Cohomological localization of $\mathcal{N} = 2$ gauge theories with matter

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ABSTRACT: We construct a large class of gauge theories with extended supersymmetry on four-dimensional manifolds with a Killing vector field and isolated fixed points. We extend previous results limited to super Yang-Mills theory to general $\mathcal{N} = 2$ gauge theories including hypermultiplets. We present a general framework encompassing equivariant Donaldson-Witten theory and Pestun's theory on S^4 as two particular cases. This is achieved by expressing fields in cohomological variables, whose features are dictated by supersymmetry and require a generalized notion of self-duality for two-forms and of chirality for spinors. Finally, we implement localization techniques to compute the exact partition function of the cohomological theories we built up and write the explicit result for manifolds with diverse topologies.

Contents

1 Introduction and summary

The analysis of $\mathcal{N} = 2$ gauge theories has led to numerous advances in our understanding of the dynamics of quantum field theories at strong coupling. Moreover, it has proven to be a fruitful arena to develop interesting connections between physics and mathematics. Two broad classes of developments in the study of $\mathcal{N} = 2$ gauge theories, namely the equivariant

topological twisting and Pestun's supersymmetric localization on the four-sphere, were unified in a single framework in [1]. That analysis only applied to $\mathcal{N} = 2$ super Yang-Mills theories; hence, here we consider its extension to theories that include both vector and hypermultiplets.

The first class of developments mentioned above, revolves around the topological twisting of $\mathcal{N} = 2$ gauge theories. Witten showed in [2] that a certain twisting of $\mathcal{N} = 2$ super Yang-Mills theory results in a four dimensional topological field theory whose correlators are the Donaldson invariants [3]. This work was later generalized by [4–12], who constructed twisted versions of $\mathcal{N} = 2$ gauge theories with vector and hypermultiplets. In particular, for the purposes of our work, we are interested in the equivariant version of the topological theory of Witten, which can be defined on manifolds admitting a torus T^2 action, and was studied in [13–16]. When specifying to the omega background, its partition function was calculated in the seminal work of Nekrasov [17, 18]. For more general non-compact toric manifolds $\mathcal{M}$, the partition function is a product of Nekrasov partition functions for each of the fixed points of the torus action on $\mathcal{M}$ [19], see also [20–22]. The extension of this analysis to compact toric manifolds has been considered both recently [23, 24] and in earlier works [25–27].

The second class of developments is related to [28], where Pestun placed an $\mathcal{N} = 2$ gauge theory on a round four-sphere preserving all eight supercharges. He was able to show that the partition function of this theory and certain supersymmetric Wilson loop observables, can be computed exactly using localization techniques. His work was later generalized to squashed four-spheres in [29, 30]. These advances led to the realization that localization techniques are a powerful tool to derive exact results in supersymmetric field theories in different dimensions (for a summary of these developments see for instance the review [31]).

The localization result for the partition function of Pestun's theory on the four-sphere is fairly simple. It consists of an integral over a real parameter of the product of two factors, each of which can be associated by equivariant localization to one of the poles of the sphere. These are the fixed points of a U(1) action generated by squaring the supercharge used for localization. One factor includes a Nekrasov partition function for instantons, the other a partition function for anti-instantons. Hence, the final result is structurally similar to the form of the partition functions stemming from equivariant topological twisting, which only involve Nekrasov partition functions for instantons. This relation was explained in [1] via the construction of a general class of $\mathcal{N} = 2$ supersymmetric Yang-Mills theories, on any four-manifold that possesses a Killing vector field with isolated fixed points. Following [1], we call *plus fixed points* those where instantons contribute to the partition function, while we call *minus fixed points* those where anti-instantons contribute. Thus, Pestun's theory on the four-sphere and the equivariant topological twist are both examples of this general construction. All the theories considered in [1] can be written in terms of cohomological (twisted) fields which helps elucidating what data, of geometrical or other origin, their supersymmetric observables can depend on. These results were formulated in a more rigorous mathematical framework in [32].

Summary of results. In the present work we extend the results of [1, 32] to theories involving hypermultiplets. The inclusion of hypermultiplets generically requires the manifold to admit a spin structure. Nevertheless, this substantially enlarges the set of supersymmetric field theories which can be analyzed using localization techniques. We reformulate the theories we construct in terms of cohomological (twisted) fields. A generalization of the notion of self-duality for two-forms was found necessary in [1] to define the cohomological SYM theory. Similarly here we find that a generalization of a Weyl spinor is necessary to define cohomological variables for the hypermultiplets. The use of such spinors is forced upon us by supersymmetry, thus establishing an interesting relation between supersymmetry and the geometry of the manifolds where our theories live. Furthermore, we employ localization techniques to compute the partition function of the cohomological theories we constructed. Computations of path integrals via a purely cohomological formulation of supersymmetry appeared for theories defined on specific manifolds in 3d [33–35], 4d [25, 32, 36–39], 5d [40–44], and 7d [45–47]. We refer to [31] and to references therein for an exhaustive bibliography. In this paper we aim to present a general formula for the partition function of $\mathcal{N} = 2$ gauge theories with matter on any simply connected Riemannian spin manifold admitting an isometry with isolated fixed points as well as non-trivial fluxes. Technically, the computation of the relevant one-loop contributions descends from the index of a transversally elliptic differential operator of Dirac-type. Combining the result of this paper with those of [1], we find

$$Z^{\mathcal{N}=2}_{\vec{\epsilon}_1,\vec{\epsilon}_2}(q,\overline{q}) = \sum_{k_i \text{ discrete}} \int_{\mathbf{h}} da_0 \, e^{-S_{\mathrm{cl}}} \prod_{i=1}^{p} Z^{\mathrm{inst}}_{\epsilon_1^{(i)},\epsilon_2^{(i)}}(a_0,k_i,q) Z^{\mathrm{VM}}_{\epsilon_1^{(i)},\epsilon_2^{(i)}}(a_0,k_i) Z^{\mathrm{HM}}_{\epsilon_1^{(i)},\epsilon_2^{(i)}}(a_0,k_i)$$

$$\times \prod_{i=p+1}^{l} Z^{\mathrm{anti\text{-}inst}}_{\epsilon_1^{(i)},\epsilon_2^{(i)}}(a_0,k_i,\overline{q}) \tilde{Z}^{\mathrm{VM}}_{\epsilon_1^{(i)},\epsilon_2^{(i)}}(a_0,k_i) \tilde{Z}^{\mathrm{HM}}_{\epsilon_1^{(i)},\epsilon_2^{(i)}}(a_0,k_i) \ , \tag{1.1}$$

where the formula above holds for a manifold with p plus points and $(l-p)$ minus fixed points. In (1.1), Z^{inst}, Z^{VM} and Z^{HM} respectively are the instantons contribution, the vector multiplet 1-loop determinant and the hypermultiplet 1-loop determinant at a plus fixed point. Analogously, $Z^{\mathrm{anti-inst}}$, $\tilde{Z}^{\mathrm{VM}}$ and $\tilde{Z}^{\mathrm{HM}}$ respectively are the anti-instantons contribution, the vector multiplet 1-loop determinant and the hypermultiplet 1-loop determinant at a minus fixed point. The integral is taken over the Cartan gauge subalgebra $\mathbf{h}$, while $\overline{q}, q$ are counting parameters labeling (anti-)instantons. The constants $\epsilon_{1,2}^{(i)}$ are real equivariant parameters. The 1-loop contributions $Z^{\mathrm{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)$ and $\tilde{Z}^{\mathrm{HM}}_{\epsilon_1^{(i)},\epsilon_2^{(i)}}(a_0,k_i)$ are Barnes double gamma functions [48]. With a specific choice of regularization (other choices are considered in section 5) we get for instance:

$$Z^{\mathrm{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+) = \prod_{\rho \in \mathcal{R}} \Gamma_2(i\,\rho(\Phi_0) + ((\epsilon_1^{(+)} + \epsilon_2^{(+)})/2)|\epsilon_1^{(+)},\epsilon_2^{(+)}) \ , \tag{1.2}$$

where we have a product over hypermultiplet representations $\mathcal{R}$, while the fugacity Φ_0 at a plus point is a combination of the Coulomb branch parameter a_0 and the function k_+ encoding the flux contribution:

$$\Phi_0 = a_0 + k_+(\epsilon_1^{(+)},\epsilon_2^{(+)}) \ . \tag{1.3}$$

As we spell out in the main text, the expression for $\tilde{Z}^{\mathrm{HM}}_{\epsilon_1^{(-)}, \epsilon_2^{(-)}}(a_0, k_-)$ and for Φ_0' valid at a minus fixed point are analogous. Hypermultiplet masses enter the partition function as a constant shift of a_0. At the level of the Lagrangian, a massless hypermultiplet is made massive by weakly gauging a U(1) flavour symmetry, where the mass coincides with the real part of a scalar from a background vector multiplet. It is important to stress that the integral in (1.1) is a formal expression. To concretely evaluate it several additional data need to be provided. Firstly, an integration contour has to be specified. One also needs to determine the precise form of the sum over flux sectors and the corresponding shifts in (1.3). These issues are interdependent and highly nontrivial. We do not investigate them in this work (for specific examples in the context of the equivariant topological twist on compact toric manifolds see [23, 24]).

Outline of the paper. In section 2, we review how to place an $\mathcal{N}=2$ field theory on Euclidean four-manifolds preserving some supersymmetry. This can be accomplished by coupling the theory to a supersymmetric rigid supergravity background, whose properties we review. In particular, we show that the usual hypermultiplet action, quadratic in derivatives, when coupled to rigid supergravity, is δ-exact up to total derivatives.

In section 3, we consider a Riemannian spin four-manifold $\mathcal{M}$ that admits a Killing vector field with isolated fixed points. We review the arguments in [1] showing that $\mathcal{M}$ admits globally well defined Killing spinors that depend on a choice of a plus/minus label for each fixed point. In order to deal with off-shell hypermultiplets, we also need to show the existence of auxiliary Killing spinors satisfying certain properties. We prove that a smooth, globally well-defined choice for these auxiliary Killing spinors exists.

In section 4, we move to reformulate the gauged hypermultiplet in terms of cohomological fields. For this we first define novel splits of the Dirac spinor bundle into two subbundles. These splits require the existence of a vector field with isolated fixed points and depend on the choice of $\pm$ at each fixed point. We also study the relation between these spinor bundles and the "flipping" sub-bundles of the bundle of two-forms studied in [1]. The appropriate cohomological fields for the hypermultiplet are elements of the novel spinor bundles we introduced. We explicitly prove that there is a smooth invertible map between cohomological variables and the usual hypermultiplet component fields. We also show how supersymmetry organizes the cohomological fields in different multiplets. Finally we rewrite the action for a gauged hypermultiplet in terms of the cohomological fields. In the reformulation of the theory in terms of cohomological variables, there is an important difference with [1], where only vector multiplets were considered. In the case of the hypermultiplet the twisted fields are spinors so that generically the four-manifold has to be spin. This is however too restrictive. For instance, depending on the flavor symmetry of the theory, it may be enough for the manifold to be spin^c.

Finally in section 5, having constructed the cohomological theory, we use it to set up the localization computation of the partition function. As an application, we apply our formula to the specific case of the squashed four-sphere. We find perfect agreement with the localization result for $\mathcal{N}=2$ matter multiplets found in [29].

Outlook. A compelling direction to be explored in a future work would be applying the technology developed in this paper to study the consequences of S-duality invariance of $\mathcal{N} = 4$ super Yang-Mills theory. Indeed, it should be possible to generalize the analysis of [49], valid for pure topological twisting, to the case of equivariant topological twisting by rearranging the twisted fields of vector and hypermultiplets in different cohomological complexes.

Another intriguing line of investigation would be deriving the cohomological complex for $\mathcal{N} = 4$ theories in three dimensions. Although this can simply be achieved by means of dimensional reduction, the outcome would be non-trivial as several fields of the three-dimensional cohomological complex would become charged under an SU(2) (Coulomb) R-symmetry. This would open up the possibility of more elaborate topological twisting, as well as of exploring three-dimensional mirror symmetry from a cohomological viewpoint.

2 $\mathcal{N} = 2$ theories on four-manifolds

In this section we review the construction of rigid $\mathcal{N} = 2$ supersymmetric fields theories on a curved four-manifold by coupling to background supergravity. Assuming the existence of appropriate Killing spinors, we write down the supersymmetry variations for fields in vector multiplets and hypermultiplets. We also present a δ-exact Lagrangian for a gauged hypermultiplet.

2.1 Review of $\mathcal{N} = 2$ rigid supergravity

In order to couple a supersymmetric field theory defined in flat space to off-shell supergravity, we have to set the fermionic fields in the supergravity multiplet to zero and freeze the bosonic supergravity fields to fixed values. If this background is invariant under some supergravity variation, the resulting theory is supersymmetric [50]. Here we consider $\mathcal{N} = 2$ theories with a conserved SU(2) R-current. Their supercurrent multiplet was studied by Sohnius [51] and the $\mathcal{N} = 2$ Poincaré supergravity to which they couple is described in [52–55]. Rigid $\mathcal{N} = 2$ supergravity backgrounds and the conditions they have to satisfy to preserve supersymmetry have been considered in [30, 56, 57].

In order to specify the supergravity background we need the following data:

- A Riemannian manifold $\mathcal{M}$ equipped with a metric g and a spin structure.[1]

- An SU(2)$_R$ connection $V_\mu{}^i{}_j$. (Here and in the following $i, j \ldots$ are SU(2)$_R$ indices.)

- Various other auxiliary fields: a one-form G_μ, a two-form $W_{\mu\nu}$, a scalar N, a closed two-form $F_{\mu\nu}$, and a scalar S_{ij} transforming as an SU(2)$_R$ triplet.

The supergravity variations are parametrized by a left-handed spinor ζ_α^i and a right-handed spinor $\bar{\chi}_i^{\dot\alpha}$, both transforming in the fundamental representation of the SU(2)$_R$ R-symmetry. Here i is the SU(2)$_R$ index and $\alpha, \dot\alpha$ are spinor indices. For a brief review of the conventions

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[1]We are going to comment on non-spin manifolds in section 4.2.2.

we use, see appendix A . Unless otherwise noted, we take these spinors to obey symplectic-Majorana reality conditions:

$$(\zeta_{i\alpha})^* = \zeta^{i\alpha}, \qquad (\bar\chi_i^{\dot\alpha})^* = \bar\chi_{\dot\alpha}^i \, . \tag{2.1}$$

Requiring the background to be invariant under the supergravity variation parametrized by ζ_α^i and $\bar\chi_i^{\dot\alpha}$, one obtains two sets of Killing spinor equations. The first set is

$$
\begin{aligned}
(D_\mu - iG_\mu)\zeta_i - \frac{i}{2}W_{\mu\rho}^+\sigma^\rho\bar\chi_i - \frac{i}{2}\sigma_\mu\bar\eta_i &= 0 \, , \\
(D_\mu + iG_\mu)\bar\chi^i + \frac{i}{2}W_{\mu\rho}^-\bar\sigma^\rho\zeta^i - \frac{i}{2}\bar\sigma_\mu\eta^i &= 0 \, ,
\end{aligned}
\tag{2.2}
$$

where D_μ is a covariant derivative that incorporates the $\mathrm{SU}(2)_R$ connection $V_\mu{}^i{}_j$. The second set is

$$
\begin{aligned}
\left(N - \frac{1}{6}R\right)\bar\chi^i &= 4i\partial_\mu G_\nu\bar\sigma^{\mu\nu}\bar\chi^i + i\big(\nabla^\mu + 2iG^\mu\big)W_{\mu\nu}^-\bar\sigma^\nu\zeta^i + i\bar\sigma^\mu\big(D_\mu + iG_\mu\big)\eta^i \, , \\
\left(N - \frac{1}{6}R\right)\zeta_i &= -4i\partial_\mu G_\nu\bar\sigma^{\mu\nu}\zeta_i - i\big(\nabla^\mu - 2iG^\mu\big)W_{\mu\nu}^+\sigma^\nu\bar\chi_i + i\sigma^\mu\big(D_\mu - iG_\mu\big)\bar\eta_i \, ,
\end{aligned}
\tag{2.3}
$$

where R is the Ricci scalar and the spinors η^i and $\bar\eta^i$ are defined as:

$$
\begin{aligned}
\eta_i &= (\mathcal{F}^+ - W^+)\zeta_i - 2G_\mu\sigma^\mu\bar\chi_i - S_{ij}\zeta^j \, , \\
\bar\eta^i &= -(\mathcal{F}^- - W^-)\bar\chi^i + 2G_\mu\bar\sigma^\mu\zeta^i - S^{ij}\bar\chi_j \, .
\end{aligned}
\tag{2.4}
$$

Here we use the notation $W^+ = \frac{1}{2}W_{\mu\nu}\sigma^{\mu\nu}$ and $W^- = \frac{1}{2}W_{\mu\nu}\bar\sigma^{\mu\nu}$ (similarly for $\mathcal{F}$).

