\langle \mathrm{e}^{-\phi(z)}\,\mathrm{e}^{-\phi(\bar z)}\right\rangle\left\langle \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel(x)}\right\rangle\left\langle \sigma_a(z)\,\sigma_{M-a}(\bar z)\right\rangle\\ \times\left\langle :\Psi^3(z)s_a(z):\,:\overline{\Psi}^3(\bar z)s_{M-a}(\bar z):\,:\overline{\Psi}^2(x)\Psi^2(x):\right\rangle .\end{aligned} \tag{4.9}$$

The fermionic correlator in the second line above can be evaluated by factorizing it in the two independent directions 2 and 3 and using the bosonization method [6]. In this way we have

$$\begin{aligned}\left\langle :\Psi^3(z)s_a(z):\,:\overline{\Psi}^3(\bar z)s_{M-a}(\bar z):\,:\overline{\Psi}^2(x)\Psi^2(x):\right\rangle = \left\langle s^2_{\nu_a}(z)\,s^2_{-\nu_a}(\bar z)\,:\overline{\Psi}^2(x)\Psi^2(x):\right\rangle\\ \times\left\langle :\Psi^3(z)s^3_{-\nu_a}(z):\,:\overline{\Psi}^3(\bar z)s^3_{\nu_a}(\bar z):\right\rangle\end{aligned} \tag{4.10}$$

where[6]

$$\begin{aligned}\left\langle s^2_{\nu_a}(z)\,s^2_{-\nu_a}(\bar z)\,:\overline{\Psi}^2(x)\Psi^2(x):\right\rangle &= \left\langle\,\mathrm{e}^{\mathrm{i}\,\nu_a\,\phi_2}(z)\,\mathrm{e}^{-\mathrm{i}\,\nu_a\,\phi_2}(\bar z)\,(-\mathrm{i}\,\partial\phi_2(x))\,\right\rangle\\ &= \frac{-\nu_a}{(z-\bar z)^{\nu_a^2-1}(z-x)(\bar z-x)}\,,\end{aligned} \tag{4.11}$$

and

$$\left\langle :\Psi^3(z)s^3_{-\nu_a}(z):\,:\overline{\Psi}^3(\bar z)s^3_{\nu_a}(\bar z):\right\rangle = \left\langle \mathrm{e}^{\mathrm{i}\,(1-\nu_a)\phi_3}(z)\,\mathrm{e}^{-\mathrm{i}\,(1-\nu_a)\phi_3}(\bar z)\right\rangle = \frac{1}{(z-\bar z)^{(1-\nu_a)^2}}\,. \tag{4.12}$$

Combining everything together in (4.9), we obtain

$$\left\langle \mathcal{V}_a^1(z)\,\mathcal{V}_{M-a}^2(\bar z)\,\mathcal{V}_{A_2}(x)\right\rangle = \frac{\kappa_2\,\nu_a}{(z-\bar z)(\bar z-x)(x-z)}\,\delta^{(2)}(\kappa_\parallel)\,. \tag{4.13}$$

Finally, inserting this into (4.8) and (4.1), we find that the coupling of $b_{12}^{(a)}$ with A_2 is

$$\left\langle \mathcal{V}_{A_2}\right\rangle_{b_{12}^{(a)};I} = -\mathrm{i}\,b_{12}^{(a)}\,\kappa_2\,\nu_a\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\,. \tag{4.14}$$

The same calculation shows that $b_{12}^{(a)}$ also couples to $\overline{A}_2$ and the result is simply obtained by replacing κ_2 with $-\bar\kappa_2$ in the above expression.

We can similarly repeat the analysis for the other components $b_{\alpha\beta}^{(a)}$. For example, taking $b_{21}^{(a)}$ we find that its only non-vanishing coupling is

$$\left\langle \mathcal{V}_{A_2}\right\rangle_{b_{21}^{(a)};I} = \mathrm{i}\,b_{21}^{(a)}\,\kappa_2\,(1-\nu_a)\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\,, \tag{4.15}$$

with a similar result for $\overline{A}_2$ in which κ_2 is replaced with $-\bar\kappa_2$. The diagonal components $b_{11}^{(a)}$ and $b_{22}^{(a)}$, instead, only couple to the complex scalars Φ and $\overline{\Phi}$ according to

$$\begin{aligned}\left\langle \mathcal{V}_{\Phi}\right\rangle_{b_{22}^{(a)};I} &= -\mathrm{i}\,b_{22}^{(a)}\,\bar\kappa_2\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\,,\\ \text{and}\quad \left\langle \mathcal{V}_{\overline{\Phi}}\right\rangle_{b_{11}^{(a)};I} &= \mathrm{i}\,b_{11}^{(a)}\,\kappa_2\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\,.\end{aligned} \tag{4.16}$$

[6] Here ϕ_2 and ϕ_3 denote the fields that bosonize the fermionic systems in the complex directions 2 and 3.

It is equally straightforward to compute the open/closed string correlators in the twisted sectors with $\widehat{a} = (M-a)$. In this case, we find again that the off-diagonal components $b_{12}^{(M-a)}$ and $b_{21}^{(M-a)}$ only interact with A_2 and $\overline{A}_2$, and that the couplings with A_2 are

$$\begin{aligned} \big\langle \mathcal{V}_{A_2} \big\rangle_{b_{12}^{(M-a)};I} &= -\mathrm{i}\, b_{12}^{(M-a)}\, \kappa_2\, (1-\nu_a) \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel)\,, \\ \text{and} \quad \big\langle \mathcal{V}_{A_2} \big\rangle_{b_{21}^{(M-a)};I} &= \mathrm{i}\, b_{21}^{(M-a)}\, \kappa_2\, \nu_a \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel)\,, \end{aligned} \tag{4.17}$$

while those with $\overline{A}_2$ follow by replacing κ_2 with $-\bar{\kappa}_2$ in the above expressions. The diagonal components $b_{11}^{(M-a)}$ and $b_{21}^{(M-a)}$ interact instead with Φ and $\overline{\Phi}$ with the following couplings:

$$\begin{aligned} \big\langle \mathcal{V}_{\Phi} \big\rangle_{b_{22}^{(M-a)};I} &= -\mathrm{i}\, b_{22}^{(M-a)}\, \bar{\kappa}_2 \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel)\,, \\ \text{and} \quad \big\langle \mathcal{V}_{\overline{\Phi}} \big\rangle_{b_{11}^{(M-a)};I} &= \mathrm{i}\, b_{11}^{(M-a)}\, \kappa_2 \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel)\,. \end{aligned} \tag{4.18}$$

As a consistency check of our results, we observe that the formulas (4.17) and (4.18) can be obtained from (4.14), (4.15) and (4.16) by simply replacing everywhere a with $(M-a)$. Thus, despite the fact that the fermionic approach we have used introduces differences in the explicit expressions for the twisted sector vertex operators, in the end, all sectors are treated on an equal footing.

