A Push Forward Construction and the Comprehensive Factorization for Internal Crossed Modules

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A push forward construction and the comprehensive factorization for internal crossed modules

Dedicated to George Janelidze on the occasion of his 60th birthday

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Abstract

In a semi-abelian category, we give a categorical construction of the push forward of an internal pre-crossed module, generalizing the pushout of a short exact sequence in abelian categories. The main properties of the push forward are discussed. A simplified version is given for action accessible categories, providing examples in the categories of rings and Lie algebras. We show that push forwards can be used to obtain the crossed module version of the comprehensive factorization for internal groupoids.

Keywords: push forward; crossed module; comprehensive factorization; semi-abelian category

MSC: 18A32; 18D35; 18G50; 20J05; 18C05

1 Introduction

Given a short exact sequence $0 \to A \to X \to Y \to 0$ in an abelian category $\mathcal{A}$, any morphism $y\colon Y' \to Y$ produces by pullback a new short exact sequence with the same kernel A. Dually, any morphism $a\colon A \to A'$ produces by pushout a new short exact sequence with the same cokernel Y. These constructions are functorial and they give rise to the classical homomorphisms:

$$\mathrm{Ext}_{\mathcal{A}}(y, A)\colon \mathrm{Ext}_{\mathcal{A}}(Y', A') \to \mathrm{Ext}_{\mathcal{A}}(Y, A')$$

$$\mathrm{Ext}_{\mathcal{A}}(Y, a)\colon \mathrm{Ext}_{\mathcal{A}}(Y, A) \to \mathrm{Ext}_{\mathcal{A}}(Y, A')$$

If the base category is a semi-abelian category $\mathcal{C}$, the first construction still produces a short exact sequence, but this is no longer true for the second one, because the pushout of a normal monomorphism is not in general a normal monomorphism.

This problem can be fixed by giving some supplementary conditions, and a *push forward* construction, that in the abelian case specializes to a pushout.

For group extensions with abelian kernel, this issue is discussed in literature, especially in connection with the different interpretations of low dimensional non abelian cohomology theories (see e.g. [18, Chapter IV] or [12, Chapter IV]). For pointed protomodular categories with semidirect products, in [9] Bourn and Janelidze proved the functoriality of the assignment:

$$(A, \xi) \mapsto \mathrm{Opext}_{\mathcal{C}}(Y, A, \xi)\,,$$

which associates with every Y-module (A, ξ) the abelian group of isomorphism classes of extensions of Y via A inducing the action ξ. Their result extends the torsor theoretical approach developed in [1] for groups and other algebraic categories.
Another viewpoint was considered by Bourn in [5], where a *direction functor* assigns a Y-module to each abelian extension of Y. In the semi-abelian context, we show that the push forward construction gives a direct proof of the functoriality of $\mathrm{Opext}_{\mathcal{C}}(Y, -)$, and the properties of the push forward allow to recover the fact, already proved in [6], that Bourn's direction functor is a cofibration.

When the base category is the category of groups, under suitable hypotheses, it is possible to push forward along a map not only a normal monomorphism, but any pre-crossed module. This yields a crossed module with the same cokernel (to the best of our knowledge, the push forward of a pre-crossed module was introduced by Noohi in [23]).
The purpose of the present work is to develop a push forward construction in the intrinsic setting of a semi-abelian category $\mathcal{C}$, where the notion of internal crossed module was introduced by Janelidze in [15].
In fact, in Theorem 2.7, we present necessary and sufficient conditions, expressed in terms of internal actions, for the push forward of a given pre-crossed module to exist in $\mathcal{C}$ (for the case of a crossed module, a push forward construction with equivalent but differently formulated conditions was independently developed by Hartl [14]). These conditions simplify when the category $\mathcal{C}$ is action accessible [10], as presented in Theorem 2.10. The last result is very useful for the construction of the push forward in many algebraic examples, like those of rings, Lie algebras, associative algebras and, more in general, any category of interest in the sense of Orzech [24].
The push forward construction, together with its main property (see Theorem 2.7, (PF)), turns out to be strongly related to the comprehensive factorization of internal functors (see [4]). Our investigation shows that push forwards can be used in order to factorize morphisms of crossed modules, so that final functors between internal groupoids can be characterized as push forward squares.

Contents

2 Push forward of pre-crossed modules

2.1 The main result

In the following, the base category $\mathcal{C}$ is supposed to be semi-abelian. For a detailed account on semi-abelian categories, semi-direct products and internal actions, the reader is addressed to [2], [8] and [3]. We just recall that, in this context, for any object Y in $\mathcal{C}$, the kernel functor:

$$\mathbf{Pt}_Y(\mathcal{C}) \xrightarrow{K} \mathcal{C}$$

is monadic. The corresponding monad on $\mathcal{C}$ is denoted by $Y\flat(-)$ (so, as an object, we will indicate $Y\flat H = \mathrm{Ker}(Y + H \xrightarrow{[1,0]} Y))$. The algebras for this monad are called (internal) Y-actions. We denote by $\mathcal{C}^Y$ the category of such algebras. The monadicity of K allows the construction of semidirect products in $\mathcal{C}$.

Lemma 2.1. *Let $p\colon A \to B$ be a split epimorphism, with chosen section s, and $f\colon E \to B$ be a morphism. Consider the following pullback diagram of split epimorphisms*

$$\begin{array}{ccc} D & \underset{s'}{\overset{p'}{\rightleftarrows}} & E \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A & \underset{s}{\overset{p}{\rightleftarrows}} & B \end{array}$$

where $s \cdot f = f' \cdot s'$. If $\xi\colon B\flat X \to X$ is the internal action determined by (p, s) on its kernel X, then $f^*(\xi) = \xi \cdot (f\flat 1_X)$ is the internal action determined by (p', s').

Proof. Trivial. □

Definition 2.2. A pre-crossed module (∂_H, ξ_H) is a morphism $\partial_H\colon H \to H_0$ together with an action $\xi_H\colon H_0\flat H \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} H_0\flat H & \xrightarrow{\ \xi\ } & H \\ {\scriptstyle 1\flat\partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ H_0\flat H_0 & \xrightarrow[\ \chi\ ]{} & H_0 \end{array}$$

where $\chi = \chi_{H_0}$ is the canonical conjugation action.

Subscripts are omitted when no confusion arises.

Pre-crossed modules correspond to reflexive graphs (see [15]). The construction is based on the semidirect product $H \rtimes_\xi H_0$: the codomain map is the canonical projection p_{H_0}, while the domain is the unique arrow $[\partial, 1_{H_0}\rangle$ that makes the two triangles commute:

$$\begin{array}{ccccc} H & \xrightarrow{\ h\ } & H \rtimes_\xi H_0 & \xleftarrow{\ i_{H_0}\ } & H_0 \\ & {\scriptstyle\partial}\searrow & \downarrow{\scriptstyle [\partial,1\rangle} & /\!\!/ & \\ & & H_0 & & \end{array}$$

In order to define an internal crossed module, we need a further condition, which is not in general the straightforward generalization of the Peiffer condition for crossed modules of groups (see e.g. [20]). In fact, Janelidze in [15] gave a definition of internal crossed module, showing the equivalence between the category of crossed modules and the one of internal groupoids.

Definition 2.3 ([15])**.** A *crossed module* is a pre-crossed module (∂, ξ) such that the following diagram commutes:

$$\begin{array}{ccc} (H_0 + H)\flat H & \xrightarrow{[1,\partial]\flat 1} & H_0\flat H \\ {\scriptstyle [1,\iota_2]^{\#}}\downarrow & & \downarrow{\scriptstyle \xi} \\ H_0\flat H & \xrightarrow[\ \xi\ ]{} & H \end{array}$$

where the arrow $[1, \iota_2]^{\#}$ is defined by the following commutative diagram:

$$\begin{array}{ccccc} (H_0 + H)\flat H & \xrightarrow{\ker[1,0]} & (H_0 + H) + H & \xrightarrow{[1,0]} & H_0 + H \\ {\scriptstyle [1,\iota_2]^{\#}}\downarrow & & \downarrow{\scriptstyle [1,\iota_2]} & & \downarrow{\scriptstyle [1,0]} \\ H_0\flat H & \xrightarrow[\ker[1,0]]{} & H_0 + H & \xrightarrow[{[1,0]}]{} & H_0 \end{array}$$

The following proposition gives a new characterization of crossed modules among pre-crossed modules.

Proposition 2.4. *Let (∂, ξ) be a pre-crossed module. Then the following conditions are equivalent:*

1. *(∂, ξ) is a crossed module;*
2. *the following diagram commutes:*

$$\begin{array}{ccc} (H \rtimes H_0)\flat H & \xrightarrow{[\partial,1\rangle\flat 1} & H_0\flat H \\ & \searrow^{\chi_|} & \Big\downarrow{\scriptstyle \xi} \\ & & H \end{array}$$

where $\chi_| = \chi_H^{H \rtimes H_0}$ is the restriction of the conjugation action of $H \rtimes H_0$ to the kernel $i_H \colon H \to H \rtimes H_0$ (it is defined since normal subobjects are clots, see e.g. [19]).

Proof. Let us consider the following commutative diagrams:

$$\begin{array}{ccccc} (H_0+H)\flat H & \xrightarrow{\ker[1,0]} & (H_0+H)+H & \xrightarrow{[1,0]} & H_0+H \\ {\scriptstyle [i_{H_0},i_H]\flat 1}\Big\downarrow & & {\scriptstyle [i_{H_0},i_H]+1}\Big\downarrow & & \Big\downarrow{\scriptstyle [i_{H_0},i_H]} \\ (H \rtimes H_0)\flat H & \xrightarrow{\ker[1,0]} & (H \rtimes H_0)+H & \xrightarrow{[1,0]} & H \rtimes H_0 \\ {\scriptstyle \chi_|}\Big\downarrow & & {\scriptstyle [i_{H\rtimes H_0},i_H]}\Big\downarrow & & \Big\| \\ H & \xrightarrow[i_H]{} & H \rtimes (H \rtimes H_0) & \xrightarrow[p_{H\rtimes H_0}]{} & H \rtimes H_0 \\ \Big\| & & {\scriptstyle [i_H,1\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle p_{H_0}} \\ H & \xrightarrow[i_H]{} & H \rtimes H_0 & \xrightarrow[p_{H_0}]{} & H_0 \end{array}$$

$$\begin{array}{ccccc} (H_0+H)\flat H & \xrightarrow{\ker[1,0]} & (H_0+H)+H & \xrightarrow{[1,0]} & H_0+H \\ {\scriptstyle [1,\iota_2]^{\#}}\Big\downarrow & & {\scriptstyle [1,\iota_2]}\Big\downarrow & & \Big\downarrow{\scriptstyle [1,0]} \\ H_0\flat H & \xrightarrow{\ker[1,0]} & H_0+H & \xrightarrow{[1,0]} & H_0 \\ {\scriptstyle \xi}\Big\downarrow & & {\scriptstyle [i_{H_0},i_H]}\Big\downarrow & & \Big\| \\ H & \xrightarrow[i_H]{} & H \rtimes H_0 & \xrightarrow[p_{H_0}]{} & H_0 \end{array}$$

