Simulation of N-Dimensional Second-Order Fluid Models with Different Absorbing, Reflecting and Mixed Barriers

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Simulation of n-dimensional brownian motion with an arbitrary reflecrting barrier

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Abstract

Second Order Fluid Models ....

Keywords: Second Order Fluid Models, Performance evaluation, Fluid Transfer

1. Introduction

Second Order Fluid Models are hard to simulate. A single reflecting barrier has been studied in [1]. Here we extend those results to simulate an arbitrarily placed barrier in n-dimensions.

2. Related Works

2.1. Browninan

The differential equations that describe a fluid model are hard to solve and the symbolic solution of the equations can be obtained only for trivial cases. In the case of transient analysis the system has an initial state which can be exploited as considered in [2, 3, 4, 5], to mention a few, while in the case of stationary analysis the equations that describe a fluid model are ordinary differential equations (ODEs) without initial condition. Indeed this problem has been solved for first order models by the analysis of first passage time probabilities, see for instance

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[6, 7, 8, 9, 10, 11] and the references therein. The key of these solutions lies in the matrix characterisation of the distribution of the phase visited at the end of a busy period of the fluid queue.

The problem remains open for Second order models (also known as modulated diffusion processes), where the solution is obtained from a set of boundary equations, ODEs and a normalising condition. For example, in case of fluid level independent transition and fluid drift, the solution of the ODE is obtained by the computation of eigenvalues and eigenvectors of a matrix [12]. Usually those approaches are very sensitive to the computation of the eigenvalues and may lead to severe numerical errors. An alternative approach using modal decomposition is proposed in [13]. Second order models are introduced in [12] and [14]. In these works the authors consider a *white noise* factor which represents the variability of the traffic during the transmission periods. The fluid level is described by a reflected Brownian motion modulated by a continuous time Markov chain (CTMC). When the CTMC is in state i, the fluid level is modelled by a reflected Brownian motion with drift r_i and variance parameter σ_i^2. The authors of [15] provide a stability analysis of such models when the modulating process is general stationary ergodic (not necessarily Markovian).
Second order models can have two different types of boundaries: absorbing or reflecting [16]. A special approximation method is proposed in [17] for approximating the absorbing boundary based on the solution of the system with reflecting boundary.

2.2. Wiener Process

2.3. Nostra roba del secondo ordine

In [18] authors address the problem of performing steady state solution of modulating diffusion processes without using neither discretisation nor singular value decomposition. The approach is similar to the one used in [19, 20] for first order models and it is focused on the boundary behaviours. This work considers models where the upper and lower boundary of each state can be either absorbing or reflecting.

2.4. Roba di Harrison

2.5. Example

Se facciamo un esempio, roba relativa all'esempio

Table 1: Notations

Notation	Description
X	Brownian Motion process
$x(t)$	position of the Brownian Motion process at time t
$x_{\max}(0, \Delta t)$	maximum reached by a Brownian Motion process in the time interval $[0, \Delta t]$
n	sample normally distributed
u	sample uniformly distributed
Notations	Description
$\mathcal{V}$	geographical region

3. Reflecting and absorbing barriers in second order fluid models

We start presenting the literature results for simulation of second order fluid models with a reflecting barrier in one dimension. In the following, we will use the notations summarized in Table 1.

3.1. Brownian Motion

In first order models the fluid level grows linearly with a deterministic rate r_i. If $x(t)$ denotes the fluid level at time t and t' is a time instant such that $t' > t$, then:

$$x(t') = x(t) + (t' - t)r_i \ , \tag{1}$$

provided that the continuous variable does not reach a boundary in the (t, t') interval. Markov modulated diffusion processes instead consider random fluid changes. In these models, we have that:

$$x(t') = x(t) + N\left((t' - t)r_i, (t' - t)\sigma_i^2\right), \tag{2}$$

that is, the fluid level change in the (t, t') interval is normal distributed with mean $(t' - t)r_i$ and variance $(t' - t)\sigma_i^2$. Note that Equation (2) is valid if the continuous variable does not reach a boundary in the (t, t') interval.