4.1.2 Results

We are finally in a position to write down the complete expression for the open string fields emitted by a fractional D3-brane of type I in the presence of background values for the scalars of the NS/NS twisted sectors. This is given by summing over all components of $b_{\alpha\beta}^{(\widehat{a})}$ and over all twisted sectors:

$$\big\langle \mathcal{V}_{\text{open}} \big\rangle_I = \sum_{\widehat{a}=1}^{M-1} \sum_{\alpha,\beta=1}^{2} \big\langle \mathcal{V}_{\text{open}} \big\rangle_{b_{\alpha\beta}^{(\widehat{a})};I}\,. \tag{4.19}$$

As we have seen, the components of the gauge field along the parallel direction 1 and the complex scalars Φ_r do not couple to any NS/NS twisted field, while we have a non-vanishing source for A_2, Φ and their complex conjugates. For A_2 the above formula gives

$$\begin{aligned} \big\langle \mathcal{V}_{A_2} \big\rangle_I = -\mathrm{i}\, \kappa_2 \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \Big[\nu_a\, \omega^{-Ia}\, b_{12}^{(a)} - (1-\nu_a)\, \omega^{-Ia}\, b_{21}^{(a)} \\ - \nu_a\, \omega^{Ia}\, b_{21}^{(M-a)} + (1-\nu_a)\, \omega^{Ia}\, b_{12}^{(M-a)} \Big]\, \delta^{(2)}(\kappa_\parallel)\,. \end{aligned} \tag{4.20}$$

Taking into account the relations (2.38), it is easy to realize that the quantity in square brackets is purely imaginary. A similar result holds for $\overline{A}_2$ with κ_2 replaced by $-\bar{\kappa}_2$.

For the complex scalars Φ and $\overline{\Phi}$ we have instead

$$\begin{aligned} \big\langle \mathcal{V}_{\Phi} \big\rangle_I &= -\mathrm{i}\, \bar{\kappa}_2 \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \Big[\omega^{-Ia}\, b_{22}^{(a)} + \omega^{Ia}\, b_{22}^{(M-a)} \Big]\, \delta^{(2)}(\kappa_\parallel)\,, \\ \big\langle \mathcal{V}_{\overline{\Phi}} \big\rangle_I &= \mathrm{i}\, \kappa_2 \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \Big[\omega^{-Ia}\, b_{11}^{(a)} + \omega^{Ia}\, b_{11}^{(M-a)} \Big]\, \delta^{(2)}(\kappa_\parallel)\,. \end{aligned} \tag{4.21}$$

4.2 Correlators with R/R twisted fields

We now turn to the calculation of the interactions between the massless open string fields and the twisted R/R potentials. For definiteness we only consider non-vanishing background values for the scalars $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(M-a)}$, since they are the only ones that turn out to be relevant for the description of the continuous parameters of surface defects. Thus, the closed string vertex operators we consider are

$$\mathcal{C}^{(a)}\,C_{A\dot{B}}\,\mathcal{V}_a^A(z)\,\widetilde{\mathcal{V}}_a^{\dot{B}}(\bar{z}) \quad \text{and} \quad \mathcal{C}^{(M-a)}\,C_{\dot{A}B}\,\mathcal{V}_{M-a}^{\dot{A}}(z)\,\widetilde{\mathcal{V}}_{M-a}^{B}(\bar{z})\ . \tag{4.22}$$

By inspecting the fermionic structure of these vertex operators and comparing it with that of the open string vertices, one realizes that only the longitudinal component of the gauge field A_1 and its conjugate $\overline{A}_1$ can have a non-vanishing coupling.

Let us start by considering the interaction between A_1 and $\mathcal{C}^{(a)}$. This is given by

$$\big\langle\mathcal{V}_{A_1}\big\rangle_{\mathcal{C}^{(a)},I} = \mathcal{C}^{(a)}\,C_{A\dot{B}}\int\frac{dz\,d\bar{z}\,dx}{dV_{\text{proj}}}\,\big\langle\mathcal{V}_a^A(z)\,\widetilde{\mathcal{V}}_a^{\dot{B}}(\bar{z})\,\mathcal{V}_{A_1}(x)\big\rangle_I\,, \tag{4.23}$$

where the projective invariant volume element is defined in (4.2). Using the reflection rules (3.23) for the R/R fields, the integrand of (4.23) becomes

$$\big\langle\mathcal{V}_a^A(z)\,\widetilde{\mathcal{V}}_a^{\dot{B}}(\bar{z})\,\mathcal{V}_{A_1}(x)\big\rangle_I = \sin(\pi\nu_a)\,\omega^{-Ia}\,(\Gamma_1\Gamma_2)^{\dot{B}}_{\ \dot{C}}\,\big\langle\mathcal{V}_a^A(z)\,\mathcal{V}_{M-a}^{\dot{C}}(\bar{z})\,\mathcal{V}_{A_1}(x)\big\rangle\ . \tag{4.24}$$

Using the explicit form of the vertex operators given in (2.47a), (2.52b) and (3.34), and taking into account the points discussed at the beginning of this section, the above correlator can be written as follows:

$$\begin{aligned}\big\langle\mathcal{V}_a^A(z)\,\mathcal{V}_{M-a}^{\dot{C}}(\bar{z})\,\mathcal{V}_{A_1}(x)\big\rangle = \kappa_1\big\langle\mathrm{e}^{-\frac{1}{2}\phi(z)}\,\mathrm{e}^{-\frac{3}{2}\phi(\bar{z})}\big\rangle\,\big\langle\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\big\rangle\,\big\langle\sigma_a(z)\,\sigma_{M-a}(\bar{z})\big\rangle&\\ \times\,\big\langle r_a(z)\,r_{M-a}(\bar{z})\big\rangle\,\big\langle S^A(z)\,S^{\dot{C}}(\bar{z})\ :\overline{\Psi}^1\Psi^1:(x)\big\rangle\ .&\end{aligned} \tag{4.25}$$

Each factor in this expression can be easily computed using standard conformal field theory methods. The new ingredients with respect to the calculations in the NS/NS sectors are the following two-point functions:

$$\begin{aligned}\big\langle\mathrm{e}^{-\frac{1}{2}\phi(z)}\,\mathrm{e}^{-\frac{3}{2}\phi(\bar{z})}\big\rangle &= \frac{1}{(z-\bar{z})^{\frac{3}{4}}}\,,\\ \big\langle r_a(z)\,r_{M-a}(\bar{z})\big\rangle &= \frac{1}{(z-\bar{z})^{\frac{1}{2}-2\nu_a(1-\nu_a)}}\,,\\ \text{and}\quad\big\langle S^A(z)\,S^{\dot{C}}(\bar{z})\ :\overline{\Psi}^1\Psi^1:(x)\big\rangle &= \frac{\mathrm{i}}{2}\,\frac{(\Gamma_1\Gamma_2C^{-1})^{A\dot{C}}}{(z-\bar{z})^{-\frac{1}{4}}(z-x)(\bar{z}-x)}\ .\end{aligned} \tag{4.26}$$

Putting everything together, we have

$$\big\langle\mathcal{V}_a^A(z)\,\mathcal{V}_{M-a}^{\dot{C}}(\bar{z})\,\mathcal{V}_{A_1}(x)\big\rangle = -\frac{\mathrm{i}}{2}\,\frac{(\Gamma_1\Gamma_2C^{-1})^{A\dot{C}}}{(z-\bar{z})(\bar{z}-x)(x-z)}\,\delta^{(2)}(\kappa_\parallel)\ . \tag{4.27}$$