We have that

$$\xi \cdot [1, \iota_2]^{\#} = \chi_| \cdot [i_{H_0}, i_H]\flat 1 \, ,$$

since they are the restriction to kernels of the same morphism between extensions. Now, if Condition 2 holds, then

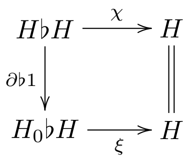
$$\xi \cdot [1, \iota_2]^{\#} = \xi \cdot [\partial, 1\rangle \flat 1 \cdot [i_{H_0}, i_H] \flat 1 = \xi \cdot [\partial, 1] \flat 1 \,,$$

i.e. (∂, ξ) is a crossed module. The converse implication follows from the fact that $[i_{H_0}, i_H]$ is a regular epimorphism and $-\flat 1$ preserves such morphisms (see [15]). □

Remark 2.5. If we pre-compose diagram 2 of the above proposition with the morphism

$$i_H \flat 1 \colon H \flat H \to (H \rtimes H_0) \flat H$$

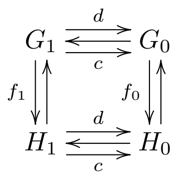
we obtain the Peiffer condition (see [15, 19]):



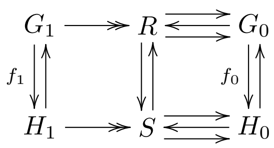
which is weaker, in general, than the above equivalent conditions. However, if the category $\mathcal{C}$ satisfies the "Smith is Huq" property, the Peiffer condition turns out to be sufficient to characterize internal crossed modules among pre-crossed modules (see [22]).

The following technical lemma will be useful in the proof of Theorem 2.7.

Lemma 2.6. *Let $\mathcal{C}$ be a regular Mal'tsev category. Consider a split epimorphic discrete fibration and cofibration of internal reflexive graphs in $\mathcal{C}$, i.e. a split epimorphism $\underline{G} \to \underline{H}$ of reflexive graphs*



where both the squares $df_1 = f_0 d$ and $cf_1 = f_0 c$ are pullbacks.
Let R and S be the supports of $\underline{G}$ and $\underline{H}$ respectively (R and S are equivalence relations since they are reflexive relations and $\mathcal{C}$ is a Mal'tsev category). Then the diagram above factorizes through a discrete fibration between R and S.



Proof. Trivial by Proposition 3.4 in [11]. □

Now we are ready to prove our main result on the push forward construction.

Theorem 2.7. *In a semi-abelian category, we consider a pre-crossed module* $(\partial\colon H \to H_0, \xi)$*, a morphism* $p\colon H \to G$*, and an internal action* $\alpha\colon H_0 \flat G \to G$ *such that the following diagram commutes:*

$$\begin{array}{ccc} H_0\flat H & \xrightarrow{\ \xi\ } & H \\ {\scriptstyle 1\flat p}\downarrow & & \downarrow{\scriptstyle p} \\ H_0\flat G & \xrightarrow[\ \alpha\ ]{} & G \end{array} \qquad (1)$$

or, equivalently, the arrow $(p \rtimes 1)\colon H \rtimes_\xi H_0 \to G \rtimes_\alpha H_0$ *exists. Moreover we require that the following diagram commutes:*

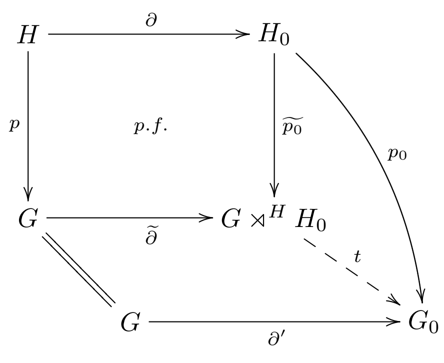
$$\begin{array}{ccc} (H \rtimes_\xi H_0)\flat G & \xrightarrow{\ d\flat 1\ } & H_0\flat G \\ {\scriptstyle (p\rtimes 1)\flat 1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ (G \rtimes_\alpha H_0)\flat G & \xrightarrow[\ \chi_{|}\ ]{} & G \end{array} \qquad (2)$$

where $d = [\partial, 1\rangle\colon H \rtimes_\xi H_0 \to H_0$ *is the domain map determined by the pre-crossed module* (∂, ξ)*, and* $\chi_| = \chi_G^{G\rtimes H_0}$.
Then there exist:

1. *an object* $G \rtimes^H H_0$*;*
2. *a crossed module* $(\widetilde{\partial}\colon G \to G \rtimes^H H_0, \widetilde{\xi})$*, with* $\operatorname{coker}(\widetilde{\partial}) \cong \operatorname{coker}(\partial)$*;*
3. *a morphism* $\widetilde{p_0}\colon H_0 \to G \rtimes^H H_0$*, such that* $(p, \widetilde{p_0})$ *is a morphism of pre-crossed modules.*

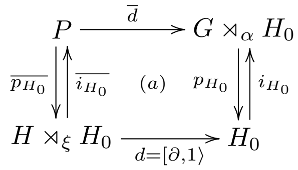
This construction is characterized by the following property:

(PF) *for any morphism* (p, p_0) *from* (∂, ξ) *to a crossed module* $(\partial'\colon G \to G_0, \xi')$*, with* $p_0^*(\xi') = \alpha$*, there exists a unique arrow* $t\colon G \rtimes^H H_0 \to G_0$ *such that* $t\widetilde{p_0} = p_0$ *and* $(1_G, t)$ *is a morphism of crossed modules:*

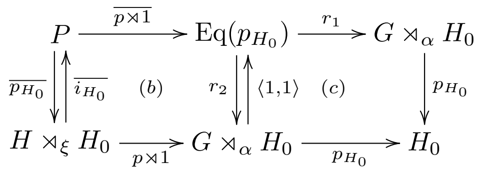


The object $G \rtimes^H H_0$ *is called* generalized semi-direct product*, and the crossed module* $\widetilde{\partial}$ *is called the* push forward *of* ∂ *along* p*.*

Proof. Consider the following pullback of split epimorphisms:

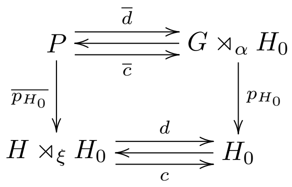


By Lemma 2.1, the hypothesis given by the commutative diagram (2) says that in the pullback (b) below we can choose the same point $(\overline{p_{H_0}}, \overline{i_{H_0}})$ as before:

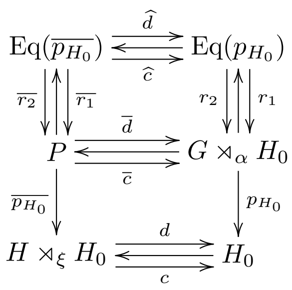


(3)

$(\mathrm{Eq}(p_{H_0})$ stands for the kernel pair of $p_{H_0})$. The downward pullbacks (b) and (c) paste together, so that (a) and $(b)+(c)$ give a discrete fibration and cofibration of reflexive graphs:

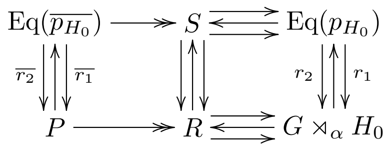


where the codomain map $c\colon H \rtimes_\xi H_0 \to H_0$ is indeed the projection onto H_0, i.e. the composition $p_{H_0} \cdot (p \rtimes 1)$ in diagram (3) above, so that $\overline{c} = r_1 \cdot \overline{p \rtimes 1}$. Now consider the kernel pairs of $\overline{p_{H_0}}$ and p_{H_0}, with the induced morphisms between them:

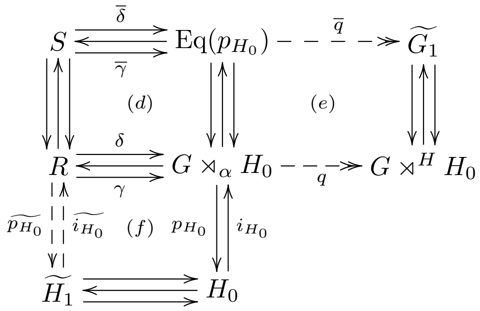


As the lower commutative squares are pullbacks, the four commutative squares at the top are also pullbacks. Since the base category is semi-abelian, it is Mal'tsev and regular, hence by Lemma 2.6 the split discrete fibrations/cofibrations

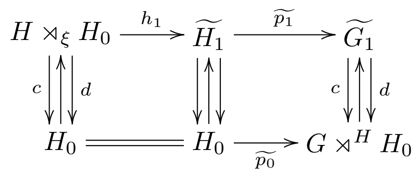
$(\overline{r_1}, r_1)$ and $(\overline{r_2}, r_2)$ between the reflexive graphs $(P, G \rtimes_\alpha H_0)$ and $(\mathrm{Eq}(\overline{p_{H_0}}), \mathrm{Eq}(p_{H_0}))$ factorize through discrete fibrations between their supports R and S:



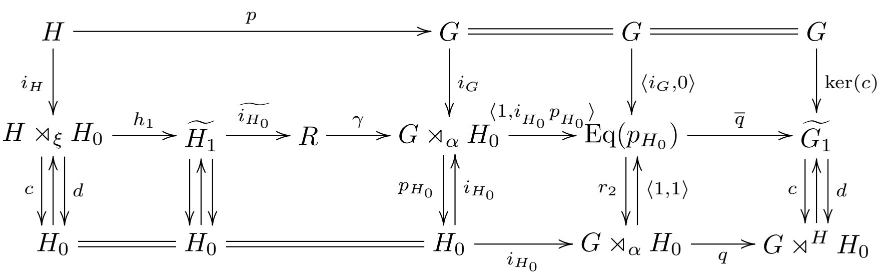
that is, all the commutative squares in the diagram are pullbacks. In particular R and $\mathrm{Eq}(p_{H_0})$ are connected, S is their centralizing double relation and the right hand side is the core of a profunctor (see [7]). Indeed, as the category is Barr-exact, R is an effective equivalence relations on $G \rtimes_\alpha H_0$ and S is effective on both $\mathrm{Eq}(p_{H_0})$ and R, so we can take their coequalizers q, $\overline{q}$ and $\widetilde{p_{H_0}}$:



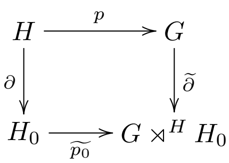
It is proved in [7] that this way we obtain two groupoids $\underline{\widetilde{H}} = (\widetilde{H_1}, H_0)$ and $\underline{\widetilde{G}} = (\widetilde{G_1}, G \rtimes^H H_0)$ and a profunctor $(p_{H_0}, q)\colon \underline{\widetilde{H}} \nrightarrow \underline{\widetilde{G}}$. Since p_{H_0} is split epimorphic, this profunctor is representable and determines an internal functor of groupoids $(\widetilde{p_1}, \widetilde{p_0})$, where $\widetilde{p_0} = q \cdot i_{H_0}$ and $\widetilde{p_1} = \overline{q} \cdot \langle 1, i_{H_0} p_{H_0} \rangle \cdot \gamma \cdot \widetilde{i_{H_0}}$ (see [21, Proposition 5.7]). Moreover, by the universal property of the coequalizer $\overline{p_{H_0}}$, there is a morphism $h_1\colon H \rtimes_\xi H_0 \to \widetilde{H_1}$, and $(h_1, 1_{H_0})$ is a morphism of reflexive graphs. By composing with the given functor of groupoids we get a morphism of reflexive graphs:



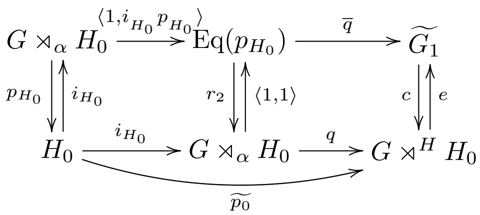
The following diagram shows that p is the restriction of the arrow $\widetilde{p_1} \cdot h_1$ to the kernels of the codomain maps:



It is always possible to choose $\ker(c)$ in such a way that $\ker(c) = \overline{q}\cdot\langle 1, i_{H_0} p_{H_0}\rangle\cdot i_G$, so the only thing to prove is that $\gamma\cdot\widetilde{i_{H_0}}\cdot h_1\cdot i_H = i_G\cdot p$. But $\gamma\cdot\widetilde{i_{H_0}}\cdot h_1 = \overline{c}\cdot\overline{i_{H_0}} = r_1\cdot\langle 1,1\rangle\cdot(p\rtimes 1) = p\rtimes 1$ (see diagram (3)), hence $\gamma\cdot\widetilde{i_{H_0}}\cdot h_1\cdot i_H = (p\rtimes 1)\cdot i_H = i_G\cdot p$. As a consequence, the normalization of $(\widetilde{p_1}, \widetilde{p_0})$ yields a morphism of pre-crossed modules which is the desired push forward:



Now, combining the following pullbacks (the one on the right is part of (e), while the left one is a commutative square where the two parallel split epimorphisms have isomorphic kernels):

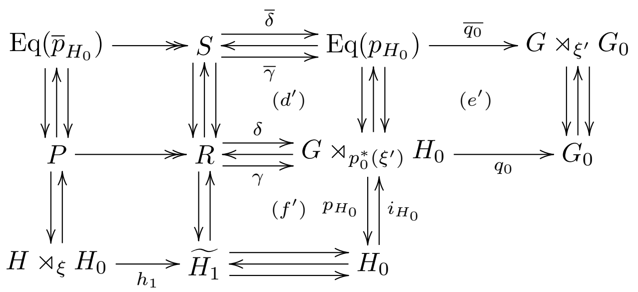


one gets that $\widetilde{p_0}^*(\widetilde{\xi}) = \alpha$.

Finally, let us consider a morphism of pre-crossed modules

$$(p, p_0)\colon (\partial, \xi, H, H_0) \to (\partial', \xi', G, G_0),$$

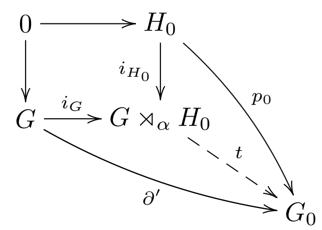
where (∂', ξ') is a crossed module, with $p_0^*(\xi') = \alpha$. This determines a representable profunctor $(p_{H_0}, q_0)\colon \underline{\widetilde{H}} \nrightarrow \underline{G}$:



Since $p_0^*(\xi') = \alpha$, the square (f') coincides with the square (f), hence the square (d') coincides with the square (d). Moreover q_0 coequalizes δ and γ, similarly $\overline{q_0}$ coequalizes $\overline{\delta}$ and $\overline{\gamma}$. The property (PF) of the push forward easily follows from the universal property of the coequalizers q and $\overline{q}$. $\square$

Remark 2.8. 1. Applying Theorem 2.7 with $p = 1_H$ and $\alpha = \xi$, we recover the result of Proposition 2.4.

2. If $H = 0$, the conditions in Theorem 2.7 reduce to the request of existence of the action α, and the above construction is nothing but a semi-direct product with its characterizing property (see Theorem 1.3 in [15]):



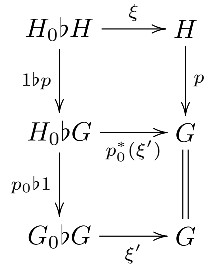
whence the use of the notation $G \rtimes^H H_0$.

3. In the case where (∂, ξ) is a crossed module, the push forward is a morphism of crossed modules. We call it *push forward of crossed modules* and we write $(\widetilde{\partial}, \widetilde{\xi}) = p_*(\partial, \xi)$.

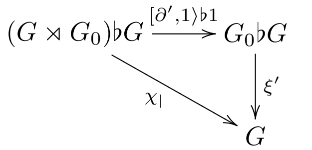
4. Whenever there exists a morphism of pre-crossed modules

$$(p, p_0)\colon (\partial, \xi, H, H_0) \to (\partial', \xi', G, G_0)\,,$$

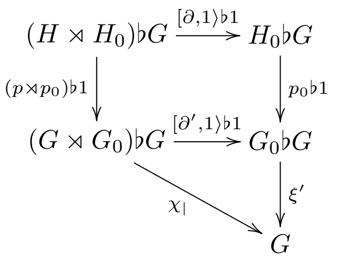
with (∂', ξ') a crossed module, it is always possible to compute the push forward of ∂ along p, with respect to the action $p_0^*(\xi')\colon H_0 \flat G \to G$. Indeed, the first condition for the construction of the push forward follows from the fact that (p, p_0) is a pre-crossed module morphism, as the commutative diagram below shows:



On the other hand, since (∂', ξ') is a crossed module, by Proposition 2.4, the following diagram is commutative:



and by composition, we get the second condition of Theorem 2.7:



This also shows that the hypotheses of Theorem 2.7 are necessary.

2.2 The case of action accessible categories

In many algebraic contexts, the construction of the push forward can be performed under milder hypotheses on the arrows and the actions involved. In particular, we restrict our attention to semi-abelian action accessible categories (see [10]). Here the following result holds:

Lemma 2.9. *In an action accessible category, let be given two subobjects of the same object:*

$$X \overset{x}{\rightarrowtail} Z \overset{y}{\leftarrowtail} Y\,,$$

with x a normal monomorphism. If $\overline{y}$ is the normal closure of y, then x and y cooperate if and only if x and $\overline{y}$ cooperate, in other words:

$$[X, Y] = 0 \quad \Leftrightarrow \quad [X, \overline{Y}] = 0$$

where $\overline{Y}$ is the normal closure of Y in Z.

(For details on cooperating morphisms and commutators see, for example, [2]).

Proof. This property follows from the fact that, in action accessible categories, normal subobjects admit normal centralizers (see [10]). □

In this case, we can state the same result as in Theorem 2.7 with the weaker condition expressed by diagram (4) below. This is done by pre-composing diagram (2) with $i_H \flat 1 \colon H \flat G \to (H \rtimes_\xi H_0) \flat G$.

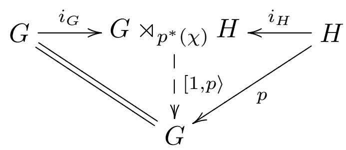
Theorem 2.10 (Push forward, action accessible case)**.** *In a semi-abelian action accessible category, we consider a pre-crossed module $(\partial \colon H \to H_0, \xi)$, a morphism $p \colon H \to G$, and an internal action $\alpha \colon H_0 \flat G \to G$ such that the following diagrams commute:*

$$\begin{array}{ccc} H_0 \flat H & \xrightarrow{\ \xi\ } & H \\ {\scriptstyle 1\flat p}\downarrow & & \downarrow{\scriptstyle p} \\ H_0 \flat G & \xrightarrow[\ \alpha\ ]{} & G \end{array}$$

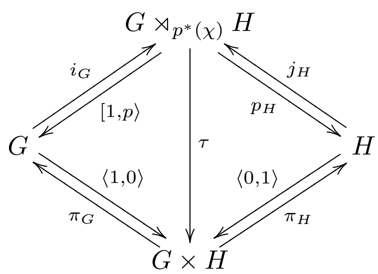
$$\begin{array}{ccc} H \flat G & \xrightarrow{\ \partial\flat 1\ } & H_0 \flat G \\ {\scriptstyle p\flat 1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \flat G & \xrightarrow[\ \chi\ ]{} & G \end{array} \tag{4}$$

Then there exists a push forward $(\widetilde{\partial} \colon G \to G \rtimes^H H_0, \widetilde{\xi})$, with the properties listed in Theorem 2.7.

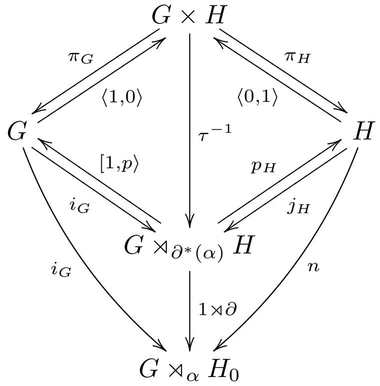
Proof. By the definition of the conjugation action χ_G, there is an isomorphism between $G \rtimes_\chi G$ and the product $G \times G$. Simply by composition, we get a similar situation for $G \rtimes_{p^*(\chi)} H$. More precisely, there is a unique arrow $[1, p\rangle$ making the following diagram commute:



and we have an isomorphism $\tau = \langle [1, p\rangle, p_H \rangle$ of punctual spans:



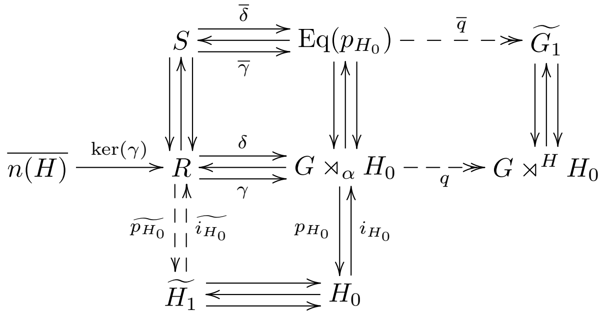
where $j_H = \ker[1, p\rangle$.
Since by hypothesis diagram (4) commutes, it follows that $p^*(\chi) = \partial^*(\alpha)$ and $G \rtimes_{p^*(\chi)} H = G \rtimes_{\partial^*(\alpha)} H$.
Now, from the following commutative diagram



we deduce that the morphisms $n = (1 \rtimes \partial) \cdot j_H$ and i_G cooperate in $G \rtimes_\alpha H_0$, so that $[n(H), G] = 0$. Moreover, by Lemma 2.9:

$$[n(H), G] = 0 \quad \Rightarrow \quad \left[\overline{n(H)}, G\right] = 0$$

where $m\colon \overline{n(H)} \to G \rtimes_\alpha H_0$ is the normal closure of $n(H)$ in $G \rtimes_\alpha H_0$, and $h\colon H \to \overline{n(H)}$ is such that $m \cdot h = n$.
But, in the action accessible context, the so called "Smith is Huq" property holds, i.e. two normal subobjects cooperate if and only if the corresponding equivalence relations are connected (see [10]). As a consequence, if R is the equivalence relation corresponding to $\overline{n(H)}$, we have that R and $\mathrm{Eq}(p_{H_0})$ are connected. Hence, by means of their centralizing double relation S, we get the core of a profunctor. Thus, we can apply the same technique as in the proof of Theorem 2.7, obtaining a profunctor



which represents the crossed module morphism on the right hand side of the following commutative diagram:

$$
\begin{array}{ccccc}
H & \xrightarrow{h} & \overline{n(H)} & \xrightarrow{\overline{p}} & G \\
{\scriptstyle \partial}\downarrow & & {\scriptstyle \overline{\partial}}\downarrow & & \downarrow{\scriptstyle \widetilde{\partial}} \\
H_0 & = & H_0 & \xrightarrow[\widetilde{p_0}]{} & G \rtimes^H H_0
\end{array}
$$

With an argument similar to the one used in the proof of Theorem 2.7, one can show that $\overline{p} \cdot h = p$. It remains to prove that the whole rectangle above is a morphism of pre-crossed modules, as the following commutative diagram shows (the arrow $p \rtimes 1$ exists by equivariance of p):

$$
\begin{array}{ccccccc}
H \rtimes_\xi H_0 & \xrightarrow{p \rtimes 1} & G \rtimes_\alpha H_0 & \xrightarrow{\langle 1, i_{H_0} p_{H_0} \rangle} & \mathrm{Eq}(p_{H_0}) & \xrightarrow{\overline{q}} & \widetilde{G_1} \\
{\scriptstyle p_{H_0}}\downarrow\uparrow{\scriptstyle i_{H_0}} & & {\scriptstyle p_{H_0}}\downarrow\uparrow{\scriptstyle i_{H_0}} & & {\scriptstyle r_2}\downarrow\uparrow{\scriptstyle \langle 1,1 \rangle} & & {\scriptstyle c}\downarrow\uparrow{\scriptstyle e} \\
H_0 & = & H_0 & \xrightarrow[i_{H_0}]{} & G \rtimes_\alpha H_0 & \xrightarrow[q]{} & G \rtimes^H H_0
\end{array}
$$

□

2.3 A useful construction for the push forward of pre-crossed modules

The proof of Theorem 2.7 takes place in the denormalized context of reflexive graphs, while the theorem itself is stated mainly in terms of pre-crossed modules (only diagram (2) uses the reflexive graph structure determined by the pre-crossed module ∂). For this reason, it is useful to show a construction *a posteriori*, that takes place in the normalized context and may be used in applications. In fact, we will see how the object $G \rtimes^H H_0$ can be computed as a cokernel.

As in Section 2.2, we can consider the condition given by the commutativity of diagram (4) and the consequent isomorphism $G \rtimes_{\partial^*(\alpha)} H \cong G \times H$. In particular, we are interested in the arrow $(1 \rtimes \partial) \cdot j_H$.

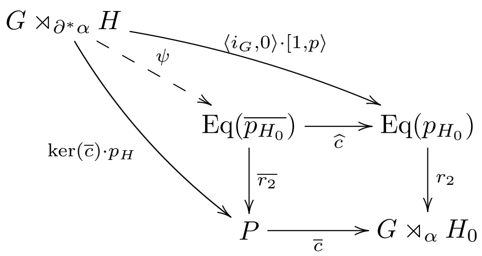
Lemma 2.11. *Under the hypotheses of Theorem 2.7, the composite*

$$H \xrightarrow{j_H} G \rtimes_{\partial^*(\alpha)} H \xrightarrow{1 \rtimes \partial} G \rtimes_\alpha H_0$$

is a pre-crossed module. More precisely, it is the normalization $\overline{d} \cdot \ker(\overline{c})$ *of the reflexive graph* $(P, G \rtimes_\alpha H_0)$ *introduced in the proof of the theorem.*

Proof. Throughout this proof, we will refer to the notation and the diagrams used in the proof of Theorem 2.7 and $\ker(\overline{c})$ will be the unique arrow such that $\overline{p_{H_0}} \cdot \ker(\overline{c}) = i_H \colon H \to H \rtimes H_0$.

Consider the arrow ψ, defined by the universal property of the pullback $r_2 \cdot \widehat{c} = \overline{c} \cdot \overline{r_2}$:



that is, ψ is the unique arrow such that $\overline{r_2} \cdot \psi = \ker(\overline{c}) \cdot p_H$ and $\widehat{c} \cdot \psi = \langle i_G, 0\rangle \cdot [1, p\rangle$.
First of all, we can prove that:

$$\Delta_P \cdot \ker(\overline{c}) = \psi \cdot j_H, \qquad \widehat{e} \cdot \langle i_G, 0\rangle = \psi \cdot i_G,$$

where $\widehat{e}$ is the common section of $\widehat{d}$ and $\widehat{c}$. To this end, it suffices to compose on the left with the jointly monic pair $(\overline{r_2}, \widehat{c})$.
The second step is to prove the equality:

$$\overline{i_{H_0}} \cdot i_H = \overline{r_1} \cdot \psi \cdot i_H,$$

by composition on the left with the arrows $\overline{p_{H_0}}$ and $\overline{c}$, which are jointly monic as projections of the pullback $(b) + (c)$ in diagram (3).
Now, by composing on the right with the jointly epic pair (i_G, i_H), we can prove that:

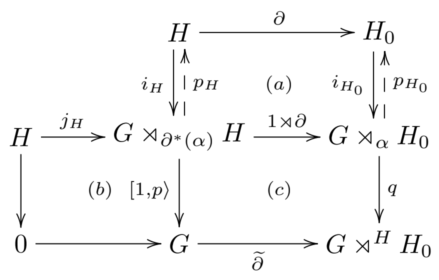
$$1 \rtimes \partial = \overline{d} \cdot \overline{r_1} \cdot \psi.$$

Finally we have:

$$(1 \rtimes \partial) \cdot j_H = \overline{d} \cdot \overline{r_1} \cdot \psi \cdot j_H = \overline{d} \cdot \overline{r_1} \cdot \Delta_P \cdot \ker(\overline{c}) = \overline{d} \cdot \ker(\overline{c})$$

and the proof is completed. □

As a consequence of Lemma 2.11, we have that $q = \mathrm{coker}((1 \rtimes \partial) \cdot j_H)$, since it is the coequalizer of $\overline{d}$ and $\overline{c}$, and this gives an alternative way to compute $G \rtimes^H H_0$.
Now, consider the following commutative diagram:



(5)

Since the sequence $(j_H, [1,p\rangle)$ is short exact, the square (b) is a pullback and a pushout. The rectangle $(b)+(c)$ is also a pushout, then also (c) is a pushout. On the other hand, the pasting of (a) and (c) is the push forward of ∂ along $p = [1,p\rangle \cdot i_{H_0}$. Consequently, the push forward $\widetilde{\partial}$ can be alternatively obtained as the pushout of $1 \rtimes \partial$ along $[1,p\rangle$.
It is also possible to describe the crossed module structure assigned to the map $\widetilde{\partial}$. This is done by observing that the action $(G \rtimes^H H_0)\flat G \to G$ is the unique arrow making the following diagram commute:

$$
\begin{array}{ccc}
(G \rtimes_\alpha H_0)\flat G & & \\
{\scriptstyle q\flat 1}\downarrow & \searrow^{\chi_G^{G\rtimes H_0}} & \\
(G \rtimes^H H_0)\flat G & \dashrightarrow & G
\end{array}
$$

where the map $q\flat 1_G$ is a regular epimorphism since q is and $-\flat 1_G$ preserves such maps (see [15]).

2.4 A characterization

In this section we present a result providing easy-to-handle conditions in order to check whether a commutative square is a push forward.

Proposition 2.12. *Let $\mathcal{C}$ be a semi-abelian category, $(\partial\colon H \to H_0, \xi)$ a pre-crossed module in $\mathcal{C}$ and let*

$$
\begin{array}{ccc}
H & \xrightarrow{\partial} & H_0 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}_0} \\
G & \xrightarrow[\widetilde{\partial}]{} & G \rtimes^H H_0
\end{array}
$$

be a push forward of ∂ along p. Then the restriction $p_|$ of p to kernels is a regular epimorphism and

$$\mathrm{Ker}(p_|) = \mathrm{Ker}(\partial) \cap \mathrm{Ker}(p).$$

Proof. We first decompose the push forward square into the rectangle (a) and the rectangle (c) of diagram (5). Since (a) is a pullback, we can just consider the restriction of $[1,p\rangle$ to the kernels of the horizontal arrows in (c).
We factorize the maps $1 \rtimes \partial$ and $\widetilde{\partial}$ into regular epimorphisms followed by monomorphisms. We get the commutative diagram:

$$
\begin{array}{ccccc}
G \rtimes H & \xrightarrow{e} & \bullet & \xrightarrow{m} & G \rtimes H_0 \\
\downarrow & (c_1) & \downarrow{\scriptstyle \ell} & (c_2) & \downarrow \\
G & \xrightarrow{\widehat{e}} & \bullet & \xrightarrow{\widetilde{m}} & G \rtimes^H H_0
\end{array}
$$

with $m \cdot e = 1 \rtimes \partial$ and $\widetilde{m} \cdot \widehat{e} = \widetilde{\partial}$.

