

On the D(-1)/D7-brane systems

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ABSTRACT: We study non-perturbative effects in supersymmetric $U(N)$ gauge theories in eight dimensions realized by means of D(-1)/D7-brane systems with non-trivial world-volume fluxes turned on. Using an explicit string construction in terms of vertex operators, we derive the action for the open strings ending on the D(-1)-branes and exhibit its BRST structure. The space of vacua for these open strings is shown to be in correspondence with the moduli space of generalized ADHM gauge connections which trigger the non-perturbative corrections in the eight-dimensional theory. These corrections are computed via localization and turn out to depend on the curved background used to localize the integrals on the instanton moduli space, and vanish in flat space. Finally, we show that for specific choices of the background the instanton partition functions reduce to weighted sums of the solid partitions of the integers.

KEYWORDS: D-branes, Extended Supersymmetry, Solitons Monopoles and Instantons, Field Theories in Higher Dimensions

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1 Introduction

Non-perturbative effects in gauge theories can be rephrased in the language of string theory by considering the effects induced by branes of lower dimensions distributed along the world-volume of branes of higher dimensions. The prime example of this construction is represented by the D(-1)/D3-brane system which realizes the ADHM moduli space of gauge instantons in four-dimensional gauge theories [1–7]. The massless sector of the open strings with at least one end-point on the D(-1)-branes accounts for the moduli of the gauge instanton solutions in four dimensions, and their effective action can be recovered by computing string scattering amplitudes on disk diagrams connecting the two stacks of branes [3–6].

This analysis can be generalized to other brane systems. In particular, new stringy non-perturbative corrections to four-dimensional gauge theories can be obtained by adding Euclidean D3-branes wrapping non-trivial cycles of the internal space. When orientifold planes are inserted, these exotic instanton configurations [8–15] can generate non-perturbative effects in the effective action which are prohibited in perturbation theory, like for instance certain Majorana mass terms or Yukawa couplings, which may be relevant for phenomenological applications.

These methods can also be used to study non-perturbative effects in eight-dimensional field theories by considering systems of D7-branes in presence of D-instantons. Indeed, the prepotential generated by exotic instantons in $O(N)$ gauge theories in eight dimensions has been studied in [16, 17] by means of a system made of D(-1) and D7-branes on top of an orientifold O7-plane, allowing to explicitly test the non-perturbative heterotic/Type I string duality [18]. More general Dp/Dp' systems can be also considered after turning on fluxes for the Neveu-Schwarz B -field, in such a way that supersymmetry is restored in the vacuum [1, 19].

The peculiarity of all these exotic or higher dimensional instanton systems is that they lack the bosonic moduli connected to the size and the gauge orientation of the instanton configurations. Moreover, they possess extra fermionic zero-modes, in addition to those connected with the broken supersymmetries, which can make their contributions to the effective action vanish if they are not properly lifted or removed. The problems related to the presence of these extra fermionic zero-modes can be cured in various ways, for example by adding orientifold or orbifold projections, or more generically by considering curved backgrounds. However, in all these cases the non-perturbative configurations always remain point-like since no bosonic moduli can account for a non-zero size. The fundamental reason for this is that in all these exotic configurations there are eight directions with mixed Dirichlet/Neumann boundary conditions, differently from the standard cases in which there are only four mixed directions.

The non-perturbative contributions to the gauge theory effective action are obtained after an explicit evaluation of the instanton partition functions which are expressed as integrals over the instanton moduli space. This is possible thanks to the localization techniques introduced in [20] for the $\mathcal{N} = 2$ super Yang-Mills theories in four dimensions, building also on previous results in [21]. In the end, the instanton partition functions are reduced to integrals of rational functions which can be performed using standard complex analysis methods. Actually, the poles of these rational functions that contribute to the integrals can be put in one-to-one correspondence with sets of Young tableaux which in turn are related to the partitions of the integer numbers [20, 22–26]. When this approach is used for gauge theories in dimensions higher than four, higher dimensional Young tableaux appear and a connection with the planar and solid partitions of the integers arises [24, 27]. The instanton partition functions are thus expressed as weighted sums over integer, plane or solid partitions. Therefore, studying these systems may be interesting not only for the string theory applications we mentioned above, but also for mathematical reasons, related directly to such weighted sums (the generating function of solid partitions is not yet known in closed form) but also, for instance, in the study of Donaldson-Thomas invariants of Calabi-Yau four-folds [28].

Recently, the instanton partition functions of a pure supersymmetric $U(N)$ gauge theory in nine dimensions and of its conformal extension with a $U(N)$ flavor symmetry have been analyzed from this point of view in [27, 29] and shown to be given by a simple all-instanton plethystic exponential formula. More recently, these systems have been generalized to non-conformal set-ups with a $U(M)$ flavor symmetry in [30] where also the instanton contributions to the chiral correlators have been computed. In all cases it turns out that the non-perturbative sectors are described by sets of moduli that share some features of both the standard and exotic instantons. Thus, in [27, 29] it has been conjectured that they might arise from the open strings of a D0/D8-brane system with a Neveu-Schwarz B -field and with anti-D8 branes in the background.

The aim of this paper is to provide an explicit string theory derivation of the moduli space and of the instanton partition functions studied in [27, 29]. We consider a D(-1)/D7-brane system in Type II B string theory and do not introduce any anti-branes which would make the configuration unstable, but instead turn on a magnetic flux along the world-volume of the D7-branes.¹ More precisely, we start from a stack of $(N + M)$ D7-branes to describe a gauge theory in eight dimensions; then we introduce a constant magnetic field on the first N D7-branes to break the gauge group to $U(N) \times U(M)$, maintaining stability and supersymmetry. Finally, we add k D(-1)-branes. The open strings with at least one end-point on the D-instantons describe the moduli space of the non-perturbative configurations. The spectrum of the open strings that start and end on the D(-1)-branes is standard and includes, among others, the bosonic moduli describing the positions of the instantonic branes in the world-volume of the seven-branes. The spectrum of the mixed strings stretching between the D(1) and the D7-branes is instead peculiar because, despite the presence of eight directions with mixed Dirichlet/Neumann boundary conditions, due to the presence of the magnetic flux, it comprises a set of bosonic moduli that can be associated to the size and orientations of the instanton configurations in eight dimensions.

We also provide a detailed analysis of the vertex operators associated to all moduli and use them to derive the effective instanton action from disk amplitudes. After introducing vacuum expectation values for the scalar fields on the D7-branes and turning on a closed-string background with Ramond-Ramond fluxes, which is known to mimic the so-called Ω -background in the moduli space [31], we compute the instanton partition function using localization. Our results agree with those in [27, 29]; in particular for a suitable choice of the background the instanton partition function reduces to a weighted sum over the solid partitions of the integers. Our derivation also suggests a possible consistent generalization of the results of [27, 29] in which the Ω background is less constrained; we will investigate this possibility in a future work.

Finally, we show that the moduli space of vacua of the matrix theory defined on the D(-1)-branes is compatible with an ADHM construction of instanton connections in eight dimensions, which has also been recently discussed in [32]. In the concluding section we comment on the significance of our results for the eight-dimensional gauge theory, which

¹Notice that this magnetic flux is an open string background and thus is not represented by a Neveu-Schwarz B -field which belongs to the closed string sector.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| — | — | — | — | — | — | — | — | * | * |

Table 1. The D7-branes are aligned along the first eight directions.

receives non-perturbative corrections from these instanton configurations only in its U(1) part and only in curved space. Our notations and conventions, together with some more technical material, are collected in the appendices.

2 The D7-brane system and its open strings

In Type II B string theory, we consider a stack of $(N + M)$ D7-branes aligned along the directions $\mu, \nu, \dots = 1, \dots, 8$ (with Euclidean signature). The directions 9 and 10 are, instead, transverse.

On the world-volume of these D7-branes there is an eight-dimensional gauge theory with group $U(N + M)$ and sixteen supercharges. The part of the effective action which only depends on the gauge field strength can be written as

$$S_{D7} = S_2 + S_4 + \dots \quad (2.1)$$

Here S_2 is the quadratic Yang-Mills action in eight dimensions

$$S_2 = \frac{1}{2g_{\text{YM}}^2} \int d^8x \operatorname{tr}(F^2) \quad (2.2)$$

with a dimensionful gauge coupling constant

$$g_{\text{YM}}^2 \equiv 4\pi g_s (2\pi\sqrt{\alpha'})^4, \quad (2.3)$$

(g_s is the string coupling and $\sqrt{\alpha'}$ is the string length), while S_4 is a quartic action of the form

$$S_4 = -\frac{1}{4!\lambda^4} \int d^8x \operatorname{tr}(t_8 F^4) - \frac{i\vartheta}{4!(2\pi)^4} \int d^8x \operatorname{tr}(F \wedge F \wedge F \wedge F) \quad (2.4)$$

where

$$\lambda^4 \equiv 4\pi^3 g_s \quad (2.5)$$

is a dimensionless coupling, and ϑ is the vacuum angle, which in string theory is identified with the scalar field C_0 of the RR sector according to

$$\vartheta = 2\pi C_0. \quad (2.6)$$

The eight-index tensor t_8 appearing in the first term of (2.4) is such that [33]²

$$\begin{aligned} \operatorname{tr}(t_8 F^4) &\equiv \frac{1}{16} t_8^{\mu_1 \mu_2 \dots \mu_7 \mu_8} \operatorname{tr}(F_{\mu_1 \mu_2} \dots F_{\mu_7 \mu_8}) \\ &= \operatorname{tr} \left(F_{\mu\nu} F^{\nu\rho} F^{\lambda\mu} F_{\rho\lambda} + \frac{1}{2} F_{\mu\nu} F^{\rho\nu} F_{\rho\lambda} F^{\mu\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \frac{1}{8} F_{\mu\nu} F_{\rho\lambda} F^{\mu\nu} F^{\rho\lambda} \right). \end{aligned} \quad (2.7)$$

²For an explicit definition of t_8 , see for instance appendix B of [34].

Finally, the ellipses in (2.1) stand for α' corrections containing at least five field strengths or their covariant derivatives.

The gauge field strength $F_{\mu\nu}$ is actually part of a scalar superfield $\Phi(x, \theta)$ in eight dimensions, defined as

$$\Phi(x, \theta) = \phi(x) + \sqrt{2} \theta \Lambda(x) + \frac{1}{2} \theta \sigma^{\mu\nu} \theta F_{\mu\nu}(x) + \dots \quad (2.8)$$

where θ is the fermionic superspace coordinate, ϕ is a complex scalar and Λ is an eight-dimensional chiral fermion.³ In terms of the superfield Φ , the quartic action (2.4) (plus its supersymmetric completion) can be written as

$$S_4 = \tau \int d^8x d^8\theta \operatorname{tr}(\Phi^4) + \text{c.c.} \quad (2.9)$$

where τ is the complexified string coupling

$$\tau = C_0 + \frac{i}{g_s} . \quad (2.10)$$

In principle the quartic action (2.9) can receive quantum corrections and takes the general form

$$S'_4 = \int d^8x d^8\theta \mathcal{F}(\Phi, \tau) + \text{c.c.} \quad (2.11)$$

where $\mathcal{F}(\Phi, \tau)$ is the (holomorphic) prepotential.

We now introduce a (constant) background field on the first N D7-branes. The other M D7-branes, which remain without background field, will be called from now on D7'-branes. In this way the initial gauge symmetry group $U(N + M)$ is broken to $U(N) \times U(M)$. In particular, we consider a background that corresponds to a constant flux on the D7-brane world-volume described by

$$2\pi\alpha' F^{(0)} = \begin{pmatrix} 0 & +f_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +f_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +f_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +f_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -f_4 & 0 \end{pmatrix} \mathbb{1}_{N \times N} . \quad (2.12)$$

In the following we will specify the four parameters f_1, \dots, f_4 in such a way to preserve some supersymmetry, but for the time being we may consider them as arbitrary.⁴ The background (2.12) gives rise to four different types of open strings, as represented in figure 1: the 7/7 strings starting and ending on the N D7-branes with fluxes, the 7'/7' strings starting and ending on the M D7'-branes without fluxes, and the 7/7' or 7'/7 strings which start

³See appendix A for our conventions on spinors and Dirac matrices.

⁴We notice that the flux breaks Lorentz invariance in the eight-dimensional space, so that the preserved supersymmetry can be also viewed as a two-dimensional supersymmetry in the transverse space.

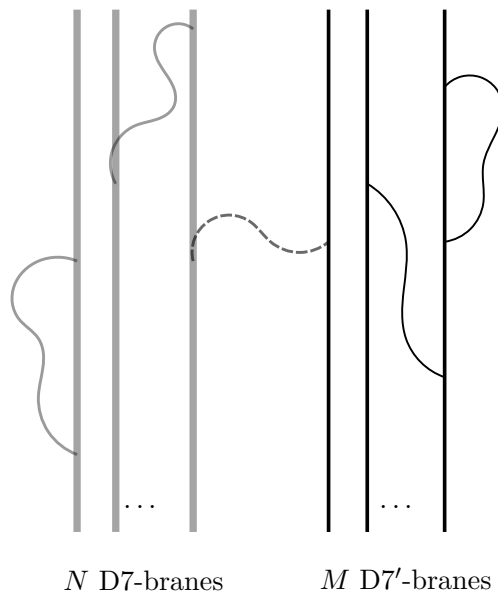


Figure 1. A background flux is turned on on the N D7 branes. The D7/D7 and D7'/D7' open strings are untwisted, the mixed ones (depicted as dashed) are instead twisted.

and end on branes of different type. The 7/7 and 7'/7' strings are untwisted and contain exactly the same physical states of the original strings. In particular they give rise to the same effective action (2.1) for the group $U(N)$ (in presence of the background (2.12)) and for the group $U(M)$, respectively. The 7/7' or 7'/7 strings, instead, are twisted and their spectrum is completely different, depending on the values of the fluxes.

2.1 The D7/D7' and D7'/D7 strings

To analyze the spectrum of the D7/D7' and D7'/D7 open strings, it is convenient to group the space-time coordinates in pairs and introduce the complex combinations

$$z^I = \frac{x^{2I-1} + ix^{2I}}{\sqrt{2}} \quad \text{and} \quad \bar{z}_I = \frac{x^{2I-1} - ix^{2I}}{\sqrt{2}} \tag{2.13}$$

for $I = 1, 2, 3, 4, 5$. The directions with $I = 1, 2, 3, 4$ are longitudinal to branes, while the direction with $I = 5$ is transverse. In this complex notation, the background (2.12) implies that in the 7/7' sector, the complex string coordinates $\{Z^I(z), \bar{Z}_I(z)\}$ and their fermionic partners $\{\Psi^I(z), \bar{\Psi}_I(z)\}$ with $I = 1, 2, 3, 4$ are twisted with a twist parameter θ_I given by⁵

$$e^{2\pi i \theta_I} = \frac{1 - if_I}{1 + if_I} . \tag{2.14}$$

This means that they have the following monodromy properties around the origin of the world-sheet:

$$Z^I(e^{2\pi i} z) = e^{2\pi i \theta_I} Z^I(z), \quad \bar{Z}_I(e^{2\pi i} z) = e^{-2\pi i \theta_I} \bar{Z}_I(z), \tag{2.15}$$

⁵In the 7'/7 sector we have to send f_I to $-f_I$.

and

$$\Psi^I(e^{2\pi i z}) = \pm e^{2\pi i \theta_I} \Psi^I(z), \quad \bar{\Psi}_I(e^{2\pi i z}) = \pm e^{-2\pi i \theta_I} \bar{\Psi}_I(z). \quad (2.16)$$

Here z is a point in the upper half complex plane and the sign in (2.16) is $+$ for the Neveu-Schwarz (NS) boundary conditions and $-$ for the Ramond (R) ones. In [35], the bosonic and fermionic conformal field theory for generic twist parameters has been worked out in detail and we summarize the most relevant features in appendix B. Here we just recall that the monodromy properties (2.15) of the bosonic coordinates do not allow for the existence of zero-modes associated to momentum in these directions.

The bosonic and fermionic coordinates $\{Z^5(z), \bar{Z}_5(z)\}$ and $\{\Psi^5(z), \bar{\Psi}_5(z)\}$ are instead untwisted. They have the standard mode expansion of untwisted fields, but with Dirichlet/Dirichlet boundary conditions since they are transverse to both type of branes. Due to these boundary conditions there are no bosonic zero-modes in these directions either. Thus, when all four θ_I 's are different from zero, the momentum is not defined in any direction, and the corresponding string excitations represent non-dynamical degrees of freedom.