The spinors ζ^i and $\bar\chi_i$ can be used to construct various bilinears. Restricting our attention to singlets of the $\mathrm{SU}(2)$ R-symmetry, we have the scalars

$$s = 2\zeta^i\zeta_i, \qquad \tilde{s} = 2\bar\chi^i\bar\chi_i, \tag{2.5}$$

and the vector field

$$v^\mu = 2\bar\chi^i\bar\sigma^\mu\zeta_i \, . \tag{2.6}$$

The reality conditions (2.1) imply that v^μ, s and $\tilde{s}$ are real and that s and $\tilde{s}$ are nowhere negative. The vector v and the scalars s, $\tilde{s}$ satisfy $||v||^2 = s\tilde{s}$, hence they are not independent. Using the Killing spinor equations we can show that v^μ is a Killing vector and that s, $\tilde{s}$ are constant along the orbits of v. We assume that both $\sqrt{s}$ and $\sqrt{\tilde{s}}$ are smooth in a neighborhood of the fixed points.

2.2 Supersymmetric multiplets

Here we assume that the Killing spinor equations introduced in the previous subsection are satisfied in some supergravity background. We present the structure of the supersymmetry variations for vector multiplets and hypermultiplets coupled to this background.

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2.2.1 Vector multiplet

The $\mathcal{N} = 2$ vector multiplet contains a complex scalar field X, a gauge field A_μ, two gauginos $\lambda_{i\alpha}$ and $\tilde{\lambda}^i_{\dot\alpha}$ that transform in the fundamental of $SU(2)_R$ and an auxiliary scalar field D_{ij} transforming as a triplet of $SU(2)_R$. All these fields (except A_μ) transform in the adjoint representation of the gauge group. The supersymmetry variations are given by:

$$\delta\bar{X} = \bar{\chi}^i\bar{\lambda}_i\,, \qquad \delta X = -\zeta_i\lambda^i\,,$$

$$\delta A_\mu = i\zeta_i\sigma_\mu\bar{\lambda}^i + i\bar{\chi}^i\bar{\sigma}_\mu\lambda_i\,,$$

$$\delta D_{ij} = i\zeta_i\sigma^\mu(D_\mu + iG_\mu)\bar{\lambda}_j - i\bar{\chi}_i\bar{\sigma}^\mu(D_\mu - iG_\mu)\lambda_j + 2i[X,\bar{\chi}_i\bar{\lambda}_j] + 2i[\bar{X},\zeta_i\lambda_j] + (i \leftrightarrow j)\,,$$

$$\delta\lambda_i = -2i(D_\mu - 2iG_\mu)X\sigma^\mu\bar{\chi}_i + 2(F^+ - \bar{X}\,W^+)\zeta_i + D_{ij}\zeta^j + 2i[X,\bar{X}]\zeta_i - 2X\eta_i\,,$$

$$\delta\bar{\lambda}^i = 2i(D_\mu + 2iG_\mu)\bar{X}\bar{\sigma}^\mu\zeta^i + 2(F^- - X\,W^-)\bar{\chi}^i - D^{ij}\bar{\chi}_j - 2i[X,\bar{X}]\bar{\chi}^i + 2\bar{X}\bar{\eta}^i\,. \quad (2.7)$$

We used the shorthand notation $F^+ = \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}$ and $F^- = \frac{1}{2}F_{\mu\nu}\bar{\sigma}^{\mu\nu}$ where $F_{\mu\nu}$ is the field strength for the gauge field A_μ.

The square supersymmetry variation of a field Ψ in the vector multiplet results in a translation along the vector field v defined in (2.6), together with a gauge transformation and an $SU(2)_R$ transformation

$$\delta^2\Psi = i\mathcal{L}_v\Psi + iv^\mu V_\mu \circ \Psi + i\Lambda^{(R)}\circ\Psi - i[\Phi,\Psi]\,. \quad (2.8)$$

Here $\mathcal{L}_v$ is the Lie derivative along v, and $\circ$ denotes that Ψ is acted upon according to which $SU(2)_R$ representation it belongs. The gauge transformation parameter is

$$\Phi = iv^\mu A_\mu + s\bar{X} + \tilde{s}X\,, \quad (2.9)$$

and $\Lambda^{(R)}$ is a $SU(2)_R$ transformation parameter given by:

$$\Lambda^{(R)}_{ij} = \bar{\chi}_i\bar{\sigma}^\mu(D_\mu - iG_\mu)\zeta_j - \zeta_i\sigma^\mu(D_\mu + iG_\mu)\bar{\chi}_j + (i \leftrightarrow j)\,. \quad (2.10)$$

2.2.2 Hypermultiplet

For the following we will embed the gauge group in $Sp(k)$ and consider a hypermultiplet in the fundamental of this $Sp(k)$. The hypermultiplet contains a scalar q_{ni} (where the index n transforms under $Sp(k)$ and runs over $n = 1,\ldots,2k$, while the index i transforms under the fundamental of $SU(2)_R$), and a pair of spinors $\psi_{\alpha n}$ and $\bar{\psi}_{\dot\alpha n}$. Additionally there are auxiliary fields $F_{n\check{i}}$ that are necessary for the off-shell closure of the supersymmetry algebra. The $F_{n\check{i}}$ transform in the fundamental of an $SU(2)_{\check{R}}$ symmetry that is generically distinct from the SU(2) R-symmetry.

We take the Grassmann-even fields in the hypermultiplet to satisfy the following reality conditions:

$$(q_{ni})^* = q^{ni}\,, \qquad (F_{n\check{i}})^* = F^{n\check{i}}\,. \quad (2.11)$$

The supersymmetry variations of the hypermultiplet components are:

$$\delta q_{ni} = \zeta_i \psi_n + \bar{\chi}_i \bar{\psi}_n \,,$$

$$\delta \psi_n = 2i(D_\mu q_{ni})\sigma^\mu \bar{\chi}_i + i q_{ni}\sigma^\mu \left(D_\mu + iG_\mu\right)\bar{\chi}_i + 4i\bar{X}_n{}^m q_{mi}\zeta^i + 2i F_{ni}\check{\zeta}^{\check{\imath}} \,,$$

$$\delta \bar{\psi}^n = 2i(D_\mu q^{ni})\bar{\sigma}^\mu \zeta_i + i q^{ni}\bar{\sigma}^\mu \left(D_\mu - iG_\mu\right)\zeta_i + 4i X^n{}_m q^{mi}\bar{\chi}_i + 2i F^{n\check{\imath}}\check{\bar{\chi}}_{\check{\imath}} \,, \qquad (2.12)$$

$$\delta F_{n\check{\imath}} = \check{\zeta}_{\check{\imath}}\left[\sigma^\mu\left(D_\mu - iG_\mu\right)\bar{\psi}_n - 2X_n{}^m\psi_m + 2(\lambda^j)_n{}^m q_{mj} - iW^+\psi_n\right]$$

$$\qquad\qquad + \check{\bar{\chi}}_{\check{\imath}}\left[\bar{\sigma}^\mu\left(D_\mu + iG_\mu\right)\psi_n + 2\bar{X}_n{}^m\bar{\psi}_m - 2(\bar{\lambda}^j)_n{}^m q_{mj} + iW^-\bar{\psi}_n\right]\,,$$

where $X_n{}^m = X^\alpha t^\alpha{}_n{}^m$ (and similarly for other vector multiplet components) and the derivative D_μ is covariant with respect both to the gauge symmetry and the R-symmetry. Finally, we introduced the checked spinors $\check{\zeta}_{\check{\imath}}$ and $\check{\bar{\chi}}_{\check{\imath}}$, which need to satisfy the constraints:

$$\zeta_i\check{\zeta}_{\check{\jmath}} - \bar{\chi}_i\check{\bar{\chi}}_{\check{\jmath}} = 0 \,, \qquad\qquad \check{\zeta}_{\check{\imath}}\check{\zeta}^{\check{\imath}} = \bar{\chi}^i\bar{\chi}_i \,,$$

$$\check{\bar{\chi}}^{\check{\imath}}\bar{\sigma}^\mu\check{\zeta}_{\check{\imath}} + \bar{\chi}^i\bar{\sigma}^\mu\zeta_i = 0 \,, \qquad\qquad \check{\bar{\chi}}_{\check{\imath}}\check{\bar{\chi}}^{\check{\imath}} = \zeta^i\zeta_i \,, \qquad (2.13)$$

for the off-shell closure of the supersymmetry algebra. Unless otherwise noted, we assume that $\check{\zeta}_{\check{\imath}}$ and $\check{\bar{\chi}}_{\check{\imath}}$ satisfy symplectic-Majorana reality conditions:

$$(\check{\zeta}^{\check{\imath}}_\alpha)^* = \check{\zeta}_{\check{\imath}}^\alpha \,, \qquad (\check{\bar{\chi}}^{\check{\imath}\dot\alpha})^* = \check{\bar{\chi}}_{\check{\imath}\dot\alpha} \,. \qquad (2.14)$$

Except when acting on the auxiliary fields $F_{n\check{\imath}}$, the square supersymmetry variation of a field Ψ of the hypermultiplet, is given by:

$$\delta^2\Psi = i\mathcal{L}_v\Psi + iv^\mu V_\mu \circ \Psi + i\Lambda^{(R)}\circ \Psi + \mathcal{G}_\Phi \diamond \Psi \,. \qquad (2.15)$$

This includes a translation along $v^\mu = 2\bar{\chi}^i\bar{\sigma}^\mu\zeta_i$, an $\mathrm{SU}(2)_R$ transformation and a gauge transformation. The gauge transformation parameter Φ and the $\mathrm{SU}(2)_R$ transformation parameter $\Lambda^{(R)}$ are as in (2.9) and (2.10). The squared supersymmetry variation of the auxiliary fields similarly includes $\mathrm{SU}(2)_{\check{R}}$ transformations:

$$\delta^2\Psi = i\mathcal{L}_v\Psi + iv^\mu\check{V}_\mu \circ \Psi + i\Lambda^{(\check{R})}\circ \Psi + \mathcal{G}_\Phi \diamond \Psi \,. \qquad (2.16)$$

Here $\check{V}_\mu$ is a background connection for $\mathrm{SU}(2)_{\check{R}}$ and the $\mathrm{SU}(2)_{\check{R}}$ transformation parameter $\Lambda^{(\check{R})}$ is given by:

$$\Lambda^{(\check{R})}_{\check{\imath}\check{\jmath}} = 2\check{\zeta}_{\check{\imath}}\sigma^\mu\left(\check{D}_\mu - iG_\mu\right)\check{\bar{\chi}}_{\check{\jmath}} + 2i\check{\zeta}_{\check{\imath}}W^+\check{\zeta}_{\check{\jmath}}$$

$$\qquad - 2\check{\bar{\chi}}_{\check{\imath}}\bar{\sigma}^\mu\left(\check{D}_\mu + iG_\mu\right)\check{\zeta}_{\check{\jmath}} + 2i\check{\bar{\chi}}_{\check{\imath}}W^-\check{\bar{\chi}}_{\check{\jmath}} + \left(\check{\imath} \leftrightarrow \check{\jmath}\right) \,. \qquad (2.17)$$

The derivative $\check{D}_\mu$ is covariant with respect to the background $\mathrm{SU}(2)_{\check{R}}$ connection $\check{V}_\mu$. Note that in (2.16), the connection $\check{V}_\mu$ cancels between the terms $iv^\mu\check{V}_\mu \circ \Psi$ and $i\Lambda^{(\check{R})}\circ \Psi$.

2.3 Hypermultiplet Lagrangian

Having completed the coupling to rigid supergravity, we can write a supersymmetric Lagrangian $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ for the hypermultiplet:

$$\mathcal{L}_B = +\frac{1}{2}(D^\mu q^{ni})(D_\mu q_{ni}) - \frac{i}{2}q^n{}_i(D^{ij})_n{}^m q_{mj} + \frac{1}{2}F^{n\check{i}}F_{n\check{i}}$$
$$- \left(\frac{R}{12} + \frac{N}{4}\right) q^{ni}q_{ni} + q^{ni}\{\bar{X}, X\}_n{}^m q_{mi}\,,\tag{2.18a}$$

$$\mathcal{L}_F = -\frac{i}{2}\psi^n\sigma^\mu\left(D_\mu - iG_\mu\right)\bar{\psi}_n + \frac{i}{2}\psi^n X_n{}^m\psi_m + \frac{i}{2}\bar{\psi}^n\bar{X}_n{}^m\bar{\psi}_m$$
$$- i\psi^n(\lambda^i)_n{}^m q_{mi} - i\bar{\psi}^n(\bar{\lambda}^i)_n{}^m q_{mi} - \frac{1}{4}\left(\psi^n W^+\psi_n + \bar{\psi}^n W^-\bar{\psi}_n\right)\,.\tag{2.18b}$$

It turns out that (up to total derivatives) this Lagrangian is itself the supersymmetry variation of some Grassmann-odd V_G:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F = \delta V_G\,,\tag{2.19}$$

where V_G is given by the following expression:

$$V_G = \frac{1}{2(s+\tilde{s})}\Big[2i(D_\mu + iG_\mu)(q_{ni}\zeta^i)\sigma^\mu\bar{\psi}^n - 2i(D_\mu - iG_\mu)(q_{ni}\bar{\chi}^i)\bar{\sigma}^\mu\psi^n$$
$$+ 2iF_{n\check{i}}(\check{\bar{\chi}}^{\check{i}}\bar{\psi}^n - \check{\zeta}^{\check{i}}\psi^n) - 4iq_{mi}(X^m{}_n\bar{\chi}^i\bar{\psi}^n + \bar{X}^m{}_n\zeta^i\psi^n)$$
$$- 2q_{ni}(\bar{\chi}^i W^-\bar{\psi}^n + \zeta^i W^+\psi^n) - \frac{2}{s+\tilde{s}}v^\nu\mathcal{F}_{\mu\nu}q_{ni}(\bar{\chi}^i\bar{\sigma}^\mu\psi^n - \zeta^i\sigma^\mu\bar{\psi}^n)$$
$$- 4iq^{ni}\left[(\lambda_i)_n{}^m\zeta_j + (\bar{\lambda}_i)_n{}^m\bar{\chi}_j\right]q_m{}^j\Big]\,.\tag{2.20}$$

The expression for V_G in (2.20) is in agreement with and generalizes a similar one found in [29], to which it reduces if we assume that $s + \tilde{s}$ is a constant (in that case $\mathcal{F}_{\mu\nu} = 0$).

3 Construction of Killing spinors

In the last section we have considered gauged hypermultiplets coupled to a rigid $\mathcal{N} = 2$ supergravity background. We assumed that this background preserves supersymmetry, i.e. that there are nonzero solutions to the Killing spinor equations (2.2) and (2.3). A general class of backgrounds allowing solutions to these equations was studied in [1]. Here we will briefly review their main properties.

Consider a Euclidean orientable four-manifold $\mathcal{M}$ with metric g and a spin structure. It was shown in [1] that the Killing equations can be satisfied by spinors ζ^i and $\bar{\chi}_i$ that are both non-vanishing provided that the metric admits a Killing vector v whose fixed points are isolated. In addition to v, the supergravity background is specified by a choice of a real, nowhere-negative scalar s on $\mathcal{M}$ that is constant along orbits of v. Moreover s has to approach a positive constant K at a subset of the fixed points of v and needs to go to zero as $||v||^2/K$ at the remaining fixed points. At these fixed points of v, the scalar $\tilde{s} = ||v||^2/s$ approaches K. Hence, the fixed points of v are separated into a set where $\tilde{s} = 0$ with $s = K$ and a second set where $s = 0$ and $\tilde{s} = K$. The fixed points satisfying $\tilde{s} = 0$ are plus fixed points, whereas those satisfying $s = 0$ are minus fixed points. This is consistent with [1].