Inserting this into (4.24) and (4.23), and performing the Γ-matrix algebra, we finally obtain

$$\big\langle\mathcal{V}_{A_1}\big\rangle_{\mathcal{C}^{(a)},I} = -2\mathrm{i}\,\kappa_1\,\sin\pi\nu_a\,\omega^{-Ia}\,\mathcal{C}^{(a)}\,\delta^{(2)}(\kappa_\parallel)\ . \tag{4.28}$$

In a very similar way we find

$$\langle \mathcal{V}_{A_1} \rangle_{\mathcal{C}^{(M-a)},I} = -2\mathrm{i}\,\kappa_1\, \sin\pi\nu_a\, \omega^{Ia}\, \mathcal{C}^{(M-a)}\, \delta^{(2)}(\kappa_\parallel)\ . \tag{4.29}$$

Thus, the full amplitude becomes

$$\langle \mathcal{V}_{A_1} \rangle_{I} = -2\mathrm{i}\,\kappa_1 \sum_{a=1}^{\frac{M-1}{2}} \Big[\sin\pi\nu_a \big(\omega^{-Ia}\, \mathcal{C}^{(a)} + \omega^{Ia}\, \mathcal{C}^{(M-a)}\big)\Big]\, \delta^{(2)}(\kappa_\parallel)\ . \tag{4.30}$$

Taking into account that $\mathcal{C}^{(M-a)} = \mathcal{C}^{(a)\,\star}$, as it follows from (2.61), we see that the expression inside the square brackets is real.

5 Continuous parameters of surface defects

We are now ready to identify the twisted closed string background that leads to a monodromy surface defect in the gauge theory on the world-volume of the fractional D3-branes. It is convenient to decompose the twisted fields of the NS/NS sectors into irreducible representations of the unbroken $\mathrm{SU}(2)_+$ symmetry group of the orbifolded space (see the discussion in section 2). In each twisted sector $\widehat{a}$, this can be done by writing

$$b^{(\widehat{a})}_{\alpha\beta} = \mathrm{i}\, b^{(\widehat{a})}_{\mathrm{s}}\, \epsilon_{\alpha\beta} + b^{(\widehat{a})}_{+}\, (\epsilon\tau_+)_{\alpha\beta} + b^{(\widehat{a})}_{-}\, (\epsilon\tau_-)_{\alpha\beta} + b^{(\widehat{a})}_{3}\, (\epsilon\tau_3)_{\alpha\beta} \tag{5.1}$$

where ϵ is defined in (2.31) and $\tau_\pm = (\tau_1 \pm \mathrm{i}\tau_2)/2$. In the $M = 2$ case studied in [1] it was found that only the singlet component $b^{(\widehat{a})}_{\mathrm{s}}$ (which we denoted b in that reference) acted as a source for the gauge field. This can also be seen from (4.20) by setting $\nu_1 = \frac{1}{2}$ and $\omega = -1$ for the only twisted sector that is present when $M = 2$. For the general $M > 2$ case, however, we see that the gauge field couples to both the scalars $b^{(\widehat{a})}_{\mathrm{s}}$ and $b^{(\widehat{a})}_{3}$. Since we wish to have a uniform description of surface defects for all values of M, in what follows, we will set $b^{(\widehat{a})}_{3} = 0$ and only turn on the background value for $b^{(\widehat{a})}_{\mathrm{s}}$. Furthermore, we also turn on the doublet components $b^{(\widehat{a})}_{\pm}$ which source the scalar fields Φ and $\overline{\Phi}$. This means that, in terms of the initial fields $b^{(\widehat{a})}_{\alpha\beta}$, our background reads

$$\begin{aligned} &b^{(\widehat{a})}_{12} = -b^{(\widehat{a})}_{21} = -\mathrm{i}\, b^{(\widehat{a})}_{\mathrm{s}}\,, \\ &b^{(\widehat{a})}_{22} = b^{(\widehat{a})}_{+}\,, \qquad b^{(\widehat{a})}_{11} = -b^{(\widehat{a})}_{-}\,, \end{aligned} \tag{5.2}$$

with $\big(b^{(\widehat{a})}_{\mathrm{s}}\big)^* = b^{(M-\widehat{a})}_{\mathrm{s}}$ and $\big(b^{(\widehat{a})}_{+}\big)^* = b^{(M-\widehat{a})}_{-}$ for all twisted sectors, as follows from the relations (2.38).

Inserting these background values in (4.20) and (4.21), we have

$$\langle \mathcal{V}_{A_2} \rangle_{I} = -\kappa_2\, b_I\, \delta^{(2)}(\kappa_\parallel)\,, \tag{5.3}$$

and

$$\langle \mathcal{V}_{\Phi} \rangle_{I} = -\mathrm{i}\,\bar{\kappa}_2\, b^+_I\, \delta^{(2)}(\kappa_\parallel)\,, \quad \langle \mathcal{V}_{\overline{\Phi}} \rangle_{I} = -\mathrm{i}\,\kappa_2\, b^-_I\, \delta^{(2)}(\kappa_\parallel)\,, \tag{5.4}$$

where we have defined the combinations

$$b_I = \sum_{a=1}^{\frac{M-1}{2}} \sin\pi\nu_a \left[\omega^{-Ia}\, b_{\rm s}^{(a)} + \omega^{Ia}\, b_{\rm s}^{(M-a)}\right] = \sum_{\widehat{a}=1}^{M-1} \sin\left(\frac{\pi\widehat{a}}{M}\right) \omega^{-I\widehat{a}}\, b_{\rm s}^{(\widehat{a})}\,, \tag{5.5}$$

and

$$b_I^{\pm} = \sum_{a=1}^{\frac{M-1}{2}} \sin\pi\nu_a \left[\omega^{-Ia}\, b_{\pm}^{(a)} + \omega^{Ia}\, b_{\pm}^{(M-a)}\right] = \sum_{\widehat{a}=1}^{M-1} \sin\left(\frac{\pi\widehat{a}}{M}\right) \omega^{-I\widehat{a}}\, b_{\pm}^{(\widehat{a})}\,. \tag{5.6}$$

Notice that b_I is real, while $(b_I^+)^* = b_I^-$. It is interesting to note that a similar change of basis for profiles of closed string fields between the fractional branes (labelled by irreducible representations) and the twisted sectors (labelled by conjugacy classes) has been observed previously for fractional branes at orbifolds in [11].