As mentioned above, we are interested in supersymmetric configurations. This means that the background (2.12) must be such that

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 \in \mathbb{Z}. \quad (2.17)$$

Of course, if $\theta_I = 0$ for all I , the background vanishes and all sixteen supercharges are conserved on the branes. If two of the twists are zero, the system preserves eight supercharges while if only one of the θ_I 's vanishes the system has four conserved supercharges. If all twists are non-zero and satisfy (2.17), the system preserves the minimal amount of supersymmetry, namely only two supercharges are conserved. This is the case we will consider. For simplicity, we assume that $0 < \theta_I < \frac{1}{2}$ for all I 's.⁶ With this choice, the supersymmetry condition (2.17) becomes

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1. \quad (2.18)$$

In this set-up, the condition that selects the physical states is

$$\begin{cases} \hat{N} = \frac{1}{2} \left(1 - \sum_{I=1}^4 \theta_I \right) & \text{in the NS sector,} \\ \hat{N} = 0 & \text{in the R sector.} \end{cases} \quad (2.19)$$

where \hat{N} is the sum of the bosonic and fermionic number operators (see (B.6) and (B.8)). Using the supersymmetry condition (2.18), we see that both the NS and R physical states must actually obey

$$\hat{N} = 0. \quad (2.20)$$

Since the number operator is non-negative, this requirement can be satisfied only by the vacuum.

⁶The reason to exclude the value $\frac{1}{2}$ is merely to avoid the appearance of fermionic zero-modes in the NS sector.

In the NS sector, the physical vacuum $|\Omega\rangle_{\text{NS}}$ is related to the $\text{SL}(2, \mathbb{R})$ invariant Fock vacuum $|0\rangle$ by the action of four bosonic twist fields $\sigma_{\theta_I}^I(z)$ of conformal dimensions

$$h(\sigma_{\theta_I}^I) = \frac{1}{2} \theta_I(1 - \theta_I) \tag{2.21}$$

and four fermionic twist fields $s_{\theta_I}^I(z)$ of conformal dimensions

$$h(s_{\theta_I}) = \frac{1}{2} \theta_I^2 \tag{2.22}$$

as follows [36]

$$|\Omega\rangle_{\text{NS}} = \lim_{z \rightarrow 0} \left(\prod_{I=1}^4 \sigma_{\theta_I}^I(z) s_{\theta_I}^I(z) \right) e^{-\varphi(z)} |0\rangle . \tag{2.23}$$

Here $\varphi(z)$ is the bosonic field that is used to describe the superghost system in the bosonized formalism [37] and the coefficient -1 in the exponent indicates that we have chosen the standard (-1) -superghost picture. Using (2.21), (2.22) and the supersymmetry condition (2.18), one can check that the vertex operator creating the physical NS vacuum out of the Fock vacuum is a conformal field of total weight 1.

The fermionic twist fields $s_{\theta_I}^I(z)$ can be written in the bosonized formalism as

$$s_{\theta_I}^I(z) = e^{i\theta_I \varphi^I(z)} \tag{2.24}$$

where $\varphi^I(z)$ is the field that bosonizes the fermionic string coordinates $\{\Psi^I(z), \bar{\Psi}_I(z)\}$ according to [37]

$$\Psi^I(z) = e^{i\varphi^I(z)}, \quad \bar{\Psi}_I(z) = e^{-i\varphi^I(z)} . \tag{2.25}$$

Of course, analogous bosonization formulas hold also for the untwisted coordinates $\{\Psi^5(z), \bar{\Psi}_5(z)\}$. Using these expressions, it is easy to realize that the sum of the charges of the five bosonic fields φ_I appearing in the vertex operator corresponding to the NS vacuum minus the superghost charge is an even integer, namely

$$\sum_{I=1}^4 \theta_I + 0 - (-1) = 2 . \tag{2.26}$$

Denoting by F this combination of charges, we have

$$(-1)^F |\Omega\rangle_{\text{NS}} = +|\Omega\rangle_{\text{NS}} . \tag{2.27}$$

Let us now consider the R sector. In this case the physical vacuum is created out of the Fock vacuum by the same bosonic twist fields $\sigma_{\theta_I}(z)$ as before, but in the fermionic part we have the following twist fields

$$s_{\theta_I - \frac{1}{2}}^I(z) = e^{i(\theta_I - \frac{1}{2})\varphi^I(z)} \tag{2.28}$$

for $I = 1, \dots, 4$ and the spin field

$$S^\pm(z) = e^{\pm \frac{1}{2}\varphi^5(z)} \tag{2.29}$$

in the fifth complex untwisted direction. The presence of this spin field implies that the R vacuum is actually a two-component SO(2) spinor. Altogether, working in the standard $(-\frac{1}{2})$ -superghost picture, in the R sector we find two physical vacuum states which in the bosonized formalism are given by

$$|\Omega, \pm\rangle_{\text{R}} = \lim_{z \rightarrow 0} \left(\prod_{I=1}^4 \sigma_{\theta_I}^I(z) e^{i(\theta_I - \frac{1}{2})\varphi^I(z)} \right) S^{\pm}(z) e^{-\frac{1}{2}\varphi(z)} |0\rangle. \quad (2.30)$$

The combination of charges that before we denoted by F takes the value 0 on $|\Omega, +\rangle_{\text{R}}$ and the value -1 on $|\Omega, -\rangle_{\text{R}}$. Thus we have

$$(-1)^F |\Omega, \pm\rangle_{\text{R}} = \pm |\Omega, \pm\rangle_{\text{R}}, \quad (2.31)$$

showing that $(-1)^F$ actually measures the chirality of R vacuum.

If we define the GSO projector as

$$P_{\text{GSO}} = \frac{1 + (-1)^F}{2}, \quad (2.32)$$

then, using (2.27) and (2.31), we see that it selects the vacuum in the NS sector and the component with positive chirality in the R sector, realizing in this way a supersymmetric physical spectrum containing just two states, the bosonic vacuum $|\Omega\rangle_{\text{NS}}$ and the fermionic vacuum $|\Omega, +\rangle_{\text{R}}$. We stress once more that these states do not carry any momentum and thus represent non-dynamical degrees of freedom since they do not propagate in space. Moreover, they transform in the fundamental of $U(N)$ and in the anti-fundamental of $U(M)$, since they arise from oriented open strings stretching between the N D7-branes and the M D7'-branes.

We conclude this analysis with a couple of comments. First, the GSO-projection in the NS appears to be different with respect to the one we are used to in the standard untwisted case when one removes the (tachyonic) vacuum and selects states with an odd number of fermionic oscillators. However, if one reformulates this standard GSO projection in terms of F , one can realize that it is precisely given by (2.32). Secondly, the analysis we have described for the D7/D7' strings can be repeated for the D7'/D7 strings without any difficulty, and also in that case one finds that the physical states are a non-dynamical scalar, corresponding to the NS vacuum, and the chiral component of a non-dynamical fermion, corresponding the chiral component of the R vacuum. Of course, these states of the 7'/7 sector transform in the anti-fundamental of $U(N)$ and in the fundamental of $U(M)$.

2.2 The no-force condition

We have seen that the condition (2.18) guarantees a supersymmetric physical spectrum. To complete our analysis we now show that the same condition also implies that the partition functions of the mixed strings of the 7/7' and 7'/7 sectors vanish, thus leading to a no-force condition.

Let us first consider the 7/7' open strings whose partition function is

$$Z_{7/7'} = \int_0^{\infty} \frac{dt}{2t} \text{Tr}_{7/7'} \left(P_{\text{GSO}} e^{-2\pi t L_0} \right) \quad (2.33)$$

where the GSO projector is defined in (2.32) and L_0 is the total open string Hamiltonian consisting of the orbital part and of the ghost and superghost parts (see for instance [37] for details). The trace in (2.33) is computed over all excitations of the $7/7'$ strings with a $+$ sign in the bosonic NS sector and a $-$ sign in the fermionic R sector. This calculation is pretty standard (see for example [34, 38] and references therein), but we would like to point out just a couple of subtle issues that arise in the present case. First of all, in the $7/7'$ sector, there are eight (real) twisted directions and two (real) untwisted directions. This is precisely the case in which the odd spin-structure $R(-1)^F$ gives a non-vanishing contribution to the string partition function due to a cancellation between the fermionic zero-modes in the two untwisted directions and the bosonic zero-modes of the superghost-sector [38–40]. Secondly, the presence of the background (2.12) along the D7-brane world-volume modifies the contribution of the zero-modes of the eight twisted bosonic coordinates by multiplying the eight-dimensional volume V_8 by the factor $\prod_I f_I$.

Taking this into account and using the GSO projection (2.32), we find

$$Z_{7/7'} = \frac{NM V_8}{(4\pi^2 \alpha')^4} \prod_{I=1}^4 f_I \int_0^\infty \frac{dt}{2t} \frac{1}{2} \left[\prod_{I=1}^4 \frac{\vartheta_3(\theta_I t)}{\vartheta_1(\theta_I t)} + \prod_{I=1}^4 \frac{\vartheta_4(\theta_I t)}{\vartheta_1(\theta_I t)} - \prod_{I=1}^4 \frac{\vartheta_2(\theta_I t)}{\vartheta_1(\theta_I t)} - 1 \right] \quad (2.34)$$

where the factor NM accounts for the color multiplicity of each string excitation and the four terms in the square brackets correspond to the contributions of the NS, $NS(-1)^F$, R and $R(-1)^F$ spin-structures, respectively. Here, as usual, we have defined:

$$\vartheta_1(z) \equiv \vartheta_{[1]}^1(iz|it), \quad \vartheta_2(z) \equiv \vartheta_{[0]}^1(iz|it), \quad \vartheta_3(z) \equiv \vartheta_{[0]}^0(iz|it), \quad \vartheta_4(z) \equiv \vartheta_{[1]}^0(iz|it) \quad (2.35)$$

where $\vartheta_b^a(iz|it)$ are the Jacobi ϑ -functions with characteristics (see for example [41]). They satisfy the Jacobi-Riemann identity

$$\prod_{I=1}^4 \vartheta_3(z_I) - \prod_{I=1}^4 \vartheta_4(z_I) - \prod_{I=1}^4 \vartheta_2(z_I) - \prod_{I=1}^4 \vartheta_1(z_I) = -2 \prod_{I=1}^4 \vartheta_1(Z/2 - z_I) \quad (2.36)$$

with $Z = \sum_I z_I$. Applying this identity in our case where $z_I = \theta_I t$ and using the periodicity properties of the Jacobi ϑ -functions, we can recast the previous identity in the following form

$$\prod_{I=1}^4 \vartheta_3(\theta_I t) + \prod_{I=1}^4 \vartheta_4(\theta_I t) - \prod_{I=1}^4 \vartheta_2(\theta_I t) - \prod_{I=1}^4 \vartheta_1(\theta_I t) = 0, \quad (2.37)$$

when the twist parameters satisfy the supersymmetry relation (2.18). Thus, inserting this relation into (2.34), we find

$$Z_{7/7'} = 0. \quad (2.38)$$

In a completely similar way one can show that also for the other orientation the partition function vanishes:

$$Z_{7'/7} = 0. \quad (2.39)$$

This fact, together with the vanishing of the partition function in the $7/7$ and $7'/7'$ sectors due to the BPS property of these branes, implies that our brane system is stable when the condition (2.18) is satisfied, and that one can pile up an arbitrary number of D7 and D7'-branes since the net force in all channels is zero.

3 Adding D-instantons

To study some non-perturbative features in the effective theory defined on the world-volume of the D7 and D7'-branes, we add a stack of k D-instantons, *i.e.* D-branes with Dirichlet boundary conditions in all directions also known as D(-1)-branes. This addition gives rise to new open string sectors: the (-1)/(-1) strings starting and ending on the D-instantons, the (-1)/7 and 7/(-1) strings which stretch between the D-instantons and the D7-branes, and finally (-1)/7' and 7'/(-1) strings which connect the D-instantons and the D7'-branes. We now analyze the physical states in all these sectors.

3.1 The D(-1)/D(-1) strings

The open strings stretching between two D(-1)-branes are untwisted and have Dirichlet/Dirichlet boundary conditions in all ten directions. Thus, they do not carry any momentum and describe non-propagating degrees of freedom. To describe the physical states in this sector, it is convenient to distinguish the eight directions, that are longitudinal to the D7 and D7'-branes from the two ones that are transverse, and organize the open string states in representations of $\text{Spin}(8) \times \text{Spin}(2)$.

In the NS sector, the only physical states are those with one fermionic oscillator, corresponding to the ten-dimensional vector. Using the same complex notation introduced in the previous section, we can pack the eight (real) components of this vector along the D7 or D7'-branes into four complex variables which we denote by B^I , together with their complex conjugates \bar{B}_I , with $I = 1, 2, 3, 4$. Their corresponding vertex operators, in the (-1)-superghost picture, are⁷

$$V_{B^I} = \frac{B^I}{\sqrt{2}} \bar{\Psi}_I(z) e^{-\varphi(z)} \quad \text{and} \quad V_{\bar{B}_I} = \frac{\bar{B}_I}{\sqrt{2}} \Psi^I(z) e^{-\varphi(z)}. \quad (3.1)$$

The remaining two components of the vector, along the transverse directions to the branes, give rise to the variables ξ and χ , whose vertex operators are

$$V_\xi = \xi \bar{\Psi}_5(z) e^{-\varphi(z)} \quad \text{and} \quad V_\chi = \frac{\chi}{2} \Psi^5(z) e^{-\varphi(z)}. \quad (3.2)$$

All these vertex operators are conformal fields of dimension 1 and possess an even F -charge, as one can see using the bosonization formulas (2.25); thus they are preserved by the GSO projection (2.32).

In the R sector the only physical state is the vacuum. This state is actually a sixteen-component spinor of $\text{Spin}(10)$, which the GSO projection fixes to be anti-chiral. To describe these sixteen components we use the formalism of spin fields [37] and introduce the notation

$$S^{\pm\pm\pm\pm}(z) = e^{\pm\frac{1}{2}\varphi^1(z)} e^{\pm\frac{1}{2}\varphi^2(z)} e^{\pm\frac{1}{2}\varphi^3(z)} e^{\pm\frac{1}{2}\varphi^4(z)} \quad (3.3)$$

to denote the spin fields in the first four complex directions, which have to be used together with the spin fields $S^\pm(z)$ in the fifth direction already introduced in (2.29). The physical

⁷The normalization factors in these and in all the following vertices are chosen for later convenience so that the final formulas look simpler.

ground state of the R sector, in the $(-\frac{1}{2})$ -superghost picture, is therefore given by

$$\lim_{z \rightarrow 0} S^{\pm\pm\pm\pm}(z) S^\pm(z) e^{-\frac{1}{2}\varphi(z)} |0\rangle \quad (3.4)$$

with a total odd number of $-$ signs in the spin fields in order to have an even F -charge and be preserved by the GSO projection. It is convenient to distinguish the cases in which the last spin field is $S^+(z)$ or $S^-(z)$.

If we have $S^+(z)$, the spin fields in the first four directions must have an odd number of minus signs. Clearly, there are eight possibilities. Using a notation that resembles the one used for the components of the vector field in the NS sector, we denote these eight spin fields by $S^I(z)$ and $\bar{S}_I(z)$ with $I = 1, \dots, 4$ (see (A.26) for their detailed definition). The components of the fermionic R ground state corresponding to these fields are denoted by \bar{M}_I and M^I , respectively, and are described by the following vertex operators of weight 1:

$$V_{M^I} = M^I \bar{S}_I(z) S^+(z) e^{-\frac{1}{2}\varphi(z)} \quad \text{and} \quad V_{\bar{M}_I} = S^I(z) S^+(z) e^{-\frac{1}{2}\varphi(z)} \bar{M}_I . \quad (3.5)$$

If instead we have $S^-(z)$, the spin fields in the first four directions must have an even number of minus signs and there are again eight possibilities. We denote these spin fields by $S(z)$, $\bar{S}(z)$ and $S^{IJ}(z) = -S^{JI}(z)$ with $I, J = 1, \dots, 4$ (see (A.23) for details). We can also introduce the conjugate fields $\bar{S}_{IJ}(z)$, but they are not independent since the following relation holds

$$\bar{S}_{IJ}(z) = \frac{1}{2} \epsilon_{IJKL} S^{KL}(z) . \quad (3.6)$$

Using this notation, we see that the eight components of the R ground states with $S^-(z)$, which we denote by λ^{IJ} , λ and η , are described by the following vertex operators

$$V_{\lambda^{IJ}} = \frac{\lambda^{IJ}}{\sqrt{2}} \bar{S}_{IJ}(z) S^-(z) e^{-\frac{1}{2}\varphi(z)} , \quad (3.7)$$

and

$$V_\lambda = \lambda \frac{\bar{S}(z) - S(z)}{\sqrt{2}i} S^-(z) e^{-\frac{1}{2}\varphi(z)} , \quad V_\eta = \eta \frac{\bar{S}(z) + S(z)}{\sqrt{2}} S^-(z) e^{-\frac{1}{2}\varphi(z)} . \quad (3.8)$$

It is easy to verify that these vertex operators are conformal fields of weight 1. Furthermore, the fields λ and η are real, while the fields λ^{IJ} are complex with their complex conjugates $\bar{\lambda}_{IJ}$ given by

$$\bar{\lambda}_{IJ} = \frac{1}{2} \epsilon_{IJKL} \lambda^{KL} , \quad (3.9)$$

as a consequence of the relation (3.6).