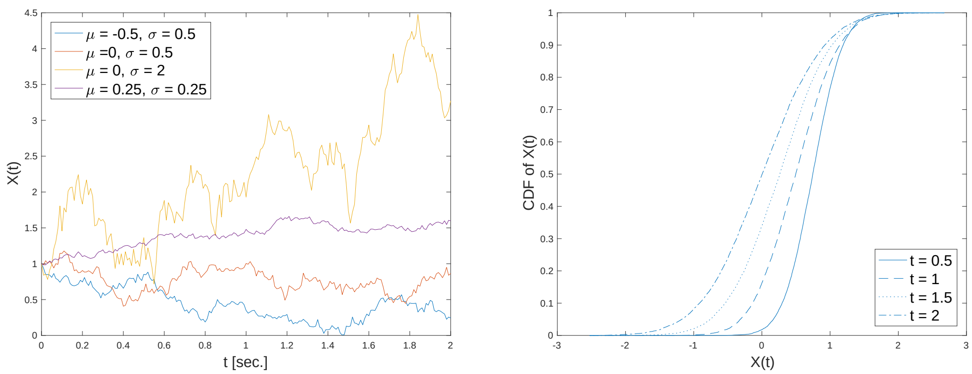
3.2. Simulation of Brownian Motion

Let us call $x(t)$ the position of the Brownian Motion process at time t, starting from $x(0) = x_0$. If we consider a step Δt, we then have:

$$x(t + \Delta t) = x(t) + N(\mu\Delta t, \sigma^2\Delta t) \tag{3}$$

where $N(\mu, \sigma)$ is a sample from a Normal distribution characterized by average μ and variance σ^2.

-¿AA¡-Algoritmo per generare la traccia di un modo Browninano Figure 1 a) shows four Brownian Motion traces, starting from $x_0 = 1$, and for a time interval $t \in [0, 2]$, with different drifts ($\mu = \{-0.5, 0, 0, 0.25\}$) and standard deviations ($\sigma = \{0.5, 0.5, 2, 0.25\}$). Figure 1 b) shows the distribution of the position of the process with $\mu = -0.5$ and $\sigma = 0.5, x_0 = 1$, at four different time instants, namely $t \in \{0.5, 1, 1.5, 2\}$.



a) b)

Figure 1: a) **Traces of a Brownian Motion, with different parameters, all starting** $x_0 = 1$**.** b) **Distribution of the position at** $t \in \{0.5, 1, 1.5, 2\}$ **for the case with** $\mu = -0.5, \sigma = 0.5, x_0 = 1$**.**

Brownian motion with an reflecting barrier (non constrained process - minus - the current barrier position)

Figure 2 shows the evolution of the unconstrained process, of the barrier and of the resulting reflected process.

Simulating a reflected process poses a challenge, since during each time interval Δt the process might reach values that are below the one obtained at the end of the sampling period. This is shown in Figure 3.

Let us call $x_{\max}(0, \Delta t)$ the maximum reached by a process in the time interval $[0, \Delta t]$, i.e.:

$$x_{\max}(0, \Delta t) = \max_{0 \leq t \leq \Delta t} x(t) \tag{4}$$

If we consider a process characterised by zero mean and unitary variance ($\mu = 0$ and $\sigma = 1$), which starts at $x(0) = 0$, it can be proven (see [1]) that:

$$F(m|y) = \text{Prob}\left(x_{\max}(0, \Delta t) \leq m | x(\Delta t) = y\right) = 1 - e^{\frac{-2m(m-y)}{\Delta t}}, m \geq y \tag{5}$$

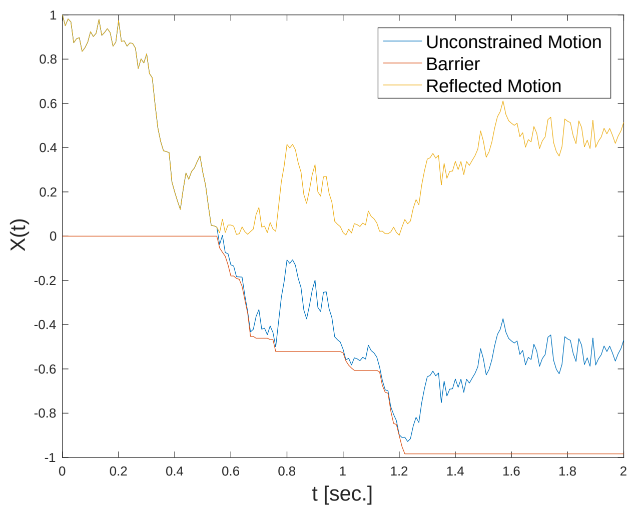


Figure 2: **The components of a reflected Brownian Motion process with $\mu = -0.5$, $\sigma = 0.5$ and $x_0 = 1$: the unconstrained process, the barrier, and the reflected process**