As explained in detail in [1], these amplitudes are interpreted as a source for the corresponding open string field (see also figure 1), whose profile in configuration space is obtained by taking the Fourier transform, after attaching the massless propagator along the D3-brane world-volume:

$$\frac{1}{k^2} = \frac{1}{2\big(|\kappa_\parallel|^2 + |\kappa_\perp|^2\big)}\,. \tag{5.7}$$

For example, for the gauge field A_2 we have

$$A_{2;I} = \mathcal{FT}\left[\frac{\big\langle \mathcal{V}_{A_2}\big\rangle_I}{k^2}\right]. \tag{5.8}$$

In appendix B we show how to organize the calculation of this Fourier transform in terms of the generalized plane-waves $\mathcal{E}_I$ that transform covariantly with charge I under the orbifold group. Applying these methods to the present case, we see that since the source (5.3) is proportional to κ_2, which has charge (-1), only the term proportional to $\mathcal{E}_1$ remains so that (5.8) becomes

$$\begin{aligned} A_{2;I} &= \int \frac{d^2\kappa_\parallel d^2\kappa_\perp}{(2\pi)^2}\, \frac{\big\langle \mathcal{V}_{A_2}\big\rangle_I}{2(\kappa_\parallel^2+\kappa_\perp^2)}\, \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot z_\parallel}\, \mathcal{E}_1 \\ &= -b_I\, \frac{1}{M} \sum_{J=0}^{M-1} \omega^{-J} \int \frac{d^2\kappa_\perp}{(2\pi)^2}\, \frac{\kappa_2}{2|\kappa_\perp|^2}\, \mathrm{e}^{\mathrm{i}\,(\omega^{-J}\kappa_2\,\bar{z}_2 + \omega^{J}\,\bar{\kappa}_2\, z_2)} = -\frac{\mathrm{i}\, b_I}{4\pi\bar{z}_2}\,, \end{aligned} \tag{5.9}$$

where the last equality is a consequence of the fact that all M terms in the sum are actually all equal to each other and equal to $\mathrm{i}/(4\pi\bar{z}_2)$.

Combining this result with the one for the complex conjugate component $\overline{A}_2$, we find that the gauge field on the I-th fractional D3-brane has the following profile:

$$\mathbf{A}_I = A\cdot dx = A_{2;I}\, d\bar{z}_2 + \overline{A}_{2;I}\, dz_2 = -\frac{\mathrm{i}\, b_I}{4\pi}\left(\frac{d\bar{z}_2}{\bar{z}_2} - \frac{dz_2}{z_2}\right) = -\frac{b_I}{2\pi}\, d\theta\,, \tag{5.10}$$

where θ is the polar angle in the $\mathbb{C}_{(2)}$-plane transverse to the surface defect.

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The only other open string field that has a non-vanishing profile in the twisted NS/NS background we have chosen is the complex scalar Φ. The analogous calculation takes the following form:

$$\begin{aligned}\Phi_I &= \mathcal{FT}\Big[\frac{\big\langle\mathcal{V}_\Phi\big\rangle_I}{k^2}\Big] = \int\frac{d^2\kappa_\parallel d^2\kappa_\perp}{(2\pi)^2}\,\frac{\big\langle\mathcal{V}_\Phi\big\rangle_I}{2(\kappa_\parallel^2+\kappa_\perp^2)}\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot z_\parallel}\,\mathcal{E}_{M-1}\\ &= -\mathrm{i}\,b_I^+\,\frac{1}{M}\sum_{J=0}^{M-1}\omega^J\int\frac{d^2\kappa_\perp}{(2\pi)^2}\,\frac{\bar\kappa_2}{2|\kappa_\perp|^2}\,\mathrm{e}^{\mathrm{i}\,(\omega^J\kappa_2\,\bar z_2+\omega^J\,\bar\kappa_2\,z_2)} = \frac{b_I^+}{4\pi z_2}\,.\end{aligned} \tag{5.11}$$

If we now consider a general configuration with n_I fractional D3-branes of type I for all values of I, as in the KT proposal [4], we obtain the following profiles:

$$\mathbf{A} = -\frac{d\theta}{2\pi}\begin{pmatrix} b_0\,\mathbb{1}_{n_0} & 0 & \cdots & 0\\ 0 & b_1\,\mathbb{1}_{n_1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_{M-1}\,\mathbb{1}_{n_{M-1}}\end{pmatrix}, \tag{5.12}$$

and

$$\mathbf{\Phi} = \frac{1}{4\pi\,z_2}\begin{pmatrix} b_0^+\,\mathbb{1}_{n_0} & 0 & \cdots & 0\\ 0 & b_1^+\,\mathbb{1}_{n_1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_{M-1}^+\end{pmatrix}. \tag{5.13}$$

These are precisely the profiles of a GW surface defect in the $\mathcal{N}=4$ theory corresponding to the breaking of $\mathrm{U}(N)$ group to the Levi subgroup $\mathrm{U}(n_0)\times\ldots\times\mathrm{U}(n_{M-1})$, provided the continuous parameters $(\alpha_I,\beta_I,\gamma_I)$ that conventionally parametrize the singular profiles near the defect are related to the background values of the NS/NS twisted scalars as follows:

$$\alpha_I = -\frac{b_I}{2\pi}\,,\qquad \beta_I = \frac{\mathrm{Re}(b_I^+)}{2\pi}\,,\qquad \gamma_I = \frac{\mathrm{Im}(b_I^+)}{2\pi}\,. \tag{5.14}$$

If the original gauge group is $\mathrm{SU}(N)$, the corresponding field profiles are obtained by removing the overall trace from each of the above expressions.

We now turn to discussing the coupling of the open string fields with the twisted scalars in the R/R sector. As we have seen in section 4.2, we only need to consider the coupling with the longitudinal component A_1 of the gauge field. This is given in (4.30), which we rewrite as

$$\big\langle\mathcal{V}_{A_1}\big\rangle_I = -2\mathrm{i}\,\kappa_1 c_I\,\delta^{(2)}(\kappa_\parallel) \tag{5.15}$$

where

$$c_I = \sum_{a=1}^{\frac{M-1}{2}}\sin\pi\nu_a\,\Big[\,\omega^{-Ia}\,\mathcal{C}^{(a)}+\omega^{Ia}\,\mathcal{C}^{(M-a)}\Big] = \sum_{\widehat a=1}^{M-1}\sin\Big(\frac{\pi\widehat a}{M}\Big)\,\omega^{-I\widehat a}\,\mathcal{C}^{(\widehat a)}\,. \tag{5.16}$$

This real quantity is the R/R counterpart of b_I defined in (5.5) for the NS/NS sectors.

At face value, the coupling (5.15) is vanishing because of the δ-function. However, as was explained in the $\mathbb{Z}_2$ in [1], if we multiply this amplitude and its complex conjugate with the corresponding gauge field polarizations, the resulting sum can be interpreted as an interaction term between the R/R scalars and the longitudinal components of the gauge field strength. Indeed,

$$\overline{A}_{1,I}\,\big\langle\mathcal{V}_{A_1}\big\rangle_I + A_{1,I}\,\big\langle\mathcal{V}_{\overline{A}_1}\big\rangle_I = -2\,\mathrm{i}\,c_I\big(\bar{\kappa}_1 A_1 - \kappa_1\overline{A}_1\big)\,\delta^2(\kappa_\parallel) = 2\,\mathrm{i}\,c_I\,\widetilde{F}_I\,\delta^2(\kappa_\parallel)\,, \tag{5.17}$$

where $\widetilde{F}_I$ is the gauge field strength on the Ith fractional brane (along the defect), in momentum space. Performing the Fourier transform, this expression becomes an effective interaction term localized on the surface defect:

$$\frac{\mathrm{i}\,c_I}{2\pi}\int d^2z_\parallel\,F_I\,, \tag{5.18}$$

where F_I is the gauge field strength in configuration space, on the Ith fractional brane. If this has a non-trivial first Chern class, then this effective interaction can be understood as the $2d$ topological θ-term that can be included in the path integral definition of the theory with surface defect. When a generic configuration with n_I D3-branes of type I is considered, the following phase factor is therefore introduced in the path integral

$$\exp\left(\mathrm{i}\sum_{I=0}^{M-1}\frac{c_I}{2\pi}\int d^2z_\parallel\,\mathrm{Tr}_{\mathrm{U}(n_I)}F_I\right), \tag{5.19}$$

leading to the following identification of the η-parameters of the surface defect:

$$\eta_I = \frac{c_I}{2\pi}\;. \tag{5.20}$$

This completes the identification of all the parameters of the generic GW monodromy defect with the background values of the twisted scalars in the $\mathbb{Z}_M$ orbifold. We note that these formulas generalize those in [1] and exactly reduce to them when $M=2$. We also remark that if we write the parameters b_I, $b_I^\pm$ and c_I as sums over all twisted sectors, their relation with the parameters of the surface defects holds also for even M. In this case, in fact, beside the twisted sectors we have described at length in this paper, there is also a sector with twist $\frac{1}{2}$ whose contribution is exactly the same as in the $M=2$ case. For this reason, therefore, we see that the restriction we made at the beginning to restrict to odd values of M does not lead to any loss of generality.

We end this section by observing that the identifications (5.14) and (5.20), namely

$$\{\alpha_I,\beta_I,\gamma_I,\eta_I\} = \left\{-\frac{b_I}{2\pi},\frac{\mathrm{Re}(b_I^+)}{2\pi},\frac{\mathrm{Im}(b_I^+)}{2\pi},\frac{c_I}{2\pi}\right\}, \tag{5.21}$$

are consistent with the behavior of the GW parameters under S-duality, as given in [2]. In fact, even though our world-sheet analysis has been at the orbifold fixed point, it is possible to blow-up the $\mathbb{Z}_M$-singularity into an ALE space and provide an interpretation to the twisted scalars of the orbifold theory as massless moduli in the low-energy supergravity (see for instance [9, 18]). In such a geometric approach, the combinations b_I and c_I, which are made of the singlets $b_s^{(\widehat{a})}$ and $\mathcal{C}^{(\widehat{a})}$ from each twisted sector as shown in (5.5) and (5.16), arise by integrating, respectively, the NS/NS 2-form $B_{(2)}$ and the R/R 2-form $C_{(2)}$ of Type II B supergravity around the exceptional cycles ω_I of the blown-up ALE space. Therefore, from (5.21) we read

$$\alpha_I = -\frac{1}{2\pi}\int_{\omega_I} B_{(2)}\,, \qquad \eta_I = \frac{1}{2\pi}\int_{\omega_I} C_{(2)}\,. \tag{5.22}$$

Using the S-duality action on the 2-forms, with simple manipulations [1] one can show that this identification implies that α_I and η_I indeed transform in the expected way.

Similarly, the $b_I^{\pm}$ parameters can be identified with the (string frame) metric moduli corresponding to the complex structure of the blown-up exceptional cycle ω_I. As such they inherit the S-duality transformation properties from the (string frame) metric, which are precisely the ones expected for the parameters β_I and γ_I of the GW defects.

We finally remark that when $M > 2$ also the scalars $b_3^{(\widehat{a})}$ can couple to the gauge fields, differently from what happens in the $M = 2$ case [1]. To have a uniform description for all M we have therefore chosen to set $b_3^{(\widehat{a})} = 0$ in each twisted sector. As we have just seen, this choice has allowed us to identify a perturbative closed string realization of the generic GW defects that is fully consistent with S-duality. However, our approach offers the possibility of considering more general backgrounds with also $b_3^{(\widehat{a})}$ turned on, and it would be interesting to further investigate their meaning and implications for the world-volume theory on the D3-branes and their defects.

6 Discussion

The present work extends the analysis of [1], where the main ideas of our approach to a string theoretic realization of the GW surface defects were already anticipated. Here we have concentrated on the technical ingredients necessary to implement those ideas in the case of a generic half-BPS surface defect. Therefore we think it is useful to recapitulate at this point our motivations and the main features of our construction, and highlight some new perspectives and potential future developments.

The study of defects in quantum field theories is an important subject from many different points of view. For example, a proper understanding of conformal defects is a crucial step towards a complete classification of higher dimensional conformal field theories. In this context, much progress has been made in elucidating the kinematic constraints that the residual symmetry of conformal defects imposes on the observables of the theory, leading to their parametrization by some set of conformal data [19–23]. The kinematics is even more constrained for superconformal defects where stringent relations between the two-point functions of the displacement operator and the stress tensor one-point function for surface defects exist [24].

The general symmetry structure helps to tackle the dynamics of defects also in (super) Yang-Mills theories. The line defects corresponding to Wilson or 't Hooft lines represent a widely studied set of observables. Surface defects, whose definition is more delicate,

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are also extremely interesting, especially with regard to the duality properties of the theory. Groundbreaking work on conformal surface defects in gauge theories was carried out in [2, 3], where half-BPS monodromy defects in $\mathcal{N} = 4$ super Yang-Mills theories were characterized and their S-duality properties clarified. Many developments followed, giving such defects an holographic realization in type IIB supergravity [25, 26], extending the study to generic $\mathcal{N} = 2$ theories [27] and taking advantage of 6d and M-theory embeddings [28] and of localization techniques [4, 29–32].

What we have done in this work is to directly realize the GW monodromy defects within perturbative Type II B string theory using fractional D3-branes on orbifolds. This realization was already suggested in [4] where it was shown that the instanton contributions to the effective theory of surface defects are organized in terms of chain-saw quivers and described as D-instanton corrections to a system of fractional D3-branes with two world-volume directions extended along the orbifold background (see also [31, 33]). Here we have taken this picture seriously and showed that such a D3-brane configuration with a partially longitudinal orbifold action is a GW defect. It was already clear from the KT construction that the discrete data of a GW defect are represented by the order M of the orbifold group and by the numbers n_I of D3-branes assigned to the I-th irreducible representation of $\mathbb{Z}_M$. What was missing, however, was the identification of the non-trivial profiles of the gauge fields around the defect and their continuous monodromy parameters. Here we have filled this gap showing for a generic defect how these continuous data are encoded within the D-brane configuration.

As we already pointed out in [1], our description in terms of closed string background fields has some similarities with the holographic realization of surface defects as bubbling geometries of Type II B supergravity that asymptote to $AdS_5 \times S^5$ [25, 26]; indeed, in that realization, like in ours, the continuous parameters of the defects are mapped to integrals of the NS/NS and R/R 2-forms over suitable cycles. Our construction, however, is based on an exactly solvable string background — D3-branes on an orbifold — in which explicit world-sheet computations are possible. Moreover, we are on the gauge theory side of the holographic correspondence: the branes have not dissolved into geometry and the open string degrees of freedom are explicitly present. It would be a worthwhile exercise to relate our D-branes on orbifolds to the bubbling geometries of [25, 26].