Since there are k D-instantons, all variables appearing as polarizations in the above vertex operators are $(k \times k)$ matrices transforming in the adjoint representation of $U(k)$.

3.2 The D(-1)/D7 and D7/D(-1) strings

We now consider the open strings stretching between the D-instantons and the D7-branes. In this case there are two facts that one has to consider, namely that the first four complex space directions have mixed Dirichlet/Neumann boundary conditions and that there is a

background field on the D7-branes given by (2.12). The combined net effect is that the first four complex directions have an effective twist parameter given by $(\frac{1}{2} - \theta_I)$.

Taking this into account, one can see that the condition which selects the physical states in the NS sector is

$$\widehat{N} + \frac{1}{2} - \frac{1}{2} \sum_{I=1}^4 \theta_I = 0 \tag{3.10}$$

where \widehat{N} is the total number operator. Inserting the constraint (2.18), we see that the physical states in the NS sector must have $\widehat{N} = 0$, which is possible only for the vacuum. Thus in the spectrum we have only one state whose polarization we denote by w . Its corresponding vertex operator, in the (-1) -superghost picture, is

$$V_w = \frac{w}{\sqrt{2}} \left(\prod_{I=1}^4 \sigma_{\frac{1}{2} - \theta_I}^I(z) e^{i(\frac{1}{2} - \theta_I)\varphi^I(z)} \right) e^{-\varphi(z)} \tag{3.11}$$

which is a conformal field of weight 1 when the condition (2.18) is satisfied. The same condition also guarantees that the total F -charge is even so that this vertex is preserved by the GSO projection.

In the R sector, the condition for physical states implies $\widehat{N} = 0$. Thus, also in this sector the vacuum is the only physical state. Due to the R boundary conditions in the untwisted fifth complex direction, the ground state is degenerate and is a two-component spinor denoted by μ^\pm . The corresponding vertex operator in the $(-\frac{1}{2})$ -superghost picture is

$$V_{\mu^\pm} = \mu^\pm \left(\prod_{I=1}^4 \sigma_{\frac{1}{2} - \theta_I}^I(z) e^{-i\theta_I\varphi^I(z)} \right) S^\pm(z) e^{-\frac{1}{2}\varphi(z)} \tag{3.12}$$

which is a conformal field of weight 1. Exploiting the supersymmetry condition (2.18), it is easy to see that

$$(-1)^F V_{\mu^\pm} = \pm V_{\mu^\pm} \tag{3.13}$$

so that only μ^+ is selected by the GSO projection. To simplify the notation, we will denote μ^+ simply as μ .

This analysis can be easily repeated for the strings with the opposite orientation stretching between the D7-branes and the D-instantons. The result is that in the NS sector we have only one GSO-even physical state \bar{w} whose vertex operator is

$$V_{\bar{w}} = \frac{\bar{w}}{\sqrt{2}} \left(\prod_{I=1}^4 \sigma_{\theta_I - \frac{1}{2}}^I(z) e^{i(\theta_I - \frac{1}{2})\varphi^I(z)} \right) e^{-\varphi(z)}, \tag{3.14}$$

while in the R sector the only GSO-even physical state is $\bar{\mu}$ whose vertex operator is

$$V_{\bar{\mu}} = \bar{\mu} \left(\prod_{I=1}^4 \sigma_{\theta_I - \frac{1}{2}}^I(z) e^{i\theta_I\varphi^I(z)} \right) S^+(z) e^{-\frac{1}{2}\varphi(z)}. \tag{3.15}$$

Taking into account the multiplicity of the D-instantons and of the D7-branes, the variables w and μ are $(k \times N)$ matrices transforming in the $(\mathbf{k}, \bar{\mathbf{N}})$ representation of $U(k) \times U(N)$, while the conjugate variables \bar{w} and $\bar{\mu}$ are $(N \times k)$ matrices transforming in the $(\bar{\mathbf{k}}, \mathbf{N})$ representation of the same group.

3.3 The D(-1)/D7' and D7'/D(-1) strings

We now consider the strings connecting the D-instantons and the D7'-branes. The only but important difference with the previous case is that now there is no background field on the 7-branes so that the string fields in the first four complex directions have standard Dirichlet/Neumann boundary conditions without extra twists. Therefore, we can formally set $\theta_I = 0$ in the previous formulas. If we do this, we immediately realize that in the NS sector the physical states must satisfy the condition

$$\widehat{N} + \frac{1}{2} = 0, \tag{3.16}$$

which has no solution. Thus, there are no physical states in the NS part of the spectrum. This fact was already observed in [34] where the D(-1)/D7-brane system was explored in connection with the first studies of exotic instantons in eight dimensions.

In the R sector, instead, the condition for physical states is satisfied by the degenerate ground state that describes a two-component spinor which we denote by μ'^{\pm} . The corresponding vertex operator is

$$V_{\mu'^{\pm}} = \mu'^{\pm} \left(\prod_{I=1}^4 \sigma_{\frac{1}{2}}^I(z) \right) S^{\pm}(z) e^{-\frac{1}{2}\varphi(z)} \tag{3.17}$$

where $\sigma_{\frac{1}{2}}^I$ is the appropriate bosonic twist field when the I -th coordinate has Dirichlet/Neumann boundary conditions. This twist field has conformal dimension $\frac{1}{8}$, and thus the vertex operator (3.17) has total weight 1. Furthermore, it is easy to check

$$(-1)^F V_{\mu'^{\pm}} = \mp V_{\mu'^{\pm}}, \tag{3.18}$$

so that only μ'^{-} , which we will simply call μ' , survives the GSO projection.

For the strings with opposite orientation that stretch between the D7'-branes and the D-instantons, we find a similar structure with no physical states in the NS sector, and only one physical state in the R sector which we denote $\bar{\mu}'$. Its vertex operator is

$$V_{\bar{\mu}'} = \bar{\mu}' \left(\prod_{I=1}^4 \sigma_{-\frac{1}{2}}^I(z) \right) S^{-}(z) e^{-\frac{1}{2}\varphi(z)} \tag{3.19}$$

where $\sigma_{-\frac{1}{2}}^I$ is the bosonic twist field corresponding to an I -th coordinate with Neumann/Dirichlet boundary conditions, that is conjugate to the twist field $\sigma_{\frac{1}{2}}^I$.

Taking into account the multiplicity of the branes, we have that μ' is a $(k \times M)$ matrix transforming the $(\mathbf{k}, \bar{\mathbf{M}})$ representation of $U(k) \times U(M)$, while $\bar{\mu}'$ is $(M \times k)$ matrix transforming the $(\bar{\mathbf{k}}, \mathbf{M})$ representation of this group.

3.4 D-instanton partition functions

We now study the partition function of the open strings with at least one end-point on the D-instantons. In the $(-1)/(-1)$ sector, we simply have

$$Z_{(-1)/(-1)} = 0 \tag{3.20}$$

as a consequence of the BPS condition satisfied by the D(-1)-branes.

Let us now consider the $(-1)/7'$ and the $7'/(-1)$ sectors. Recalling that the D7'-branes do not have any background field on their world-volume, the calculation of the partition functions $Z_{(-1)/7'}$ and $Z_{7'/(-1)}$ is exactly the same as the one described in detail in [34], which we briefly recall here. For the strings starting from the D-instantons we have

$$Z_{(-1)/7'} = kM \int_0^\infty \frac{dt}{2t} \frac{1}{2} \left[\left(\frac{\vartheta_2(0)}{\vartheta_4(0)} \right)^4 + \left(\frac{\vartheta_1(0)}{\vartheta_4(0)} \right)^4 - \left(\frac{\vartheta_3(0)}{\vartheta_4(0)} \right)^4 - 1 \right] \quad (3.21)$$

where the prefactor accounts for the multiplicity of the branes, and the four terms inside the square brackets arise from the four spin structures with eight directions with Dirichlet/Neumann boundary conditions. We notice in particular that the second term, corresponding to the $\text{NS}(-1)^F$ sector, actually vanishes since $\vartheta_1(0) = 0$, while the fourth term, corresponding to the $\text{R}(-1)^F$ spin structure, is non-zero due to the cancellation of the fermionic zero-modes in the two directions with Dirichlet/Dirichlet boundary conditions with the bosonic zero-modes of the superghost system [38]. Exploiting the abstruse identity

$$\vartheta_3(0)^4 - \vartheta_4(0)^4 - \vartheta_2(0)^4 = 0, \quad (3.22)$$

we easily conclude that

$$Z_{(-1)/7'} = -kM \int_0^\infty \frac{dt}{2t}. \quad (3.23)$$

The partition function for the other orientation corresponds to placing the D-instantons “on the other side” of the 7-brane, which is obtained with a parity transformation that reverses the sign of the odd spin structure $\text{R}(-1)^F$ [38]. Then we have

$$Z_{7'/(-1)} = kM \int_0^\infty \frac{dt}{2t} \frac{1}{2} \left[\left(\frac{\vartheta_2(0)}{\vartheta_4(0)} \right)^4 + \left(\frac{\vartheta_1(0)}{\vartheta_4(0)} \right)^4 - \left(\frac{\vartheta_3(0)}{\vartheta_4(0)} \right)^4 + 1 \right]. \quad (3.24)$$

The abstruse identity (3.22) now implies that⁸

$$Z_{7'/(-1)} = 0. \quad (3.25)$$

Let us now consider the contribution of the open strings connecting the D-instantons to the D7-branes. In this case, due to the presence of the background (2.12) on the 7-branes, we have an extra twist to take into account with respect to the previous cases. In particular, the partition function $Z_{(-1)/7}$ is obtained from (3.21) by twisting the ϑ -functions by $-\theta_I t$ in each complex direction and replacing M by N , which is the number of D7-branes. In other words we have

$$Z_{(-1)/7} = kN \int_0^\infty \frac{dt}{2t} \frac{1}{2} \left[\prod_{I=1}^4 \frac{\vartheta_2(-\theta_I t)}{\vartheta_4(-\theta_I t)} + \prod_{I=1}^4 \frac{\vartheta_1(-\theta_I t)}{\vartheta_4(-\theta_I t)} - \prod_{I=1}^4 \frac{\vartheta_3(-\theta_I t)}{\vartheta_4(-\theta_I t)} - 1 \right]. \quad (3.26)$$

Since the twists satisfy the condition (2.18), the Jacobi-Riemann identity (2.37), together with the following parity properties

$$\vartheta_1(-z) = -\vartheta_1(z), \quad \vartheta_2(-z) = +\vartheta_2(z), \quad \vartheta_3(-z) = +\vartheta_3(z), \quad \vartheta_4(-z) = +\vartheta_4(z), \quad (3.27)$$

⁸In the T-dual set-up, namely in the D0/D8-brane system, the vanishing of the partition function can be interpreted as due to the creation of a fundamental string, see for example [38, 40, 42].

implies that

$$Z_{(-1)/7} = 0 . \tag{3.28}$$

The partition function for the other orientation is obtained from (3.24) by twisting the ϑ -functions with $\theta_I t$ and replacing M with N , namely

$$Z_{7/(-1)} = kN \int_0^\infty \frac{dt}{2t} \frac{1}{2} \left[\prod_{I=1}^4 \frac{\vartheta_2(\theta_I t)}{\vartheta_4(\theta_I t)} + \prod_{I=1}^4 \frac{\vartheta_1(\theta_I t)}{\vartheta_4(\theta_I t)} - \prod_{I=1}^4 \frac{\vartheta_3(\theta_I t)}{\vartheta_4(\theta_I t)} + 1 \right] . \tag{3.29}$$

Using the Jacobi-Riemann identity, we easily find

$$Z_{7/(-1)} = kN \int_0^\infty \frac{dt}{2t} . \tag{3.30}$$

Summing all contributions (3.20), (3.23), (3.25), (3.28) and (3.30), we conclude that the total partition function of the open strings with at least one end-point on the D-instantons is

$$Z_{\text{tot}} = k(N - M) \int_0^\infty \frac{dt}{2t} . \tag{3.31}$$

If we set

$$N = M , \tag{3.32}$$

the total partition function vanishes and no forces are present. This no-force condition is equivalent to requiring that the brane system be conformal invariant.⁹

4 The instanton moduli and their action

In the previous sections we have described a system made of N D7-branes with flux, M D7'-branes without flux and k D-instantons. The flux on the D7-branes is subject to the condition (2.18) with all θ_I different from zero in order to preserve the minimal amount of supersymmetry. For simplicity, from now on we choose the twists to be all equal, namely

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{1}{4} . \tag{4.1}$$

With this choice the four complex directions are treated symmetrically and many formulas simplify. For example, the fermionic twist fields to be used in the vertex operators of the $(-1)/7$ strings (see (3.11) and (3.12)) become

$$\Sigma(z) = e^{\frac{i}{4}(\varphi^1(z)+\varphi^2(z)+\varphi^3(z)+\varphi^4(z))} \quad \text{and} \quad \bar{\Sigma}(z) = e^{-\frac{i}{4}(\varphi^1(z)+\varphi^2(z)+\varphi^3(z)+\varphi^4(z))} \tag{4.2}$$

in the NS and R sectors, respectively. The same twist fields are used also in the vertex operators of the $7/(-1)$ strings (see (3.14) and (3.15)), but in this case $\Sigma(z)$ is used in the

⁹If $N \neq M$ the total open-string partition function with D-instantons does not vanish and the gauge theory on the D7-brane becomes not-conformal in the sense that the dimensionless coupling constant λ of the quartic action (2.4) runs with a β -function coefficient proportional to $(N - M)$. Furthermore, as we will show in later sections, when $N \neq M$ the instanton partition function Z_k is not dimensionless and the instanton counting parameter q carries dimensions of mass ^{$(N-M)$} , in such a way that $q^k Z_k$ is dimensionless.

| Sector | Moduli | Vertex Operators | Statistics | Dimensions |
|-----------|------------------|--|------------|----------------------------------|
| (-1)/(-1) | B^I, \bar{B}_I | $\frac{B^I}{\sqrt{2}} \bar{\Psi}_I e^{-\varphi}, \frac{\bar{B}_I}{\sqrt{2}} \Psi^I e^{-\varphi}$ | bosonic | (length) ⁻¹ |
| | ξ, χ | $\xi \bar{\Psi}_5 e^{-\varphi}, \frac{\chi}{2} \Psi^5 e^{-\varphi}$ | bosonic | (length) ⁻¹ |
| | M^I, \bar{M}_I | $M^I \bar{S}_I S^+ e^{-\frac{1}{2}\varphi}, S^I S^+ e^{-\frac{1}{2}\varphi} \bar{M}_I$ | fermionic | (length) ^{-\frac{3}{2}} |
| | λ^{IJ} | $\frac{\lambda^{IJ}}{\sqrt{2}} \bar{S}_{IJ} S^- e^{-\frac{1}{2}\varphi}$ | fermionic | (length) ^{-\frac{3}{2}} |
| | λ | $\lambda \frac{\bar{S}-S}{\sqrt{2}i} S^- e^{-\frac{1}{2}\varphi}$ | fermionic | (length) ^{-\frac{3}{2}} |
| | η | $\eta \frac{\bar{S}+S}{\sqrt{2}} S^- e^{-\frac{1}{2}\varphi}$ | fermionic | (length) ^{-\frac{3}{2}} |
| (-1)/7 | w | $\frac{w}{\sqrt{2}} \Delta \Sigma e^{-\varphi}$ | bosonic | (length) ⁻¹ |
| | μ | $\mu \Delta \bar{\Sigma} S^+ e^{-\frac{1}{2}\varphi}$ | fermionic | (length) ^{-\frac{3}{2}} |
| 7/(-1) | \bar{w} | $\frac{\bar{w}}{\sqrt{2}} \bar{\Delta} \bar{\Sigma} e^{-\varphi}$ | bosonic | (length) ⁻¹ |
| | $\bar{\mu}$ | $\bar{\mu} \bar{\Delta} \Sigma S^+ e^{-\frac{1}{2}\varphi}$ | fermionic | (length) ^{-\frac{3}{2}} |
| (-1)/7' | μ' | $\mu' \Delta' S^- e^{-\frac{1}{2}\varphi}$ | fermionic | (length) ^{-\frac{3}{2}} |
| 7'/(-1) | $\bar{\mu}'$ | $\bar{\mu}' \bar{\Delta}' S^- e^{-\frac{1}{2}\varphi}$ | fermionic | (length) ^{-\frac{3}{2}} |

Table 2. The physical moduli corresponding to open strings with at least one end-point on the D-instantons, and their vertex operators in the canonical superghost pictures, (-1) in the NS sector and (- $\frac{1}{2}$) in the R sector. The last two columns contain their statistics and their scaling dimensions.