where the comparison map ℓ is a regular epimorphism.
Let us consider the diagram below:

$$\begin{array}{ccccccccc}
\mathrm{Ker}(e_|) & \longrightarrow & H & \xrightarrow{e_|} & \mathrm{Ker}(\ell) & \xrightarrow{m_|} & \mathrm{Ker}(q) & \longrightarrow & 0 \\
{\scriptstyle j_{H|}}\downarrow & & {\scriptstyle j_H}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Ker}(e) & \longrightarrow & G \rtimes H & \overset{e}{\twoheadrightarrow} & \bullet & \overset{m}{\rightarrowtail} & G \rtimes H_0 & \twoheadrightarrow & C \\
{\scriptstyle [1,p\rangle_|}\downarrow & & {\scriptstyle [1,p\rangle}\downarrow & (c_1) & {\scriptstyle \ell}\downarrow & (c_2) & \downarrow{\scriptstyle q} & & \| \\
\mathrm{Ker}(\widehat{e}) & \longrightarrow & G & \underset{\widehat{e}}{\twoheadrightarrow} & \bullet & \underset{\widetilde{m}}{\rightarrowtail} & G \rtimes^H H_0 & \twoheadrightarrow & C
\end{array}$$

which extends $(c_1) + (c_2)$ in the following way. First we take the kernels of the (vertical) arrows $[1, p\rangle, \ell$ and q, and consider the restrictions $e_|$ and $m_|$ of e and m respectively. Since $m \cdot e \cdot j_H = (1 \rtimes \partial) \cdot j_H$ is a pre-crossed module (see Lemma 2.11), it factorizes into a regular epimorphism followed by a *normal* monomorphism, the last being the kernel of $\mathrm{coker}(m \cdot e \cdot j_H) = q$. So the composite $m_| \cdot e_|$ is a regular epimorphism.
Then we observe that the monomorphisms m and $\widetilde{m}$ are normal, since they come from a factorization of two pre-crossed modules, and they have isomorphic cokernels just because $(c_1) + (c_2)$ is a pushout. This allows us to complete the diagram with the last column on the right, in such a way that the four squares on the right commute. In fact, in the four squares on the right, the three vertical sequences and the bottom and middle horizontal sequences are short exact. The 3×3 lemma (see e.g. [2]) ensures that also the upper horizontal sequence is short exact, so that $m_|$ is an isomorphism.
On the other hand, we already observed that the composite $m_| \cdot e_|$ is a regular epimorphism, and then $e_|$ itself is a regular epimorphism. This implies immediately that the square (c_1) is a pushout.
Now we can take the kernels of e, $\widetilde{e}$ and $e_|$. Call $j_{H|}$ and $[1, p\rangle_|$ the restrictions of j_H and $[1, p\rangle$ respectively. By applying 3×3 lemma again, since the horizontal sequences are short exact likewise the middle and the rightmost vertical ones, we can conclude that also the leftmost vertical sequence is short exact.
This fact concludes the proof. Indeed $[1, p\rangle_|$, and then $p_|$, are regular epimorphisms, and $\mathrm{Ker}(p_|) = \mathrm{Ker}(e_|) = \mathrm{Ker}(e) \cap H = \mathrm{Ker}(1 \rtimes \partial) \cap \mathrm{Ker}([1, p\rangle) = \mathrm{Ker}(\partial) \cap \mathrm{Ker}(p)$. □

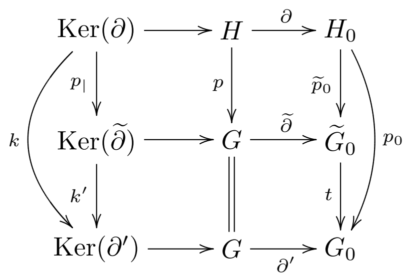
Theorem 2.13. *Let $\mathcal{C}$ be a semi-abelian category, and let*

$$\begin{array}{ccc}
H & \xrightarrow{\partial} & H_0 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_0} \\
G & \xrightarrow[\partial']{} & G_0
\end{array}$$

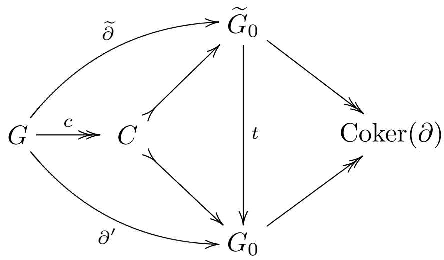
be a pre-crossed module morphism, with (∂', ξ') a crossed module.
Then the following are equivalent:

- (p, p_0) *makes* ∂' *the push forward of* ∂ *along* p *with respect to the action* $p_0^*(\xi')$*;*
- (p, p_0) *induces an isomorphism between the cokernels and a regular epimorphism between the kernels of* ∂ *and* ∂'*.*

Proof. The *only if* part is a consequence of Theorem 2.7 (2) and Proposition 2.12 above.
For what concerns the other implication, thanks to Remark 2.8 (3), we can compute the push forward $\widetilde{\partial}$ of ∂ along p with respect to the action $p_0^*(\xi')$. By comparison, we obtain a factorization of (p, p_0) which is represented by the two squares on the right in the diagram below. Then we take the kernels of ∂, $\widetilde{\partial}$ and ∂', and we complete the diagram with the restrictions $p_|$ and k':



Since $k = k' \cdot p_|$ is a regular epimorphism by hypothesis, then also k' is. But k' is also a monomorphism, hence it is an isomorphism.
Now, we have proved that $\widetilde{\partial}$ and ∂' have isomorphic kernels, so that they factor through the same (isomorphic) cokernel(s), i.e. the regular epimorphism c in the commutative diagram below:



Since $\widetilde{\partial}$ and ∂' are crossed modules, the two images through c are normal, so that the two triangles on the right form a morphism of short exact sequences. By the short five lemma, t is an isomorphism. □

Corollary 2.14. *Push forwards are closed under composition.*

Proof. Trivial by Theorem 2.13. □

3 Examples

In this section we present some examples of explicit computation of the push forward in algebraic varieties (see [23] for the case of groups). The examples below take place in action accessible categories, so we will refer to the conditions given in Theorem 2.10. Moreover, following [16], internal actions can be described by appropriate set-theoretical functions satisfying suitable conditions.

3.1 Rings

Following the notation of Theorem 2.10, let $(\partial\colon H \to H_0, \xi)$ be a pre-crossed module in the category of rings. The action ξ can be given by the assignment of two bilinear maps:

$$\begin{array}{ll} H_0 \times H \to H, & (h_0, h) \mapsto h_0 \cdot h\,, \\ H \times H_0 \to H, & (h, h_0) \mapsto h \cdot h_0\,, \end{array}$$

satisfying the following identities (for all $h, h' \in H$ and $h_0, h_0' \in H_0$):

$$\begin{array}{ll} (h_0 h_0') \cdot h = h_0 \cdot (h_0' \cdot h)\,, & (h_0 \cdot h)h' = h_0 \cdot (hh')\,, \\ (h_0 \cdot h) \cdot h_0' = h_0 \cdot (h \cdot h_0')\,, & (h \cdot h_0)h' = h(h_0 \cdot h')\,, \\ (h \cdot h_0) \cdot h_0' = h \cdot (h_0 h_0')\,, & (hh') \cdot h_0 = h(h' \cdot h_0)\,. \end{array}$$

The pre-crossed module condition says that:

$$\partial(h_0 \cdot h) = h_0 \cdot \partial(h) \qquad \text{and} \qquad \partial(h \cdot h_0) = \partial(h) \cdot h_0.$$

In the same way, let an action α of H_0 on G be given, together with a ring homomorphism $p\colon H \to G$, equivariant with respect to the actions ξ and α, i.e. such that for all $h \in H$ and $h_0 \in H_0$:

$$p(h_0 \cdot h) = h_0 \cdot p(h) \qquad \text{and} \qquad p(h \cdot h_0) = p(h) \cdot h_0.$$

Moreover, the condition given by diagram (4) requires that $p^*(\chi_G) = \partial^*(\alpha)$, i.e. for all $h \in H$ and $g \in G$:

$$\partial(h) \cdot g = p(h)g \qquad \text{and} \qquad g \cdot \partial(h) = gp(h).$$

Under these hypotheses, we can compute the push forward of ∂ along p in the following way.
Consider the semi-direct product $G \rtimes_\alpha H_0$, which is given by the set $G \times H_0$ endowed with the operations:

$$\begin{array}{l} (g, h_0) + (g', h_0') = (g + g', h_0 + h_0') \\ (g, h_0) \cdot (g', h_0') = (gg' + h_0 \cdot g' + g \cdot h_0', h_0 h_0') \end{array}$$

and the ring homomorphism:

$$n = (1 \rtimes \partial) \cdot j_H\colon H \to G \rtimes H_0, \qquad h \mapsto (-p(h), \partial(h)).$$

The image $n(H)$ is an ideal of $G \rtimes H_0$ since, for all $h \in H$, $g \in G$ and $h_0 \in H_0$:

$$\begin{aligned}(g, h_0) \cdot (-p(h), \partial(h)) &= (-gp(h) - h_0 \cdot p(h) + g \cdot \partial(h), h_0\partial(h)) = (-p(h_0 \cdot h), \partial(h_0 \cdot h))\,,\\ (-p(h), \partial(h)) \cdot (g, h_0) &= (-p(h)g + \partial(h) \cdot g - p(h) \cdot h_0, \partial(h)h_0) = (-p(h \cdot h_0), \partial(h \cdot h_0))\,.\end{aligned}$$

So we can take the quotient homomorphism:

$$q\colon G \rtimes H_0 \to G \rtimes^H H_0 = (G \rtimes H_0)/n(H).$$

Since the conjugation action of $n(H)$ on G is trivial:

$$\begin{aligned}(g, 0) \cdot (-p(h), \partial(h)) &= (-gp(h) + g \cdot \partial(h), 0) = (0, 0)\,,\\ (-p(h), \partial(h)) \cdot (g, 0) &= (-p(h)g + \partial(h) \cdot g, 0) = (0, 0)\,,\end{aligned}$$

the conjugation action of $G \rtimes H_0$ on G is well defined on cosets, thus giving an action of $G \rtimes^H H_0$ on G. This is the action $\widetilde{\xi}$ making the morphism $q \cdot i_G = \widetilde{\partial}$ a crossed module.

3.2 Lie and Leibniz algebras

Consider now the category of Lie algebras over a fixed field. As above, ∂ is a pre-crossed module. Here, the action ξ is a bilinear map:

$$H_0 \times H \to H, \qquad (h_0, h) \mapsto [h_0, h]$$

satisfying the following identities (for all $h, h' \in H$ and $h_0, h_0' \in H_0$):

$$\begin{aligned}[][h_0, h_0'], h] &= [h_0, [h_0', h]] - [h_0', [h_0, h]]\\ [h_0, [h, h']] &= [[h_0, h], h'] - [[h_0, h'], h]\end{aligned}$$

and the pre-crossed module condition says that:

$$\partial[h_0, h] = [h_0, \partial(h)]\,.$$

An action α of H_0 on G and a Lie algebra homomorphism $p\colon H \to G$ are given, and the equivariance of p says that for all $h \in H$ and $h_0 \in H_0$:

$$p[h_0, h] = [h_0, p(h)]\,.$$

Finally, the request $p^*(\chi_G) = \partial^*(\alpha)$ means that for all $h \in H$ and $g \in G$:

$$[\partial(h), g] = [p(h), g]\,.$$

Under these hypotheses, we can compute the push forward of ∂ along p as above. The semi-direct product $G \rtimes_\alpha H_0$ is given by the set $G \times H_0$ endowed with the operations:

$$\begin{aligned}(g, h_0) + (g', h_0') &= (g + g', h_0 + h_0')\\ [(g, h_0), (g', h_0')] &= ([g, g'] + [h_0, g'] - [h_0', g], [h_0, h_0'])\end{aligned}$$

and we have the homomorphism:

$$n = (1 \rtimes \partial) \cdot j_H \colon H \to G \rtimes H_0, \qquad h \mapsto (-p(h), \partial(h)) \,.$$

The image $n(H)$ is an ideal of $G \rtimes H_0$, indeed, for all $h \in H$, $g \in G$ and $h_0 \in H_0$:

$$[(g, h_0), (-p(h), \partial(h))] = (-[g, p(h)] - [h_0, p(h)] - [\partial(h), g], [h_0, \partial(h)]) = (-p[h_0, h], \partial[h_0, h])$$

and we can take the quotient homomorphism:

$$q \colon G \rtimes H_0 \to G \rtimes^H H_0 = (G \rtimes H_0)/n(H).$$

Since the conjugation action of $n(H)$ on G is trivial:

$$[(-p(h), \partial(h)), (g, 0)] = (-[p(h), g] + [\partial(h), g], 0) = (0, 0)$$

again, there is a well defined action of $G \rtimes^H H_0$ on G making the morphism $q \cdot i_G$ a crossed module.