3.1 Killing spinors

In this subsection, we employ the Killing vector v and the scalar s to construct spinors ζ^i_α and $\bar\chi^i_{\dot\alpha}$ that satisfy the reality conditions $(\zeta_{i\alpha})^* = \zeta^{i\alpha}$ and $(\bar\chi^{\dot\alpha}_i)^* = \bar\chi^i_{\dot\alpha}$ and such that

$$\zeta^i\zeta_i = \frac{s}{2}\,, \qquad \bar\chi^i\bar\chi_i = \frac{\tilde{s}}{2}\,, \qquad \bar\chi^i\bar\sigma^\mu\zeta_i = \frac{1}{2}v^\mu\,. \tag{3.1}$$

We cover the manifold with charts U_k, such that every fixed point of v belongs to a single distinct chart and we make a choice of vielbein $e^a{}_k$ in each chart. We can also assume that there are no overlaps between charts containing different fixed points of v. We consider the spinors ζ^i and $\bar\chi_i$ given by the following expressions in every chart:

$$\zeta^i_\alpha = \frac{\sqrt{s}}{2}\,\delta^i_\alpha\,, \qquad \bar\chi_i = \frac{1}{s}v^\mu\bar\sigma_\mu\zeta_i\,. \tag{3.2}$$

These spinors satisfy the reality conditions and the constraints (3.1).

In going from chart to chart, ζ transforms under $\mathrm{SU}(2)_l \times_{\mathbb{Z}_2} \mathrm{SU}(2)_R$. For the form of ζ above to be valid in each chart, we have to undo the $\mathrm{SU}(2)_l$ transformation by an appropriate $\mathrm{SU}(2)_R$ transformation. The expression for $\bar\chi$ will then also be valid in each chart because it is directly related to ζ.

Unfortunately (3.2) is singular in the charts containing a fixed point of v where $s = 0$. To fix this problem, in going from a chart where $s \neq 0$ everywhere to a chart where there is a fixed point of v with $s = 0$, we can act with a further $\mathrm{SU}(2)_R$ transformation:

$$U_i{}^j = i\frac{v^\mu}{||v||}\sigma_{\mu i}{}^j\,. \tag{3.3}$$

As a consequence, in charts containing a fixed point where $s = 0$, the spinors are:

$$\bar\chi^{\dot\alpha}_i = -i\frac{\sqrt{\tilde{s}}}{2}\delta^{\dot\alpha}_i\,, \qquad \zeta_i = -\frac{1}{\tilde{s}}v^\mu\sigma_\mu\bar\chi_i\,. \tag{3.4}$$

This specifies regular spinors ζ and $\bar\chi$ on $\mathcal{M}$ that satisfy the reality conditions and for which the relations (3.1) are satisfied. Moreover there is a choice of smooth background supergravity fields for which the spinors ζ and χ, that we just constructed, satisfy the Killing spinor equations (2.2)(2.3) (see [1]). The resulting expressions for the background fields are presented in appendix B.

3.2 Auxiliary Killing spinors

An additional element we need to consider is the construction of smooth auxiliary spinors $\check\zeta^{\check{i}}_\alpha$ and $\check{\bar\chi}^{\dot\alpha}_{\check{i}}$, since they are used for the off-shell extension of the supersymmetry transformations for the $\mathcal{N} = 2$ hypermultiplet. These spinors satisfy the reality conditions:

$$(\check\zeta^\alpha_{\check{i}})^* = \check\zeta^{\check{i}}_\alpha\,, \qquad (\check{\bar\chi}^{\check{i}\dot\alpha})^* = \check{\bar\chi}_{\check{i}\dot\alpha}\,, \tag{3.5}$$

as well as the following constraints:

$$\zeta_i\check\zeta_{\check{j}} - \bar\chi_i\check{\bar\chi}_{\check{j}} = 0\,, \qquad\qquad \check\zeta_{\check{i}}\check\zeta^{\check{i}} = \frac{\tilde{s}}{2}\,,$$

$$\check{\bar\chi}^{\check{i}}\bar\sigma^\mu\check\zeta_{\check{i}} = -\frac{1}{2}v^\mu\,, \qquad\qquad \check{\bar\chi}_{\check{i}}\check{\bar\chi}^{\check{i}} = \frac{s}{2}\,. \tag{3.6}$$

These spinors transform under $\mathrm{SU}(2)_l \times_{\mathbb{Z}_2} \mathrm{SU}(2)_{\check{R}}$ and $\mathrm{SU}(2)_r \times_{\mathbb{Z}_2} \mathrm{SU}(2)_{\check{R}}$ respectively. The $\mathrm{SU}(2)_{\check{R}}$ bundle is not generally identified with the $\mathrm{SU}(2)_R$ bundle. The constraints in (3.6) determine $\check{\zeta}$ and $\check{\chi}$ uniquely up to local $\mathrm{SU}(2)_{\check{R}}$ transformations.

The construction of these spinors parallels that of ζ and χ. On patches U_k that do not contain fixed points where $s = 0$, we define the spinors as follows:

$$\check{\bar{\chi}}_{\check{i}}^{\dot{\alpha}} = \frac{\sqrt{s}}{2}\delta_{\check{i}}^{\dot{\alpha}}, \qquad \check{\zeta}_{\check{i}\,\alpha} = -\frac{1}{s}v^\mu \left(\sigma_\mu \check{\bar{\chi}}_{\check{i}}\right)_\alpha .\tag{3.7}$$

It can be checked using the expressions for ζ^i and $\bar{\chi}_i$ in (3.2), that these satisfy the constraints (3.6). In this case in going from patch to patch, we have to undo $\mathrm{SU}(2)_r$ transformations by appropriate $\mathrm{SU}(2)_{\check{R}}$ transformations.

On a patch U_l that includes a fixed point of v with $s = 0$, we take instead:

$$\check{\zeta}_\alpha^{\check{i}} = i\frac{\sqrt{\tilde{s}}}{2}\delta_\alpha^{\check{i}}, \qquad \check{\bar{\chi}}_{\check{i}}^{\dot{\alpha}} = \frac{1}{\tilde{s}}v^\mu \left(\bar{\sigma}_\mu \check{\zeta}_{\check{i}}\right)^{\dot{\alpha}} ,\tag{3.8}$$

which also satisfy the constraints (3.6) with ζ^i and $\bar{\chi}_i$ in (3.4). As before, when transitioning to one of these patches, there is an extra $\mathrm{SU}(2)_{\check{R}}$ transformation given by:

$$U_{\check{i}}^{\check{j}} = i\frac{v^\mu}{||v||}(\sigma_\mu)_{\check{i}}^{\check{j}} .\tag{3.9}$$

Hence, we have found smooth solutions to the constraints (3.6). There is a different construction of solutions to these constraints that is often used in the literature. This is given by:

$$\check{\zeta}_\alpha^{\check{i}} = i\sqrt{\frac{\tilde{s}}{s}}\,\delta_i^{\check{i}}\,\zeta_\alpha^i, \qquad \check{\bar{\chi}}_{\check{i}}^{\dot{\alpha}} = i\sqrt{\frac{s}{\tilde{s}}}\,\delta_{\check{i}}^i\,\bar{\chi}_i^{\dot{\alpha}} .\tag{3.10}$$

In this case $\mathrm{SU}(2)_{\check{R}}$ is identified with $\mathrm{SU}(2)_R$; however the resulting spinors are singular at the fixed points of v.

4 Twisted supersymmetry

In this section we rewrite the component fields for the vector multiplet and hypermultiplet in terms of twisted variables. The case of the vector multiplet was considered in [1]. The construction of these twisted variables is intimately connected with the geometry of $\mathcal{M}$. In particular, for the vector multiplet it relies on a novel decomposition of the bundle of two-forms on $\mathcal{M}$ which we review. For the hypermultiplet we will introduce a corresponding decomposition of spinors on $\mathcal{M}$.

4.1 Flipping projectors

Consider the bundle of two-forms $\Lambda^2(\mathcal{M})$. On an orientable Euclidean manifold with metric g, we can split $\Lambda^2(\mathcal{M})$ in two orthogonal sub-bundles, consisting of self-dual and anti self-dual two-forms. Self-dual two-forms play an essential role in Donaldson-Witten theory and its equivariant extension [2, 13]. As shown in [1], on a manifold equipped with a vector field v with isolated fixed points, one can introduce a different splitting of the bundle of

two-forms. This splitting arises naturally in the study of the $\mathcal{N} = 2$ theories on $\mathcal{M}$ we presented in section 2.

Let v be a real vector field on $\mathcal{M}$ (with isolated fixed points) and for what follows we will take it to be the Killing vector. The s and $\tilde{s}$, which are the spinor bilinears in (2.5), determine the $\pm$ fixed points. We can define projectors $P_\pm$ on the bundle of two-forms as follows:

$$P_+ = \frac{1}{2(s^2 + \tilde{s}^2)} \left((s + \tilde{s})^2 \mathbb{1} + (s^2 - \tilde{s}^2) \star - 4\kappa \wedge \iota_v \right) , \qquad P_- = \mathbb{1} - P_+ , \qquad (4.1)$$

where we use $\mathbb{1}$ to indicate the identity operator throughout the paper. Here, $\kappa = g(v)$ is the one-form dual to the Killing vector v. The objects $P_\pm$ are well defined projectors as they satisfy $(P_\pm)^2 = P_\pm$, $P_+ P_- = 0$ and $P_+ + P_- = \mathbb{1}$. Especially, P_+ induces a split of the bundle of two-forms into two orthogonal subbundles. The image of P_+ consists of two-forms that are self dual at the fixed points of v where $\tilde{s} = 0$, and anti-self dual at those fixed points where $s = 0$. We refer to P_+ as a *flipping* projector.

Projectors $Z_\pm, \tilde{Z}_\pm$ analogous to $P_\pm$ can be defined for spinors on $\mathcal{M}$. Consider a Dirac spinor Ψ on $\mathcal{M}$ (see appendix A for a summary of our conventions). Its left-handed and right-handed components are ψ_α and $\bar{\psi}^{\dot{\alpha}}$. Given two such spinors $\Psi_{1,2}$ we can define the SO(4) invariant product $\bar{\Psi}_2 \Psi_1 = \psi_2 \psi_1 + \bar{\psi}_2 \bar{\psi}_1$. The projectors $L = \frac{1}{2}(\mathbb{1} + \gamma_5)$ and $R = \frac{1}{2}(\mathbb{1} - \gamma_5)$ on the left- and right-handed components of Ψ are compatible with the product of two spinors, that is:

$$\bar{\Psi}_2 L \Psi_1 = \overline{L \Psi_2} \, \Psi_1 , \qquad \bar{\Psi}_2 R \Psi_1 = \overline{R \Psi_2} \, \Psi_1 . \qquad (4.2)$$

If a vector field v with isolated fixed points related to s and $\tilde{s}$ exists on $\mathcal{M}$, we can define a new projector acting on Dirac spinors

$$Z_+ = \frac{1}{2} \left(\mathbb{1} + \frac{s - \tilde{s}}{s + \tilde{s}} \gamma_5 - \frac{2}{s + \tilde{s}} v^\mu \gamma_5 \gamma_\mu \right) . \qquad (4.3)$$

We have $Z_+^2 = Z_+$ and $\bar{\Psi}_2 Z_+ \Psi_1 = \overline{Z_+ \Psi_2} \, \Psi_1$, the projector Z_+ is then compatible with the inner product. Starting from Z_+ we can find more projectors

$$Z_- = \mathbb{1} - Z_+ , \qquad \tilde{Z}_+ = \gamma_5 Z_+ \gamma_5 , \qquad \tilde{Z}_- = \mathbb{1} - \tilde{Z}_+ . \qquad (4.4)$$

The image of Z_+ (or $\tilde{Z}_+$) comprises spinors that are left-handed at the plus fixed points of v and right-handed at the minus fixed points. There is a direct relation between the Killing spinors ζ_i and $\bar{\chi}_i$ and the projectors we introduced. Indeed we can construct a Dirac spinor $\mathfrak{z}_i$

$$\mathfrak{z}_i = \begin{pmatrix} \zeta_i \\ \bar{\chi}_i \end{pmatrix} , \qquad (4.5)$$

which satisfies $Z_+ \mathfrak{z}_i = \mathfrak{z}_i$ as a consequence of (3.2) or (3.4). This is a first indication of the strict relation between supersymmetry and the flipping bundles constructed here. Similarly we construct a Dirac spinor $\check{\mathfrak{z}}_{\check{i}}$ out of the auxiliary spinors $\check{\zeta}_{\check{i}}$, $\check{\bar{\chi}}_{\check{i}}$:

$$\check{\mathfrak{z}}_{\check{i}} = \begin{pmatrix} \check{\zeta}_{\check{i}} \\ \check{\bar{\chi}}_{\check{i}} \end{pmatrix} , \qquad (4.6)$$

which, using (3.7), is seen to satisfy $\tilde{Z}_- \check{\mathfrak{z}}_{\tilde{\imath}} = \check{\mathfrak{z}}_{\tilde{\imath}}$. Let $\Psi_{1,2}$ be spinors such that $Z_+ \Psi_{1,2} = \Psi_{1,2}$. We can construct the two-form

$$\omega_{\mu\nu} = \bar{\Psi}_2 \gamma_{\mu\nu} \Psi_1 = \psi_2 \sigma_{\mu\nu} \psi_1 + \bar{\psi}_2 \bar{\sigma}_{\mu\nu} \bar{\psi}_1 \,, \tag{4.7}$$

which satisfies $P_+ \omega = \omega$ with P_+ defined above in (4.1). This establishes a relation between the bundle of two-forms in the image of P_+ and the spinor bundle in the image of Z_+. In the same way, starting from two spinors such that $\tilde{Z}_- \Psi_{1,2} = \Psi_{1,2}$ the two-form $\bar{\Psi}_2 \gamma_{\mu\nu} \Psi_1$ is in the image of $P_- = \mathbb{1} - P_+$ (similar relations can be found using Z_- or $\tilde{Z}_+$).

4.2 Cohomological fields

Here we briefly review the rewriting of the vector multiplet in terms of cohomological (or twisted) fields introduced in [1]. We then proceed to constructing the appropriate twisted fields for the case of the hypermultiplet.

4.2.1 Vector multiplet

The vector multiplet comprises a complex scalar X, the gauge field A, an auxiliary scalar D_{ij} and gauginos λ_i, $\bar{\lambda}_i$. There is an invertible map between these component fields and cohomological (twisted) fields. We present the details of the map in appendix C. The twisted fields arrange themselves into various multiplets. There is one long multiplet made of the gauge field A, a scalar ϕ, and a Grassmann one-form Ψ both in the adjoint of the gauge group. Supersymmetry acts on these fields as follows:

$$\delta\phi = \iota_v \Psi, \quad \delta\Psi = \iota_v F + i d_A \phi, \quad \delta A = i \Psi \,. \tag{4.8}$$

Here F is the field strength of A and ι_v denotes contraction with the vector field v.

The rest of the twisted fields arrange in two short multiplets. One is formed by a scalar φ and a Grassmann scalar η and the second is formed by a Grassmann two-form χ satisfying $P_+ \chi = \chi$ and a second two-form H also satisfying $P_+ H = H$. All these fields are in the adjoint of the gauge group. Supersymmetry acts as follows:

$$
\begin{aligned}
\delta\varphi &= i\eta \,, & \delta\eta &= \iota_v d_A \varphi - [\phi, \varphi] \,, \\
\delta\chi &= H \,, & \delta H &= i\mathcal{L}_v^A \chi - i[\phi, \chi] \,.
\end{aligned} \tag{4.9}
$$

By construction, the twisted fields above do not transform under $SU(2)_R$. The forms χ and H are well defined only on orientable manifolds, while the latter does not need to be spin as the vector multiplet cohomological complex does not contain spinors.

With canonical reality conditions on the scalars $X^* = \bar{X}$, the field φ is real. On the other hand, the reality properties of ϕ involve a non-trivial shift depending on $(s, \tilde{s}, \varphi)$:

$$\phi^* = \phi + i(s - \tilde{s})\varphi \,. \tag{4.10}$$

4.2.2 Hypermultiplet

The hypermultiplet cohomological fields are fermions $\mathfrak{q}, \mathfrak{b}, \mathfrak{c}, \mathfrak{h}$ transforming in the fundamental of $Sp(k)$. The fields $\mathfrak{q}$ and $\mathfrak{h}$ are Grassmann-even, while $\mathfrak{b}$ and $\mathfrak{c}$ are Grassmann-odd.