R sector and $\bar{\Sigma}(z)$ in the NS sector. Both $\Sigma(z)$ and $\bar{\Sigma}(z)$ are conformal fields of weight $\frac{1}{8}$. It is also convenient to define the bosonic twist fields

$$\Delta(z) = \prod_{I=4} \sigma_{\frac{1}{4}}^I(z) \quad \text{and} \quad \bar{\Delta}(z) = \prod_{I=4} \sigma_{-\frac{1}{4}}^I(z) \quad (4.3)$$

to be used for the (-1)/7 and 7/(-1) strings respectively. These fields have conformal dimension $\frac{3}{8}$ and are conjugate to each other. Similarly, for the strings of type (-1)/7' and 7'/(-1) we define the following pair of conjugate twist fields

$$\Delta'(z) = \prod_{I=4} \sigma_{\frac{1}{2}}^I(z) \quad \text{and} \quad \bar{\Delta}'(z) = \prod_{I=4} \sigma_{-\frac{1}{2}}^I(z), \quad (4.4)$$

which have conformal weight $\frac{1}{2}$.

Using these notations, we summarize for future reference all physical states of the open strings with at least one end-point on the D-instantons and their vertex operators in table 2.

4.1 The moduli action

The action of the $(-1)/(-1)$ moduli can be obtained by dimensionally reducing the $\mathcal{N} = 1$ super Yang-Mills action from ten to zero dimensions and, in the notations previously introduced, it reads as follows

$$\begin{aligned}
 S_{(-1)} = & \frac{1}{g_0^2} \text{tr} \left\{ \frac{1}{2} [B^I, \bar{B}_I] [B^J, \bar{B}_J] - [B^I, B^J] [\bar{B}_I, \bar{B}_J] - [\xi, B^I] [\chi, \bar{B}_I] - [\chi, B^I] [\xi, \bar{B}_I] \right. \\
 & + \frac{1}{2} [\chi, \xi]^2 + \frac{1}{2} \lambda [\chi, \lambda] + \frac{1}{2} \eta [\chi, \eta] + \frac{1}{4} \lambda^{IJ} [\chi, \lambda^{KL}] \epsilon_{IJKL} \\
 & - 2 M^I [\xi, \bar{M}_I] + \lambda^{IJ} \left([\bar{B}_I, \bar{M}_J] - [\bar{B}_J, \bar{M}_I] + \epsilon_{IJKL} [B^K, M^L] \right) \\
 & \left. + (\eta + i\lambda) [\bar{B}_I, M^I] + (\eta - i\lambda) [B^I, \bar{M}_I] \right\} \tag{4.5}
 \end{aligned}$$

where

$$g_0^2 = \frac{g_s}{4\pi^3 \alpha'^2} \tag{4.6}$$

is the Yang-Mills coupling constant in zero dimensions.

The action involving the moduli of the mixed sectors is

$$S_{\text{mixed}} = \frac{1}{g_0^2} \text{tr} \left\{ \bar{w} (\chi \xi + \xi \chi) w - 2 \bar{\mu} \xi \mu - \bar{\mu} (\eta + i\lambda) w - \bar{w} (\eta - i\lambda) \mu + \bar{\mu}' \chi \mu' \right\} . \tag{4.7}$$

Using standard conformal field theory methods, one can verify that all cubic couplings in the above actions can be obtained by computing the 3-point amplitudes among the vertex operators introduced in the previous section. If one applies this method to compute the 4-point amplitudes and obtain the quartic couplings, one encounters divergent integrals over the vertex insertion points, as it has also been recently pointed out in [43]. To circumvent this problem, we adopt the strategy already used in [6, 31, 34] and introduce auxiliary fields to disentangle the quartic interactions. For our purposes it is enough to consider the first two quartic terms in (4.5). The minimal set of auxiliary fields which are needed to decouple these interactions comprises seven auxiliary fields, denoted by D and D^{IJ} with $D^{JI} = -D^{IJ}$ ($I, J = 1, \dots, 4$). Notice that D is real while the six D^{IJ} 's are complex with their complex conjugates \bar{D}_{IJ} given by

$$\bar{D}_{IJ} = \frac{1}{2} \epsilon_{IJKL} D^{KL} . \tag{4.8}$$

Thus D^{IJ} correspond to six real degrees of freedom. Using these auxiliary fields, to which we assign canonical scaling dimensions of $(\text{length})^{-2}$, the first two terms of (4.5) can be replaced by

$$\frac{1}{g_0^2} \text{tr} \left\{ \frac{1}{4} D^{IJ} D^{KL} \epsilon_{IJKL} - D^{IJ} \left([\bar{B}_I, \bar{B}_J] + \frac{1}{2} \epsilon_{IJKL} [B^K, B^L] \right) + \frac{1}{2} D^2 - i D [B^I, \bar{B}_I] \right\} . \tag{4.9}$$

Indeed, eliminating the auxiliary fields through their equations of motion and using the Jacobi identity, one can show that (4.9) reduces exactly to the first two terms of (4.5).

We observe that the cubic terms in (4.9) can be obtained by computing the couplings among the vertex operators V_{B^I} and $V_{\bar{B}^I}$, given in (3.1), and the following vertex operators for the auxiliary fields

$$\begin{aligned} V_{D^{IJ}} &= -\frac{D^{IJ}}{2} \left(\bar{\Psi}_I(z) \bar{\Psi}_J(z) + \frac{1}{2} \epsilon_{IJKL} \Psi^K(z) \Psi^L(z) \right), \\ V_D &= -\frac{iD}{2} \Psi^I(z) \bar{\Psi}_I(z). \end{aligned} \tag{4.10}$$

These vertices are in the (0)-superghost picture, as appropriate for auxiliary fields, and are conformal operators of weight 1.

The vertex V_D has a non-vanishing coupling also with the vertex operators V_w and $V_{\bar{w}}$ of the mixed bosonic moduli w and \bar{w} ; this coupling leads to the following term

$$\frac{1}{g_0^2} \text{tr} \{ -iD w \bar{w} \} \tag{4.11}$$

that has to be added to the moduli action (4.7). Furthermore, we find convenient to add also the term

$$\frac{1}{g_0^2} \text{tr} \{ \bar{h}' h' \} \tag{4.12}$$

where h' and its conjugate \bar{h}' are auxiliary fields with scaling dimensions of (length)⁻². Even if these fields look trivial since they do not interact with any other moduli, it is useful to introduce them for reasons that will be clear in a moment. Also these auxiliary fields can be described by vertex operators in the (0)-superghost picture, which are

$$V_{h'} = h' \Delta'(z) \frac{\bar{S}(z) + S(z)}{\sqrt{2}} \quad \text{and} \quad V_{\bar{h}'} = \bar{h}' \bar{\Delta}'(z) \frac{\bar{S}(z) + S(z)}{\sqrt{2}}. \tag{4.13}$$

These operators are conformal fields of weight 1.

We now introduce ADHM-like variables by means of the following rescalings

$$B^I \rightarrow g_0 B^I, \quad M^I \rightarrow g_0 M^I, \quad w \rightarrow g_0 w, \quad \mu \rightarrow g_0 \mu, \quad \mu' \rightarrow g_0 \mu', \quad h' \rightarrow g_0 h' \tag{4.14}$$

with analogous ones for their conjugates, in such a way that the rescaled bosons B^I and w have dimensions of length, the rescaled fermions μ and μ' have dimensions of (length)^{1/2}, and the rescaled auxiliary field h' becomes dimensionless. After these rescalings, adding all contributions one finds that the instanton moduli action can be written as

$$S_{\text{inst}} = \frac{1}{g_0^2} S_G + S_K + S_D \tag{4.15}$$

where

$$S_G = \text{tr} \left\{ \frac{1}{4} D^{IJ} D^{KL} \epsilon_{IJKL} + \frac{1}{2} D^2 + \frac{1}{2} [\chi, \xi]^2 + \frac{1}{2} \lambda [\chi, \lambda] + \frac{1}{2} \eta [\chi, \eta] + \frac{1}{4} \lambda^{IJ} [\chi, \lambda^{KL}] \epsilon_{IJKL} \right\}, \quad (4.16a)$$

$$S_K = \text{tr} \left\{ - [\xi, B^I] [\chi, \bar{B}_I] - [\chi, B^I] [\xi, \bar{B}_I] - 2 M^I [\xi, \bar{M}_I] - 2 \bar{\mu} \xi \mu + \bar{w} \xi \chi w + \bar{w} \chi \xi w + \bar{\mu}' \chi \mu' + \bar{h}' h' \right\}, \quad (4.16b)$$

$$S_D = \text{tr} \left\{ -i D \left([B^I, \bar{B}_I] + w \bar{w} \right) - D^{IJ} \left([\bar{B}_I, \bar{B}_J] + \frac{1}{2} \epsilon_{IJKL} [B^K, B^L] \right) + \lambda^{IJ} \left([\bar{B}_I, \bar{M}_J] - [\bar{B}_J, \bar{M}_I] + \epsilon_{IJKL} [B^K, M^L] \right) + (\eta + i\lambda) \left([\bar{B}_I, M^I] + w \bar{\mu} \right) + (\eta - i\lambda) \left([B^I, \bar{M}_I] - \mu \bar{w} \right) \right\}. \quad (4.16c)$$

In the field theory limit $\alpha' \rightarrow 0$, or equivalently in the strong-coupling limit $g_0 \rightarrow \infty$, the term S_G can be discarded and the fields D and D^{IJ} become Lagrange multipliers for the bosonic constraints

$$[B^I, \bar{B}_I] + w \bar{w} = 0, \quad [\bar{B}_I, \bar{B}_J] + \frac{1}{2} \epsilon_{IJKL} [B^K, B^L] = 0, \quad (4.17)$$

while the fields $(\eta \pm i\lambda)$ and λ^{IJ} become Lagrange multipliers for the fermionic constraints

$$[\bar{B}_I, M^I] + w \bar{\mu} = 0, \quad [\bar{B}_I, \bar{M}_J] - [\bar{B}_J, \bar{M}_I] + \epsilon_{IJKL} [B^K, M^L] = 0. \quad (4.18)$$

In section 6 we will show that the same equations arise from the ADHM construction for instantons in eight dimensions. This fact allows us to regard the open strings ending on the D(-1) branes as instanton moduli.

4.2 Introducing vacuum expectation values and the ε -background

We now generalize the action (4.15) by introducing the interactions with external backgrounds.

From the string theory point of view, these backgrounds correspond either to vacuum expectation values of massless fields propagating on the world-volume of the D7 and D7'-branes, or to vacuum expectation values of bulk fields in the closed string sectors. Therefore, one can obtain their interactions with the instanton moduli by following the procedure described in [6, 16, 31], namely by computing the correlation functions among the vertex operators of the moduli and the vertex operators of the external background fields. However, there is also an alternative route to obtain these interactions which exploits the symmetries of the brane system.

Let us give some details. The moduli actions (4.16) are invariant by construction under the symmetry group $U(k)$ of the D-instantons, and the moduli χ and ξ can be

interpreted as the parameters of infinitesimal $U(k)$ transformations.¹⁰ In our conventions, these transformations are

$$\mathbb{T}_\chi[\bullet] = \begin{cases} -[\chi, \bullet] & \text{if } \bullet \text{ is an adjoint modulus,} \\ -\chi \bullet & \text{if } \bullet \text{ is a fundamental modulus,} \\ \bullet \chi & \text{if } \bullet \text{ is an anti-fundamental modulus,} \end{cases} \quad (4.19)$$

and similarly for \mathbb{T}_ξ . Using this notation, for example the terms $-\xi[B^I][\chi, \bar{B}_I]$ and $\bar{\mu}' \chi \mu'$ appearing in (4.16b) can be rewritten respectively as

$$-\mathbb{T}_\xi[B^I] \mathbb{T}_\chi[\bar{B}_I] \quad \text{and} \quad -\bar{\mu}' \mathbb{T}_\chi[\mu'] . \quad (4.20)$$

All other χ - and ξ -dependent terms in (4.16) can be treated in the same way.

The idea is to extend this approach to all symmetries of the brane system. The actions (4.16) are invariant under the $U(N)$ transformations of the N D7-branes and under the $U(M)$ transformations of the M D7'-branes, in which only the mixed moduli transform according to the representations defined by the open string construction. The actions (4.16) are also invariant under the $SU(4)$ transformations related to the rotations in the four complex planes indexed by I . Let us denote by ε_I the parameters of such rotations and define the quantities

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 , \quad (4.21)$$

and

$$\varepsilon_{IJ} = \varepsilon_I + \varepsilon_J - \frac{\varepsilon}{2} . \quad (4.22)$$

Then, it is easy to show that the actions (4.16) are invariant under the following transformations

$$\begin{aligned} (B^I, M^I) &\rightarrow e^{+i\varepsilon_I} (B^I, M^I), \\ (\bar{B}_I, \bar{M}_I) &\rightarrow e^{-i\varepsilon_I} (\bar{B}_I, \bar{M}_I), \\ (\lambda^{IJ}, D^{IJ}) &\rightarrow e^{+i\varepsilon_{IJ}} (\lambda^{IJ}, D^{IJ}), \end{aligned} \quad (4.23)$$

with all other moduli unchanged, provided $\varepsilon = 0$. This is a $SU(4)$ symmetry whose origin is quite clear. Indeed, the initial $SO(8)$ symmetry of the D-brane system is broken to $SO(7)$ on the D-instantons [16, 21], and this symmetry is further reduced to $SO(6) \simeq SU(4)$ by the complex structure we have introduced. Therefore, the four parameters ε_I subject to the condition $\varepsilon = 0$ can be understood as the parameters of the $SU(4)$ transformations along the Cartan directions (see appendix A).

We now exploit the $U(N)$, $U(M)$ and $SU(4)$ symmetries and replace the $U(k)$ transformations \mathbb{T}_χ and \mathbb{T}_ξ according to

$$\begin{aligned} \mathbb{T}_\chi[\bullet] &\rightarrow \mathbb{T}_{(\chi, a, m, \varepsilon_I)}[\bullet], \\ \mathbb{T}_\xi[\bullet] &\rightarrow \mathbb{T}_{(\xi, \bar{a}, \bar{m}, \bar{\varepsilon}_I)}[\bullet] \end{aligned} \quad (4.24)$$

¹⁰The use of χ or ξ is fixed by a neutrality condition with respect to the charge in the fifth complex plane of the initial string construction.

where $\mathbb{T}_{(\chi,a,m,\varepsilon_I)}$ and $\mathbb{T}_{(\xi,\bar{a},\bar{m},\bar{\varepsilon}_I)}$ denote $U(k) \times U(N) \times U(M) \times SU(4)$ infinitesimal transformations parametrized by χ , a , m and ε_I , and by ξ , \bar{a} , \bar{m} and $\bar{\varepsilon}_I$ respectively, in the appropriate representations. For example, applying this rule the terms in (4.20) get shifted and become

$$\begin{aligned}
 -\mathbb{T}_{(\xi,\bar{a},\bar{m},\bar{\varepsilon}_I)}[B^I] \mathbb{T}_{(\chi,a,m,\varepsilon_I)}[\bar{B}_I] &= -(-[\xi, B^I] + \bar{\varepsilon}_I B^I)(-[\chi, \bar{B}_I] - \varepsilon_I \bar{B}_I), \\
 -\bar{\mu}' \mathbb{T}_{(\chi,a,m,\varepsilon_I)}[\mu'] &= -\bar{\mu}'(-\chi \mu' + \mu' m).
 \end{aligned}
 \tag{4.25}$$

Proceeding systematically in this way, we can generate new terms in the moduli action. Taking for simplicity the $U(N)$ and $U(M)$ parameters only along the Cartan directions, these new terms correspond to shifting the actions (4.16) as follows¹¹

$$S_G \rightarrow S'_G = S_G + \text{tr} \left\{ \sum_{I < J, K < L} \varepsilon_{IJ} \lambda^{IJ} \lambda^{KL} \epsilon_{IJKL} \right\},
 \tag{4.26a}$$

$$\begin{aligned}
 S_K \rightarrow S'_K &= S_K + \text{tr} \left\{ 2 \sum_I \left(\varepsilon_I B^I [\xi, \bar{B}_I] + \bar{\varepsilon}_I B^I [\chi, \bar{B}_I] + \bar{\varepsilon}_I \varepsilon_I B^I \bar{B}_I - \bar{\varepsilon}_I M^I \bar{M}_I \right) \right. \\
 &\quad \left. - 2 a \bar{w} \xi w - 2 \bar{a} \bar{w} (\chi - a) w + 2 \bar{a} \bar{\mu} \mu - \bar{\mu}' \mu' m \right\},
 \end{aligned}
 \tag{4.26b}$$

$$S_D \rightarrow S'_D = S_D.
 \tag{4.26c}$$

As mentioned above, these new terms can also be obtained from the interactions that the instanton moduli have with the background fields on the D7 and D7'-branes or with the bulk fields of the closed string sectors. In particular, the terms with a , \bar{a} and m arise when a vacuum expectation value is given to the adjoint scalars ϕ and $\bar{\phi}$ on the N D7-branes (see (2.8)) and to the analogue scalars on the M D7'-branes. One can explicitly check that all such terms arise from the correlators among the vertex operators of the instanton moduli and the vertex operators of these adjoint scalars. The latter have the same structure as the vertices of ξ and χ in (3.2), and thus it is not surprising that they produce similar couplings in the moduli action.