In the case of Leibniz algebras (see [17] for definitions), we still have a bracket operation, satisfying a weaker version of the Jacobi identity, and which is not antisymmetric in general. As a consequence, actions are given by pairs of bilinear maps, like in the case of rings, and the conditions involving actions are a bit more complicated. However the homomorphism n is defined exactly as above and the push forward construction is the same.

4 Push forward of extensions

4.1 Abelian extensions

Let $\mathcal{C}$ be a semi-abelian category. Given an object Y in $\mathcal{C}$, following Beck's terminology (see [1]), we call Y*-module* an abelian group in the slice category $\mathcal{C} \downarrow Y$, that can be interpreted as a totally disconnected groupoid in $\mathcal{C}$, with Y as object of objects:

$$A \rtimes_\xi Y \underset{p_Y}{\overset{p_Y}{\underset{\xleftarrow{i_Y}}{\rightrightarrows}}} Y \,. \tag{6}$$

A Y-module morphism is then an internal functor of the form:

$$\begin{array}{ccc} A \rtimes_\xi Y & \underset{p_Y}{\overset{p_Y}{\underset{\xleftarrow{i_Y}}{\rightrightarrows}}} & Y \\ {\scriptstyle a \rtimes 1}\downarrow & & \| \\ A' \rtimes_{\xi'} Y & \underset{p_Y}{\overset{p_Y}{\underset{\xleftarrow{i_Y}}{\rightrightarrows}}} & Y \end{array}$$

which is uniquely determined by a morphism $a\colon A \to A'$ in $\mathcal{C}$ making the following commute:

$$\begin{array}{ccc} Y\flat A & \xrightarrow{\xi} & A \\ {\scriptstyle 1\flat a}\downarrow & & \downarrow{\scriptstyle a} \\ Y\flat A' & \xrightarrow[\xi']{} & A' \end{array}$$

Classically, a Y-module is simply an abelian group in $\mathcal{C}$, together with an action ξ of Y on A. In general, this notion is weaker than the one given above; however, if the "Smith is Huq" property holds (as, for example, in action accessible categories), the two definitions are equivalent. On the other hand, the notion of Y-module morphism remains the same. Throughout the section, with abuse of notation, we will write (A, ξ) for the Y-module (6).

Now, consider an *abelian extension* in $\mathcal{C}$, that is a short exact sequence:

$$0 \longrightarrow A \xrightarrow{i} X \xrightarrow{f} Y \longrightarrow 0$$

with the kernel pair $\mathrm{Eq}(f)$ connected with itself. In other words, f is a Mal'tsev object in $\mathcal{C} \downarrow Y$ (see e.g. [2]). Again, if the "Smith is Huq" property holds, this is the same as an extension with abelian kernel. Such an extension is associated with a Y-module structure (A, ξ), called the *direction* of the Mal'tsev object (see [5]), which can be computed as the pushout of the split monomorphism Δ along f:

$$\begin{array}{ccc} \mathrm{Eq}(f) & \longrightarrow & D(f) \\ {\scriptstyle r_2}\downarrow\uparrow{\scriptstyle \Delta} & & \downarrow\uparrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Indeed, the downward commutative square is a pullback, so that $D(f) \cong A \rtimes_{\xi} Y$ for an action ξ of Y on A.

In [6, Theorem 2.1] Bourn proved that, in any efficiently homological category, the direction functor is a cofibration. In the following proposition we will give a different proof of the same result, by means of the push forward construction in the semi-abelian context. The special case of central extensions is also given in [13, Corollary 3.3].

Proposition 4.1. *Given an abelian extension f as above, with direction (A, ξ) and a morphism $a\colon (A, \xi) \to (A', \xi')$ of Y-modules, then there exists an abelian extension:*

$$0 \longrightarrow A' \xrightarrow{i'} X' \xrightarrow{f'} Y \longrightarrow 0$$

with direction (A', ξ'), and a morphism of extensions $(a, x, 1_Y)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & X & \xrightarrow{f} & Y & \longrightarrow & 0 \\
 & & {\scriptstyle a}\downarrow & & {\scriptstyle x}\downarrow & & \| & & \\
0 & \longrightarrow & A' & \xrightarrow[i']{} & X' & \xrightarrow[f']{} & Y & \longrightarrow & 0
\end{array}$$

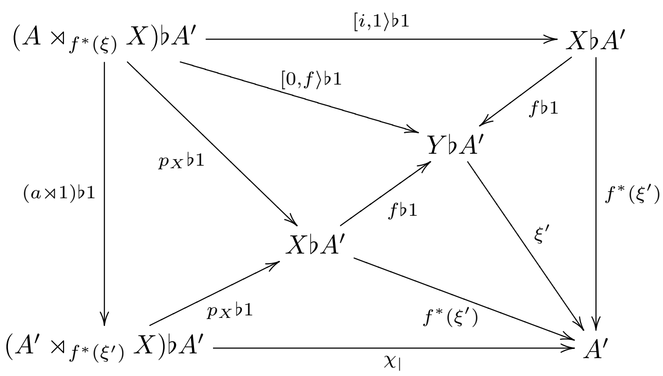
Moreover, for any other morphism of extensions $(a, \overline{x}, 1_Y)\colon (f, i) \to (f'', i'')$ with (A', ξ') direction of f'', there is a unique arrow t such that $t \cdot x = \overline{x}$ and $t \cdot i' = i''$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & X & \xrightarrow{f} & Y & \longrightarrow & 0 \\
 & & {\scriptstyle a}\downarrow & & {\scriptstyle x}\downarrow \ \searrow{\scriptstyle \overline{x}} & & \| & & \\
0 & \longrightarrow & A' & \xrightarrow[i']{} & X' & \xrightarrow[f']{} & Y & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle t}\downarrow & & \| & & \\
0 & \longrightarrow & A' & \xrightarrow[i'']{} & X'' & \xrightarrow[f'']{} & Y & \longrightarrow & 0
\end{array} \qquad (7)$$

Proof. It suffices to show that the push forward of i along a exists (with respect to the action $f^*(\xi')\colon X\flat A' \to A'$), and construct in such a way the square $x \cdot i = i' \cdot a$. Indeed, if it is the case, Theorem 2.13 shows that the crossed module morphism (a, x) induces an isomorphism between the cokernels and a regular epimorphism between the kernels. Hence, $\mathrm{Coker}(i') = Y$ and $\mathrm{Ker}(i') = 0$, so that i' is a monomorphism (and it is normal, being a crossed module).
So, we only have to prove that the conditions for the construction of the push forward are fulfilled. The equivariance of a comes from the fact that a is a morphism of Y-modules:

$$\begin{array}{ccccc}
X\flat A & \xrightarrow{f\flat 1} & Y\flat A & \xrightarrow{\xi} & A \\
{\scriptstyle 1\flat a}\downarrow & & {\scriptstyle 1\flat a}\downarrow & & \downarrow{\scriptstyle a} \\
X\flat A' & \xrightarrow[f\flat 1]{} & Y\flat A' & \xrightarrow[\xi']{} & A'
\end{array}$$

To check the second condition, we have to prove the commutativity of the following diagram:



where the only commutativity which is not obvious is the one of the lower triangle $f^*(\xi') \cdot p_X\flat 1 = \chi_|$. This equality depends on the fact that A' is a Y-module, so that the groupoid

$$A' \rtimes_{f^*(\xi')} X \underset{p_X}{\overset{p_X}{\underset{\longleftarrow i_X}{\rightrightarrows}}} X$$

coincides with its direction. Hence, following the construction of [5], we have a discrete fibration:

$$\begin{array}{ccc} \mathrm{Eq}(p_X) & \longrightarrow & A' \rtimes_{f^*(\xi')} X \\ r_2 \downarrow \uparrow \Delta \downarrow r_1 & & p_X \downarrow \uparrow i_X \downarrow p_X \\ A' \rtimes_{f^*(\xi')} X & \xrightarrow{\ p_X\ } & X \end{array}$$

and this implies that $f^*(\xi') \cdot p_X\flat 1 = \chi_|$. The factorization property (7) depends on the fact that the square $i' \cdot a = x \cdot i$ is a push forward and $(\overline{x})^*((f'')^*(\xi')) = f^*(\xi') : X\flat A' \to A'$. $\square$

Let us denote OPEXT(Y, A, ξ) the groupoid of extensions of Y via A inducing the action ξ. In the semi-abelian context, we can construct Baer sums of abelian extensions as in [6]. Given two extensions in OPEXT(Y, A, ξ):

$$E_1\colon \qquad 0 \longrightarrow A \xrightarrow{\ i_1\ } X_1 \xrightarrow{\ f_1\ } Y \longrightarrow 0 \ ,$$

$$E_2\colon \qquad 0 \longrightarrow A \xrightarrow{\ i_2\ } X_2 \xrightarrow{\ f_2\ } Y \longrightarrow 0 \ ,$$

the Baer sum $E_1 + E_2$ is obtained as:

$$\begin{array}{lccccccccc} & 0 & \longrightarrow & A \times A & \xrightarrow{i_1 \times_Y i_2} & X_1 \times_Y X_2 & \longrightarrow & Y & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle (1,1)} & & \downarrow & & \| & & \\ E_1 + E_2\colon & 0 & \longrightarrow & A & \longrightarrow & X_1 \otimes_Y X_2 & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

where the left hand square is a pushout, which is also a push forward, since it fulfills the conditions of Theorem 2.13.
This construction yields an abelian group structure on the set $\mathrm{Opext}(Y, A, \xi)$ of isomorphism classes.
Starting from a Y-module morphism a, the above proposition allows us to define a functor:

$$a_* \colon \mathrm{OPEXT}(Y, A, \xi) \to \mathrm{OPEXT}(Y, A', \xi') \,.$$

It is proved in [9] that, in pointed protomodular categories with semidirect products, the functor a_* provides a group homomorphism when passing to isomorphism classes. This way the authors obtained the functoriality of the assignment:

$$\mathrm{Opext}_{\mathcal{C}}(Y, -) \colon \mathbf{Ab}(\mathcal{C} \downarrow Y) \to \mathbf{Ab}$$

which sends every Y-module (A, ξ) to the abelian group of isomorphism classes of extensions of Y via A inducing the action ξ. In the next proposition we show how to recover this result by means of the properties of push forwards.