We can invert Equation 5 and use it to generate samples for the maximum using the *Inverse Sampling Transform* method:

$$F^{-1}(u|y) = \frac{1}{2}\left(y + \sqrt{y^2 - 2\Delta t \ln(1-u)}\right) \tag{6}$$

Exploiting the symmetry properties of the Normal distribution, given two samples $\mathbf{n} \sim N(0,1)$ distributed according to a standard normal distribution, and $\mathbf{u} \sim \text{Unif}(0,1)$ uniformly distributed between 0 and 1, we can compute both $x(t+\Delta t)$ and $\min_{t,t+\Delta t} x(t)$:

$$x(t+\Delta t) = x(t) - \mathbf{n} \cdot \sigma \cdot \sqrt{\Delta t} + \mu \cdot \Delta t \tag{7}$$

$$\min_{t,t+\Delta t} x(t) = x(t) - F^{-1}(\mathbf{u}|\mathbf{n})\sigma \cdot \sqrt{\Delta t} + \mu \cdot \Delta t \tag{8}$$

3.3. Reflecting barrier in one dimension

Application to the simulation of a second order fluid model with a reflecting barrier in one dimension placed at 0. Figure 4 a) shows four Brownian Motion traces, with a reflecting barrier at $x = 0$, starting from $x_0 = 1$, and for a time

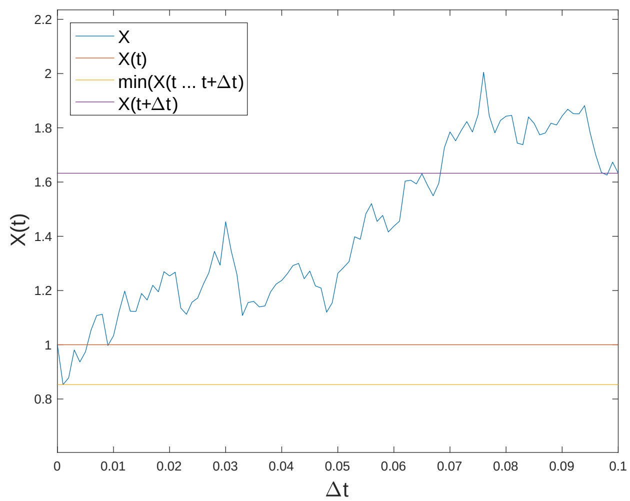


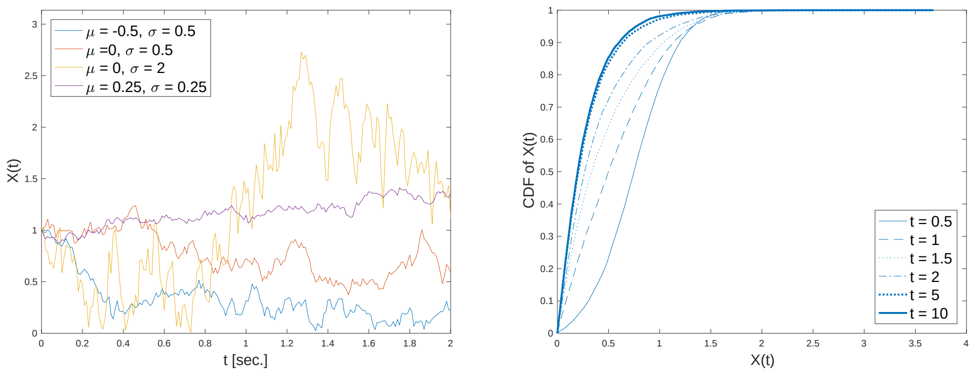
Figure 3: **The evolution of Brownian Motion with respect to a barrier at 0 in a time interval $\Delta t = 0.1$ sec., with $\mu = 0$ and $\sigma = 2$: the process, the starting and ending point, and the minimum value reached during the evolution.**

interval $t \in [0, 2]$, with different drifts ($\mu = \{-0.5, 0, 0, 0.25\}$) and standard deviations ($\sigma = \{0.5, 0.5, 2, 0.25\}$). Figure 4 b) shows the distribution of the position of the process with $\mu = -0.5$ and $\sigma = 0.5$, $x_0 = 1$, at four different time instants, namely $t \in \{0.5, 1, 1.5, 2\}$.