The terms in (4.26) that depend on ε_I and $\bar{\varepsilon}_I$ can instead be generated by introducing a graviphoton background corresponding to the 3-form field strength $\mathcal{W} = d(C_2 - \tau B_2)$ of Type II B string theory.¹² In our set-up, the 3-form field strength has both holomorphic and anti-holomorphic components along the fifth complex direction, namely it is of the form $\mathcal{W}_{\mu\nu z}$ and $\bar{\mathcal{W}}_{\mu\nu \bar{z}}$ and is skew-symmetric along the μ and ν directions:¹³

$$\mathcal{W}_{12z} = \varepsilon_1, \quad \mathcal{W}_{34z} = \varepsilon_2, \quad \mathcal{W}_{56z} = \varepsilon_3, \quad \mathcal{W}_{78z} = \varepsilon_4,
 \tag{4.27a}$$

$$\bar{\mathcal{W}}_{12\bar{z}} = \bar{\varepsilon}_1, \quad \bar{\mathcal{W}}_{34\bar{z}} = \bar{\varepsilon}_2, \quad \bar{\mathcal{W}}_{56\bar{z}} = \bar{\varepsilon}_3, \quad \bar{\mathcal{W}}_{78\bar{z}} = \bar{\varepsilon}_4.
 \tag{4.27b}$$

¹¹Notice that no dependence on \bar{m} arises with this method, since there are no terms depending both on ξ and on the primed moduli in the original action.

¹²Here C_2 and B_2 are the 2-forms in the R-R and NS-NS sectors, respectively, and τ is the axio-dilaton (2.10).

¹³As we will show later in section 4.3, if $\sum_I \varepsilon_I = 0$ the BRST structure of the moduli action is deformed but not spoiled by the background, meaning that supersymmetry is not broken. Notice that in complex notation, the 3-form \mathcal{W} in (4.27a) is of type (2,1), while the 3-form $\bar{\mathcal{W}}$ in (4.27b) of type (1,2).

As shown in [31], to obtain the ε_I -dependent terms of the moduli action, it is enough to consider the R-R part of the 3-form field-strength \mathcal{W} , whose vertex operator was discussed in detail in [16]. In the notation for the spin fields we are adopting now, the vertex operator for the holomorphic components is proportional to

$$V_{\mathcal{W}}(z, \bar{z}) = \sum_I \left(\varepsilon_I - \frac{\varepsilon}{2} \right) \left[\bar{S}_I(z) S^+(z) e^{-\frac{1}{2}\varphi(z)} \times \tilde{S}^I(\bar{z}) \tilde{S}^+(\bar{z}) e^{-\frac{1}{2}\tilde{\varphi}(\bar{z})} - S^I(z) S^+(z) e^{-\frac{1}{2}\varphi(z)} \times \tilde{S}_I(\bar{z}) \tilde{S}^+(\bar{z}) e^{-\frac{1}{2}\tilde{\varphi}(\bar{z})} \right] \quad (4.28)$$

where the tilded and untilded operators denote the right and left-moving parts respectively. On the other hand, the vertex for the anti-holomorphic part is proportional to

$$V_{\overline{\mathcal{W}}}(z, \bar{z}) = \sum_{I < J} \bar{\varepsilon}_{IJ} \left[\bar{S}_{IJ}(z) S^-(z) e^{-\frac{1}{2}\varphi(z)} \times \tilde{S}^{IJ}(\bar{z}) \tilde{S}^-(\bar{z}) e^{-\frac{1}{2}\tilde{\varphi}(\bar{z})} \right] \quad (4.29)$$

where $\bar{\varepsilon}_{IJ}$ is defined in analogy with (4.22). Setting $\varepsilon = \bar{\varepsilon} = 0$ and computing mixed open/closed string amplitudes on disks with insertions of the moduli vertex operators on the boundary and of the vertex operators $V_{\mathcal{W}}$ and $V_{\overline{\mathcal{W}}}$ in the interior,¹⁴ one can correctly reproduce all terms in (4.26) that depend on ε_I and $\bar{\varepsilon}_I$. If we release the constraint $\varepsilon = 0$, the following extra term

$$\frac{1}{g_0^2} \text{tr} \{ i \varepsilon (\lambda \eta - D \xi) \} \quad (4.30)$$

is generated by the vertex (4.28). As we will see in the next section, this term plays a crucial role when one integrates over the instanton moduli space and its presence turns out to be necessary in the intermediate steps, even if the end result will not exhibit any *explicit* dependence on ε . The full moduli action we will consider is therefore

$$S'_{\text{inst}} = \frac{1}{g_0^2} \left(S'_G + \text{tr} \{ i \varepsilon (\lambda \eta - D \xi) \} \right) + S'_K + S'_D \quad (4.31)$$

with S'_G , S'_K and S'_D given in (4.26).

4.3 The BRST structure of the moduli spectrum

We now show that all moduli, except for χ , can be organized in doublets with respect to a BRST charge \mathcal{Q} which is a real combination of two super-symmetries preserved in the brane system. More specifically, we take

$$\mathcal{Q} = \frac{Q + \bar{Q}}{\sqrt{2}} \quad (4.32)$$

where

$$Q = \oint \frac{dz}{2\pi i} S(z) S^-(z) e^{-\frac{1}{2}\varphi(z)} \quad \text{and} \quad \bar{Q} = \oint \frac{dz}{2\pi i} \bar{S}(z) S^-(z) e^{-\frac{1}{2}\varphi(z)} \quad (4.33)$$

with $S(z)$ and $\bar{S}(z)$ given in (A.22).

¹⁴See for example [44] for details on mixed amplitudes involving R-R backgrounds.

To obtain a BRST transformation of the form

$$\delta X \equiv \mathcal{Q} X = Y, \quad (4.34)$$

we can follow the methods described in [6, 16], namely we evaluate the commutator of \mathcal{Q} with the vertex operator for Y and from the resulting expression read the form of the BRST variation according to

$$[\mathcal{Q}, V_Y(w)] = V_{\delta X}(w). \quad (4.35)$$

We can explicitly verify that there is no fermionic vertex operator which under the action of \mathcal{Q} can reproduce the structure of the vertex for χ . This implies that

$$\mathcal{Q} \chi = 0, \quad (4.36)$$

so that the interpretation of χ as a parameter for the $U(k)$ transformations can be maintained. On the contrary, we find

$$\mathcal{Q} \xi = \eta. \quad (4.37)$$

The choice of the BRST charge \mathcal{Q} introduces therefore an asymmetry between χ and ξ , which up to this point played a quite similar role.

To obtain the BRST transformation of Y , we impose the nilpotency of \mathcal{Q} up to symmetries, namely

$$\mathcal{Q} Y = \mathcal{Q}^2 X = \mathbb{T}_{(\chi, a, m, \varepsilon_I)}[X]. \quad (4.38)$$

Following this method, for the moduli of the $(-1)/(-1)$ sector we explicitly find

$$\begin{aligned} \mathcal{Q} B^I &= M^I, & \mathcal{Q} M^I &= -[\chi, B^I] + \varepsilon_I B^I, \\ \mathcal{Q} \bar{B}_I &= \bar{M}_I, & \mathcal{Q} \bar{M}_I &= -[\chi, \bar{B}_I] - \varepsilon_I \bar{B}_I, \\ \mathcal{Q} \lambda^{IJ} &= D^{IJ}, & \mathcal{Q} D^{IJ} &= -[\chi, \lambda^{IJ}] + \varepsilon_{IJ} \lambda^{IJ}, \\ \mathcal{Q} \lambda &= D, & \mathcal{Q} D &= -[\chi, \lambda], \\ \mathcal{Q} \xi &= \eta, & \mathcal{Q} \eta &= -[\chi, \xi]. \end{aligned} \quad (4.39)$$

with the understanding the repeated indices are not summed. For the mixed moduli, instead, we obtain

$$\begin{aligned} \mathcal{Q} w &= \mu, & \mathcal{Q} \mu &= -\chi w + w a, \\ \mathcal{Q} \bar{w} &= \bar{\mu}, & \mathcal{Q} \bar{\mu} &= \bar{w} \chi - a \bar{w}, \\ \mathcal{Q} \mu' &= h', & \mathcal{Q} h' &= -\chi \mu' + \mu' m, \\ \mathcal{Q} \bar{\mu}' &= \bar{h}', & \mathcal{Q} \bar{h}' &= \bar{\mu}' \chi - m \bar{\mu}'. \end{aligned} \quad (4.40)$$

These BRST transformations allow us to build doublets of the form $(X, \mathcal{Q} X)$. They are

$$(B^I, M^I), \quad (\bar{B}_I, \bar{M}_I), \quad (\xi, \eta), \quad (\lambda^{IJ}, D^{IJ}), \quad (\lambda, D) \quad (4.41)$$

in the $(-1)/(-1)$ sector, and

$$(w, \mu), \quad (\bar{w}, \bar{\mu}), \quad (\mu', h'), \quad (\bar{\mu}', \bar{h}') \quad (4.42)$$

in the mixed sectors. Furthermore, we can check that the action (4.31) is BRST invariant and Q -exact. Indeed,

$$Q S'_{\text{inst}} = 0, \quad S'_{\text{inst}} = Q \Lambda \tag{4.43}$$

where

$$\begin{aligned} \Lambda = \text{tr} \left\{ \frac{1}{g_0^2} \left(\frac{1}{4} \lambda^{IJ} D^{KL} \epsilon_{IJKL} + \frac{1}{2} \lambda D - \frac{1}{2} \eta [\chi, \xi] - i \varepsilon \lambda \xi \right) \right. \\ + \xi \left([B^I, \bar{M}_I] + [\bar{B}_I, M^I] + w \bar{\mu} - \mu \bar{w} \right) + \bar{a} (\bar{w} \mu - \bar{\mu} w) \\ - \lambda^{IJ} \left([\bar{B}_I, \bar{B}_J] + \frac{1}{2} \epsilon_{IJKL} [B^K, B^L] \right) - i \lambda \left([B^I, \bar{B}_I] + w \bar{w} \right) \\ \left. - \sum_I \bar{\varepsilon}_I (B^I \bar{M}_I - M^I \bar{B}_I) + \frac{1}{2} \mu' \bar{h}' + \frac{1}{2} h' \bar{\mu}' \right\}. \end{aligned} \tag{4.44}$$

We observe that the dependence of the moduli action on \bar{a} , $\bar{\varepsilon}_I$ and ε is entirely encoded in the fermion Λ on which the BRST charge acts. This fact will be important in the following, and in particular it suggests that the end-result of the localization process will not carry any *explicit* dependence on these parameters.

5 The instanton partition function

The effects of the D-instantons in the D7/D7' system are encoded in the gran-canonical partition function

$$Z = \sum_{k=0}^{\infty} q^k Z_k \tag{5.1}$$

where $q = e^{2\pi i \tau}$ is the instanton fugacity and Z_k is the partition function in the k -instanton sector (with the convention that $Z_0 = 1$). The latter is defined by the following integral

$$Z_k = \int d\mathcal{M}_k e^{-S'_{\text{inst}}} . \tag{5.2}$$

where \mathcal{M}_k denotes the set of moduli with k instantons. Actually, as we have seen in the previous section, all such moduli except for χ are paired in BRST doublets each one containing a boson and a fermion. We have in fact the following structure

$$d\mathcal{M}_k = d\chi (dB^I dM^I) (d\bar{B}_I d\bar{M}_I) (d\lambda^{IJ} dD^{IJ}) (d\lambda dD) (d\xi d\eta) (dw d\mu) (d\bar{w} d\bar{\mu}) (d\mu' dh') (d\bar{\mu}' d\bar{h}') . \tag{5.3}$$

Therefore, any rescaling of the doublets cancels between bosons and fermions and does not affect the end result. We exploit this feature and rescale all doublets with a factor of $1/x$. In the limit $x \rightarrow \infty$ a drastic simplification occurs since only the quadratic terms of the action S'_{inst} are relevant in the limit, while the cubic and quartic terms become subleading

and can be neglected. In other words the moduli action becomes

$$\begin{aligned}
 S'_{\text{inst}} = & \frac{1}{x^2 g_0^2} \text{tr} \left\{ \frac{1}{4} D^{IJ} D^{KL} \epsilon_{IJKL} + \frac{1}{2} D^2 + \frac{1}{2} [\chi, \xi]^2 + \frac{1}{2} \lambda [\chi, \lambda] + \frac{1}{2} \eta [\chi, \eta] \right. \\
 & \left. + \frac{1}{4} \lambda^{IJ} [\chi, \lambda^{KL}] \epsilon_{IJKL} + \sum_{I < J, K < L} \epsilon_{IJ} \lambda^{IJ} \lambda^{KL} \epsilon_{IJKL} + i \epsilon (\lambda \eta - D \xi) \right\} \\
 & + \frac{1}{x^2} \text{tr} \left\{ \bar{\mu}' (\chi - m) \mu' + \bar{h}' h' - 2 \bar{a} (\bar{w} (\chi - a) w - \bar{\mu} \mu) \right. \\
 & \left. + 2 \sum_I \bar{\epsilon}_I (B^I [\chi, \bar{B}_I] + \epsilon_I B^I \bar{B}_I - M^I \bar{M}_I) \right\} + O(x^{-3}) .
 \end{aligned} \tag{5.4}$$

Notice that this quadratic action is invariant under the transformations (4.23) even for $\epsilon \neq 0$. In other words, in the scaling limit we are considering, the SU(4) symmetry of the initial action could be promoted to a U(4) symmetry.

The integrations over all moduli but χ are straightforward since all integrals are Gaussian and can be readily performed. From the primed doublets (μ', h') and $(\bar{\mu}', \bar{h}')$ we obtain¹⁵

$$\prod_{r=1}^M \det (\chi - m_r)_{\text{fund}} \tag{5.5}$$

where m_r are the eigenvalues of m and the determinant is computed in the fundamental representation of U(k). Integrating over the mixed fundamental doublets (w, μ) and $(\bar{w}, \bar{\mu})$ yields

$$\prod_{u=1}^N \det (\chi - a_u)_{\text{fund}}^{-1} \tag{5.6}$$

where a_u are the eigenvalues of a . We observe that in order to get this result it is essential to have the \bar{a} parameter in the action which provides an effective “mass” for the fermions μ and $\bar{\mu}$. However, this \bar{a} dependence cancels out in the final result between the numerator and the denominator.

The integration over the adjoint doublet (λ^{IJ}, D^{IJ}) gives

$$\prod_{I < J=1}^3 \det (\chi - \epsilon_{IJ})_{\text{adj}} \tag{5.7}$$

while from the integration over (B^I, M^I) and (\bar{B}_I, \bar{M}_I) we get

$$\prod_{I=1}^4 \det (\chi - \epsilon_I)_{\text{adj}}^{-1} \tag{5.8}$$

where the determinants are computed in the adjoint of U(k). In this case it is essential to have the $\bar{\epsilon}_I$ terms in the action since they provide a “mass” for the fermions M^I and \bar{M}_I . This is necessary to guarantee a non-vanishing result but, as before, the final answer (5.8) does not depend on these parameters.

¹⁵Here and in the following we report the results of the integrations up to numerical factors that can always be reabsorbed in the overall normalization of the instanton partition function.

The last integrations to be performed are those on the doublets (λ, D) and (ξ, η) . One can explicitly verify that these Gaussian integrations yield just a non-vanishing numerical constant. In fact, the entire dependence on χ and ε cancels between the contribution of the fermions λ and η and the contribution of the bosons D and ξ . Again it is important in the intermediate steps to keep the ε -dependence; in particular the term $\varepsilon \lambda \eta$ in the action acts as an effective “mass” for the components of λ and η along the Cartan directions and it is crucial to have such a mass in order to have a non-zero result. However, the final result does not depend on ε , which can therefore be safely set everywhere to zero at the end or kept at a non-zero value, if one wishes to do so.