Moreover $\mathfrak{q}$ and $\mathfrak{c}$ are in the image of the projector Z_+ defined in (4.3), while $\mathfrak{b}$ and $\mathfrak{h}$ are in the image of $\tilde{Z}_-$ defined in (4.4):

$$Z_+\mathfrak{q} = \mathfrak{q}\,, \qquad Z_+\mathfrak{c} = \mathfrak{c}\,, \qquad \tilde{Z}_-\mathfrak{b} = \mathfrak{b}\,, \qquad \tilde{Z}_-\mathfrak{h} = \mathfrak{h}\,. \tag{4.11}$$

The fields $\mathfrak{q}$ and $\mathfrak{c}$ are related to the component fields of section 2.3 by the following map:

$$\mathfrak{q}_n = \mathfrak{z}^i q_{ni} = \begin{pmatrix} \zeta^i q_{ni} \\ \bar{\chi}^i q_{ni} \end{pmatrix} \quad \text{and} \quad \mathfrak{c}_n = -\frac{s+\tilde{s}}{4} Z_+ \begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} s\psi_n - v^\mu \sigma_\mu \bar{\psi}_n \\ \tilde{s}\bar{\psi}_n + v^\mu \bar{\sigma}_\mu \psi_n \end{pmatrix}\,, \tag{4.12}$$

where we show explicitly their left- and right-handed components. The index n transforms in the fundamental of $\mathrm{Sp}(k)$.

There are two remaining cohomological fields. The first $\mathfrak{b}_n$ is given by:

$$\mathfrak{b}_n = \frac{s+\tilde{s}}{4}\tilde{Z}_-\gamma_5 \begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \tilde{s}\psi_n + v^\mu \sigma_\mu \bar{\psi}_n \\ -s\bar{\psi}_n + v^\mu \bar{\sigma}_\mu \psi_n \end{pmatrix}\,. \tag{4.13}$$

The last field $\mathfrak{h}$ is related to the variation of $\mathfrak{b}$ under supersymmetry $\mathfrak{h} = -i\,\delta\mathfrak{b}$:

$$\mathfrak{h}_n = \frac{s+\tilde{s}}{2}\mathfrak{z}^{\check{\imath}} F_{n\check{\imath}} + \tilde{Z}_- \left(\frac{s+\tilde{s}}{2}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{q}_n + iv^\mu G_\mu \mathfrak{q}_n - i\frac{(s+\tilde{s})}{2}\varphi_n{}^m \mathfrak{q}_m \right)\,. \tag{4.14}$$

In this formula T is a combination of supergravity background fields and derivatives of Killing spinor bilinears,

$$T_\mu = \frac{s-\tilde{s}}{(s+\tilde{s})}G_\mu + \frac{s\tilde{s}}{(s+\tilde{s})^2}b_\mu + i\frac{\partial_\mu(s^2+\tilde{s}^2)}{2(s+\tilde{s})^2}\,, \tag{4.15}$$

where b is a one-form satisfying $\iota_v b = 0$, which parametrizes remaining freedom in choosing the supergravity background (see appendix B).

The map from the hypermultiplet components to the cohomological variables has a smooth inverse. For the scalar q_{ni}, this is

$$q_{ni} = -\frac{4}{s+\tilde{s}}\bar{\mathfrak{z}}_i\,\mathfrak{q}_n\,, \tag{4.16}$$

where $\mathfrak{z}$ is the Dirac spinor built out of the Killing spinors defined in (4.5). The map from $\mathfrak{b}, \mathfrak{c}$ to the ordinary component fields $\psi, \bar{\psi}$ is

$$\begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = \frac{4}{s+\tilde{s}}(\gamma_5 \mathfrak{b}_n - \mathfrak{c}_n)\,, \tag{4.17}$$

and the one for the ordinary auxiliary field is

$$F_{n\check{\imath}} = \frac{8}{(s+\tilde{s})^2}\left[\bar{\bar{\mathfrak{z}}}_{\check{\imath}}\mathfrak{h}_n - \frac{s+\tilde{s}}{2}\bar{\bar{\mathfrak{z}}}_{\check{\imath}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{q}_n - iv^\mu G_\mu(\bar{\bar{\mathfrak{z}}}_{\check{\imath}}\mathfrak{q}_n) + i\frac{(s+\tilde{s})}{2}\varphi_n{}^m(\bar{\bar{\mathfrak{z}}}_{\check{\imath}}\mathfrak{q}_m) \right]\,. \tag{4.18}$$

Equations (2.11) and (B.1) imply the following reality conditions on Grassmann-even spinors:

$$(\mathfrak{h}_n)^* = -\bar{\mathfrak{h}}^n - [(s+\tilde{s})(D_\mu + iT_\mu)\bar{\mathfrak{q}}^n\gamma^\mu - 2iv^\mu G_\mu\bar{\mathfrak{q}}^n - i(s+\tilde{s})\bar{\mathfrak{q}}^m\varphi_m{}^n]\,\tilde{Z}_- \ ,$$
$$(\mathfrak{q}_n)^* = \bar{\mathfrak{q}}^n. \tag{4.19}$$

The twisted fields for the hypermultiplet we defined above are spinors; consequently, a spin structure is necessary in order to define them. This is different from the case of the vector multiplet whose twisted fields can be defined provided the manifold is orientable and admits a U(1) action. However, there are cases where the requirement of the manifold to be spin may be relaxed. For instance, if the theory under consideration has a U(1) flavor symmetry, the twisted fields are sections of the product of the spin bundle and powers of a unitary line bundle L. If the manifold admits a spin^c structure, such products can be well defined even if L and the spin bundle do not exist. Depending on the charges under the flavor symmetry, it may be then possible to define the twisted theory on a spin^c manifold (in four dimensions any closed orientable four manifold is spin^c). The requirement that the manifold be spin may be relaxed in other circumstances and is therefore dependent on the specific theory under consideration.

The cohomological fields we introduced split into two separate multiplets under supersymmetry:

$$\delta\mathfrak{q} = \mathfrak{c} \ , \qquad \delta\mathfrak{c} = (i\mathcal{L}_v - \mathcal{G}_\Phi)\mathfrak{q} \ ,$$
$$\delta\mathfrak{b} = i\mathfrak{h} \ , \qquad \delta\mathfrak{h} = (\mathcal{L}_v + i\mathcal{G}_\Phi)\mathfrak{b} \ , \tag{4.20}$$

where, just like in (28) of [1], the supersymmetry algebra encodes gauge transformations $\mathcal{G}_\Phi$ with respect to the field $\Phi = (i\iota_v A + \phi)$. Here A and ϕ are fields in the twisted vector multiplet (see appendix C). As in section 2.3, the object $\mathcal{G}_\Phi$ acts on fields according to their representation, e.g.

$$(\mathcal{G}_\Phi\,\mathfrak{q})_n = i\Phi_n{}^m\mathfrak{q}_m = i\left[i(\iota_v A)_n{}^m + \phi_n{}^m\right]\mathfrak{q}_m \ . \tag{4.21}$$

Few comments are in order:

- It follows from (4.8) that $\mathcal{G}_\Phi$ is δ-closed, which ensures the closure of the algebra (4.20). Indeed, δ acts on the cohomological fields as an equivariant differential, namely:

$$\delta^2 = i\mathcal{L}_v - \mathcal{G}_\Phi \ . \tag{4.22}$$

- The action of supersymmetry commutes with the projectors $Z_\pm, \tilde{Z}_\pm$. This is because the vector field that enters in their definition is the same as the Killing vector that appears in the supersymmetry variations (in general it is sufficient for the two vector fields to commute).

- The gauge field A and the scalar field ϕ always appear in the combination Φ. When discussing localization we will see that, after gauge fixing, Φ will be a Coulomb branch modulus appearing in one-loop determinants. If Φ is part of a dynamical

vector multiplet, it will be integrated over. Φ could also arise from a background vector multiplet, in which case it will remain as a free parameter in supersymmetric observables. For instance it can be identified with a flavour fugacity.

4.3 Hypermultiplet Lagrangian in cohomological fields

We have shown in section 2.3 that the usual quadratic Lagrangian for a hypermultiplet coupled to a rigid $\mathcal{N} = 2$ supergravity background is the δ variation of V_G in (2.20). In terms of the cohomological fields $(\mathfrak{q}, \mathfrak{b}, \mathfrak{c}, \mathfrak{h})$, the deformation term V_G reads as follows:

$$
V_G = \frac{8}{(s+\tilde{s})^3} \left\{ i\bar{\mathfrak{c}}\mathcal{L}_v\mathfrak{q} + i\bar{\mathfrak{q}}\left[\phi + i(s-\tilde{s})\varphi\right]\mathfrak{c} - i(\partial_\mu v_\nu)\bar{\mathfrak{c}}\,\gamma^{\mu\nu}\mathfrak{q} \right.
$$
$$
- (s+\tilde{s})\left(G_\mu - \frac{s^2 - \tilde{s}^2}{64}b_\mu\right)\bar{\mathfrak{c}}\gamma^\mu\mathfrak{q} + i(s+\tilde{s})\bar{\mathfrak{b}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{q}
$$
$$
\left. - (s+\tilde{s})\bar{\mathfrak{b}}\,\varphi\,\mathfrak{q} - 2\iota_v G\bar{\mathfrak{b}}\mathfrak{q} - \frac{i}{4}(s+\tilde{s})^2\bar{\mathfrak{q}}\chi\mathfrak{q} - i\bar{\mathfrak{h}}\mathfrak{b} \right\} , \tag{4.23}
$$

where, in the expression above, $\mathrm{Sp}(k)$ gauge/flavor indices are contracted as $\overline{\Psi}_1\Psi_2 = (\overline{\Psi}_1)_m\Psi_2^m$. Here we used the short-hand notation:

$$
\chi = \frac{1}{2}\chi_{\mu\nu}\gamma^{\mu\nu} . \tag{4.24}
$$

By taking a δ variation of V_G, we obtain the Lagrangian for the hypermultiplet $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$:

$$
\mathcal{L}_B = \frac{8}{(s+\tilde{s})^3} \left\{ -\mathcal{L}_v\bar{\mathfrak{q}}\mathcal{L}_v\mathfrak{q} - D_\mu v_\nu\bar{\mathfrak{q}}\gamma^{\mu\nu}\mathcal{L}_v\mathfrak{q} - i(s-\tilde{s})\bar{\mathfrak{q}}\,\varphi\,\mathcal{L}_v\mathfrak{q} \right.
$$
$$
+ i(s+\tilde{s})\left(G_\mu - \frac{s^2 - \tilde{s}^2}{64}b_\mu\right)(\bar{\mathfrak{q}}\gamma^\mu\mathcal{L}_v\mathfrak{q} + \bar{\mathfrak{q}}\gamma^\mu\phi\,\mathfrak{q})
$$
$$
- \partial_\mu v_\nu\bar{\mathfrak{q}}\gamma^{\mu\nu}\phi\,\mathfrak{q} - \frac{1}{2}\bar{\mathfrak{q}}\left[\frac{i}{2}(s+\tilde{s})^2 H^+ + \{\phi, \phi\} + i(s-\tilde{s})\{\varphi, \phi\}\right]\mathfrak{q}
$$
$$
\left. - (s+\tilde{s})\bar{\mathfrak{h}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{q} - 2i\iota_v G\bar{\mathfrak{h}}\mathfrak{q} + i(s+\tilde{s})\bar{\mathfrak{q}}\,\varphi\,\mathfrak{h} + \bar{\mathfrak{h}}\mathfrak{h} \right\}, \tag{4.25}
$$

and

$$
\mathcal{L}_F = \frac{8}{(s+\tilde{s})^3} \left\{ -i\bar{\mathfrak{b}}\mathcal{L}_v\mathfrak{b} + i\bar{\mathfrak{b}}\phi\mathfrak{b} - i\bar{\mathfrak{c}}\mathcal{L}_v\mathfrak{c} - i(s+\tilde{s})\bar{\mathfrak{b}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{c} + 2\iota_v G\bar{\mathfrak{b}}\mathfrak{c} \right.
$$
$$
+ (s+\tilde{s})\left(G_\mu - \frac{s^2 - \tilde{s}^2}{64}b_\mu\right)\bar{\mathfrak{c}}\,\sigma^\mu\mathfrak{c} + i\bar{\mathfrak{c}}\left[\phi + i(s-\tilde{s})\varphi\right]\mathfrak{c} + i\partial_\mu v_\nu\bar{\mathfrak{c}}\,\sigma^{\mu\nu}\mathfrak{c}
$$
$$
+ (s+\tilde{s})\bar{\mathfrak{c}}\,\varphi\,\mathfrak{b} + i(s+\tilde{s})\bar{\mathfrak{q}}\gamma^\mu\Psi_\mu\mathfrak{b} + i(s+\tilde{s})\bar{\mathfrak{q}}\,\eta\,\mathfrak{b}
$$
$$
\left. + i\bar{\mathfrak{q}}\left[2\iota_v\Psi - (s-\tilde{s})\eta + \frac{1}{2}(s+\tilde{s})^2\chi\right]\mathfrak{c} \right\} . \tag{4.26}
$$

4.4 Non-equivariant topological twist

In our setup, pure topological twisting is recovered by turning off either $(\bar\chi_{i\dot\alpha}, \check\zeta_{i\alpha})$ or $(\zeta_{i\alpha}, \check\chi_{i\dot\alpha})$. In terms of Killing spinor bilinears, this corresponds to respectively setting either $\tilde s = 0$ and $s = $ constant or $\tilde s = $ constant and $s = 0$. The Killing vector v vanishes in both cases, yielding a non-equivariant differential δ satisfying $\delta^2 = 0$. In turn, the hypermultiplet cohomological complex fulfills $\delta\mathfrak{h} = \delta\mathfrak{c} = 0$, while Dirac spinors become pure Weyl spinors:

$$(\mathfrak{q}, \mathfrak{b}, \mathfrak{c}, \mathfrak{h})_{\tilde s=0} = (L\,\mathfrak{q}, R\,\mathfrak{b}, L\,\mathfrak{c}, R\,\mathfrak{h}) , \qquad (\mathfrak{q}, \mathfrak{b}, \mathfrak{c}, \mathfrak{h})_{s=0} = (R\,\mathfrak{q}, L\,\mathfrak{b}, R\,\mathfrak{c}, L\,\mathfrak{h}) . \tag{4.27}$$

Consequently, the deformation term V_G becomes

$$\begin{aligned}
\mathcal{V}_G &= V_G|_{\tilde s=0} = \frac{8}{s^2}\left[\bar{\mathfrak{b}}(i\gamma^\mu D_\mu - \varphi)\mathfrak{q} - \frac{i\,s}{4}\bar{\mathfrak{q}}\chi\mathfrak{q} - i\bar{\mathfrak{h}}\mathfrak{b}\right] , \\
\tilde{\mathcal{V}}_G &= V_G|_{s=0} = \frac{8}{\tilde s^2}\left[\bar{\mathfrak{b}}(i\gamma^\mu D_\mu - \varphi)\mathfrak{q} - \frac{i\,\tilde s}{4}\bar{\mathfrak{q}}\chi\mathfrak{q} - i\bar{\mathfrak{h}}\mathfrak{b}\right] .
\end{aligned} \tag{4.28}$$

The objects $\mathcal{V}_G$ and $\tilde{\mathcal{V}}_G$ are deformation terms reproducing the features of topologically twisted hypermultiplets, see for instance [58] and references therein.

5 Cohomological localization

In this section we perform supersymmetric localization in the cohomological language described previously. The localization argument is the standard one: let Z_{hyper} be the hypermultiplet partition function, which is a path integral with functional weights $e^{-S_{\text{hyper}}}$. We then make the replacement $S_{\text{hyper}} \to S_{\text{hyper}} + t\,S_{\text{loc}}$, with t being a real parameter and S_{loc} a δ-exact positive definite deformation action. The resulting partition function $Z(t)$ naively depends on t and is such that $Z(0) = Z_{\text{hyper}}$. In fact, $Z(t)$ is independent of t because both the path integral measure and the integrand are invariant under δ. This implies that $Z(\infty) = Z_{\text{hyper}}$, meaning that the latter can be computed exactly via saddle point method by using the integrand $e^{-S_{\text{loc}}}$. Consequently, Z_{hyper} is given by $e^{-S_{\text{loc}}}$ evaluated on the locus of BPS field configurations, times the corresponding 1-loop determinant. Actually, Z_{hyper} can elegantly be obtained from the index of a transversally elliptic operator, as we shall see. After gauge fixing, the supersymmetry transformation δ becomes an equivariant differential squaring to $\delta^2 = i\mathcal{L}_v - \mathcal{G}_{a_0}$, where a_0 is a constant valued in the Cartan subalgebra of the gauge group G. A priori, then, Z_{hyper} will be a function of a_0. If the manifold admits non-trivial fluxes with magnetic charges $\mathfrak{m}_i$, the value of the Coulomb branch parameter a_0 is suitably shifted by $\mathfrak{m}_i$, as in the case of $S^2 \times S^2$ [25].