Proposition 4.2. *Following the notation above, we have:*

$$a_*(E_1 + E_2) \cong a_*(E_1) + a_*(E_2) \,.$$

So that, passing to isomorphism classes we obtain a group homomorphism:

$$a_* \colon \mathrm{Opext}(Y, A, \xi) \to \mathrm{Opext}(Y, A', \xi') \,.$$

Proof. Let us compute $a_*(E_1)$ and $a_*(E_2)$ by taking the push forwards of the normal monomorphisms i_1 and i_2 along a:

$$\begin{array}{ccc} A & \xrightarrow{i_1} & X_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g_1} \\ A' & \xrightarrow[k_1]{} & Z_1 \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{i_2} & X_2 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g_2} \\ A' & \xrightarrow[k_2]{} & Z_2 \end{array}$$

and consider the following commutative diagrams:

$$\begin{array}{ccc} A \times A & \xrightarrow{i_1 \times i_2} & X_1 \times_Y X_2 \\ {\scriptstyle (1,1)}\downarrow & (i) & \downarrow \\ A & \longrightarrow & X_1 \otimes_Y X_2 \\ {\scriptstyle a}\downarrow & (ii) & \downarrow \\ A' & \longrightarrow & Q \end{array} \qquad \begin{array}{ccc} A \times A & \xrightarrow{i_1 \times i_2} & X_1 \times_Y X_2 \\ {\scriptstyle a \times a}\downarrow & (iii) & \downarrow{\scriptstyle g_1 \times g_2} \\ A' \times A' & \xrightarrow[k_1 \times k_2]{} & Z_1 \times_Y Z_2 \\ {\scriptstyle (1,1)}\downarrow & (iv) & \downarrow \\ A' & \longrightarrow & Z_1 \otimes_Y Z_2 \end{array}$$

where the squares (i) and (iv) are Baer sum constructions, (ii) is obtained as a push forward and (iii) turns out to be a push forward by Theorem 2.13. Since, by Corollary 2.14, the composite of two push fowards is again a push forward, then $(i) + (ii)$ and $(iii) + (iv)$ are push forwards and $Q \cong Z_1 \otimes_Y Z_2$ because $a \cdot (1, 1) = (1, 1) \cdot (a \times a)$. $\square$

4.2 General case

In the previous section, we showed that a morphism a of Y-modules is sufficient to produce the push foward of an abelian extension, providing a way to lift the morphism a to a morphism of abelian extensions.
But the push forward construction holds, under suitable hypotheses, also for general (not necessarily abelian) extensions, and Theorem 2.13 ensures that the push forward of an extension is again an extension. So, the conditions (1) and (2) of Theorem 2.7, in the special case where ∂ is a normal monomorphism, can be regarded as conditions to generalize the above construction. More precisely, the following result holds.

Proposition 4.3. *Given an extension:*

$$0 \longrightarrow K \xrightarrow{i} X \xrightarrow{f} Y \longrightarrow 0\,,$$

a morphism $k\colon K \to K'$ and an action α of X on K', such that the following diagrams commute:

$$\begin{array}{ccc} X\flat K & \xrightarrow{\chi_|} & K \\ {\scriptstyle 1\flat k}\downarrow & & \downarrow{\scriptstyle k} \\ X\flat K' & \xrightarrow[\alpha]{} & K' \end{array} \qquad \begin{array}{ccc} K \rtimes_{\chi_|} X)\flat K' & \xrightarrow{[k,1\rangle\flat 1} & X\flat K' \\ {\scriptstyle (k\rtimes 1)\flat 1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ (K' \rtimes_{\alpha} X)\flat K' & \xrightarrow[\chi_|]{} & K' \end{array}$$

then there exists an extension:

$$0 \longrightarrow K' \xrightarrow{i'} X' \xrightarrow{f'} Y \longrightarrow 0$$

and a morphism of extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle x}\downarrow & & \| & & \\ 0 & \longrightarrow & K' & \xrightarrow[i']{} & X' & \xrightarrow[f']{} & Y & \longrightarrow & 0 \end{array}$$

5 Push forward and the comprehensive factorization for internal crossed modules

In this section we will establish a link between the push forward construction developed in Section 2 and the *comprehensive* factorization system in $\mathbf{XMod}(\mathcal{C})$, for $\mathcal{C}$ a semi-abelian category.

5.1 Factorization systems for internal crossed modules

Factorization systems for internal groupoids have been studied in [4, 11], so that it is quite natural to express them in terms of crossed modules.
Let $\mathcal{C}$ be a Barr-exact category. In [4] the author studies two factorization systems for $\mathbf{Gpd}(\mathcal{C})$, and he shows the way they are related to each other.
The first one consists of the classes of π_0-invertible and π_0-cartesian morphisms, where the reflection

$$\pi_0 \colon \mathbf{Gpd}(\mathcal{C}) \longrightarrow \mathcal{C}$$

functorially extends the assignment $\pi_0(\underline{G}) = \mathrm{Coeq}(d_{\underline{G}}, c_{\underline{G}})$, for any groupoid $\underline{G}$. Indeed, π_0 is a fibration up to isomorphism (fibred reflection in [4]), and its definition, rather than being an *ad hoc* one, comes from a universal construction. Actually, the category $\mathbf{Gpd}(\mathcal{C})$ is monadic over the category $\mathbf{Pt}(\mathcal{C})$ (of split epimorphisms with a chosen section) and π_0 is the extension to the algebras of the forgetful functor

$$(\ )_0 \colon \mathbf{Pt}(\mathcal{C}) \longrightarrow \mathcal{C}\,,$$

that sends a split epimorphism p, with section s, to the codomain of p.
The second factorization system consists of the classes D and F of *discrete fibrations* and of *final functors* respectively, and it is the lifting to the algebras of the factorization system determined on $\mathbf{Pt}(\mathcal{C})$ by the fibration $(\ )_0$.

For what concerns the first factorization system, the constructions involved in the case of groupoids are fairly easy to translate. Actually, the main fact is that, for a groupoid $\underline{G}$, the coequalizer $\pi_0(\underline{G})$ amounts to the cokernel of the map underlying the corresponding crossed module $(\partial_{\underline{G}}, \xi_{\underline{G}})$. Thanks to this fact, one simply translates the problem in terms of groupoids, then one normalizes the resulting construction. The outcome is a factorization of a given morphism of crossed modules

$$(f, f_0) \colon \underline{G} \to \underline{H}$$

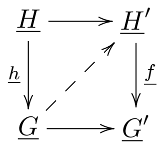
described in the diagram below:

$$\begin{array}{ccccc}
H & \xrightarrow{\quad f \quad} & G & = \!= \!= \!= & G \\
{\scriptstyle \partial_{\underline{H}}}\downarrow & (a) & {\scriptstyle \partial'}\,\vdots & (b) & \downarrow{\scriptstyle \partial_{\underline{G}}} \\
H_0 & \overset{}{\underset{f_0'}{- - \to}} & G_0' & \xrightarrow[f_0'']{} & G_0 \\
{\scriptstyle \mathrm{coker}(\partial_{\underline{H}})}\downarrow & & \downarrow & (c) & \downarrow{\scriptstyle \mathrm{coker}(\partial_{\underline{G}})} \\
\mathrm{Coker}(\partial_{\underline{H}}) & = \!= \!= & \mathrm{Coker}(\partial_{\underline{H}}) & \xrightarrow[\mathrm{coker}(\underline{f})]{} & \mathrm{Coker}(\partial_{\underline{G}})
\end{array}$$

Here the bottom right square (c) is a pullback by construction, so that the comparison map f_0' gives $f_0'' f_0' = f_0$. Moreover, it can be shown that the comparison map $\partial' = \langle 0, \partial_{\underline{G}} \rangle$ inherits a crossed module structure, such that (a) and (b) are morphisms, and the vertical unlabelled arrow is the cokernel of ∂'.
Hence, we obtain the desired factorization of (f, f_0) into the π_0-invertible morphism (f, f_0') followed by the π_0-cartesian morphism $(1_G, f_0'')$.

5.2 Final morphisms of internal crossed modules

The comprehensive factorization of a functor has been introduced in the set-theoretical context by Street and Walters in [25]. An internal version has been developed by Bourn in [4] for internal groupoids in a Barr-exact category, further extended in [11] to the efficiently regular setting.
The notion of final morphism of crossed modules we are going to introduce is a mere translation of the one defined by Bourn. Nevertheless, the technique adopted here is different: indeed, it arises from the property (PF) of push forward.
Let us recall from [4] that, for a Barr-exact category $\mathcal{C}$, the class of final functors is the class F of all internal functors $\underline{h}\colon \underline{H} \to \underline{G}$ satisfying the following property: for any commutative diagram (of solid arrows) in $\mathbf{Gpd}(\mathcal{C})$



with $\underline{f}$ a discrete fibration, there exists a (dashed) diagonal such that both triangles above commute. The pair (F, D) constitutes a factorization system.

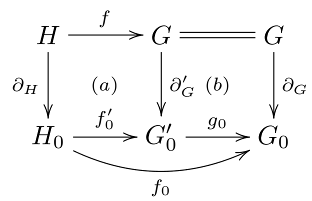
The notion of discrete fibration can be easily translated in terms of crossed module morphism: a morphism $\underline{f} = (f, f_0)$ is (i.e. it corresponds to) a discrete fibration if and only if f is an isomorphism.
For what concerns final morphisms, they are clearly defined by the diagonalization property as above, but in order to make this notion easier to handle, we are going to give a characterization.