Collecting all terms, we have

$$Z_k = \int d\chi \frac{\prod_{r=1}^M \det(\chi - m_r)_{\text{fund}}}{\prod_{u=1}^N \det(\chi - a_u)_{\text{fund}}} \frac{\prod_{I < J=1}^3 \det(\chi - \varepsilon_{IJ})_{\text{adj}}}{\prod_{I=1}^4 \det(\chi - \varepsilon_I)_{\text{adj}}} . \tag{5.9}$$

Exploiting the invariance of the integrand under $U(k)$ transformations, we can bring χ to the Cartan sub-algebra and, after introducing the Vandermonde determinant, integrate over the k eigenvalues χ_i ($i = 1, \dots, k$). Thus, we can rewrite (5.9) in the following way

$$Z_k = \int \prod_{i=1}^k d\chi_i \prod_{i=1}^k \frac{\prod_{r=1}^M (\chi_i - m_r)}{\prod_{u=1}^N (\chi_i - a_u)} \prod_{i,j=1}^k \frac{\chi_{ij}^{1-\delta_{ij}} \prod_{I < J=1}^3 (\chi_{ij} - \varepsilon_{IJ})}{\prod_{I=1}^4 (\chi_{ij} - \varepsilon_I)} \tag{5.10}$$

with $\chi_{ij} = \chi_i - \chi_j$. For $\varepsilon = 0$, the formula (5.10) agrees with the one proposed in [29] where it was conjectured to possibly derive from a system with branes and anti-branes. This instanton partition function has been recently considered also in [30, 32].

From now on we set $\varepsilon = 0$ and for simplicity present explicit results only for $N = M$. In this case the instanton partition function Z_k becomes dimensionless (χ_i, m_u, a_u and ε_I have all dimensions of a mass) and the theory behaves as if it were conformal. With these positions we can rewrite (5.10) as follows

$$Z_k = \mathcal{V}^k \int \prod_{i=1}^k d\chi_i \prod_{i=1}^k \prod_{u=1}^N \frac{(\chi_i - m_u)}{(\chi_i - a_u)} \prod_{i < j=1}^k \frac{\chi_{ij}^2 \prod_{I \neq J=1}^4 (\chi_{ij} - \varepsilon_{IJ})}{\prod_{I=1}^4 (\chi_{ij} - \varepsilon_I)} \tag{5.11}$$

where

$$\mathcal{V} = - \frac{(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 + \varepsilon_3)(\varepsilon_2 + \varepsilon_3)}{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4} \tag{5.12}$$

with $\varepsilon_4 = -\varepsilon_1 - \varepsilon_2 - \varepsilon_3$. Even if it is not at all obvious, we will show that Z_k in (5.11) depends on a_u, m_u and ε_I only through the combination \mathcal{VM} where

$$\mathcal{M} = \sum_{u=1}^N (m_u - a_u) . \tag{5.13}$$

Moreover, we will find that our explicit results for Z_k are compatible with the small q expansion of

$$Z = [M(-q)]^{\mathcal{VM}} \tag{5.14}$$

where

$$M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 \dots \quad (5.15)$$

is the MacMahon function which counts the number of the planar partitions of the integers. This result was also derived in [27] using a conjectured plethystic exponential formula.

From (5.14) we may deduce that the non-perturbative prepotential is

$$\mathcal{F} = \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \log Z = -(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 + \varepsilon_3)(\varepsilon_2 + \varepsilon_3) \mathcal{M} \log M(-q) . \quad (5.16)$$

5.1 The $k = 1$ case

At $k = 1$ the instanton partition function is

$$Z_1 = \mathcal{V} \int d\chi \prod_{u=1}^N \frac{(\chi - m_u)}{(\chi - a_u)} . \quad (5.17)$$

As usual, the integral over χ is computed by moving the poles a_u in the upper-half complex plane and using the residue theorem after closing the integration path at infinity [20, 21]. The result is simply

$$Z_1 = \mathcal{V} \sum_{u=1}^N \frac{\prod_{v=1}^N (a_u - m_v)}{\prod_{v \neq u=1}^N (a_u - a_v)} = -\mathcal{V} \mathcal{M} . \quad (5.18)$$

5.2 Explicit results at higher instanton numbers

To evaluate Z_k at a generic instanton number k , we follow the prescription of [21]. This corresponds to move again the poles a_u in the upper-half complex plane, assign to each ε_I a small imaginary part in such a way that

$$\text{Im}(\varepsilon_4) \gg \text{Im}(\varepsilon_3) \gg \text{Im}(\varepsilon_2) \gg \text{Im}(\varepsilon_1) \gg \text{Im}(a_u) , \quad (5.19)$$

and then integrate over each χ_i in lexicographic order by closing each contour in the upper-half plane. With this prescription,¹⁶ the poles that contribute to the result can be put in one-to-one correspondence with an N -array of four-dimensional Young tableaux $\{Y_u\}$ (with $u = 1, \dots, N$) containing a total number of k boxes. Each box in the diagram Y_u is labeled by four positive integers (n_1, n_2, n_3, n_4) and is associated to a pole for one of the integration variables in (5.10) located at

$$\chi^{(n_1, n_2, n_3, n_4)} = a_u + (n_1 - 1) \varepsilon_1 + (n_2 - 1) \varepsilon_2 + (n_3 - 1) \varepsilon_3 + (n_4 - 1) \varepsilon_4 . \quad (5.20)$$

For example, for $N = 1$ and $k = 2$ we have

$$Z_2 = \mathcal{V}^2 \int d\chi_1 d\chi_2 \frac{(\chi_1 - m)(\chi_2 - m)(\chi_1 - \chi_2)^2 \prod_{I \neq J=1}^4 (\chi_1 - \chi_2 - \varepsilon_I - \varepsilon_J)}{(\chi_1 - a)(\chi_2 - a)(\chi_1 - \chi_2 - \varepsilon_1)(\chi_1 - \chi_2 - \varepsilon_2)(\chi_1 - \chi_2 - \varepsilon_3)(\chi_1 - \chi_2 - \varepsilon_4)} \quad (5.21)$$

and we see that the poles for (χ_1, χ_2) are at

$$(a, a + \varepsilon_1), \quad (a, a + \varepsilon_2), \quad (a, a + \varepsilon_3), \quad (a, a + \varepsilon_4), \quad (5.22)$$

¹⁶This is fully equivalent to the so-called Jeffrey-Kirwan prescription [45].

which indeed correspond to the four four-dimensional Young tableaux with two boxes. These four tableaux are also associated to the four solid partitions of 2. This is a general feature of the $N = 1$ theory in which the poles that contribute to the integrals of Z_k are in one-to-one correspondence with the solid partitions of k .

For $N = 2$ and $k = 2$, instead, the poles for (χ_1, χ_2) turn out to be at

$$(a_1, a_2), \quad (a_u, a_u + \varepsilon_1), \quad (a_u, a_u + \varepsilon_2), \quad (a_u, a_u + \varepsilon_3), \quad (a_u, a_u + \varepsilon_4) \quad (5.23)$$

with $u = 1, 2$. The first pole location corresponds to a pair of tableaux with one box each; the other locations describe pairs of tableaux, one with two boxes and the other with no boxes. This example can be systematically generalized to other values of N and k .

The evaluation of the residues is straightforward, even if tedious. When we sum up all contributions, remarkable simplifications occur and, as anticipated, the dependence on a_u , m_u and ε_I in the final result is only through the combination \mathcal{VM} . We have explicitly checked this fact up to $k = 5$, and found

$$\begin{aligned} Z_2 &= \frac{1}{2} (\mathcal{VM})^2 + \frac{5}{2} \mathcal{VM}, \\ Z_3 &= -\frac{1}{6} (\mathcal{VM})^3 - \frac{5}{2} (\mathcal{VM})^2 - \frac{10}{3} \mathcal{VM}, \\ Z_4 &= \frac{1}{24} (\mathcal{VM})^4 + \frac{5}{4} (\mathcal{VM})^3 + \frac{155}{24} (\mathcal{VM})^2 + \frac{21}{4} \mathcal{VM}, \\ Z_5 &= -\frac{1}{120} (\mathcal{VM})^5 - \frac{5}{12} (\mathcal{VM})^4 - \frac{115}{24} (\mathcal{VM})^3 - \frac{163}{12} (\mathcal{VM})^2 - \frac{26}{5} \mathcal{VM}. \end{aligned} \quad (5.24)$$

These results allow us to infer the formula (5.14).

Let us remark that, once individuated the poles through the prescription (5.19), the evaluation of the residues does not introduce any sign ambiguity in summing up the contributions associated to the various partitions of the type considered in [27, 29]. A particular sign choice is automatically picked up, and is exactly the one that leads to the correct exponentiation of the result. This choice is in agreement with the one pointed out in [30].

We mentioned above that the poles contributing to the integrals of Z_k for the rank 1 theory are in one-to-one correspondence with the solid partitions of k . However, it is the generating function of the planar partitions $M(q)$ that appears in the end result (5.14). To understand this fact, let us specialize to $N = 1$ with $a = 0$ and $m = \varepsilon_4$, and then take the limit $\varepsilon_4 \rightarrow 0$. This limit clearly implies that $\mathcal{VM} \rightarrow 1$. It turns out that in this scaling all contributions to Z_k corresponding to Young tableaux which contain at least one box with $n_4 > 1$ are set to zero, while all contributions associated to tableaux in which all boxes have $n_4 = 1$ are uniformly weighted with $(-1)^k$. In other words, in this limit the gran-canonical partition function (5.1) can be written as a weighted sum over the solid partitions π :

$$Z = \sum_{\pi} w_{\pi} q^{|\pi|} \quad (5.25)$$

where $|\pi|$ is the size of π , and

$$w_{\pi} = \begin{cases} (-1)^{|\pi|} & \text{if } Y(\pi) \text{ does not extend in the } \varepsilon_4 \text{ direction,} \\ 0 & \text{if } Y(\pi) \text{ does extend in the } \varepsilon_4 \text{ direction,} \end{cases} \quad (5.26)$$

with $Y(\pi)$ denoting the four-dimensional Young diagram associated to π . Given the form of the weight function (5.26), it is clear that only the solid partitions π which are also planar with respect to the ε_4 -direction, contribute in the sum. Therefore, (5.25) actually coincides with the partition function of the planar partitions with uniform weight (up to an alternating sign), which are counted by the MacMahon function, *i.e.*

$$Z = M(-q) \tag{5.27}$$

in agreement with (5.14) when $\mathcal{VM} = 1$. This argument can be extended to more general cases along the lines discussed in [28].

6 ADHM construction

We have anticipated that the physical states of the open strings with at least one end-point on a D(-1)-brane can be put in correspondence with the moduli of instantonic configurations in eight dimensions. In this section we would like to elaborate on this.

There are different definitions of eight-dimensional gauge instantons, which are briefly reviewed in a similar (although not identical) brane context in [34]. The definition appropriate to our current set-up is the one in which these configurations correspond to Yang Mills connections whose field strength satisfies the (anti) self-duality condition

$$*(F \wedge F) = \pm F \wedge F . \tag{6.1}$$

When this condition holds, the quartic action S_4 for the gauge fields on the D7 branes, given in (2.4), drastically simplifies. Indeed, in this case

$$\text{tr} (t_8 F^4) = \pm \frac{1}{2} \text{tr} (F \wedge F \wedge F \wedge F) , \tag{6.2}$$

and hence

$$S_4 = \begin{cases} -2\pi i \bar{\tau} k , \\ -2\pi i \tau k , \end{cases} \tag{6.3}$$

where τ is defined in (2.10) and

$$k = \frac{1}{4!(2\pi)^4} \int \text{tr} (F \wedge F \wedge F \wedge F) \tag{6.4}$$

is the fourth Chern number. This precisely matches the action of k anti D-instantons or k D-instantons, respectively.

In analogy with the familiar case of the four-dimensional instanton, one way to realize the conditions (6.1) is to exploit the duality properties of the eight-dimensional Dirac matrices. Indeed, the chiral blocks $\sigma^{\mu\nu}$ (see (A.16)), satisfy

$$\sigma^{[\mu_1\mu_2} \sigma^{\mu_3\mu_4]} = + \frac{1}{4!} \epsilon^{\mu_1\mu_2\dots\mu_7\mu_8} \sigma_{\mu_5\mu_6} \sigma_{\mu_7\mu_8} \tag{6.5}$$

where in the left-hand side the indices are all anti-symmetrized, while the anti-chiral blocks $\bar{\sigma}^{\mu\nu}$ obey

$$\bar{\sigma}^{[\mu_1\mu_2} \bar{\sigma}^{\mu_3\mu_4]} = -\frac{1}{4!} \epsilon^{\mu_1\mu_2\dots\mu_7\mu_8} \bar{\sigma}_{\mu_5\mu_6} \bar{\sigma}_{\mu_7\mu_8} . \quad (6.6)$$

A field strength $F_{\mu\nu}$ proportional to $\sigma_{\mu\nu}$ or to $\bar{\sigma}_{\mu\nu}$ would thus enjoy the property (6.1).

To obtain such a field strength we can follow the ADHM construction in four dimensions (for definiteness we choose the minus sign in (6.1)). We first introduce the ADHM matrix¹⁷

$$\Delta = \begin{pmatrix} \alpha_{[N \times 8k]} \\ (B - x)_{[8k \times 8k]} \end{pmatrix} \quad (6.7)$$

where α is a $N \times 8k$ matrix, while B and x are $8k \times 8k$ matrices given by

$$B = (B_\mu)_{[k \times k]} \otimes \bar{\sigma}^\mu , \quad x = x_\mu \mathbb{1}_{[k \times k]} \otimes \bar{\sigma}^\mu . \quad (6.8)$$

The matrices α and B_μ contain the bosonic moduli of the configuration, while x_μ are the coordinates in eight dimensions. Then, we define the gauge connection A_μ , expressed as a $N \times N$ matrix, according to

$$A_\mu = U^\dagger \partial_\mu U , \quad (6.9)$$

where

$$U = \begin{pmatrix} u_{[N \times N]} \\ v_{[8k \times N]} \end{pmatrix} \quad (6.10)$$

is a $(N + 8k) \times N$ matrix subject to the following conditions

$$\Delta^\dagger U = U^\dagger \Delta = 0 , \quad U^\dagger U = \mathbb{1}_{[N \times N]} . \quad (6.11)$$

The corresponding field strength is

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 2 \partial_{[\mu} U^\dagger \partial_{\nu]} U + 2 U^\dagger \partial_{[\mu} U U^\dagger \partial_{\nu]} U \\ &= 2 \partial_{[\mu} U^\dagger (\mathbb{1} - U U^\dagger) \partial_{\nu]} U . \end{aligned} \quad (6.12)$$

We now introduce a $8k \times 8k$ matrix M such that

$$\mathbb{1} - U U^\dagger = \Delta M \Delta^\dagger \quad (6.13)$$

or equivalently, thanks to the conditions (6.11), such that

$$\Delta^\dagger \Delta = M^{-1} . \quad (6.14)$$

Plugging (6.13) into (6.12) leads to

$$\begin{aligned} F_{\mu\nu} &= 2 \partial_{[\mu} U^\dagger \Delta M \Delta^\dagger \partial_{\nu]} U = 2 U^\dagger \partial_{[\mu} \Delta M \partial_{\nu]} \Delta^\dagger U \\ &= 2 U^\dagger \begin{pmatrix} 0 & 0 \\ 0 & \bar{\sigma}_{[\mu} M \sigma_{\nu]} \end{pmatrix} U = 2 \bar{v} \bar{\sigma}_{[\mu} M \sigma_{\nu]} v \end{aligned} \quad (6.15)$$

¹⁷Since the construction involves matrices of various sizes, sometimes we find useful to indicate explicitly their sizes with the notation $A_{[n \times n]}$ for a matrix A of size $n \times n$.

where in the second line, with an abuse of notation, we have denoted $\mathbb{1}_{[k \times k]} \otimes (\bar{\sigma}_\mu)_{[8 \times 8]}$ and $\mathbb{1}_{[k \times k]} \otimes (\bar{\sigma}_\mu)_{[8 \times 8]}$ simply as $\bar{\sigma}_\mu$ and σ_ν .