5.1 BPS locus

On the BPS locus, the fermionic fields $\mathfrak{b}, \mathfrak{c}$ and their supersymmetric variations are vanishing. In particular, this yields $\mathfrak{h} = 0$ and

$$(i\mathcal{L}_v - \mathcal{G}_\Phi)\,\mathfrak{q} = 0 . \tag{5.1}$$

Since the field Φ is complex and the Killing vector v is real, by imposing reality conditions on the above equations we end up with a trivial BPS locus: $\mathfrak{q} = 0$.

5.2 One-loop determinant from index theorem

The deformation term we employ is

$$V_{\text{loc}} = \frac{1}{4\,\bar{\mathfrak{z}}^i\mathfrak{z}_i}(\delta\Psi_n)^*\Psi_n = \frac{8}{(s+\tilde{s})^3}[(\delta\mathfrak{b}_n)^*\mathfrak{b}_n + (\delta\mathfrak{c}_n)^*\mathfrak{c}_n]\ . \tag{5.2}$$

By construction, V_{loc} leads to a positive definite deformation Lagrangian $\mathcal{L}_{\text{loc}} = \delta V_{\text{loc}}$. Explicitly,

$$\begin{aligned}
V_{\text{loc}} = \frac{8}{(s+\tilde{s})^3}\Big\{ &-\overline{\mathfrak{b}}\delta\mathfrak{b} + \overline{\mathfrak{b}}\left[i(s+\tilde{s})\gamma^\mu(D_\mu + iT_\mu) - 2\iota_v G - g(s+\tilde{s})\varphi\right]\mathfrak{q} \\
&+ (\overline{\delta\mathfrak{q}})\left[i\mathcal{L}_v - i\Phi - i(2\phi + i(s-\tilde{s})\varphi)\right]\mathfrak{q}\Big\}\ .
\end{aligned} \tag{5.3}$$

The deformation term V_{loc} can be recast in the quadratic form

$$V_{\text{loc}} = \frac{8}{(s+\tilde{s})^3}\left(\overline{\delta\mathfrak{q}},\ \overline{\mathfrak{b}}\right)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} \mathfrak{q} \\ \delta\mathfrak{b} \end{pmatrix}\ , \tag{5.4}$$

with entries

$$\begin{aligned}
D_{00} &= i\mathcal{L}_v - i\Phi - i[2\phi + i(s-\tilde{s})\varphi]\ , & D_{01} &= 0\ , \\
D_{10} &= i(s+\tilde{s})\gamma^\mu(D_\mu + iT_\mu) - 2\iota_v G - g(s+\tilde{s})\varphi\ , & D_{11} &= -\mathbb{1}\ .
\end{aligned} \tag{5.5}$$

The operator D_{10} implicitly fulfills $D_{10} = \tilde{Z}_- D_{10} Z_+$ because it acts on $\mathfrak{q}$ and $\mathfrak{b}$, which are Dirac spinors satisfying the projection conditions $Z_+\mathfrak{q} = \mathfrak{q}$ and $\tilde{Z}_-\mathfrak{b} = \mathfrak{b}$. Furthermore, D_{10} is a transversally elliptic operator. Ellipticity of a differential operator $\mathscr{D}$ amounts to the invertibility of the corresponding symbol $\sigma[\mathscr{D}]$, where the latter is obtained by keeping only the highest derivative terms in $\mathscr{D}$ and making the substitution $\partial_\mu \to ip_\mu$. If $\sigma[\mathscr{D}]$ is invertible for any $p^\mu \neq 0$, then $\mathscr{D}$ is elliptic. For instance, the Dirac operator $\mathscr{D} = -i\gamma^\mu\partial_\mu$ has symbol $\sigma[\mathscr{D}] = \gamma^\mu p_\mu$, which is everywhere invertible in flat Euclidean space. As for D_{10}, its symbol is

$$\sigma[D_{10}] = \frac{8\,p_\mu}{(s+\tilde{s})^2}\,\tilde{Z}_-\gamma^\mu Z_+\ . \tag{5.6}$$

At plus fixed points we have

$$\sigma[D_{10}]|_{\tilde{s}=0,v=0} = \frac{8\,p_\mu}{s^2}\,R\gamma^\mu L \to \frac{8\,p_\mu}{s^2}\,\overline{\sigma}^\mu = \sigma\left[-\frac{8i}{s^2}\overline{\sigma}^\mu\partial_\mu\right]\ , \tag{5.7}$$

while at minus fixed points we find

$$\sigma[D_{10}]|_{s=0,v=0} = \frac{8\,p_\mu}{\tilde{s}^2}\,L\gamma^\mu R \to \frac{8\,p_\mu}{\tilde{s}^2}\,\sigma^\mu = \sigma\left[-\frac{8i}{\tilde{s}^2}\sigma^\mu\partial_\mu\right]\ . \tag{5.8}$$

Consequently, the symbol $\sigma[D_{10}]$ coincides with that of a chiral Dirac operator whenever $\tilde{s} = 0$ or $s = 0$, ensuring ellipticity of D_{10}. On the other hand,

$$\sigma[D_{10}]|_{s=\tilde{s}} = \frac{p_\mu}{2s^2}\left(1 - \frac{v_\nu}{s}\gamma_5\gamma^\nu\right)\gamma^\mu\left(1 - \frac{v_\rho}{s}\gamma_5\gamma^\rho\right) = \frac{4}{s^2}\,\tilde{Z}_-\gamma_5\gamma^{\mu\nu}p_\mu v_\nu\ , \tag{5.9}$$

implying that D_{10} is not elliptic in the patch where $s = \tilde{s}$, because $\sigma\left[D_{10}\right]|_{s=\tilde{s}} = 0$ for $p^\mu = v^\mu \neq 0$. The operator D_{10} is then transversally elliptic with respect to the Killing vector v, as the symbol $\sigma\left[D_{10}\right]$ is everywhere invertible for any $p^\mu \neq 0$ such that $p \cdot v = 0$.

The hypermultiplet contribution to the partition function is encoded into the index of the transversally elliptic operator D_{10} [28, 29]:

$$\text{ind}(D_{10})(t) = \sum_{x\,:\,\tilde{x}=x} \frac{\text{Tr}_{\mathfrak{q}} e^{-it\,\mathcal{H}} - \text{Tr}_{\mathfrak{b}} e^{-it\,\mathcal{H}}}{\det(1 - \partial\tilde{x}/\partial x)} \,, \qquad \mathcal{H} = \delta^2 = i\mathcal{L}_v - \mathcal{G}_\Phi \,, \qquad (5.10)$$

where $t \in \mathbb{R}$, while $\tilde{x}$ is the image of the coordinates x under the torus action induced by $\mathcal{H}$ and the sum is over the fixed points $\tilde{x} = x$. In a neighborhood of a plus fixed point, the metric becomes flat and the manifold can be parametrized by a pair of complex coordinates (z_1, z_2). In (5.10), bosons $\mathfrak{q}$ contributes to $\text{ind}(D_{10})(t)$ with a plus sign, whereas fermions $\mathfrak{b}$ contributes with a minus sign.

The Killing vector v in a neighborhood of $\tilde{s} = 0$ reads

$$v = i\epsilon_1^{(+)}(z_1\partial_{z_1} - \bar{z}_1\partial_{\bar{z}_1}) + i\epsilon_2^{(+)}(z_2\partial_{z_2} - \bar{z}_2\partial_{\bar{z}_2}) \,, \qquad (5.11)$$

with $\epsilon_1^{(+)}$ and $\epsilon_2^{(+)}$ being real parameters. Hence, $e^{-it\,\mathcal{H}}$ is a $\text{U}(1) \times \text{U}(1)$ action attaching to (z_1, z_2) phases q_i with $i = 1, 2$:

$$z_i \to \tilde{z}_i = q_i z_i \,, \qquad q_i = \exp(i\,\epsilon_i^{(+)}\,t) \,, \qquad i = 1, 2 \,. \qquad (5.12)$$

In a neighborhood of a plus point, the denominator entering the index formula (5.10) is

$$\det\left(1 - \frac{\partial\tilde{z}_i}{\partial z_j}\right) = (1 - q_1)(1 - \bar{q}_1)(1 - q_2)(1 - \bar{q}_2) \,, \qquad (5.13)$$

where $\bar{q}_i = q_i^{-1}$ is the complex conjugate/inverse of the phase q_i. We now need the action of $\mathcal{H}$ upon the spinors $\mathfrak{q}$ and $\mathfrak{b}$. We then embed $\text{U}(1) \times \text{U}(1)$ into $\text{SU}(2)_+ \times \text{SU}(2)_- \sim \text{Spin}(4)$ to see how spinors transform under $\mathcal{H}$. We define

$$\mathbf{z} = x_\mu\gamma^\mu = \begin{pmatrix} 0 & 0 & \bar{z}_2 & \bar{z}_1 \\ 0 & 0 & z_1 & -z_2 \\ -z_2 & -\bar{z}_1 & 0 & 0 \\ -z_1 & \bar{z}_2 & 0 & 0 \end{pmatrix} \,. \qquad (5.14)$$

Then, coordinates $\mathbf{z}$ and spinors $\Psi = \{\mathfrak{q}, \mathfrak{b}\}$ transform as

$$\mathbf{z} \to g\,\mathbf{z}\,g^{-1} \,, \qquad \Psi \to g^{-1}\Psi \,, \qquad g = \text{diag}\left(\sqrt{\bar{q}_1\bar{q}_2}, \sqrt{q_1 q_2}, \sqrt{\bar{q}_1 q_2}, \sqrt{q_1\bar{q}_2}\right) \,, \qquad (5.15)$$

with $g \in \text{SU}(2)_+ \times \text{SU}(2)_-$ being the spinor representation of the torus action $\mathcal{L}_v$. At a plus point, $\mathfrak{q}$ is left-handed and $\mathfrak{b}$ is right-handed: $\mathfrak{q} = L\,\mathfrak{q}$ and $\mathfrak{b} = R\,\mathfrak{b}$. Then, the action of $\mathcal{L}_v$ upon $\mathfrak{q}, \mathfrak{b}$ is

$$\mathfrak{q}_+ \to \sqrt{q_1 q_2}\,\mathfrak{q}_+ \,, \qquad \mathfrak{q}_- \to \sqrt{\bar{q}_1\bar{q}_2}\,\mathfrak{q}_- \,, \qquad \widetilde{\mathfrak{b}}^{\dot{+}} \to \sqrt{q_1\bar{q}_2}\,\widetilde{\mathfrak{b}}^{\dot{+}} \,, \qquad \widetilde{\mathfrak{b}}^{\dot{-}} \to \sqrt{\bar{q}_1 q_2}\,\widetilde{\mathfrak{b}}^{\dot{-}} \,. \qquad (5.16)$$

The operator $\mathcal{H}$ also contains $\mathcal{G}_\Phi$, which acts non-trivially on $\mathfrak{q}, \mathfrak{b}$. Thus, for a hypermultiplet in the representation $\mathcal{R}$ of the gauge group, the index formula (5.10) yields

$$\mathrm{ind}(D_{10})|_{\text{plus point}} = \frac{\sqrt{q_1 q_2}}{(1-q_1)(1-q_2)} \sum_{\rho \in \mathcal{R}} e^{-t\,\rho(\Phi_0)} \ . \tag{5.17}$$

This index translates into a functional determinant by expanding $\mathrm{ind}(D_{10})|_{\text{plus point}}$ in powers of q_1, q_2 and converting the corresponding series into an infinite product. The latter requires regularization, which is a delicate matter. For instance, in cases linked to five-dimensional manifolds, the regularization was established in [37, 42–44, 59]. Here, we will examine diverse regularizations.

We define:

$$\left[\frac{1}{1-q_i}\right]_+ = \sum_{n \geq 0} q_i^n \ , \qquad \left[\frac{1}{1-q_i}\right]_- = -\sum_{n \leq -1} q_i^n = -\sum_{n \geq 0} q_i^{-n-1} \ . \tag{5.18}$$

The difference between the two regularizations is

$$\left[\frac{1}{1-q_i}\right]_+ - \left[\frac{1}{1-q_i}\right]_- = \sum_{n \in \mathbb{Z}} q_i^n \ , \tag{5.19}$$

which becomes a periodic Dirac delta supported at $t = 0$ if $q_i = e^{it}$. Whenever both q_1 and q_2 appear, we write $[\ldots]_{\pm\pm}$, where the first (respectively, the second) subscript refers to the regularization of q_1 (q_2). Indeed, the index $\mathrm{ind}(D_{10})|_{\text{plus point}}$ depends on both q_1 and q_2, and we have four possible series expansions:

$$\begin{aligned}
\left[\mathrm{ind}(D_{10})|_{\text{plus point}}\right]_{++} &= +\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} q_1^{n_1 + \frac{1}{2}} q_2^{n_2 + \frac{1}{2}} e^{-t\,\rho(\Phi_0)} \ , \\
\left[\mathrm{ind}(D_{10})|_{\text{plus point}}\right]_{+-} &= -\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} q_1^{n_1 + \frac{1}{2}} q_2^{-n_2 - \frac{1}{2}} e^{-t\,\rho(\Phi_0)} \ , \\
\left[\mathrm{ind}(D_{10})|_{\text{plus point}}\right]_{-+} &= -\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} q_1^{-n_1 - \frac{1}{2}} q_2^{n_2 + \frac{1}{2}} e^{-t\,\rho(\Phi_0)} \ , \\
\left[\mathrm{ind}(D_{10})|_{\text{plus point}}\right]_{--} &= +\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} q_1^{-n_1 - \frac{1}{2}} q_2^{-n_2 - \frac{1}{2}} e^{-t\,\rho(\Phi_0)} \ .
\end{aligned} \tag{5.20}$$

Here, ρ is a sum over weights in the representation $\mathcal{R}$ and Φ_0 is

$$\Phi_0 = a_0 + k_+(\epsilon_1^{(+)}, \epsilon_2^{(+)}) \ , \tag{5.21}$$

with a_0 being a Coulomb branch moduli and $k_+(\epsilon_1^{(+)}, \epsilon_2^{(+)})$ parametrizing the flux contribution at the plus fixed point. As in [1, 29], the index $\mathrm{ind}(D_{10})|_{\text{plus point}}$ translates into a 1-loop determinant given by an infinite product. For each regularization, we have

$$\begin{aligned}
\left[Z^{\text{HM}}_{\epsilon_1^{(+)}, \epsilon_2^{(+)}}(a_0, k_+)\right]_{++} &= \prod_{\rho \in \mathcal{R}} \prod_{n_1, n_2 \in \mathbb{N}} \left[\epsilon_1^{(+)}\left(n_1 + \frac{1}{2}\right) + \epsilon_2^{(+)}\left(n_2 + \frac{1}{2}\right) + i\,\rho(\Phi_0)\right]^{-1} \ , \\
&= \prod_{\rho \in \mathcal{R}} \Gamma_2(i\,\rho(\Phi_0) + ((\epsilon_1^{(+)} + \epsilon_2^{(+)})/2)|\epsilon_1^{(+)}, \epsilon_2^{(+)}) \ ,
\end{aligned} \tag{5.22}$$

as well as

$$\left[Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\right]_{+-} = \prod_{\rho\in\mathcal{R}}\prod_{n_1,n_2\in\mathbb{N}}\left[\epsilon_1^{(+)}\left(n_1+\frac{1}{2}\right)-\epsilon_2^{(+)}\left(n_2+\frac{1}{2}\right)+i\,\rho(\Phi_0)\right]\,,$$

$$= \prod_{\rho\in\mathcal{R}}\left[\Gamma_2(i\,\rho(\Phi_0)+((\epsilon_1^{(+)}-\epsilon_2^{(+)})/2)|\epsilon_1^{(+)},-\epsilon_2^{(+)})\right]^{-1}\,,$$

$$\left[Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\right]_{-+} = \prod_{\rho\in\mathcal{R}}\prod_{n_1,n_2\in\mathbb{N}}\left[-\epsilon_1^{(+)}\left(n_1+\frac{1}{2}\right)+\epsilon_2^{(+)}\left(n_2+\frac{1}{2}\right)+i\,\rho(\Phi_0)\right]\,,$$

$$= \prod_{\rho\in\mathcal{R}}\left[\Gamma_2(i\,\rho(\Phi_0)+((-\epsilon_1^{(+)}+\epsilon_2^{(+)})/2)|-\epsilon_1^{(+)},\epsilon_2^{(+)})\right]^{-1}\,,$$

$$\left[Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\right]_{--} = \prod_{\rho\in\mathcal{R}}\prod_{n_1,n_2\in\mathbb{N}}\left[-\epsilon_1^{(+)}\left(n_1+\frac{1}{2}\right)-\epsilon_2^{(+)}\left(n_2+\frac{1}{2}\right)+i\,\rho(\Phi_0)\right]^{-1}\,,$$

$$= \prod_{\rho\in\mathcal{R}}\left[\Gamma_2(i\,\rho(\Phi_0)-((\epsilon_1^{(+)}+\epsilon_2^{(+)})/2)|-\epsilon_1^{(+)},-\epsilon_2^{(+)})\right]\,. \tag{5.23}$$