Proposition 5.1. *Let $\mathcal{C}$ be a semi-abelian category and let $\underline{G} = (\partial_G\colon G \to G_0, \xi_G)$ and $\underline{H} = (\partial_H\colon H \to H_0, \xi_H)$ be crossed modules.*
The following conditions are equivalent for a morphism $\underline{f} = (f, f_0)\colon \underline{H} \to \underline{G}$:

1. *$\underline{f}$ is final;*
2. *$\underline{f}$ is obtained as a push forward of ∂_H along f, w.r.t. the induced action*

$$H_0 \flat G \xrightarrow{f_0 \flat 1} G_0 \flat G \xrightarrow{\xi_G} G\ .$$

Proof. First we prove the implication (2) $\Rightarrow$ (1). Let $\underline{f}$ be obtained as a push forward. We can factorize it as a final functor followed by a discrete fibration:



where the action of the crossed module ∂'_G is univocally determined by the (equivariance of the) discrete fibration (b), namely:

$$\xi'_G\colon\ G'_0\flat G \xrightarrow{g_0\flat 1} G_0\flat \xrightarrow{\xi_G} G.$$

Actually, $(1_G, g_0)$ is an isomorphism of crossed modules.
In order to prove this statement, we can start by computing the push forward inside the square (a) with respect to the induced action $\xi'_G\cdot(f'_0\flat 1)$ (see Remark 2.8). Indeed,

$$\xi'_G\cdot(f'_0\flat 1) = \xi_G\cdot(g_0\flat 1)\cdot(f'_0\flat 1) = \xi_G\cdot(f_0\flat 1),$$

so that this push forward gives the same morphism $\underline{f}$ we started with. This yields the following factorization:

$$\begin{array}{ccccccc}
H & \xrightarrow{f} & G & = & G & = & G \\
{\scriptstyle\partial_H}\downarrow & (a_1) & \downarrow{\scriptstyle\partial_G}\ (a_2) & & \downarrow{\scriptstyle\partial'_G}\ (b) & & \downarrow{\scriptstyle\partial_G} \\
H_0 & \xrightarrow[f_0]{} & G_0 & \xrightarrow[g'_0]{} & G'_0 & \xrightarrow[g_0]{} & G_0
\end{array}$$

By the main property of the push forward (a_1), then $(a_2)+(b)$ must give the identity of $\underline{G}$.
Again we can factorize $(a_1) = (a) + (b)$ and get

$$\begin{array}{ccccccc}
H & \xrightarrow{f} & G & = & G & = & G \\
{\scriptstyle\partial_H}\downarrow & (a) & \downarrow{\scriptstyle\partial'_G}\ (b) & & \downarrow{\scriptstyle\partial_G}\ (a_2) & & \downarrow{\scriptstyle\partial'_G} \\
G_0 & \xrightarrow[f'_0]{} & G'_0 & \xrightarrow[g_0]{} & G_0 & \xrightarrow[g'_0]{} & G'_0
\end{array}$$

By the essential uniqueness of the factorization, the discrete fibration $(b)+(a_2)$ must be the identity of $\underline{G'}$, showing that g_0 is an isomorphism.

Now, let $\underline{f}$ be a final functor (notation as above). Indeed, it is also the push forward of ∂_H along the map f, with respect to the induced action. This is easily proved by performing the push forward. Since this is also a final functor by the first implication and since the comparison is a discrete fibration, the result follows from the essential uniqueness of the factorization. □

The following corollary is a straightforward application of Theorem 2.13.

Corollary 5.2. *A morphism of crossed modules $\underline{f}\colon \underline{H}\to\underline{G}$ is final if and only if it induces an isomorphism between the cokernels of $\partial_{\underline{H}}$ and $\partial_{\underline{G}}$ and a regular epimorphism between the kernels of $\partial_{\underline{H}}$ and $\partial_{\underline{G}}$.*

As a consequence, if we define the kernel functor $\mathrm{Ker}\colon \mathbf{XMod}(\mathcal{C}) \to \mathcal{C}$ as:

$$\underline{G} \mapsto \mathrm{Ker}(\partial_{\underline{G}}),$$

it is easy to see that Ker turns the (final functor, discrete fibration) factorization of a given morphism $\underline{g}$ into the (regular epi, mono) factorization of $\mathrm{Ker}(\underline{g})$.

Remark 5.3. It is interesting to state the denormalized version of the last corollary. Let $\mathcal{C}$ be a semi-abelian category. Recall that we classically define two functors

$$\pi_0\colon \mathbf{Gpd}(\mathcal{C}) \to \mathcal{C}, \qquad \underline{G} \mapsto \mathrm{Coeq}(d_{\underline{G}}, c_{\underline{G}})$$

$$\pi_1\colon \mathbf{Gpd}(\mathcal{C}) \to \mathcal{C}, \qquad \underline{G} \mapsto \mathrm{Ker}(d_{\underline{G}}) \cap \mathrm{Ker}(c_{\underline{G}})$$

It is immediate to state the following characterization:

Corollary 5.4. *An internal functor $\underline{f}\colon \underline{H} \to \underline{G}$ is final if and only if $\pi_0(\underline{f})$ is an isomorphism, and $\pi_1(\underline{f})$ is a regular epimorphism.*

It is interesting to wonder whether this result has a counterpart in a non-pointed environment. This subject will be investigated in a forthcoming paper.

5.3 Factorization of morphisms of extensions

Given a semi-abelian category $\mathcal{C}$, we consider the category $\mathrm{EXT}_{\mathcal{C}}$, whose objects are extensions

$$0 \longrightarrow K \xrightarrow{\ i\ } X \xrightarrow{\ f\ } Y \longrightarrow 0$$

and morphisms are the obvious triples of arrows. Now, $\mathrm{EXT}_{\mathcal{C}}$ can be seen as the (full) subcategory of $\mathbf{XMod}(\mathcal{C})$ with objects the normal monomorphisms, hence we can give an account on the two factorization systems described in the previous section, restricted to extensions.

Proposition 5.5. *In the category $\mathrm{EXT}_{\mathcal{C}}$, final morphisms coincide with π_0-invertible ones, and discrete fibrations coincide with π_0-cartesian morphisms.*

Proof. The proof easily follows from the constructions above and from the characterization of final morphisms given in Proposition 2.12. □

Hereafter, for an arrow y with codomain Y, we denote by $y^*((i,f))$ the short exact sequence given by a pullback of f along y, inducing the identity between the kernels. Dually, for an arrow k with domain K, we denote by $k_*((i,f))$, whenever it exists, the short exact sequence given by a push forward of i along k, inducing the identity between the cokernels.

Corollary 5.6. *Let us consider the following diagram of solid arrows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & Y & \longrightarrow & 0 \\
 & & {\scriptstyle k}\downarrow & & \vdots\,{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \\
0 & \longrightarrow & K' & \xrightarrow[\ i'\ ]{} & X' & \xrightarrow[\ f'\ ]{} & Y' & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are short exact. Then, there exists an arrow x making the two squares commute if and only if $k_*((i,f))$ exists, and it is isomorphic to $y^*((i',f'))$.

Proof. Let us suppose that such an x exists. By Proposition 5.5 above, we get a factorization

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & Y & \longrightarrow & 0 \\
 & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle x'} & & \| & & \\
0 & \longrightarrow & K' & \xrightarrow{\ \widetilde{i}\ } & \widetilde{X} & \xrightarrow{\ \widetilde{f}\ } & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle x''} & & \downarrow{\scriptstyle y} & & \\
0 & \longrightarrow & K' & \xrightarrow[\ i'\ ]{} & X' & \xrightarrow[\ f'\ ]{} & Y' & \longrightarrow & 0
\end{array}
$$

with $x'' \cdot x' = x$. By Proposition 2.12, the upper morphism of extensions is a push forward, while the right lower square is obviously a pullback.
The converse is trivial. □

This issue can be described somehow more explicitly when dealing with abelian extensions.

Corollary 5.7. *Let us consider the following diagram of solid arrows:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & Y & \longrightarrow & 0 \\
 & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \\
0 & \longrightarrow & A' & \xrightarrow[\ i'\ ]{} & X' & \xrightarrow[\ f'\ ]{} & Y' & \longrightarrow & 0
\end{array}
$$

where the horizontal extensions are abelian, and let

$$\xi \colon Y \flat A \to A, \qquad \xi' \colon Y' \flat A' \to A'$$

be the induced actions. Then, there exists an arrow x making the two squares commute if and only if the pair (a,y) induces a morphisms of modules and $a_*((i,f)) \cong y^*((i',f'))$.

Proof. The extension $a_*((i,f))$ exists since a obviously induces a morphism of Y-modules between (A,ξ) and $(A', y^*(\xi'))$. □

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References

[1] Beck J.M.: Triples, algebra and cohomology. Ph.D. thesis, Columbia University, 1967. Available in Reprints in Theory Appl. Categories **2**, 1–59 (2003)

[2] Borceux F., Bourn D.: Mal'cev, Protomodular, Homological and Semi-abelian Categories. Kluwer Academic Publishers (2004)

[3] Borceux F., Janelidze G., Kelly G.M.: Internal object actions. Comment. Math. Univ. Carolinae **46** (2), 235–255 (2005)

[4] Bourn D.: The shift functor and the comprehensive factorization for internal groupoids. Cahiers Top. Géom. Diff. Catég. **28**, 197–226 (1987)

[5] Bourn D.: Aspherical abelian groupoids and their directions. J. Pure Appl. Algebra **168**, 133–146 (2002)

[6] Bourn D.: Baer sums in homological categories. J. Algebra **308**, 414–443 (2007)

[7] Bourn D.: Internal profunctors and commutator theory; applications to extensions classification and categorical Galois Theory. Theory Appl. Categories **24**, 451–488 (2010)

[8] Bourn D., Janelidze G.: Protomodularity, descent and semi-direct products. Theory Appl. Categories **4**, 37–46 (1998)

[9] Bourn D., Janelidze G.: Extensions with abelian kernel in protomodular categories. Georgian Math. J. **11**, 645–654 (2004)

[10] Bourn D., Janelidze G.: Centralizers in action accessible categories. Cahiers Top. Géom. Diff. Catég. **50**, 211–232 (2009)

[11] Bourn D., Rodelo D.: Comprehensive factorization and universal I-central extensions in the Mal'cev context. J. Pure Appl. Algebra **216**, 598–617 (2012)

[12] Brown K.S.: Cohomology of groups. Springer-Verlag (1982)

[13] Gran M., Van der Linden T.: On the second cohomology group in semi-abelian categories. J. Pure Appl. Algebra **212**, 636–651 (2008)

[14] Hartl M.: Push forward of crossed modules. Abstract presented at Workshop on Category Theory, Coimbra 2012

[15] Janelidze G.: Internal crossed modules. Georgian Math. J. **10**, 99–114 (2003)

[16] Casas J.M., Datuashvili T., Ladra M.: Universal strict general actors and actors in categories of interest. Appl. Categ. Structures **18**, 85–114 (2010)

[17] Loday J.-L.: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. Enseignement Math. **39**, 269–293 (1993)

[18] Mac Lane S.: Homology. Springer-Verlag (1963)

[19] Mantovani S., Metere G.: Internal crossed modules and Peiffer condition. Theory Appl. Categories **23**, 113–135 (2010)

[20] Mantovani S., Metere G.: Normalities and commutators. J. Algebra **324**, 2568–2588 (2010)

[21] Mantovani S., Metere G., Vitale E.M.: Profunctors in Mal'tsev categories and fractions of functors. J. Pure Appl. Algebra **217** 1173–1186 (2013)

[22] Martins-Ferreira N., Van der Linden T.: A note on the "Smith is Huq" condition. Appl. Categ. Structures **20**, 175–187 (2012)

[23] Noohi B.: On weak maps between 2-groups. arXiv:math/0506313v3 (2008)

[24] Orzech G.: Obstruction theory in algebraic categories, I. J. Pure Appl. Algebra **2**, 287–314 (1972)

[25] Street R., Walters R.F.C.: The comprehensive factorization of a functor. Bull. Amer. Math. Soc. **79**, 936–941 (1973)