For simplicity we have considered surface defects in $\mathcal{N} = 4$ U(N) theories, but our analysis can be extended to cases with lower supersymmetry and/or with other gauge groups. For example, by introducing a mass deformation in two of the directions transverse to the D3-branes [31] we can realize the so-called $\mathcal{N} = 2^*$ theory, or by implementing another orbifold acting purely in directions transverse to the D3-branes we can obtain a $\mathcal{N} = 2$ theory. Furthermore, by introducing orientifold planes we can get models and defects with orthogonal or symplectic gauge groups. Exploring in this fashion these set-ups represents a logical line of development.

Let us remark once more that the orbifold that realizes the GW defects has a different behavior with respect to the orbifolds that are usually considered. In fact, as discussed in section 3.2, this orbifold not only acts on the oscillators and the Chan-Paton indices of the open string states, but also on the components of their momentum transverse to the defect.

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This action on the momentum can therefore compensate the corresponding action on the oscillators and the Chan-Paton factors, so that no state is projected out; rather, a specific momentum dependence is imposed. Therefore, on the world-volume of the fractional D3-branes we find the same field content of the $\mathcal{N} = 4$ super Yang-Mills theory. The exception to this pattern is represented by the open strings with no momentum transverse to the defect. Out of these states, the orbifold selects a subset of states and halves the amount of supersymmetry. Such states, which we did not investigate in the present work, represent the defect sector of the defect CFT. The bulk operators are instead represented by closed and open string vertices with non-zero momenta in the directions transverse to the defect. Mapping correlators of bulk and defect operators to ordinary string world-sheet diagrams could prove to be a useful tool in the investigation of the defect dynamics. This is another direction worth investigating.

The perturbative string theory realization of a non-trivial sector of the gauge theory that we have described bears many analogies with the explicit derivation of the gauge instanton profiles from D3/D-instanton systems [34–36] via the emission of open strings from disk diagrams with mixed boundary conditions [37]. The role of the instanton moduli is played in the construction of the surface defect by the insertion of the twisted closed string at zero momentum. The direct realization of instantons as a solvable D-brane background, besides its intrinsic interest, turned out to be very useful in evaluating instanton effects in deformed theories [38–40] as well as in engineering "exotic" instantons of purely stringy origin [41–43], possibly giving rise to effects otherwise prohibited in the effective field theory. Similarly, in the case of surface defects, it is possible that having a microscopic stringy realization might suggest some novel effects in the defect gauge theory. We hope to explore these and related issues in the future.

Acknowledgments

We would like to thank Abhijit Gadde, Dileep Jatkar, Igor Pesando, Naveen Prabhakar, Madhusudhan Raman and Ashoke Sen for helpful discussions and correspondence and Renjan John for collaboration at an early stage of the project. The work of A.L. is partially supported by "Fondi Ricerca Locale dell'Università del Piemonte Orientale".

A Conventions

In this appendix we list our conventions for spinors both in the $4d$ space along the $\mathbb{Z}_M$ orbifold, and in the $6d$ space transverse to it.

A.1 Spinors in $4d$

We consider a $4d$ space parametrized by the two complex coordinates z_2 and z_3, related to the four real coordinates x_m (with $m = 3, 4, 5, 6$) as in (2.1). Introducing the Pauli matrices

$$\sigma^m = \left(\tau^1, \tau^2, \tau^3, -i\,\mathbf{1}_2\right), \tag{A.1}$$

we can form the combination

$$X_{\alpha\dot\beta} = \frac{1}{\sqrt{2}}\, x_m \,(\sigma^m)_{\alpha\dot\beta} = \begin{pmatrix} \bar z_3 & \bar z_2 \\ z_2 & -z_3 \end{pmatrix} . \tag{A.2}$$

The $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ isometry group acts on X as follows

$$X \quad\longrightarrow\quad U_+\, X\, U_-^\dagger \tag{A.3}$$

where $U_\pm \in \mathrm{SU}(2)_\pm$. Therefore, the two columns of X are two doublets transforming as spinors of $\mathrm{SU}(2)_+$:

$$y_\alpha = \begin{pmatrix} \bar z_3 \\ z_2 \end{pmatrix} \quad\text{and}\quad w_\alpha = \begin{pmatrix} \bar z_2 \\ -z_3 \end{pmatrix} . \tag{A.4}$$

Raising the indices, we have

$$y^\alpha = y_\beta\, (\epsilon^{-1})^{\beta\alpha} = \begin{pmatrix} -z_2 \\ \bar z_3 \end{pmatrix} \quad\text{and}\quad w^\alpha = w_\beta\, (\epsilon^{-1})^{\beta\alpha} = \begin{pmatrix} z_3 \\ \bar z_2 \end{pmatrix} \tag{A.5}$$

where $\epsilon = -\mathrm{i}\,\tau_2$ as in (2.31). Of course the same combinations can be made with the fermionic coordinates leading to the doublets

$$\begin{pmatrix} -\Psi^2 \\ \bar\Psi^3 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \Psi^3 \\ \bar\Psi^2 \end{pmatrix} . \tag{A.6}$$

These are precisely the structures that have been used in section 2 to write the massless vertex operators of the twisted NS/NS sectors.

A.2 Spinors in 6d

We consider a 6d Euclidean space spanned by the coordinates x_M with $M \in \{1, 2, 7, 8, 9, 10\}$, in order to respect the configuration of the orbifold (1.1). The 6d Euclidean Clifford algebra is given by

$$\{\Gamma_M, \Gamma_N\} = 2\delta_{MN} , \tag{A.7}$$

and an explicit realization of the Γ matrices is given by:

$$\begin{aligned}
\Gamma_1 &= \begin{pmatrix} 0 & 0 & -\mathrm{i}\,\mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathrm{i}\,\mathbb{1}_2 \\ \mathrm{i}\,\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & \mathrm{i}\,\mathbb{1}_2 & 0 & 0 \end{pmatrix} , &\qquad \Gamma_2 &= \begin{pmatrix} 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & -\tau_3 \\ \tau_3 & 0 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 \end{pmatrix} , \\
\Gamma_7 &= \begin{pmatrix} 0 & 0 & -\tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -\tau_2 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \end{pmatrix} , &\qquad \Gamma_8 &= \begin{pmatrix} 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & -\tau_1 \\ \tau_1 & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{pmatrix} , \\
\Gamma_9 &= \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i}\,\mathbb{1}_2 \\ 0 & 0 & \mathrm{i}\,\mathbb{1}_2 & 0 \\ 0 & -\mathrm{i}\,\mathbb{1}_2 & 0 & 0 \\ \mathrm{i}\,\mathbb{1}_2 & 0 & 0 & 0 \end{pmatrix} , &\qquad \Gamma_{10} &= \begin{pmatrix} 0 & 0 & 0 & \mathbb{1}_2 \\ 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \\ \mathbb{1}_2 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{A.8}$$