3.4. Absorbing barrier in one dimension

Figure 5 a) shows four Brownian Motion traces, with an absorbing barrier at $x = 0$, starting from $x_0 = 1$, and for a time interval $t \in [0, 2]$, with different drifts ($\mu = \{-0.5, 0, 0, 0.25\}$) and standard deviations ($\sigma = \{0.5, 0.5, 2, 0.25\}$). Figure 5 b) shows the distribution of the position of the process with an absorbing barrier and $\mu = -0.5$ and $\sigma = 0.5$, $x_0 = 1$, at four different time instants, namely $t \in \{0.5, 1, 1.5, 2\}$.

-¿AA¡- Algoritmo per generare la traccia di un modo Browninano riflesso o assorbito



a) b)

Figure 4: a) **Traces of a Brownian Motion with a reflecting barrier and different parameters, all starting** $x_0 = 1$**.** b) **Distribution of the position at** $t \in \{0.5, 1, 1.5, 2, 5, 10\}$ **for the case with** $\mu = -0.5, \sigma = 0.5, x_0 = 1$**.**

4. Extending to more than one dimension

Initially we consider a reflecting barrier perpendicular to the last dimension and passing through the origin. The extension from the single dimensional case, however, is not straight forward. The most natural implementation, would be to move the barrier perpendicular to the hit direction, as shown in Figures 6 where a trace and a point cloud distribution of the result are shown.

This effect however might lead to results that are not correct. Let us for example imagine a simple fluid transfer between two containers, with no sources or sink. If we plot the value of the two fluid variables, ... see Figures 7.

The problem of reflection: the skew vector can shift the barrier when the process hits the boundary. ... See Figure 8.

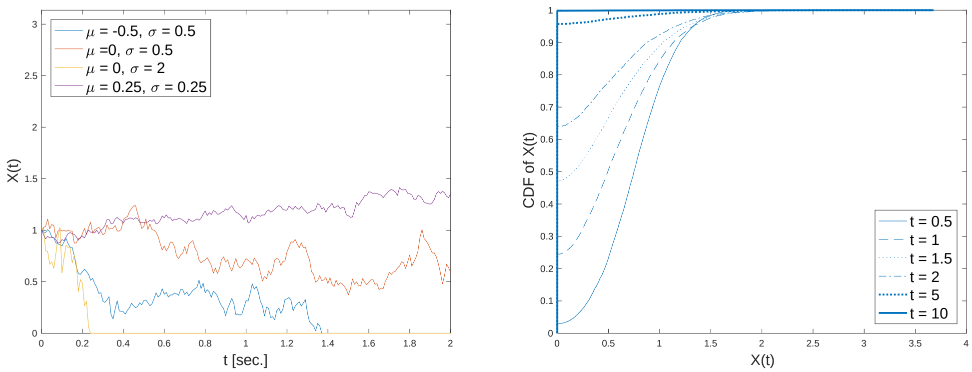
Solution: computing the entity of the displacement of the barrier, and use it to shift it according to the skew vector. ... Figures 9.

4.1. Absorbing barrier in more than one dimension

Ci sono diversi ordini di assorbimento in piu' dimensioni. ... See Figure 10.

4.2. Arbitrarily placed barriers

We start focusing on **an n-dimensional processes that have independent increments in all direction, according to an N(0,1) random variable (i.e. no**



a) b)

Figure 5: a) **Traces of a Brownian Motion with an absorbing barrier and different parameters, all starting** $x_0 = 1$**.** b) **Distribution of the position at** $t \in \{0.5, 1, 1.5, 2, 5, 10\}$ **for the case with** $\mu = -0.5, \sigma = 0.5, x_0 = 1$**.**

drift, no correlation, unitary variance).

Barrier is characterized by a position and an angle.

The process is rotated and displaced so to place the barrier in the origin, perpendicular to the last dimension.

Reflection is performed in the rotated space

The process is rotated back to restore its original characteristics. The process is shown in Figure 11.

The result is shown in Figure 12.

To support different drift vectors and covariance matrices, we perform the Cholesky decomposition .....