These are 1-loop contributions to the hypermultiplet partition function at a plus fixed point. We used Barnes multiple zeta and gamma functions [48, 60],

$$\zeta_N(s,\omega|\vec{a}) = \sum_{\vec{n}\in\mathbb{N}^N}(\omega+\vec{a}\cdot\vec{m})^{-s}\,,$$

$$\Gamma_N(\omega|\vec{a}) = \prod_{\vec{n}\in\mathbb{N}^N}(\omega+\vec{a}\cdot\vec{m})^{-1} = e^{\partial_s\zeta_N(s,\omega|\vec{a})|_{s=0}}\,. \tag{5.24}$$

to regularize the infinite product that gives $Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)$. For instance, in the case of squashed S^4 worked out in [29], the regularization chosen for the plus point contribution is $++$:

$$Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+) \equiv \left[Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\right]_{++}\,. \tag{5.25}$$

Similarly, in a neighborhood of a minus fixed point the manifold can be parametrized by a pair of complex coordinates (z_1',z_2') and the Killing vector v in a neighborhood of $s=0$ is

$$v = i\epsilon_1^{(-)}(z_1'\partial_{z_1'}-\bar{z}_1'\partial_{\bar{z}_1'})+i\epsilon_2^{(-)}(z_2'\partial_{z_2'}-\bar{z}_2'\partial_{\bar{z}_2'})\,, \tag{5.26}$$

with $\epsilon_1^{(-)}$ and $\epsilon_2^{(-)}$ being real parameters. The $U(1)\times U(1)$ action of v upon (z_1',z_2') reads

$$z_i' \to \widetilde{z}_i' = q_i'z_i'\,, \qquad q_i' = \exp(i\,\epsilon_i^{(-)}t)\,, \qquad i=1,2\,. \tag{5.27}$$

In a neighborhood of a minus point, the denominator entering the index formula (5.10) is formally the same as the one computed at the plus point:

$$\det\left(1-\frac{\partial\widetilde{z}_i'}{\partial z_j'}\right) = (1-q_1')(1-\bar{q}_1')(1-q_2')(1-\bar{q}_2')\,, \tag{5.28}$$

At $s = 0$, there is a chirality flip with respect to the plus fixed point at $\tilde{s} = 0$. Indeed, $\mathfrak{q}$ is right-handed and $\mathfrak{b}$ is left-handed: $\mathfrak{q} = R\,\mathfrak{q}$ and $\mathfrak{b} = L\,\mathfrak{b}$. Therefore,

$$\mathfrak{b}_+ \to \sqrt{q_1' q_2'}\,\mathfrak{b}_+ \,, \quad \mathfrak{b}_- \to \sqrt{\bar{q}_1' \bar{q}_2'}\,\mathfrak{b}_- \,, \quad \tilde{\mathfrak{q}}^{\dagger} \to \sqrt{q_1' \bar{q}_2'}\,\tilde{\mathfrak{q}}^{\dagger} \,, \quad \widetilde{\mathfrak{q}}^{\cdot} \to \sqrt{\bar{q}_1' q_2'}\,\widetilde{\mathfrak{q}}^{\cdot} \,. \tag{5.29}$$

By taking into account the action of $\mathcal{G}_{\Phi'}$ at a minus fixed point, the index formula (5.10) provides

$$\mathrm{ind}(D_{10})|_{\text{minus point}} = -\frac{\sqrt{q_1' q_2'}}{(1 - q_1')(1 - q_2')} \sum_{\rho \in \mathcal{R}} e^{-t\,\rho(\Phi_0')} \,. \tag{5.30}$$

Again, we have four possible series expansions:

$$\begin{aligned}
\left[\mathrm{ind}(D_{10})|_{\text{minus point}}\right]_{++} &= -\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} (q_1')^{n_1 + \frac{1}{2}} (q_2')^{n_2 + \frac{1}{2}} e^{-t\,\rho(\Phi_0')} \,, \\
\left[\mathrm{ind}(D_{10})|_{\text{minus point}}\right]_{+-} &= +\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} (q_1')^{n_1 + \frac{1}{2}} (q_2')^{-n_2 - \frac{1}{2}} e^{-t\,\rho(\Phi_0')} \,, \\
\left[\mathrm{ind}(D_{10})|_{\text{minus point}}\right]_{-+} &= +\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} (q_1')^{-n_1 - \frac{1}{2}} (q_2')^{n_2 + \frac{1}{2}} e^{-t\,\rho(\Phi_0')} \,, \\
\left[\mathrm{ind}(D_{10})|_{\text{minus point}}\right]_{--} &= -\sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} (q_1')^{-n_1 - \frac{1}{2}} (q_2')^{-n_2 - \frac{1}{2}} e^{-t\,\rho(\Phi_0')} \,,
\end{aligned} \tag{5.31}$$

where Φ_0' is

$$\Phi_0' = a_0' + k_-(\epsilon_1^{(-)}, \epsilon_2^{(-)}) \,, \tag{5.32}$$

with $k_-(\epsilon_1^{(-)}, \epsilon_2^{(-)})$ encoding the flux contribution at the minus fixed point. We now translate the index $\mathrm{ind}(D_{10})|_{\text{minus point}}$ into 1-loop determinants:

$$\begin{aligned}
\left[Z^{\mathrm{HM}}_{\epsilon_1^{(-)}, \epsilon_2^{(-)}}(a_0', k_-)\right]_{++} &= \prod_{\rho \in \mathcal{R}} \prod_{n_1, n_2 \in \mathbb{N}} \left[\epsilon_1^{(-)}\left(n_1 + \frac{1}{2}\right) + \epsilon_2^{(-)}\left(n_2 + \frac{1}{2}\right) + i\,\rho(\Phi_0')\right] \,, \\
&= \prod_{\rho \in \mathcal{R}} \left[\Gamma_2(i\,\rho(\Phi_0') + ((\epsilon_1^{(-)} + \epsilon_2^{(-)})/2)|\epsilon_1^{(-)}, \epsilon_2^{(-)})\right]^{-1} \,, \\
\left[Z^{\mathrm{HM}}_{\epsilon_1^{(-)}, \epsilon_2^{(-)}}(a_0', k_-)\right]_{+-} &= \prod_{\rho \in \mathcal{R}} \prod_{n_1, n_2 \in \mathbb{N}} \left[\epsilon_1^{(-)}\left(n_1 + \frac{1}{2}\right) - \epsilon_2^{(-)}\left(n_2 + \frac{1}{2}\right) + i\,\rho(\Phi_0')\right]^{-1} \,, \\
&= \prod_{\rho \in \mathcal{R}} \left[\Gamma_2(i\,\rho(\Phi_0') + ((\epsilon_1^{(-)} - \epsilon_2^{(-)})/2)|\epsilon_1^{(-)}, -\epsilon_2^{(-)})\right] \,, \\
\left[Z^{\mathrm{HM}}_{\epsilon_1^{(-)}, \epsilon_2^{(-)}}(a_0', k_-)\right]_{-+} &= \prod_{\rho \in \mathcal{R}} \prod_{n_1, n_2 \in \mathbb{N}} \left[-\epsilon_1^{(-)}\left(n_1 + \frac{1}{2}\right) + \epsilon_2^{(-)}\left(n_2 + \frac{1}{2}\right) + i\,\rho(\Phi_0')\right]^{-1} \,, \\
&= \prod_{\rho \in \mathcal{R}} \left[\Gamma_2(i\,\rho(\Phi_0') + ((-\epsilon_1^{(-)} + \epsilon_2^{(-)})/2)| -\epsilon_1^{(-)}, \epsilon_2^{(-)})\right] \,, \\
\left[Z^{\mathrm{HM}}_{\epsilon_1^{(-)}, \epsilon_2^{(-)}}(a_0', k_-)\right]_{--} &= \prod_{\rho \in \mathcal{R}} \prod_{n_1, n_2 \in \mathbb{N}} \left[-\epsilon_1^{(-)}\left(n_1 + \frac{1}{2}\right) - \epsilon_2^{(-)}\left(n_2 + \frac{1}{2}\right) + i\,\rho(\Phi_0')\right] \,, \\
&= \prod_{\rho \in \mathcal{R}} \left[\Gamma_2(i\,\rho(\Phi_0') - ((\epsilon_1^{(-)} + \epsilon_2^{(-)})/2)| -\epsilon_1^{(-)}, -\epsilon_2^{(-)})\right]^{-1} \,.
\end{aligned} \tag{5.33}$$

These are 1-loop contribution to the hypermultiplet partition function at a minus fixed point. For example, in the case of the squashed four-sphere studied in [29], the regularization chosen for the minus point contribution is[2] $-+$:

$$\tilde{Z}^{\text{HM}}_{\epsilon_1^{(-)},\epsilon_2^{(-)}}(a_0', k_-) \equiv \left[Z^{\text{HM}}_{\epsilon_1^{(-)},\epsilon_2^{(-)}}(a_0', k_-) \right]_{-+} . \tag{5.34}$$

Using the results of [1], we can write down the complete partition function for an equivariantly twisted $\mathcal{N} = 2$ gauge theory coupled to matter with p (respectively $(l - p)$) plus (minus) fixed points, giving formula (1.1).

5.3 Example: hypermultiplets on squashed S^4

As an example, let us apply (1.1) to the specific case of hypermultiplets on a squashed four-sphere $S^4_{\ell,\tilde{\ell}}$ studied in [29]. This manifold possesses a plus and a minus fixed point, respectively dubbed north and south pole. In fact, $S^4_{\ell,\tilde{\ell}}$ is a four-dimensional ellipsoid embedded in $\mathbb{R}^5$ according to

$$\frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\widetilde{\ell^2}} + \frac{x_5^2}{r^2} = 1 , \tag{5.35}$$

where $\ell, \widetilde{\ell}$ are the lengths of the ellipsoid axis, while r is its radius. If $\ell = \widetilde{\ell} = r$, one recovers the round four-sphere S^4, whose isometry group is SO(5). For arbitrary $\ell, \widetilde{\ell}$, the group SO(5) is broken to SO(2) × SO(2), which is a real torus action rotating (x_1, x_2) and (x_3, x_4). At the fixed points, the equivariant parameters $\epsilon_1^{(i)}$ and $\epsilon_2^{(i)}$ are related to the lengths of the ellipsoid axis as follows:

	N	S
$\epsilon_1^{(i)}$	ℓ^{-1}	ℓ^{-1}
$\epsilon_2^{(i)}$	$\tilde{\ell}^{-1}$	$-\tilde{\ell}^{-1}$

Indeed, we can make contact with the previous subsection by setting $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$, so that at the north pole of $S^4_{\ell,\tilde{\ell}}$ the torus action becomes a complex torus action U(1) × U(1) generated by a Killing vector (5.11) with $\epsilon_1^{(+)} = \ell^{-1}$ and $\epsilon_2^{(+)} = \tilde{\ell}^{-1}$. On $S^4_{\ell,\tilde{\ell}}$ we have vanishing fluxes, then $k_+ = k_- = 0$, and for a hypermultiplet in the representation $\mathcal{R}$ of the gauge group we find

$$Z^{\text{HM}}_{S^4_{\ell,\tilde{\ell}}}(a_0)|_{\text{north pole}} = \left[Z^{\text{HM}}_{\ell^{-1},\tilde{\ell}^{-1}}(a_0, 0) \right]_{++} = \prod_{\rho \in \mathcal{R}} \Gamma_2(i\,\rho(a_0) + \frac{\ell^{-1} + \tilde{\ell}^{-1}}{2} | \ell^{-1}, \tilde{\ell}^{-1}) . \tag{5.36}$$

Analogously, at the south pole we have a U(1) × U(1) generated by a Killing vector (5.26) with $\epsilon_1^{(-)} = \ell^{-1}$ and $\epsilon_2^{(-)} = -\tilde{\ell}^{-1}$. Consequently,

$$Z^{\text{HM}}_{S^4_{\ell,\tilde{\ell}}}(a_0)|_{\text{south pole}} = \left[Z^{\text{HM}}_{\ell^{-1},-\tilde{\ell}^{-1}}(a_0, 0) \right]_{-+} = \prod_{\rho \in \mathcal{R}} \Gamma_2(i\,\rho(a_0) - \frac{\ell^{-1} + \tilde{\ell}^{-1}}{2} | -\ell^{-1}, -\tilde{\ell}^{-1}) . \tag{5.37}$$

---

[2]In the notation of [29], the regularization appears to be $--$, while in ours is $-+$, as reported in the main text.

Combining the two contributions we obtain a Υ-function depending on the Coulomb branch parameter a_0 and on the lengths of the ellipsoid axis $\ell, \tilde{\ell}$:

$$Z_{S^4_{\ell,\tilde{\ell}}}^{\mathrm{HM}}(a_0) = Z_{S^4_{\ell,\tilde{\ell}}}^{\mathrm{HM}}(a_0)|_{\text{north pole}} \times Z_{S^4_{\ell,\tilde{\ell}}}^{\mathrm{HM}}(a_0)|_{\text{south pole}} ,$$

$$= \prod_{\rho \in \mathcal{R}} \Upsilon_\beta \left(i\sqrt{\ell\tilde{\ell}}\,\rho(a_0) + \frac{1}{2}(\beta + \beta^{-1}) \right)^{-1} , \qquad \beta = \sqrt{\ell/\tilde{\ell}} , \qquad (5.38)$$

which matches [29]. We recall that the definition of the Υ-function is

$$\Upsilon_\beta(x) = \prod_{m,n \in \mathbb{N}} \left(m\beta + n\beta^{-1} + x \right) \left(m\beta + n\beta^{-1} + \beta + \beta^{-1} - x \right) . \qquad (5.39)$$

5.4 Example: hypermultiplets on $S^2 \times S^2$

As another example, we apply (1.1) to hypermultiplets defined on the product of two spheres of radii $\epsilon_1^{-1}, \epsilon_2^{-1}$, which we denote by $S^2_{\epsilon_1} \times S^2_{\epsilon_2}$. Gauge theories on manifolds with such a topology were studied e.g. in [25], as well as in [37] by dimensional reduction from five-dimensional toric Sasaki-Einstein manifolds. There are four fixed points on the manifold $S^2_{\epsilon_1} \times S^2_{\epsilon_2}$, corresponding to the four combinations NN, NS, SN, SS of north (N) and south (S) poles of the two spheres. Here, we consider NN and NS being plus fixed points, while SN and SS being minus fixed points. At the fixed points, the local equivariant parameters $\epsilon_1^{(i)}$ and $\epsilon_2^{(i)}$ are related to ϵ_1 and ϵ_2 as follows:

	NN	NS	SN	SS
$\epsilon_1^{(i)}$	ϵ_1	ϵ_1	$-\epsilon_1$	$-\epsilon_1$
$\epsilon_2^{(i)}$	ϵ_2	$-\epsilon_2$	ϵ_2	$-\epsilon_2$

Using a regularization consistent with [37], the hypermultiplet contribution at plus fixed points is

$$Z_{S^2_{\epsilon_1} \times S^2_{\epsilon_2}}^{\mathrm{HM}}(\Phi_0)|_{\mathrm{NN,\ NS}} = \left[Z_{\epsilon_1,\epsilon_2}^{\mathrm{HM}}(\Phi_0|_{\mathrm{NN}}) \right]_{++} \times \left[Z_{\epsilon_1,-\epsilon_2}^{\mathrm{HM}}(\Phi_0|_{\mathrm{NS}}) \right]_{+-} ,$$

$$= \prod_{\rho \in \mathcal{R}} \frac{\Gamma_2(i\,\rho(\Phi_0|_{\mathrm{NN}}) + \frac{\epsilon_1+\epsilon_2}{2}|\epsilon_1,\epsilon_2)}{\Gamma_2(i\,\rho(\Phi_0|_{\mathrm{SS}}) + \frac{\epsilon_1+\epsilon_2}{2}|\epsilon_1,\epsilon_2)} , \qquad (5.40)$$

while the hypermultiplet contribution at minus fixed points reads

$$\tilde{Z}_{S^2_{\epsilon_1} \times S^2_{\epsilon_2}}^{\mathrm{HM}}(\Phi_0')|_{\mathrm{SN,\ SS}} = \left[Z_{-\epsilon_1,\epsilon_2}^{\mathrm{HM}}(\Phi_0'|_{\mathrm{SN}}) \right]_{++} \times \left[Z_{-\epsilon_1,-\epsilon_2}^{\mathrm{HM}}(\Phi_0'|_{\mathrm{SS}}) \right]_{+-} ,$$

$$= \prod_{\rho \in \mathcal{R}} \frac{\Gamma_2(i\,\rho(\Phi_0'|_{\mathrm{SS}}) + \frac{-\epsilon_1+\epsilon_2}{2})|-\epsilon_1,\epsilon_2)}{\Gamma_2(i\,\rho(\Phi_0'|_{\mathrm{SN}}) + \frac{-\epsilon_1+\epsilon_2}{2}|-\epsilon_1,\epsilon_2)} . \qquad (5.41)$$

In absence of fluxes, $\Phi_0 = \Phi_0' = a_0$ at any fixed point and the partition function trivializes:

$$Z_{S^2_{\epsilon_1} \times S^2_{\epsilon_2}}^{\mathrm{HM}}(a_0) = 1 . \qquad (5.42)$$

Acknowledgments

A   Notation and conventions

Here we collect the relevant formulas used in the main text and a summary of our conventions. These are based on those of [61], adapted to Euclidean signature.