It follows that the $6d$ chirality matrix $\widehat{\Gamma}$ is

$$\widehat{\Gamma} = \mathrm{i}\,\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9\Gamma_{10} = \begin{pmatrix} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \\ 0 & 0 & -\mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{pmatrix} , \tag{A.9}$$

which shows that, in this basis, a Dirac spinor is written as

$$\begin{pmatrix} S^A \\ S^{\dot{A}} \end{pmatrix} \tag{A.10}$$

where A and $\dot{A}$ label, respectively, the chiral and anti-chiral components. The charge conjugation matrix C, in this basis, is given by

$$C = \begin{pmatrix} 0 & 0 & 0 & \epsilon \\ 0 & 0 & \epsilon & 0 \\ 0 & -\epsilon & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \end{pmatrix} \tag{A.11}$$

where $\epsilon = -\mathrm{i}\,\tau_2$ as in (2.31). The charge conjugation matrix is such that

$$C\,\Gamma_M\,C^{-1} = -(\Gamma_M)^{\mathrm{t}} \ . \tag{A.12}$$

B $\mathbb{Z}_M$ in momentum space

Here we briefly comment on how to define the $\mathbb{Z}_M$ orbifold action in momentum space. Let us take the complex plane $C_{(2)}$ with coordinates z_2 and $\bar{z}_2$ on which $\mathbb{Z}_M$ acts as in (2.2), and define the momenta κ_2 and $\bar{\kappa}_2$ as in (3.27). For simplicity, however, we can drop the index 2 since in this appendix this does not cause any ambiguity.

First of all, we observe that the orbifold action on the coordinates can be equivalently read as an inverse action on the momenta. Consider for example the scalar product

$$\kappa\,\bar{z} + \bar{\kappa}\,z\,, \tag{B.1}$$

which, under the action of $\mathbb{Z}_M$ on the coordinates, is mapped to

$$\omega^{-1}\kappa\,\bar{z} + \omega\,\bar{\kappa}\,z \ . \tag{B.2}$$

Clearly, this result can also be interpreted as due to the following action of $\mathbb{Z}_M$ on the momentum variables:

$$\hat{g}\;:\;(\kappa\,,\,\bar{\kappa}) \longrightarrow (\omega^{-1}\,\kappa\,,\,\omega\,\bar{\kappa}) \tag{B.3}$$

with the coordinates held fixed.

Then, let us consider a function in momentum space, $f(\kappa,\bar{\kappa})$, and define its images under the orbifold group according to

$$\Pi_I(\kappa,\bar{\kappa}) = \frac{1}{M}\sum_{J=0}^{M-1}\omega^{-IJ}\,f\big(\omega^{-J}\kappa,\omega^{J}\,\bar{\kappa}\big) \tag{B.4}$$

where $I = 0, \ldots, M-1$, modulo M. Using (B.3), it is immediate to check that

$$\hat{g}\left[\Pi_I\right] = \omega^I \, \Pi_I \, , \tag{B.5}$$

namely that Π_I transforms in the I-th representation of $\mathbb{Z}_M$. Inverting (B.4), we get

$$f(\kappa, \bar{\kappa}) = \sum_{I=0}^{M-1} \Pi_I(\kappa, \bar{\kappa}) \, . \tag{B.6}$$

Applying these definitions to the plane wave $\mathrm{e}^{\mathrm{i}\,(\kappa\,\bar{z}+\bar{\kappa}\,z)}$, we get

$$\mathcal{E}_I = \frac{1}{M} \sum_{J=0}^{\infty} \omega^{-IJ} \, \mathrm{e}^{\mathrm{i}\,(\omega^{-J}\kappa\,\bar{z}+\omega^{J}\bar{\kappa}\,z)} \, , \tag{B.7}$$

with

$$\hat{g}\left[\mathcal{E}_I\right] = \omega^I \, \mathcal{E}_I \, . \tag{B.8}$$

These functions $\mathcal{E}_I$ have exactly the same form and properties of the functions introduced in section 3.2 when we described the $\mathbb{Z}_M$-invariant open string states. In terms of them, the plane wave can be written as

$$\mathrm{e}^{\mathrm{i}\,(\kappa\,\bar{z}+\bar{\kappa}\,z)} = \sum_{I=0}^{M-1} \mathcal{E}_I \, . \tag{B.9}$$

Let us now consider the Fourier transform of f. Using (B.6) and (B.9), we have

$$\mathcal{FT}[f](z) = \int \frac{d^2\kappa}{2\pi} \, f(\kappa, \bar{\kappa}) \, \mathrm{e}^{\mathrm{i}\,(\kappa\,\bar{z}+\bar{\kappa}\,z)} = \int \frac{d^2\kappa}{2\pi} \sum_{I,J=0}^{M-1} \Pi_I(\kappa, \bar{\kappa}) \, \mathcal{E}_J \, . \tag{B.10}$$

Since the integration measure is $\mathbb{Z}_M$-invariant, only the invariant products $\Pi_I \, \mathcal{E}_{M-I}$ survive, and thus

$$\mathcal{FT}[f](z) = \int \frac{d^2\kappa}{2\pi} \sum_{I=0}^{M-1} \Pi_I(\kappa, \bar{\kappa}) \, \mathcal{E}_{M-I} \;=\; \frac{1}{M} \int \frac{d^2\kappa}{2\pi} \sum_{I=0}^{M-1} \hat{g}^I \left[f(\kappa, \bar{\kappa}) \, \mathrm{e}^{\mathrm{i}\,(\kappa\,\bar{z}+\bar{\kappa}\,z)} \right] \, . \tag{B.11}$$

This shows that the Fourier transform leads to a well-defined function in the orbifolded theory. In particular, the Fourier transform of a function in the I-th irreducible reprentation of $\mathbb{Z}_M$ in momentum space is a function in configuration space that transforms in the representation $(M-I)$, and *viceversa*.

References

[1] S.K. Ashok, M. Billó, M. Frau, A. Lerda and S. Mahato, *Surface defects from fractional branes. Part I*, *JHEP* **07** (2020) 051 [arXiv:2005.02050] [INSPIRE].

[2] S. Gukov and E. Witten, *Gauge theory, ramification, and the geometric Langlands program*, hep-th/0612073 [INSPIRE].

[3] S. Gukov and E. Witten, *Rigid surface operators*, *Adv. Theor. Math. Phys.* **14** (2010) 87 [arXiv:0804.1561] [INSPIRE].

[4] H. Kanno and Y. Tachikawa, *Instanton counting with a surface operator and the chain-saw quiver*, *JHEP* **06** (2011) 119 [arXiv:1105.0357] [INSPIRE].

[5] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys. B* **271** (1986) 93 [INSPIRE].

[6] V. Kostelecky, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, *Conformal techniques, bosonization and tree level string amplitudes*, *Nucl. Phys. B* **288** (1987) 173 [INSPIRE].

[7] P. Di Vecchia and A. Liccardo, *D branes in string theory, I*, *NATO Sci. Ser. C* **556** (2000) 1 [hep-th/9912161] [INSPIRE].

[8] P. Di Vecchia and A. Liccardo, *D-branes in string theory. 2.*, in the proceedings of the *YITP Workshop on Developments in Superstring and M-theory*,October 27–29, Kyoto, Japan (1999), hep-th/9912275 [INSPIRE].