If $N \geq 8$, it is possible to require that the $N \times N$ matrix $2\bar{v} \bar{\sigma}_{[\mu} M \sigma_{\nu]} v$ has a single non-trivial block of size (8×8) proportional to $\bar{\sigma}_{\mu\nu}$. If this is the case, then, the field strength (6.15) satisfies the relation (6.1) with the minus sign as a consequence of the anti-self duality property (6.6). Of course, this requirement puts stringent constraints on the form of the matrices M and Δ . For $k = 1$, a solution is obtained by taking

$$M = f \mathbb{1}_{[8 \times 8]}, \quad \alpha = \begin{pmatrix} \rho \mathbb{1}_{[8 \times 8]} \\ 0_{[(N-8) \times 8]} \end{pmatrix}, \quad \bar{v} = f^{\frac{1}{2}} \alpha, \tag{6.16}$$

$$u = f^{\frac{1}{2}} \begin{pmatrix} -(B_\mu - x_\mu) \bar{\sigma}^\mu & 0_{[8 \times (N-8)]} \\ 0_{[(N-8) \times 8]} & 0_{[(N-8) \times (N-8)]} \end{pmatrix}$$

with $f = (\rho^2 + r^2)^{-1}$ and $r^2 = (x_\mu - B_\mu)(x^\mu - B^\mu)$. Through (6.15), this leads to the following field strength

$$F_{\mu\nu} = \frac{2\rho^2}{(\rho^2 + r^2)^2} \begin{pmatrix} \bar{\sigma}_{\mu\nu} & 0_{[8 \times (N-8)]} \\ 0_{[(N-8) \times 8]} & 0_{[(N-8) \times (N-8)]} \end{pmatrix}. \tag{6.17}$$

This represents an instanton solution of size ρ and center B_μ which corresponds to the embedding of the SO(8) octonionic field strength of [46, 47] into U(N) (with $N \geq 8$). One can check that the gauge connection corresponding to (6.17) is

$$A_\mu = \frac{(x^\nu - B^\nu)}{\rho^2 + r^2} \begin{pmatrix} \bar{\sigma}_{\nu\mu} & 0_{[8 \times (N-8)]} \\ 0_{[(N-8) \times 8]} & 0_{[(N-8) \times (N-8)]} \end{pmatrix}, \tag{6.18}$$

and that the fourth Chern number (6.4) is one [46, 47]. Generalizations to higher k are possible.

In a supersymmetric theory, like the one we are considering, instantons preserve a fraction of supersymmetry. A supersymmetric instanton is then characterized by the existence of a Killing spinor ϵ such that the supersymmetry variation of the gaugino Λ vanishes, namely

$$\delta\Lambda = \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon = 0, \tag{6.19}$$

which reduces to

$$F_{\mu\nu} \sigma^{\mu\nu} \epsilon = 0 \tag{6.20}$$

for a chiral ϵ . In [32] it has been shown that to solve this equation it is enough to require that

$$\Delta^\dagger \Delta \epsilon = f_{[k \times k]}^{-1} \otimes \epsilon \tag{6.21}$$

for some $(k \times k)$ matrix f . This means that the matrix M introduced in (6.14) must be such that $M \epsilon = f_{[k \times k]} \otimes \epsilon$.

To make contact with the previous sections, we write the matrices B and α appearing in the ADHM matrix Δ in terms of the bosonic moduli corresponding to the open strings in the D(-1)/D7 brane system. In particular we take the eight $k \times k$ matrices B_μ , used to

construct B as in (6.8), to be given by the neutral moduli B^I and \bar{B}_I of the D(-1)/D(-1) sector, and take the $N \times 8k$ matrix α to be related to the mixed moduli \bar{w} of the D7/D(-1) sector according to $\alpha = \bar{w}_{[N \times k]} \otimes \psi^\dagger_{[1 \times 8]}$ where ψ a reference Weyl chiral spinor [32]. Given the choices we made in the string construction, we take

$$\psi = \mathbf{e} + i \mathbf{e}' \tag{6.22}$$

with $\mathbf{e} = \delta_{a,8}$ and $\mathbf{e}' = \delta_{a,7}$. The choice on the Killing spinor \mathbf{e} reflects our choice of the preserved supersymmetry (and hence of the BRST charge) which, in turn, is related to how SO(8) has been broken to SO(7). The second choice on \mathbf{e}' reflects the choice of the complex structure with which we have broken SO(7) to SU(4) (see appendix A). With these positions it is easy to verify that

$$\psi \psi^\dagger \mathbf{e} = -\tau^7 \mathbf{e} + \mathbf{e} \quad \text{and} \quad \sigma^{\mu\nu} \mathbf{e} = -(\tau_m)^{\mu\nu} \tau^m \mathbf{e} \tag{6.23}$$

where τ^m are the seven octonionic matrices defined in (A.13). Furthermore, we have

$$\begin{aligned} \Delta^\dagger \Delta &= \alpha^\dagger \alpha + (B^\dagger - x^\dagger)(B - x) \\ &= w \bar{w} \otimes \psi \psi^\dagger + \frac{1}{2} [B_\mu, B_\nu] \otimes \sigma^{\mu\nu} + (B_\mu - x_\mu)(B^\mu - x^\mu) \otimes \mathbb{1}_{[8 \times 8]}, \end{aligned} \tag{6.24}$$

and thus projecting onto \mathbf{e} and using the relations (6.23), we deduce that in order to satisfy (6.21) one must require that

$$\frac{1}{2} [B_\mu, B_\nu] (\tau_m)^{\mu\nu} + w \bar{w} \delta_m^7 = 0. \tag{6.25}$$

In complex notation, these equations read

$$[B^I, \bar{B}_I] + w \bar{w} = 0, \quad [\bar{B}_I, \bar{B}_J] + \frac{1}{2} \epsilon_{IJKL} [B^K, B^L] = 0, \tag{6.26}$$

which are exactly the constraints reported in (4.17) and derived from the moduli action in the D(-1)/D7-brane system.

We observe that for $k = 1$ the first constraint leads to $w = \bar{w} = 0$. This implies that the ADHM data reduce to those in (6.16) for $\rho = 0$, thus describing a point-like instanton configuration with vanishing size. It would be interesting to explore what happens if one introduces a non-commutative deformation parameter in such a way that the right-hand side of the real constraint in (6.26) differs from zero, thus allowing to have configurations with non-zero size, and more generally to investigate the features of the solutions of the ADHM constraints in the general case. We leave these issues to future investigations.

7 Conclusions

In this paper we have studied a D(-1)/D7-brane system in Type IIB string theory that describes the non-perturbative sector of a $U(N)$ gauge theory in eight dimensions, and provided an explicit analysis of the massless open string states and of their vertex operators in the various sectors. The new ingredient with respect to previous work is the introduction

of a constant magnetic flux on the world-volume of the D7-branes which allows for the existence of bosonic moduli in the mixed sectors. After discussing the moduli action, we have computed the instanton partition function using localization and confirmed the results of [27] obtaining a closed form expression in terms of the MacMahon function.

It is important to remark that this instanton partition function and the corresponding prepotential given in (5.16) involve only the fields of the overall U(1) factor inside U(N). Furthermore, the prepotential is cubic in the ε_I parameters used in the localization process, showing that the gauge effective action receives non-perturbative corrections only in curved space. To see this in an explicit way, let us first turn off the parameters m_u so that the quantity \mathcal{M} in (5.13) simply reduces to $\text{tr } \phi$, where ϕ is the scalar component of the vector superfield (see (2.8)). Then let us note that

$$(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 + \varepsilon_3)(\varepsilon_2 + \varepsilon_3) \sim t_8^{\mu_1 \dots \mu_8} \mathcal{W}_{\mu_1 \mu_2} \mathcal{W}_{\mu_3 \mu_4} \mathcal{W}_{\mu_5 \mu_6} \mathcal{I}_{\mu_7 \mu_8} \quad (7.1)$$

where $\mathcal{W}_{\mu\nu} \equiv \mathcal{W}_{\mu\nu z}$ is the graviphoton field strength defined in (4.27a). The tensor $\mathcal{I}_{\mu\nu}$, instead, describes the background magnetic flux on the D7-branes and is obtained from (2.12) by setting $f_I = -1$ for all $I = 1, \dots, 4$.¹⁸ Finally, t_8 is the tensor that typically appears in higher derivative actions, like for instance (2.4).¹⁹ Suppressing all indices for simplicity, the prepotential (5.16) is therefore

$$\mathcal{F} \sim t_8 \mathcal{W}^3 \mathcal{I} \text{tr } \phi \log M(-q) . \quad (7.2)$$

To obtain the effective action, we promote ϕ to the full scalar superfield $\Phi(x, \theta)$ given in (2.8) and the graviphoton field strength $\mathcal{W}_{\mu\nu}$ to the full graviphoton superfield [31]

$$W_{\mu\nu}(x, \theta) = \mathcal{W}_{\mu\nu}(x) + \theta \chi_{\mu\nu}(x) + \frac{1}{2} \theta \sigma^{\lambda\rho} \theta R_{\mu\nu\lambda\rho}(x) + \dots \quad (7.3)$$

where $\chi_{\mu\nu}(x)$ and $R_{\mu\nu\rho\sigma}(x)$ are respectively, the self-dual parts of the gravitino field strength and of the Riemann tensor. In this way we obtain a prepotential $\mathcal{F}(\Phi, W)$ that is linear in Φ and cubic in W . The eight-dimensional effective action is then given by

$$\int d^8x d^8\theta \mathcal{F}(\Phi, W) + \text{c.c.} \quad (7.4)$$

After integration over the θ 's, our results predict an all-instanton formula for higher derivative couplings in Type II B supergravity, which are schematically of the form $t_8 t_8 R^3 \text{tr } F F_{\text{class}}$, in presence of D7-branes carrying a non-trivial world-volume flux $F_{\text{class}} \sim \mathcal{I}$ along a U(1) subgroup of U(N). It would be very interesting to further investigate the properties and the implications of these non-perturbative gravitational couplings, and also to explore the structure of the instanton partition function of our system when the graviphoton parameters ε_I do not add up to zero.

¹⁸According to (2.14), this corresponds to the choice of the twist parameters $\theta_I = 1/4$ that we have made in (4.1).

¹⁹Given the particular form of $\mathcal{W}_{\mu\nu}$ and $\mathcal{I}_{\mu\nu}$, one has

$$t_8^{\mu_1 \dots \mu_8} \mathcal{W}_{\mu_1 \mu_2} \mathcal{W}_{\mu_3 \mu_4} \mathcal{W}_{\mu_5 \mu_6} \mathcal{I}_{\mu_7 \mu_8} \sim 2 \text{tr} (\mathcal{W} \mathcal{W} \mathcal{W} \mathcal{I}) - \frac{1}{2} \text{tr} (\mathcal{W} \mathcal{W}) \text{tr} (\mathcal{W} \mathcal{I})$$

where the trace is over the space-time indices, from which the result (7.1) easily follows.

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A Notations and conventions

In this appendix we collect our notations for the various symmetry groups of the brane system and for the spinors and Dirac matrices.

From ten to eight dimensions. The symmetry of the ten-dimensional flat background in which the superstring theory is defined is broken by the presence of the seven-branes as

$$\mathrm{SO}(10) \rightarrow \mathrm{SO}(8) \otimes \mathrm{SO}(2), \tag{A.1}$$

with $\mathrm{SO}(8)$ rotating the first eight coordinates x^μ . Thus, the vector representation decomposes as

$$\mathbf{10} \rightarrow (\mathbf{8}_v, 0) \oplus (\mathbf{1}, +1) \oplus (\mathbf{1}, -1), \tag{A.2}$$

where each couple (\mathbf{R}, p) indicates a representation \mathbf{R} of $\mathrm{SO}(8)$ and a charge p under $\mathrm{U}(1) \sim \mathrm{SO}(2)$. The $\mathrm{U}(1)^4$ Cartan subgroup of $\mathrm{SO}(8)$ corresponds to the rotations in the (x^{2I-1}, x^{2I}) planes, with $I = 1, \dots, 4$, and here we will denote its four parameters as ε_I .

Under the breaking (A.1), the 32-components of the ten dimensional spinors (which can be chiral or anti-chiral) decompose into $S^A S^\pm$, where S^A are the 16 components of $\mathrm{SO}(8)$ spinors (chiral or anti-chiral) and S^\pm have charge $\pm 1/2$ with respect to $\mathrm{U}(1)$. More precisely, we have

$$\begin{aligned} \mathbf{16}_s &\rightarrow (\mathbf{8}_s, +1/2) \oplus (\mathbf{8}_c, -1/2), \\ \mathbf{16}_c &\rightarrow (\mathbf{8}_s, -1/2) \oplus (\mathbf{8}_c, +1/2). \end{aligned} \tag{A.3}$$

The ten-dimensional Dirac matrices take the form

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}, \quad \Gamma^9 = \gamma \otimes \sigma_1, \quad \Gamma^{10} = \gamma \otimes \sigma_2, \tag{A.4}$$

where γ^μ with $\mu = 1, 2, \dots, 8$ are the eight-dimensional Dirac matrices, γ is the eight-dimensional chirality matrix and the sigmas are the usual Pauli matrices. The ten-dimensional charge conjugation matrix \mathcal{C} satisfying $\mathcal{C} \Gamma^M (\mathcal{C}^{-1})^T = -(\Gamma^M)^T$ for $M = 1, \dots, 10$ is written as

$$\mathcal{C} = C \otimes \sigma_2, \tag{A.5}$$

where C is the eight-dimensional charge conjugation which is such that $C \gamma^\mu (C^{-1})^T = -(\gamma^\mu)^T$.

Eight-dimensional spinors. For the eight-dimensional spinors we can use either the “spin field” basis or the Majorana-Weyl basis, which we are going to describe.

• **The spin field basis.** In the “spin field” basis the spinor components are labeled by (twice) their SO(8) weight vectors and are

$$S^A = \left(S^{++++}, S^{-+++}, S^{+--+}, S^{-+--}, S^{++--}, S^{-+--}, S^{+---}, S^{----}, S^{+++}, S^{-++}, S^{+-+}, S^{-+-}, S^{+--}, S^{---}, S^{----} \right). \quad (\text{A.6})$$

In this basis the Dirac matrices are

$$\begin{aligned} \gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, & \gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ \gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1}, & \gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1}, \\ \gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1}, & \gamma^6 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}, \\ \gamma^7 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1, & \gamma^8 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2. \end{aligned} \quad (\text{A.7})$$

The chirality matrix is

$$\gamma = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^6 \gamma^7 \gamma^8 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3, \quad (\text{A.8})$$

while the charge conjugation matrix reads

$$C = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1. \quad (\text{A.9})$$

The chirality matrix is diagonal, and the eight components in (A.6) with an even number of minuses are chiral, *i.e.* they span the $\mathbf{8}_s$ representation of SO(8), while the eight ones with an odd number of minuses are anti-chiral, *i.e.* they span the $\mathbf{8}_c$ representation.

• **The Majorana-Weyl basis.** For SO(8) it is also possible to use a Majorana-Weyl basis. This can be achieved introducing the chiral combinations

$$S^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} iS^{+---} - iS^{-+++} \\ -S^{+---} - S^{-+++} \\ -iS^{+--+} - iS^{-+--} \\ -S^{+--+} + S^{-+--} \\ -iS^{++--} + iS^{--++} \\ -S^{++--} - S^{--++} \\ -iS^{++++} - iS^{----} \\ S^{++++} - S^{----} \end{pmatrix}, \quad (\text{A.10})$$

and the anti-chiral ones

$$S^{\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} iS^{+---} - iS^{-+++} \\ -S^{+---} - S^{-+++} \\ -iS^{+--+} - iS^{-+--} \\ -S^{+--+} + S^{-+--} \\ -iS^{++--} + iS^{--++} \\ -S^{++--} - S^{--++} \\ -iS^{++++} - iS^{----} \\ S^{++++} - S^{----} \end{pmatrix}. \quad (\text{A.11})$$

In the basis $(\mathcal{S}^\alpha, \mathcal{S}^{\dot{\alpha}})$, the chirality matrix is $\mathbb{1}_8 \otimes \sigma_3$, the charge conjugation is the identity and all Dirac matrices are purely imaginary. They take the form²⁰

$$\gamma^m = \tau^m \otimes \sigma_1, \quad \gamma^8 = \mathbb{1}_8 \otimes \sigma_2, \quad (\text{A.12})$$

where τ^m , with $m = 1, \dots, 7$, are ($-i$ times) seven-dimensional Dirac matrices. These latter have elements

$$(\tau^m)_{\alpha\beta} = -i \left(\delta_\alpha^m \delta_\beta^8 - \delta_\beta^m \delta_\alpha^8 + c_{m\alpha\beta} \right), \quad (\text{A.13})$$

where the completely anti-symmetric tensor c_{mnp} describes the octonionic structure constants:

$$c_{127} = c_{163} = c_{154} = c_{253} = c_{246} = c_{347} = c_{567} = 1, \quad (\text{A.14})$$

with all other elements being zero. The Dirac matrices in (A.12) can also be written as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (\text{A.15})$$

with $\sigma^\mu = (\tau^m, -i\mathbb{1})$ and $\bar{\sigma}^\mu = (\tau^m, i\mathbb{1})$. We also write

$$\gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \quad (\text{A.16})$$

with

$$\sigma^{mn} = \bar{\sigma}^{mn} = \tau^{mn} = \frac{1}{2} [\tau^m, \tau^n], \quad \sigma^{m8} = -\bar{\sigma}^{m8} = i\tau^m. \quad (\text{A.17})$$

This Majorana-Weyl basis is the one used in [16]. Using (A.10) and (A.11) we can rewrite all quantities involving fermions given there, in the spin field basis used in this paper.