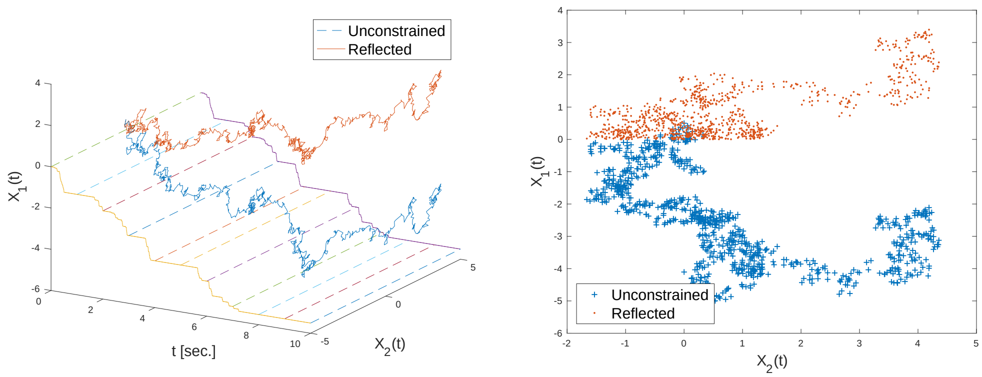
5. Analysis

We show the effect of various types of reflecting barrier

We validate our solution, by comparing the results with the ones obtained by a simulation with a finer discretization

6. Application example

if available



a) b)

Figure 6: **Position of a 2D Brownian Motion process a) trace b) point cloud distribution for the trace**

7. Conclusions and Future Work

We have ...

8. Aknowledgments

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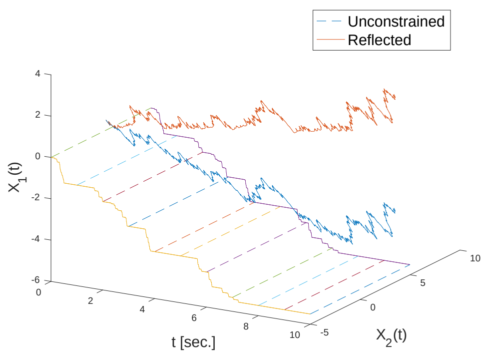
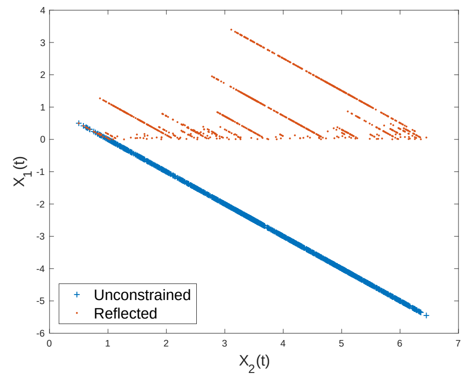


a)



b)

Figure 7: **Position of a 2D Brownian Motion process with correlated motion and no skew of the barrier a) trace b) point cloud distribution for the trace**

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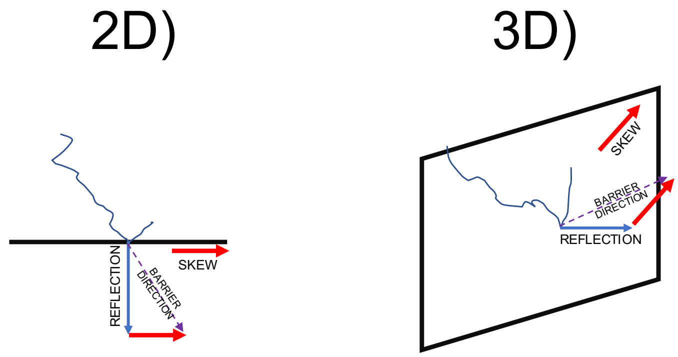

Figure 8: **Skew of the barrier during reflection in two and three dimensions.**

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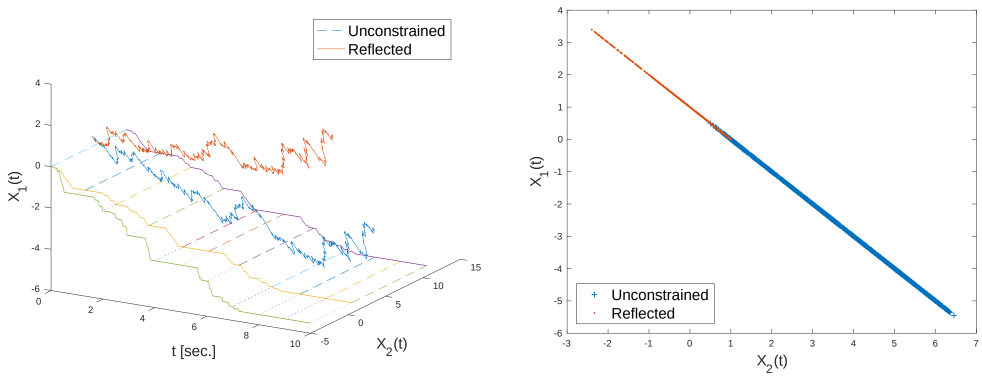

a) b)