A.1   Flat Euclidean space and Dirac spinors

We define the Levi-Civita symbol as $\epsilon_{1234} = 1$. The rotation group is $\mathrm{SO}(4) \sim \mathrm{Spin}(4) = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. Left-handed spinors are $\mathrm{SU}(2)_+$ doublets and are denoted by undotted indices ζ_α. Right-handed spinors $\bar\zeta_{\dot\alpha}$ are doublets under $\mathrm{SU}(2)_-$ and carry a bar as well as dotted indices. In Euclidean signature, $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$ are not related by complex conjugation, hence ζ and $\bar\zeta$ are independent spinors. We raise and lower dotted and undotted indices by acting on the left with the tensors $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot\alpha\dot\beta}$, where $\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot1\dot2} = \epsilon_{\dot2\dot1} = +1$. For instance, $\zeta^\alpha = \epsilon^{\alpha\beta}\zeta_\beta$ and $\bar\zeta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\zeta_{\dot\beta}$. The $\mathrm{SU}(2)_+$ invariant inner product of ζ and η is $\zeta\eta = \zeta^\alpha\eta_\alpha$. The $\mathrm{SU}(2)_-$ invariant inner product of $\bar\zeta$ and $\bar\eta$ is given by $\bar\zeta\bar\eta = \bar\zeta_{\dot\alpha}\bar\eta^{\dot\alpha}$. We introduce the sigma matrices

$$\sigma^\mu_{\alpha\dot\alpha} = (\vec\sigma, -i\mathbb{1}), \qquad \bar\sigma^{\mu\dot\alpha\alpha} = (-\vec\sigma, -i\mathbb{1}), \tag{A.1}$$

where $\vec\sigma = (\sigma^1, \sigma^2, \sigma^3)$ is a vector whose components are Pauli matrices. Four-dimensional sigma matrices satisfy the reality conditions $(\sigma_\mu)^\dagger = -\bar\sigma_\mu$. Furthermore,

$$\sigma_\mu\bar\sigma_\nu + \sigma_\nu\bar\sigma_\mu = -2\delta_{\mu\nu}, \qquad \bar\sigma_\mu\sigma_\nu + \bar\sigma_\nu\sigma_\mu = -2\delta_{\mu\nu}. \tag{A.2}$$

We also define the matrices

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu), \qquad \bar\sigma_{\mu\nu} = \frac{1}{4}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu). \tag{A.3}$$

The latter fulfill self-duality (or anti self-duality) properties:

$$\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\sigma^{\rho\lambda} = \sigma_{\mu\nu}, \qquad \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar\sigma^{\rho\lambda} = -\bar\sigma_{\mu\nu}. \tag{A.4}$$

We can use these matrices to separate the $(2,0)$ and $(0,2)$ components of a two-form ω as follows:

$$\omega^+_{\alpha\beta} = \frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}, \qquad \omega^-_{\dot\alpha\dot\beta} = \frac{1}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}. \tag{A.5}$$

A Dirac spinor Ψ contains a left-handed Weyl spinor ψ_α and a right-handed one $\tilde{\psi}^{\dot\alpha}$. They are arranged as

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \tilde{\psi}^{\dot\alpha} \end{pmatrix} . \tag{A.6}$$

Correspondingly, the adjoint spinor $\overline{\Psi}$ is

$$\overline{\Psi} = \begin{pmatrix} \psi^\alpha & \tilde{\psi}_{\dot\alpha} \end{pmatrix} = (C\Psi)^T = -\Psi^T C , \tag{A.7}$$

where T denotes transposition and C is the (skew-symmetric) charge-conjugation matrix

$$C = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix} , \qquad\qquad C^{-1} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} . \tag{A.8}$$

In general, the adjoint $\overline{\Psi}$ of a spinor Ψ, is not related to the conjugate spinor Ψ^*. The Spin(4)-invariant product between two Grassmann-odd Dirac spinors Ψ_1, Ψ_2 reads

$$\overline{\Psi}_1 \Psi_2 = -\Psi_1^T C \Psi_2 = \psi_1^\alpha \psi_{2\alpha} + \tilde{\psi}_{1\dot\alpha} \tilde{\psi}_2^{\dot\alpha} . \tag{A.9}$$

The Clifford algebra is generated by Dirac matrices γ_μ in chiral representation, namely

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar\sigma_\mu & 0 \end{pmatrix} , \qquad \{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu} , \qquad C\gamma_\mu C^{-1} = -(\gamma_\mu)^T , \tag{A.10}$$

where $g_{\mu\nu}$ is the spacetime metric. The chirality matrix γ_5 and the Spin(4) generators $\gamma_{\mu\nu}$ are

$$\gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \qquad \gamma_{\mu\nu} = \frac{1}{4}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar\sigma_{\mu\nu} \end{pmatrix} . \tag{A.11}$$

In particular, $C\gamma_5 C^{-1} = (\gamma_5)^T$, while $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$ generate $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$ respectively. The matrix γ_5 is used to construct the standard chiral projectors

$$L = \frac{1}{2}\left(\mathbb{1} + \gamma_5\right) , \qquad\qquad R = \frac{1}{2}\left(\mathbb{1} - \gamma_5\right) . \tag{A.12}$$

Bilinears of Grassmann-odd spinors satisfy

$$\overline{\Psi}_1 \Psi_2 = \overline{\Psi}_2 \Psi_1 , \qquad \overline{\Psi}_1 \gamma_\mu \Psi_2 = -\overline{\Psi}_2 \gamma_\mu \Psi_1 , \qquad \overline{\Psi}_1 \gamma_5 \Psi_2 = \overline{\Psi}_2 \gamma_5 \Psi_1 . \tag{A.13}$$

Bilinears of Grassmann-even spinors fulfill the same identities with an additional minus sign on the right-hand side.

A.2 Differential geometry

Greek letters $\mu, \nu, \ldots$ are used to denote curved indices and Latin letters $a, b, \ldots$ to denote frame indices. Let $e^a{}_\mu$ be the orthonormal vielbein corresponding to the metric $g_{\mu\nu}$. We denote the Levi-Civita connection by ∇_μ. The corresponding spin connection is

$$\omega_{\mu a}{}^b = e^b{}_\nu \nabla_\mu e_a{}^\nu . \tag{A.14}$$

The Riemann tensor is

$$R_{\mu\nu a}{}^b = \partial_\mu \omega_{\nu a}{}^b - \partial_\nu \omega_{\mu a}{}^b + \omega_{\nu a}{}^c \omega_{\mu c}{}^b - \omega_{\mu a}{}^c \omega_{\nu c}{}^b \ . \tag{A.15}$$

The Ricci tensor is $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$, and $R = R_\mu{}^\mu$ is the Ricci scalar. In these conventions a round sphere has negative Ricci scalar.

The covariant derivative acts on spinors ζ and $\bar\zeta$ as

$$\nabla_\mu \zeta = \partial_\mu \zeta + \frac{1}{2}\omega_{\mu ab}\sigma^{ab}\zeta \,, \qquad \nabla_\mu \bar\zeta = \partial_\mu \bar\zeta + \frac{1}{2}\omega_{\mu ab}\bar\sigma^{ab}\bar\zeta \ . \tag{A.16}$$

The Lie derivative of spinors along a Killing vector v is given by [62]

$$\mathcal{L}_v \zeta = v^\mu \nabla_\mu \zeta - \frac{1}{2}(\partial_\mu v_\nu)\sigma^{\mu\nu}\zeta \,, \qquad \mathcal{L}_v \bar\zeta = v^\mu \nabla_\mu \bar\zeta - \frac{1}{2}(\partial_\mu v_\nu)\bar\sigma^{\mu\nu}\bar\zeta \ . \tag{A.17}$$

B Supergravity background solutions

In this appendix, we write down explicitly the supergravity background on which the Killing spinor equations are solved by the Killing spinor constructed in section 3.1. This is a summary of the results of [1].

The background supergravity fields are the metric $g_{\mu\nu}$, a $\mathrm{SU}(2)_R$ connection $V_\mu{}^i{}_j$, a one form G_μ, a two-form $W_{\mu\nu}$, a scalar N, a closed two-form $F_{\mu\nu}$, and a scalar S_{ij} transforming as an $\mathrm{SU}(2)_R$ triplet. The Killing spinors do not determine the supergravity background completely. In the formulas below, this freedom is parametrized by two one-forms. One is G_μ and is arbitrary, the other is denoted b_μ and satisfies $\imath_v b = 0$. The reality conditions for the Killing spinors (2.1) are compatible with the following behavior of the background supergravity fields under complex conjugation:

$$(V_\mu{}^i{}_j)^* = V_\mu{}^j{}_i\,, \quad N^* = N\,, \quad G_\mu^* = -G_\mu\,, \quad W_{\mu\nu}^* = -W_{\mu\nu}\,, \quad \mathcal{F}_{\mu\nu}^* = -\mathcal{F}_{\mu\nu}\,, \quad S_{ij}^* = S^{ij} \ . \tag{B.1}$$

We will make use of various spinor bilinears. Besides the $\mathrm{SU}(2)_R$ singlets $s, \tilde s$ and v defined in (2.5) and (2.6) and used extensively throughout the paper, we introduce

$$v_\mu^{(ij)} = \zeta^i \sigma_\mu \bar\chi^j + \zeta^j \sigma_\mu \bar\chi^i\,, \qquad \Theta_{\mu\nu}^{(ij)} = \zeta^i \sigma_{\mu\nu} \zeta^j\,, \qquad \widetilde\Theta_{\mu\nu}^{(ij)} = \bar\chi^i \bar\sigma^{\mu\nu} \bar\chi^j \ . \tag{B.2}$$

We also define the combination

$$\widehat\Theta_{\mu\nu}^{ij} = 2\frac{s+\tilde s}{s^2 + \tilde s^2}\left(\Theta_{\mu\nu}^{ij} + \widetilde\Theta_{\mu\nu}^{ij}\right) \ . \tag{B.3}$$

The two-form $W_{\mu\nu}$ and the $\mathrm{SU}(2)_R$ connection $(V_\mu)_{ij}$ are given by:

$$W_{\mu\nu} = \frac{i}{s+\tilde s}(\partial_\mu v_\nu - \partial_\nu v_\mu) - \frac{2i}{(s+\tilde s)^2}\epsilon_{\mu\nu\rho}{}^\lambda v^\rho \partial_\lambda(s-\tilde s) - \frac{4}{s+\tilde s}\epsilon_{\mu\nu\rho}{}^\lambda v^\rho G_\lambda$$
$$+ \frac{s-\tilde s}{(s+\tilde s)^2}\epsilon_{\mu\nu\rho}{}^\lambda v^\rho b_\lambda + \frac{1}{s+\tilde s}(v_\mu b_\nu - v_\nu b_\mu)\,, \tag{B.4}$$

$$(V_\mu)_{ij} = \frac{4}{s+\tilde s}\left(\zeta_{(i}\nabla_\mu\zeta_{j)} + \bar\chi_{(i}\nabla_\mu\bar\chi_{j)}\right) + \frac{4}{s+\tilde s}\left(2iG_\nu - \frac{\partial_\nu(s-\tilde s)}{(s+\tilde s)}\right)(\Theta_{ij} - \widetilde\Theta_{ij})^\nu{}_\mu$$
$$+ \frac{4i}{(s+\tilde s)^2}b_\nu(\tilde s\,\Theta_{ij} + s\,\widetilde\Theta_{ij})^\nu{}_\mu \ . \tag{B.5}$$

The graviphoton field strength $\mathcal{F}_{\mu\nu}$ is:

$$\mathcal{F}_{\mu\nu} = i\partial_\mu\left(\frac{s + \tilde{s} - K}{s\tilde{s}}v_\nu\right) - i\partial_\nu\left(\frac{s + \tilde{s} - K}{s\tilde{s}}v_\mu\right). \tag{B.6}$$

Here K is the constant defined in section 3 and such that $(s + \tilde{s} - K)/(s^2\tilde{s}^2)$ is smooth at the fixed points of v. The scalars S_{ij} are:

$$S_{ij} = 4i\frac{s^2 + \tilde{s}^2}{(s + \tilde{s})^3}\widehat{\Theta}_{ij}^{\mu\nu}\partial_\mu v_\nu - 2i\frac{s + \tilde{s} - K}{(s\,\tilde{s})^2}(\tilde{s}\,\Theta_{ij} + s\,\widetilde{\Theta}_{ij})^{\mu\nu}\partial_\mu v_\nu$$
$$+ \frac{2}{s + \tilde{s}}\left(4G_\mu - \frac{s - \tilde{s}}{s + \tilde{s}}b_\mu - \frac{i}{2}\frac{s - \tilde{s}}{s\tilde{s}}\partial_\mu(s + \tilde{s})\right)v_{ij}^\mu. \tag{B.7}$$

Finally, the combination $R/6 - N$ (where R is the Ricci scalar) is given by:

$$\left(\frac{R}{6} - N\right) = \frac{s - \tilde{s}}{(s + \tilde{s})^2}\Box(s - \tilde{s}) + \frac{\partial_{[\mu}v_{\nu]}\partial^{[\mu}v^{\nu]}}{(s + \tilde{s})^2} - \frac{1}{2}\frac{s - \tilde{s}}{(s + \tilde{s})^3}\epsilon^{\mu\nu\rho\lambda}(\partial_\mu v_\nu)(\partial_\rho v_\lambda)$$
$$- \frac{4\epsilon^{\mu\nu\rho\lambda}}{(s + \tilde{s})^3}v_\mu(\partial_\nu v_\rho)\partial_\lambda(s - \tilde{s}) + \frac{2s\tilde{s}}{(s + \tilde{s})^4}\partial_\mu(s - \tilde{s})\partial^\mu(s - \tilde{s})$$
$$- 2\frac{s - \tilde{s}}{(s + \tilde{s})^3}\partial_\mu(s - \tilde{s})\partial^\mu(s + \tilde{s}) - \frac{2is\tilde{s}}{(s + \tilde{s})^2}\nabla^\mu b_\mu + \frac{2(s\tilde{s})^2}{(s + \tilde{s})^4}b_\mu b^\mu$$
$$- i\frac{s - \tilde{s}}{(s + \tilde{s})^3}\epsilon^{\mu\nu\rho\lambda}(\partial_\mu v_\nu)v_\rho b_\lambda + 3i\frac{s - \tilde{s}}{(s + \tilde{s})^3}b^\mu(\tilde{s}\partial_\mu s - s\partial_\mu\tilde{s})$$
$$+ 2i\frac{s^2 + \tilde{s}^2}{(s + \tilde{s})^4}b^\mu\partial_\mu(s\tilde{s}) - 2i\frac{s - \tilde{s}}{s + \tilde{s}}\nabla^\mu G_\mu + 4\frac{s^2 + \tilde{s}^2}{(s + \tilde{s})^2}G^\mu G_\mu + \frac{8(v^\mu G_\mu)^2}{(s + \tilde{s})^2}$$
$$+ 4s\tilde{s}\frac{s - \tilde{s}}{(s + \tilde{s})^3}G_\mu b^\mu + \frac{4i\epsilon^{\mu\nu\rho\lambda}}{(s + \tilde{s})^2}(\partial_\mu v_\nu)v_\rho G_\lambda + 4i\frac{s - \tilde{s}}{(s + \tilde{s})^3}G^\mu\partial_\mu(s\tilde{s}). \tag{B.8}$$