The U(4) symmetry. The SO(8) symmetry contains a U(4) subgroup that preserves the complex structure introduced in (2.13). The U(1)⁴ Cartan subgroup of SO(8) in U(4) and acts on the complex coordinates as

$$z^I \rightarrow e^{i\varepsilon_I} z^I. \quad (\text{A.18})$$

One has $U(4) \simeq (SU(4) \otimes U(1)) / \mathbb{Z}_4$, where U(1) is the diagonal action in the U(1)⁴ subgroup parametrized by $\varepsilon/4$, where

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4. \quad (\text{A.19})$$

The Cartan subgroup of SU(4) is spanned by the “traceless” parameters $\hat{\varepsilon}_I = \varepsilon_I - \varepsilon/4$, out of which only three are independent. We can denote the U(4) representations by \mathbf{r}_q , where

²⁰With an abuse of notation we denote these matrices again by γ^μ , even if they do not coincide with the ones in (A.7); rather they are equivalent to the latter under the change of basis in (A.10) and (A.11).

\mathbf{r} is a SU(4) representation and q a U(1) charge. With respect to U(4), the relevant SO(8) representations decompose according to the following pattern. For the vector we have²¹

$$\mathbf{8}_v \rightarrow \mathbf{4}_1 \oplus \bar{\mathbf{4}}_{-1}, \tag{A.20}$$

the two representations in the right-hand side corresponding to z^I and \bar{z}^I . For the chiral spinor we have

$$\mathbf{8}_s \rightarrow \mathbf{1}_2 \oplus \mathbf{6}_0 \oplus \mathbf{1}_{-2}. \tag{A.21}$$

The two SU(4) singlets are, respectively,²²

$$\bar{S} = S^{++++}, \quad S = -S^{----}, \tag{A.22}$$

while the states of the $\mathbf{6}_0$ representation are S^{IJ} with $I, J = 1, \dots, 4$ and $S^{IJ} = -S^{JI}$. Their identification in the spin field basis is

$$\begin{aligned} S^{12} &= S^{+-}, & S^{13} &= -S^{+-}, & S^{14} &= S^{+-}, \\ S^{23} &= S^{+-}, & S^{24} &= -S^{+-}, & S^{34} &= S^{+-}. \end{aligned} \tag{A.23}$$

Thus S^{IJ} is proportional to the spin field component which has a plus in the I -th and J -th directions. The conjugates \bar{S}_{IJ} , which are the spin fields with a minus in the I -th and J -th directions, are not independent. Indeed we have

$$\bar{S}_{IJ} = \frac{1}{2} \epsilon_{IJKL} S^{KL}. \tag{A.24}$$

The anti-chiral spinor representation decomposes as

$$\mathbf{8}_c \rightarrow \bar{\mathbf{4}}_1 \oplus \mathbf{4}_{-1}. \tag{A.25}$$

The states in the two representations in the right-hand side are respectively given by

$$\bar{S}_I = \begin{pmatrix} S^{-+++} \\ S^{+---} \\ S^{++--} \\ S^{----} \end{pmatrix}, \quad S^I = \begin{pmatrix} S^{----} \\ -S^{+---} \\ S^{++--} \\ -S^{-+++} \end{pmatrix}. \tag{A.26}$$

Using first the change of basis (A.10) and (A.11), and then the identifications (A.22), (A.23) and (A.26) we can express all quantities involving the fermions given in [16] in terms of the U(4) notation used in the main text.

²¹We fix the convention on the U(1) charge by declaring that the charge of the fundamental representation, whose states rotate by $\exp(i\varepsilon/4)$, is 1.

²²The minus signs appearing in (A.22) and later in (A.23) and (A.26) are inserted to be consistent with the action of the charge conjugation matrix (A.9) on the spin fields.

The SO(7) symmetry. The moduli action described in section 4 possesses a BRST charge which in our conventions is the last component \mathcal{S}^8 of a chiral spinor (see (4.32) and (4.33)). The SO(8) subgroup that preserves the BRST charge is SO(7), embedded in a non-standard fashion²³ in which the chiral spinor decomposes as $\mathbf{8}_s \rightarrow \mathbf{7} \oplus \mathbf{1}$, where $\mathbf{7}$ is the vector representation of SO(7). More explicitly, we have $\mathcal{S}^\alpha \rightarrow (\mathcal{S}^m, \mathcal{S}^8)$ with $m = 1, \dots, 7$. Both the anti-chiral spinor $\mathbf{8}_c$ and the vector²⁴ $\mathbf{8}_v$ become spinors of SO(7). The adjoint representation of SO(8) decomposes according to $\mathbf{28} \rightarrow \mathbf{21} + \mathbf{7}$, where $\mathbf{21}$ is the adjoint of SO(7). In particular, this means that an anti-symmetric tensor of SO(8), like for example $D_{\mu\nu}$, decomposes into

$$D_{\mu\nu}^{\mathbf{21}} = \frac{1}{2} D_{mn} (\tau^{mn})_{\mu\nu} , \quad D_{\mu\nu}^{\mathbf{7}} = D_m (\tau^m)_{\mu\nu} . \quad (\text{A.27})$$

Here D_{mn} and D_m are, respectively, an anti-symmetric tensor and a vector of SO(7), while τ^m and τ^{mn} are the matrices introduced in (A.13) and (A.17).

This is another way in which the SO(7) symmetry emerges in our brane system and in the ADHM construction described in section 6; indeed, to disentangle the quartic term proportional to $[a^\mu, a^\nu] [a_\mu, a_\nu]$ in the U(k) moduli action, thanks to the Jacobi identity, the minimal required set of auxiliary fields is not a generic anti-symmetric tensor $D_{\mu\nu}$ but its $\mathbf{7}$ part only. The U(1)³ Cartan subgroup of SO(7) is embedded into the Cartan subgroup of SO(8) by restricting the four parameters ϵ_I to be traceless, *i.e.* to satisfy the requirement $\epsilon = 0$. Only in this case is the spinor \mathcal{S}^8 invariant. Note that this is the same condition that defines the Cartan subgroup of the $\text{SU}(4) \subset \text{U}(4)$ symmetry.

The SO(6) \simeq SU(4) symmetry. The subgroup of SO(7) that is compatible with the complex structure (2.13) is the SO(6) group under which the SO(7) vector decomposes as $\mathbf{7} \rightarrow \mathbf{6} \oplus \mathbf{1}$, namely $V^m \rightarrow (V^{\hat{m}}, V^7)$, with $\hat{m} = 1, \dots, 6$ and the spinor representation of SO(7) decomposes as $\mathbf{8} \rightarrow \mathbf{4}_s \oplus \mathbf{4}_c$, the two addends being the chiral and anti-chiral spinors of SO(6). The Cartan subgroup of this SO(6) coincides with the one of SO(7).

This SO(6) group is isomorphic to the SU(4) subgroup of U(4) discussed above. Under this isomorphism, the chiral and anti-chiral spinors are mapped to the fundamental and anti-fundamental representations $\mathbf{4}$ and $\bar{\mathbf{4}}$, while a vector like $D^{\hat{m}}$ is mapped to the anti-symmetric tensor D^{IJ} , with $I, J = 1, \dots, 4$, satisfying the constraint (A.24).

B Technical details

In this appendix we collect some technical material related to the string theory and conformal field theory methods used in the main text.

Twisted coordinates and mode expansions. In [35] the conformal field theory of bosonic and fermionic fields with non-trivial monodromy properties has been described in full detail. Here we make use of the results of that reference and adapt them to the case

²³This embedding is exchanged by triality with the standard one in which $\mathbf{8}_v \rightarrow \mathbf{7} \oplus \mathbf{1}$ or with the embedding where it is the anti-chiral spinor that decomposes in such a way.

²⁴To be precise, given the vector v^μ , it is $(v^m, -v^8)$ that transforms as a spinor.

of interest. The complex bosonic and fermionic string coordinates along the longitudinal directions in the D7/D7' sectors satisfy the monodromy properties (2.15) and (2.16) around the origin of the world-sheet. After canonical quantization, these fields have the following mode expansions

$$\begin{aligned} i\partial Z^I(z) &= \sqrt{2\alpha'} \left(\sum_{n=1}^{\infty} \bar{a}_{n-\theta_I}^I z^{-n+\theta_I-1} + \sum_{n=0}^{\infty} a_{n+\theta_I}^{\dagger I} z^{n+\theta_I-1} \right), \\ i\partial \bar{Z}_I(z) &= \sqrt{2\alpha'} \left(\sum_{n=0}^{\infty} a_{I,n+\theta_I} z^{-n-\theta_I-1} + \sum_{n=1}^{\infty} \bar{a}_{I,n-\theta_I}^{\dagger} z^{n-\theta_I-1} \right), \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} \Psi^I(z) &= \sqrt{2\alpha'} \sum_{n=0+\nu}^{\infty} \left(\bar{\Psi}_{n-\theta_I}^I z^{-n+\theta_I-\frac{1}{2}} + \Psi_{n+\theta_I}^{\dagger I} z^{n+\theta_I-\frac{1}{2}} \right), \\ \bar{\Psi}_I(z) &= \sqrt{2\alpha'} \sum_{n=0+\nu}^{\infty} \left(\Psi_{I,n+\theta_I} z^{-n-\theta_I-\frac{1}{2}} + \sum_{n=0+\nu}^{\infty} \bar{\Psi}_{I,n-\theta_I}^{\dagger} z^{n-\theta_I-\frac{1}{2}} \right) \end{aligned} \tag{B.2}$$

for the fermionic coordinates with $\nu = 0$ in the R sector and $\nu = \frac{1}{2}$ in the NS sector. The bosonic oscillators satisfy the canonical commutation relations

$$\begin{aligned} [a_{I,n+\theta_I}, a_{m+\theta_J}^{\dagger J}] &= (n + \theta_I) \delta_I^J \delta_{nm} \quad \forall n, m \geq 0, \\ [\bar{a}_{n-\theta_I}^I, \bar{a}_{J,m-\theta_J}^{\dagger}] &= (n - \theta_I) \delta_J^I \delta_{nm} \quad \forall n, m \geq 1, \end{aligned} \tag{B.3}$$

while the fermionic oscillators obey the canonical anti-commutators

$$\left\{ \Psi_{I,n+\theta_I}, \Psi_{m+\theta_J}^{\dagger J} \right\} = \left\{ \bar{\Psi}_{n-\theta_I}^I, \bar{\Psi}_{J,m-\theta_J}^{\dagger} \right\} = \delta_J^I \delta_{nm} \quad \forall n, m \geq 0 + \nu. \tag{B.4}$$

Notice that if all θ_I 's are different from zero (which is the case considered in the main text), there are no bosonic oscillators with index 0, and hence the momentum can not be defined in any of these directions.

Following [35], we define the Virasoro generators L_n for the D7/D7' strings. In particular we find that the bosonic contribution to L_0 is

$$L_0^{(Z)} = \hat{N}^{(Z)} + \frac{1}{2} \sum_{I=1}^4 \theta_I (1 - \theta_I) \tag{B.5}$$

where $\hat{N}^{(Z)}$ is the bosonic number operator

$$\hat{N}^{(Z)} = \sum_{I=1}^4 \left[\sum_{n=1}^{\infty} \bar{a}_{I,n-\theta_I}^{\dagger} \bar{a}_{n-\theta_I}^I + \sum_{n=0}^{\infty} a_{n+\theta_I}^{\dagger I} a_{I,n+\theta_I} \right] + \sum_{n=1}^{\infty} (\bar{a}_{5,n}^{\dagger} \bar{a}_n^5 + a_n^{\dagger 5} a_{5,n}), \tag{B.6}$$

while the fermionic contribution to L_0 is

$$L_0^{(\Psi)} = \begin{cases} \hat{N}^{(\Psi)} + \frac{1}{2} \sum_{I=1}^4 \theta_I^2 & \text{in the NS sector,} \\ \hat{N}^{(\Psi)} - \frac{1}{2} \sum_{I=1}^4 \theta_I (1 - \theta_I) & \text{in the R sector,} \end{cases} \tag{B.7}$$

where the fermionic number operator $\widehat{N}^{(\Psi)}$ is

$$\widehat{N}^{(\Psi)} = \sum_{I=1}^4 \left[\sum_{n=1-\nu}^{\infty} (n - \theta_I) \bar{\Psi}_{I, n-\theta_I}^\dagger \bar{\Psi}_{n-\theta_I}^I + \sum_{n=0+\nu}^{\infty} (n + \theta_I) \Psi_{n+\theta_I}^{\dagger I} \Psi_{I, n+\theta_I} \right] + \sum_{n=0+\nu}^{\infty} n (\bar{\Psi}_{5, n}^\dagger \bar{\Psi}_n^5 + \Psi_n^{\dagger 5} \Psi_{5, n}) . \tag{B.8}$$

The c -numbers in (B.5) and (B.7) arise from the normal ordering of the bosonic and fermionic oscillators.

The physical states must obey the conditions

$$\begin{cases} L_0^{(Z)} + L_0^{(\Psi)} = \frac{1}{2} & \text{in the NS sector,} \\ L_0^{(Z)} + L_0^{(\Psi)} = 0 & \text{in the R sector,} \end{cases} \tag{B.9}$$

which imply eq. (2.19) of the main text.

BRST variations. We now give a few details on how the BRST transformations of the instanton moduli described in section 4.3 can be derived using the vertex operators introduced in section 3. To do so, we need the OPE's of the ten-dimensional string theory, like for example

$$S_{-1/2}^{\dot{A}}(z) S_{-1/2}^{\dot{B}}(w) \sim \frac{i(\mathcal{C} \Gamma_M)^{\dot{A}\dot{B}}}{\sqrt{2}} \frac{\psi^M(w) e^{-\varphi(w)}}{z-w} + \dots \tag{B.10}$$

with $M = 1, \dots, 10$. Here $S_{-1/2}^{\dot{A}}$ denotes a ten-dimensional spin field in the $(-\frac{1}{2})$ -superghost picture with negative chirality. In particular, we have $S^{\dot{A}} = S^A S^\pm e^{-\frac{1}{2}\varphi}$ where S^A is the eight-dimensional spin field defined in (A.6), with $+$ or $-$ depending on whether A is anti-chiral or chiral. The Dirac matrices Γ^M and the charge conjugation matrix \mathcal{C} are given in (A.4) and (A.5). Using these matrices and the notation explained in appendix A, from (B.10) we find for instance

$$(SS^- e^{-\frac{1}{2}\varphi})(z) (\bar{S}_I S^+ e^{-\frac{1}{2}\varphi})(w) \sim -\frac{\bar{\Psi}_I(w) e^{-\varphi(w)}}{z-w} + \dots . \tag{B.11}$$

Let us now derive how the BRST charge (4.32) acts on the moduli B^I and M^I . As explained in the main text, we need to compute

$$[\mathcal{Q}, V_{M^I}(w)] = \frac{1}{\sqrt{2}} [Q, V_{M^I}(w)] + \frac{1}{\sqrt{2}} [\bar{Q}, V_{M^I}(w)] . \tag{B.12}$$

The second commutator vanishes and we are left with

$$[\mathcal{Q}, V_{M^I}(w)] = \frac{1}{\sqrt{2}} [Q, V_{M^I}(w)] = -\frac{M^I}{\sqrt{2}} \oint_w \frac{dz}{2\pi i} (SS^- e^{-\frac{1}{2}\varphi})(z) (\bar{S}_I S^+ e^{-\frac{1}{2}\varphi})(w) \tag{B.13}$$

where the minus sign is due to the fermionic statistics of M^I and of the vertex operators. Using the OPE (B.11), we immediately find

$$\left[\mathcal{Q}, V_{M^I}(w) \right] = \frac{M^I}{\sqrt{2}} \bar{\Psi}_I(w) e^{-\varphi(w)}. \quad (\text{B.14})$$

Since we have obtained the vertex of B^I with a polarization M^I , we deduce that $\mathcal{Q} B^I = M^I$.

Proceeding systematically in this way, we obtain all BRST variations reported in the left columns of (4.39) and (4.40). The BRST variations in the right columns can be obtained by imposing the nilpotency of \mathcal{Q} up to the symmetries of the system as explained in the main text or, equivalently, by applying the vertex operator method in an enlarged system with auxiliary fields as discussed for example in [16].

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