Figure 9: **Position of a 2D Brownian Motion process with correlated motion and skew of the barrier a) trace b) point cloud distribution for the trace**

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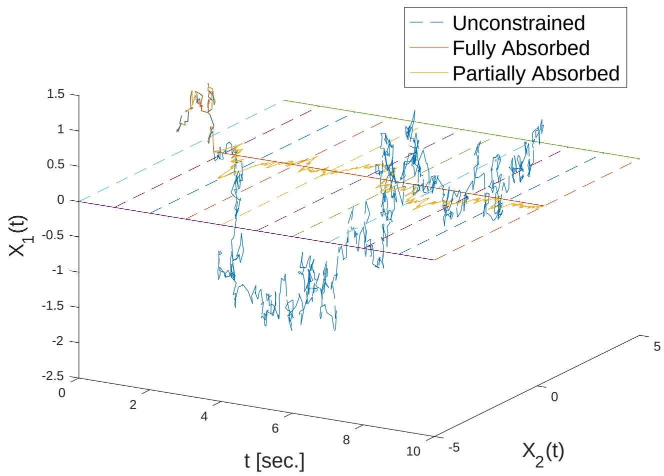


Figure 10: **Absorption in a 2D Brownian Motion process .**

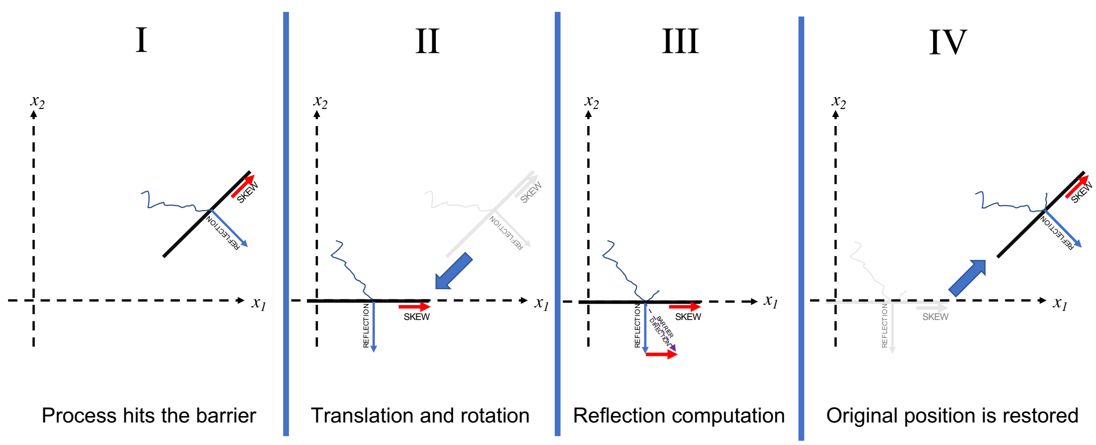
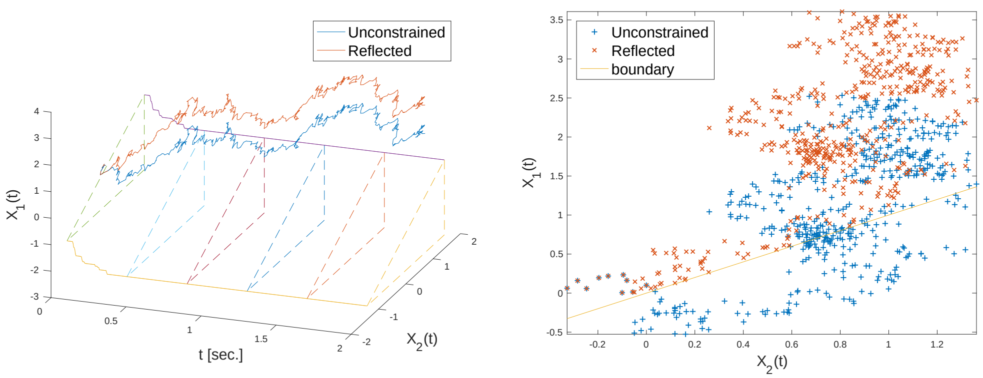


Figure 11: **Implementation of an arbitrary barrier.**



a) b)

Figure 12: **Position of a 2D Brownian Motion process reflected from an arbitrary barrier a) trace b) point cloud distribution for the trace**