C Cohomological variables for vector multiplet

The component fields in the $\mathcal{N} = 2$ vector multiplet are the complex scalar X, a gauge field A_μ, two gauginos $\lambda_{i\alpha}$ and $\tilde{\lambda}_{\dot{\alpha}}^i$ and an auxiliary scalar field D_{ij}. These can be recast in the following twisted fields (see [1]):

$$\eta = \zeta_i\lambda^i + \bar{\chi}^i\bar{\lambda}_i,$$

$$\varphi = -i(X - \bar{X}),$$

$$\Psi_\mu = \zeta_i\sigma_\mu\bar{\lambda}^i + \bar{\chi}^i\bar{\sigma}_\mu\lambda_i,$$

$$\phi = \tilde{s}X + s\bar{X},$$

$$\chi_{\mu\nu} = 2\frac{s + \tilde{s}}{s^2 + \tilde{s}^2}\left[\bar{\chi}^i\bar{\sigma}_{\mu\nu}\bar{\lambda}_i - \zeta_i\sigma_{\mu\nu}\lambda^i + \frac{1}{s + \tilde{s}}(v_\mu\Psi_\nu - v_\nu\Psi_\mu)\right], \tag{C.1}$$

$$H_{\mu\nu} = (P_+)_{\mu\nu}^{\rho\lambda}\left[\hat{\Theta}_{\rho\lambda}^{ij}D_{ij} - F_{\rho\lambda} + i\frac{X + \bar{X}}{s + \tilde{s}}(\partial_\rho v_\lambda - \partial_\lambda v_\rho) +\right.$$
$$\left. - \frac{2i}{s + \tilde{s}}\epsilon_{\rho\lambda\gamma}{}^\delta v^\gamma\left(\left(D_\delta - 2iG_\delta - i\frac{\tilde{s}}{s + \tilde{s}}b_\delta\right)X - \left(D_\delta + 2iG_\delta - i\frac{s}{s + \tilde{s}}b_\delta\right)\bar{X}\right)\right].$$

Here we used $\hat{\Theta}_{\rho\lambda}^{ij}$ defined in (B.3) and the projector P_+ defined in (4.1).

All the fields (C.1) are differential forms in the adjoint of the gauge group. They are all singlets under the $SU(2)_R$ symmetry. It follows from the definitions above that the two-forms $\chi_{\mu\nu}$ and $H_{\mu\nu}$ are in the image of the projector P_+.

The change of variables (C.1) has a smooth inverse,

$$
\begin{aligned}
X &= \frac{1}{s + \tilde{s}} \left(\phi + i\, s\, \varphi \right), \qquad \bar{X} = \frac{1}{s + \tilde{s}} \left(\phi - i\, \tilde{s}\, \varphi \right), \\
\bar{\lambda}_i &= \frac{1}{s + \tilde{s}} \left[\frac{s^2 + \tilde{s}^2}{s + \tilde{s}} \bar{\chi}^j \, \widehat{\Theta}_{ij}^{\mu\nu} \chi_{\mu\nu} + \bar{\sigma}^\mu \zeta_i \Psi_\mu + \bar{\chi}_i \eta \right], \\
\lambda_i &= \frac{1}{s + \tilde{s}} \left[\frac{s^2 + \tilde{s}^2}{s + \tilde{s}} \zeta^j \, \widehat{\Theta}_{ij}^{\mu\nu} \chi_{\mu\nu} - \sigma^\mu \bar{\chi}_i \Psi_\mu - \zeta_i \eta \right], \\
D_{ij} &= 4 \frac{s^2 + \tilde{s}^2}{(s + \tilde{s})^2} \hat{\Theta}_{ij}^{\mu\nu} (H_{\mu\nu} - \ldots),
\end{aligned}
\tag{C.2}
$$

where in the last formula the ellipsis stand for terms in $H_{\mu\nu}$ that are not proportional to D_{ij} (see (C.1)).

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References

[1] G. Festuccia, J. Qiu, J. Winding and M. Zabzine, *Twisting with a flip (the art of pestunization)*, *Commun. Math. Phys.* **377** (2020) 341 [`arXiv:1812.06473`] [INSPIRE].

[2] E. Witten, *Topological quantum field theory*, *Commun. Math. Phys.* **117** (1988) 353 [INSPIRE].

[3] S.K. Donaldson, *Polynomial invariants for smooth manifolds*, *Topology* **29** (1990) 257 [INSPIRE].

[4] J.P. Yamron, *Topological actions from twisted supersymmetric theories*, *Phys. Lett. B* **213** (1988) 325 [INSPIRE].

[5] D. Anselmi and P. Fré, *Topological twist in four-dimensions, R duality and hyperinstantons*, *Nucl. Phys. B* **404** (1993) 288 [`hep-th/9211121`] [INSPIRE].

[6] D. Anselmi and P. Fré, *Topological σ-models in four-dimensions and triholomorphic maps*, *Nucl. Phys. B* **416** (1994) 255 [`hep-th/9306080`] [INSPIRE].

[7] D. Anselmi and P. Fré, *Gauged hyper-instantons and monopole equations*, *Phys. Lett. B* **347** (1995) 247 [`hep-th/9411205`] [INSPIRE].

[8] M. Alvarez and J.M.F. Labastida, *Breaking of topological symmetry*, *Phys. Lett. B* **315** (1993) 251 [`hep-th/9305028`] [INSPIRE].

[9] M. Alvarez and J.M.F. Labastida, *Topological matter in four-dimensions*, *Nucl. Phys. B* **437** (1995) 356 [`hep-th/9404115`] [INSPIRE].

[10] J.M.F. Labastida and M. Mariño, *A topological Lagrangian for monopoles on four manifolds*, *Phys. Lett. B* **351** (1995) 146 [`hep-th/9503105`] [INSPIRE].

[11] J.M.F. Labastida and M. Mariño, *Non-Abelian monopoles on four manifolds*, *Nucl. Phys. B* **448** (1995) 373 [hep-th/9504010] [INSPIRE].

[12] S. Hyun, J. Park and J.-S. Park, *Spin-c topological QCD*, *Nucl. Phys. B* **453** (1995) 199 [hep-th/9503201] [INSPIRE].

[13] A. Losev, N. Nekrasov and S.L. Shatashvili, *Issues in topological gauge theory*, *Nucl. Phys. B* **534** (1998) 549 [hep-th/9711108] [INSPIRE].

[14] A. Lossev, N. Nekrasov and S.L. Shatashvili, *Testing Seiberg-Witten solution*, *NATO Sci. Ser. C* **520** (1999) 359 [hep-th/9801061] [INSPIRE].

[15] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97 [hep-th/9712241] [INSPIRE].

[16] G.W. Moore, N. Nekrasov and S. Shatashvili, *D particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [hep-th/9803265] [INSPIRE].

[17] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [hep-th/0206161] [INSPIRE].

[18] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, *Prog. Math.* **244** (2006) 525 [hep-th/0306238] [INSPIRE].

[19] N.A. Nekrasov, *Localizing gauge theories*, in *XIVth International Congress on Mathematical Physics*, J.C. Zambrini ed., World Scientific, Singapore (2006).

[20] L. Gottsche, H. Nakajima and K. Yoshioka, *Instanton counting and Donaldson invariants*, *J. Diff. Geom.* **80** (2008) 343 [math/0606180] [INSPIRE].

[21] L. Gottsche, H. Nakajima and K. Yoshioka, *K-theoretic Donaldson invariants via instanton counting*, *Pure Appl. Math. Quart.* **5** (2009) 1029 [math/0611945] [INSPIRE].

[22] E. Gasparim and C.-C.M. Liu, *The Nekrasov conjecture for toric surfaces*, *Commun. Math. Phys.* **293** (2010) 661 [arXiv:0808.0884] [INSPIRE].

[23] M. Bershtein, G. Bonelli, M. Ronzani and A. Tanzini, *Exact results for $\mathcal{N} = 2$ supersymmetric gauge theories on compact toric manifolds and equivariant Donaldson invariants*, *JHEP* **07** (2016) 023 [arXiv:1509.00267] [INSPIRE].

[24] M. Bershtein, G. Bonelli, M. Ronzani and A. Tanzini, *Gauge theories on compact toric surfaces, conformal field theories and equivariant Donaldson invariants*, *J. Geom. Phys.* **118** (2017) 40 [arXiv:1606.07148] [INSPIRE].

[25] A. Bawane, G. Bonelli, M. Ronzani and A. Tanzini, *$\mathcal{N} = 2$ supersymmetric gauge theories on $S^2 \times S^2$ and Liouville Gravity*, *JHEP* **07** (2015) 054 [arXiv:1411.2762] [INSPIRE].

[26] M. Sinamuli, *On $\mathcal{N} = 2$ supersymmetric gauge theories on $S^2 \times S^2$*, *JHEP* **05** (2016) 062 [arXiv:1411.4918] [INSPIRE].

[27] D. Rodriguez-Gomez and J. Schmude, *Partition functions for equivariantly twisted $\mathcal{N} = 2$ gauge theories on toric Kähler manifolds*, *JHEP* **05** (2015) 111 [arXiv:1412.4407] [INSPIRE].

[28] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[29] N. Hama and K. Hosomichi, *Seiberg-Witten theories on ellipsoids*, *JHEP* **09** (2012) 033 [*Addendum ibid.* **10** (2012) 051] [arXiv:1206.6359] [INSPIRE].

[30] V. Pestun, *Localization for $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions*, in *New dualities of supersymmetric gauge theories*, J. Teschner, Springer, Germany (2016) [arXiv:1412.7134] [INSPIRE].

[31] V. Pestun et al., *Localization techniques in quantum field theories*, *J. Phys. A* **50** (2017) 440301 [arXiv:1608.02952] [INSPIRE].

[32] G. Festuccia, J. Qiu, J. Winding and M. Zabzine, *Transversally elliptic complex and cohomological field theory*, *J. Geom. Phys.* **156** (2020) 103786 [arXiv:1904.12782] [INSPIRE].

[33] J. Kallen, *Cohomological localization of Chern-Simons theory*, *JHEP* **08** (2011) 008 [arXiv:1104.5353] [INSPIRE].

[34] B. Assel, D. Martelli, S. Murthy and D. Yokoyama, *Localization of supersymmetric field theories on non-compact hyperbolic three-manifolds*, *JHEP* **03** (2017) 095 [arXiv:1609.08071] [INSPIRE].

[35] A. Pittelli, *Supersymmetric localization of refined chiral multiplets on topologically twisted $H^2 \times S^1$*, *Phys. Lett. B* **801** (2020) 135154 [arXiv:1812.11151] [INSPIRE].

[36] C. Closset and I. Shamir, *The $\mathcal{N}=1$ chiral multiplet on $T^2 \times S^2$ and supersymmetric localization*, *JHEP* **03** (2014) 040 [arXiv:1311.2430] [INSPIRE].

[37] G. Festuccia, J. Qiu, J. Winding and M. Zabzine, *$\mathcal{N}=2$ supersymmetric gauge theory on connected sums of $S^2 \times S^2$*, *JHEP* **03** (2017) 026 [arXiv:1611.04868] [INSPIRE].

[38] A. Bawane, S. Benvenuti, G. Bonelli, N. Muteeb and A. Tanzini, *$\mathcal{N}=2$ gauge theories on unoriented/open four-manifolds and their AGT counterparts*, *JHEP* **07** (2019) 040 [arXiv:1710.06283] [INSPIRE].

[39] P. Longhi, F. Nieri and A. Pittelli, *Localization of 4d $\mathcal{N}=1$ theories on $\mathbb{D}^2 \times \mathbb{T}^2$*, *JHEP* **12** (2019) 147 [arXiv:1906.02051] [INSPIRE].

[40] J. Källén and M. Zabzine, *Twisted supersymmetric 5D Yang-Mills theory and contact geometry*, *JHEP* **05** (2012) 125 [arXiv:1202.1956] [INSPIRE].

[41] J. Källén, J. Qiu and M. Zabzine, *The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere*, *JHEP* **08** (2012) 157 [arXiv:1206.6008] [INSPIRE].

[42] J. Qiu and M. Zabzine, *5D Super Yang-Mills on $Y^{p,q}$ Sasaki-Einstein manifolds*, *Commun. Math. Phys.* **333** (2015) 861 [arXiv:1307.3149] [INSPIRE].

[43] J. Qiu and M. Zabzine, *Factorization of 5D super Yang-Mills theory on $Y^{p,q}$ spaces*, *Phys. Rev. D* **89** (2014) 065040 [arXiv:1312.3475] [INSPIRE].

[44] J. Qiu and M. Zabzine, *On twisted $N=2$ 5D super Yang-Mills theory*, *Lett. Math. Phys.* **106** (2016) 1 [arXiv:1409.1058] [INSPIRE].

[45] J.A. Minahan and M. Zabzine, *Gauge theories with 16 supersymmetries on spheres*, *JHEP* **03** (2015) 155 [arXiv:1502.07154] [INSPIRE].

[46] K. Polydorou, A. Rocén and M. Zabzine, *7D supersymmetric Yang-Mills on curved manifolds*, *JHEP* **12** (2017) 152 [arXiv:1710.09653] [INSPIRE].

[47] N. Iakovidis, J. Qiu, A. Rocén and M. Zabzine, *7D supersymmetric Yang-Mills on hypertoric 3-Sasakian manifolds*, *JHEP* **06** (2020) 026 [arXiv:2003.12461] [INSPIRE].

[48] E. Friedman and S. Ruijsenaars, *Shintani–Barnes ζ and γ functions*, *Adv. Math.* **187** (2004) 362.

[49] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys. B* **431** (1994) 3 [`hep-th/9408074`] [INSPIRE].

[50] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [`arXiv:1105.0689`] [INSPIRE].

[51] M.F. Sohnius, *The multiplet of currents for $N = 2$ extended supersymmetry*, *Phys. Lett. B* **81** (1979) 8 [INSPIRE].

[52] B. de Wit, J.W. van Holten and A. Van Proeyen, *Transformation rules of $N = 2$ supergravity multiplets*, *Nucl. Phys. B* **167** (1980) 186 [INSPIRE].

[53] B. de Wit, J.W. van Holten and A. Van Proeyen, *Structure of $N = 2$ supergravity*, *Nucl. Phys. B* **184** (1981) 77 [*Erratum ibid.* **222** (1983) 516] [INSPIRE].

[54] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of $N = 2$ supergravity-matter systems*, *Nucl. Phys. B* **255** (1985) 569 [INSPIRE].

[55] D.Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press, Cambridge U.K. (2012).

[56] C. Klare and A. Zaffaroni, *Extended supersymmetry on curved spaces*, *JHEP* **10** (2013) 218 [`arXiv:1308.1102`] [INSPIRE].

[57] D. Butter, G. Inverso and I. Lodato, *Rigid 4D $\mathcal{N} = 2$ supersymmetric backgrounds and actions*, *JHEP* **09** (2015) 088 [`arXiv:1505.03500`] [INSPIRE].

[58] J. Labastida and M. Mariño, *Topological quantum field theory and four manifolds*, Springer, Germany (2005).

[59] J. Qiu, L. Tizzano, J. Winding and M. Zabzine, *Gluing Nekrasov partition functions*, *Commun. Math. Phys.* **337** (2015) 785 [`arXiv:1403.2945`] [INSPIRE].

[60] G. Felder and A. Varchenko, *The elliptic gamma function and* $\mathrm{sl}(3, z) \ltimes z^3$, *Adv. Math.* **156** (2000) 44.

[61] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1992).

[62] Y. Kosmann, *Dérivées de Lie des spineurs*, *Ann. Mat. Pura Appl.* **91** (1971) 317.