[9] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167 [INSPIRE].

[10] M. Billó, B. Craps and F. Roose, *Orbifold boundary states from Cardy's condition*, *JHEP* **01** (2001) 038 [hep-th/0011060] [INSPIRE].

[11] M. Billó, L. Gallot and A. Liccardo, *Classical geometry and gauge duals for fractional branes on ALE orbifolds*, *Nucl. Phys. B* **614** (2001) 254 [hep-th/0105258] [INSPIRE].

[12] M. Bertolini, P. Di Vecchia and R. Marotta, $N = 2$ *four-dimensional gauge theories from fractional branes*, hep-th/0112195 [INSPIRE].

[13] M. Bertolini, M. Billó, A. Lerda, J.F. Morales and R. Russo, *Brane world effective actions for D-branes with fluxes*, *Nucl. Phys. B* **743** (2006) 1 [hep-th/0512067] [INSPIRE].

[14] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys. B* **282** (1987) 13 [INSPIRE].

[15] M. Billóet al., *Microscopic string analysis of the D0-D8-brane system and dual R-R states*, *Nucl. Phys. B* **526** (1998) 199 [hep-th/9802088] [INSPIRE].

[16] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys. B* **324** (1989) 581 [INSPIRE].

[17] A. Hashimoto and I.R. Klebanov, *Scattering of strings from D-branes*, *Nucl. Phys. B Proc. Suppl.* **55** (1997) 118 [hep-th/9611214] [INSPIRE].

[18] D. Anselmi, M. Billó, P. Fré, L. Girardello and A. Zaffaroni, *ALE manifolds and conformal field theories*, *Int. J. Mod. Phys. A* **9** (1994) 3007 [hep-th/9304135] [INSPIRE].

[19] D.M. McAvity and H. Osborn, *Conformal field theories near a boundary in general dimensions*, *Nucl. Phys. B* **455** (1995) 522 [cond-mat/9505127] [INSPIRE].

JHEP08(2020)058

[20] M. Billò, V. Gonçalves, E. Lauria and M. Meineri, *Defects in conformal field theory*, *JHEP* **04** (2016) 091 [arXiv:1601.02883] [INSPIRE].

[21] A. Gadde, *Conformal constraints on defects*, *JHEP* **01** (2020) 038 [arXiv:1602.06354] [INSPIRE].

[22] E. Lauria, M. Meineri and E. Trevisani, *Radial coordinates for defect CFTs*, *JHEP* **11** (2018) 148 [arXiv:1712.07668] [INSPIRE].

[23] E. Lauria, M. Meineri and E. Trevisani, *Spinning operators and defects in conformal field theory*, *JHEP* **08** (2019) 066 [arXiv:1807.02522] [INSPIRE].

[24] L. Bianchi and M. Lemos, *Superconformal surfaces in four dimensions*, *JHEP* **06** (2020) 056 [arXiv:1911.05082] [INSPIRE].

[25] J. Gomis and S. Matsuura, *Bubbling surface operators and S-duality*, *JHEP* **06** (2007) 025 [arXiv:0704.1657] [INSPIRE].

[26] N. Drukker, J. Gomis and S. Matsuura, *Probing $\mathcal{N}=4$ SYM with surface operators*, *JHEP* **10** (2008) 048 [arXiv:0805.4199] [INSPIRE].

[27] D. Gaiotto, *Surface operators in $\mathcal{N}=2$ 4d gauge theories*, *JHEP* **11** (2012) 090 [arXiv:0911.1316] [INSPIRE].

[28] O. Chacaltana, J. Distler and Y. Tachikawa, *Nilpotent orbits and codimension-two defects of 6d $\mathcal{N}=(2,0)$ theories*, *Int. J. Mod. Phys. A* **28** (2013) 1340006 [arXiv:1203.2930] [INSPIRE].

[29] H. Awata, H. Fuji, H. Kanno, M. Manabe and Y. Yamada, *Localization with a surface operator, irregular conformal blocks and open topological string*, *Adv. Theor. Math. Phys.* **16** (2012) 725 [arXiv:1008.0574] [INSPIRE].

[30] L.F. Alday and Y. Tachikawa, *Affine* SL(2) *conformal blocks from 4d gauge theories*, *Lett. Math. Phys.* **94** (2010) 87 [arXiv:1005.4469] [INSPIRE].

[31] S.K. Ashok, M. Billó, E. Dell'Aquila, M. Frau, R.R. John and A. Lerda, *Modular and duality properties of surface operators in $\mathcal{N}=2*$ gauge theories*, *JHEP* **07** (2017) 068 [arXiv:1702.02833] [INSPIRE].

[32] A. Gorsky, B. Le Floch, A. Milekhin and N. Sopenko, *Surface defects and instanton-vortex interaction*, *Nucl. Phys. B* **920** (2017) 122 [arXiv:1702.03330] [INSPIRE].

[33] S. Nawata, *Givental J-functions, quantum integrable systems, AGT relation with surface operator*, *Adv. Theor. Math. Phys.* **19** (2015) 1277 [arXiv:1408.4132] [INSPIRE].

[34] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys. B* **460** (1996) 335 [hep-th/9510135] [INSPIRE].

[35] M.R. Douglas, *Gauge fields and D-branes*, *J. Geom. Phys.* **28** (1998) 255 [hep-th/9604198] [INSPIRE].

[36] M.B. Green and M. Gutperle, *D instanton induced interactions on a D3-brane*, *JHEP* **02** (2000) 014 [hep-th/0002011] [INSPIRE].

[37] M. Billó, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, *Classical gauge instantons from open strings*, *JHEP* **02** (2003) 045 [hep-th/0211250] [INSPIRE].

[38] M. Billó, M. Frau, I. Pesando and A. Lerda, *$\mathcal{N}=1/2$ gauge theory and its instanton moduli space from open strings in RR background*, *JHEP* **05** (2004) 023 [hep-th/0402160] [INSPIRE].

JHEP08(2020)058

[39] M. Billó, M. Frau, S. Sciuto, G. Vallone and A. Lerda, *Non-commutative (D)-instantons*, *JHEP* **05** (2006) 069 [hep-th/0511036] [INSPIRE].

[40] M. Billó, M. Frau, F. Fucito and A. Lerda, *Instanton calculus in RR background and the topological string*, *JHEP* **11** (2006) 012 [hep-th/0606013] [INSPIRE].

[41] R. Blumenhagen, M. Cvetič and T. Weigand, *Spacetime instanton corrections in 4D string vacua: the seesaw mechanism for D-brane models*, *Nucl. Phys. B* **771** (2007) 113 [hep-th/0609191] [INSPIRE].

[42] L.E. Ibáñez and A.M. Uranga, *Neutrino Majorana masses from string theory instanton effects*, *JHEP* **03** (2007) 052 [hep-th/0609213] [INSPIRE].

[43] M. Billó, L. Ferro, M. Frau, F. Fucito, A. Lerda and J.F. Morales, *Non-perturbative effective interactions from fluxes*, *JHEP* **12** (2008) 102 [arXiv:0807.4098] [INSPIRE].

JHEP08(2020)058