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Supersymmetric localization of refined chiral multiplets on topologically twisted $H^2 \times S^1$

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ABSTRACT

We derive the partition function of an $\mathcal{N} = 2$ chiral multiplet on topologically twisted $H^2 \times S^1$. The chiral multiplet is coupled to a background vector multiplet encoding a real mass deformation. We consider an $H^2 \times S^1$ metric containing two parameters: one is the S^1 radius, while the other gives a fugacity q for the angular momentum on H^2. The computation is carried out by means of supersymmetric localization, which provides a finite answer written in terms of q-Pochammer symbols and multiple Zeta functions. Especially, the partition function of normalizable fields reproduces three-dimensional holomorphic blocks.

Contents

1. Introduction and conclusions

Localization techniques have considerably improved our understanding of quantum field theory as they allow for the exact computation of interesting physical observables. They were first applied to topological field theories [1] and then extended to supersymmetric gauge theories in diverse dimensions [2–6]. A consistent definition of supersymmetric quantum field theories on curved manifolds [7–11] was crucial in enlarging the applicability of localization, which keeps producing several non-perturbative results such as tests of the AdS/CFT correspondence and other supersymmetric dualities [6,12–20]. The literature on the subject is gargantuan and we refer the reader to the recent review [21] and to the references therein.

Localization on compact manifolds is largely investigated, see e.g. [22–31]. Less understood is localization on compact manifolds with boundary [32–35]; even less the case of non-compact hyperbolic manifolds [14,36–40]. In this paper we localize the partition function Z_{chi} of a chiral multiplet with arbitrary R-charge r on $H^2 \times S^1$. The model is topologically twisted as the R-symmetry

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background is chosen in order to cancel the spin connection, allowing for covariantly constant Killing spinors. We consider a chiral multiplet coupled to a background vector multiplet inducing a real mass deformation. In analogy with [23], once specified the action functional $S_{\rm chi}[\Psi]$ and the boundary conditions Ψ_∂ for the chiral multiplet fields Ψ, the observable $Z_{\rm chi}$ admits a path-integral representation as well as a canonical quantization definition in terms of a trace over the Hilbert space $\mathcal{H}\left[H^2\right]$ of states on H^2:

$$\begin{aligned} Z_{\rm chi} &= \int_{\Psi_\partial} [d\Psi]\, e^{-S_{\rm chi}[\Psi]} = {\rm Tr}_{\mathcal{H}[H^2]}\,(-1)^{\mathscr{F}}\, e^{2\pi {\rm i}\,\mathscr{H}} \\ &= {\rm Tr}_{\mathcal{H}[H^2]}\,(-1)^{\mathscr{F}}\, q^{P_\chi}\, t^{J_F}\,, \end{aligned} \tag{1.1}$$

where $\mathscr{F}$ is the fermion number and $\mathscr{H}$ the translation operator along the S^1 orthogonal to H^2. The second equality descends from the supersymmetry algebra $Q^2 = -\mathscr{H} + \alpha\, P_\chi - {\rm i}\, u\, J_F$, with P_χ being the (R-symmetry twisted) angular momentum on H^2, J_F generating flavor symmetries and $q = e^{2\pi {\rm i}\alpha}$, $t = e^{2\pi u}$ being fugacities thereof. The case $\alpha = 0$, $q = 1$ was studied in [14].

As in [39], the answer for $Z_{\rm chi}$ strongly depends on boundary conditions and on the normalizability of states contributing to the partition function. Indeed, if we include both normalizable and non-normalizable contributions, we obtain

$$\begin{aligned} &r \neq 1 : Z_{\rm chi} = e^{{\rm i}\pi\,\mathcal{A}_{\rm chi}}\,\frac{(t\,q^{1-\frac{r}{2}};q)}{(t^{-1}\,q^{\frac{r}{2}};q)}\,, \\ &r = 1 : Z_{\rm chi} = 1\,;\ (z;x) := \prod_{m\geq 0}(1 - z\,x^m)\,, \end{aligned} \tag{1.2}$$

with $u = L\,\beta\left(\sigma + {\rm i}\,v'\right)$. Here, σ is a real mass deformation and v' a particular component of a background field corresponding to a flavor symmetry $\mathrm{U}(1)_F$. Moreover, L is the H^2 radius, β the ratio between the S^1 radius and L and $\alpha \in \mathbb{R}$ a real parameter deforming the $H^2 \times S^1$ metric. The phase factor $\mathcal{A}_{\rm chi}$ is given in terms of double zeta functions,

$$\begin{aligned} &\mathcal{A}_{\rm chi} = \mathcal{A}_B - \mathcal{A}_\phi\,, \qquad \mathcal{A}_B = \zeta_2(0, \alpha - \frac{\alpha r}{2} + {\rm i}u|1,\alpha)\,, \\ &\mathcal{A}_\phi = \zeta_2(0, \frac{\alpha r}{2} - {\rm i}u|1,\alpha)\,. \end{aligned} \tag{1.3}$$

In particular, $Z_{\rm chi} = 1$ at $r = 1$ because no state satisfies supersymmetric boundary conditions for that specific value of the R-charge. Forby, the absolute value of $Z_{\rm chi}$ for $r \neq 1$ is the plethystic exponential [41] of a *single letter partition function* $f_r(t,q)$:

$$f_r(t,q) = \frac{t^{-1}\,q^{\frac{r}{2}} - t\,q^{1-\frac{r}{2}}}{1-q}\,, \qquad Z_{\rm chi} = e^{{\rm i}\,\mathcal{A}_{\rm chi}}\,{\rm P.E.}[\,f_r(t,q)]\,. \tag{1.4}$$

If we shrink the S^1 radius by taking the limit $\beta \to 0$, the single letter reduces to $f_r(1,q) = (q^{\frac{r}{2}} - q^{1-\frac{r}{2}})/(1-q)$. Notice that $Z_{\rm chi}$ does not depend on the S^1 radius β in absence of background vector multiplets, becoming a continuous function of r.

On the other hand, if we exclude non-normalizable contributions, $Z_{\rm chi}$ reads

$$\begin{aligned} &r < 1 : Z_{\rm chi} = e^{{\rm i}\pi\,\mathcal{A}_B}\,(t\,q^{1-\frac{r}{2}};q)\,, \qquad r = 1 : Z_{\rm chi} = 1\,, \\ &r > 1 : Z_\phi = \frac{e^{{\rm i}\pi\,\mathcal{A}_\phi}}{(t^{-1}\,q^{\frac{r}{2}};q)}\,. \end{aligned} \tag{1.5}$$

Equations (1.5) are reminiscent of what happens in topologically twisted theories on $\mathcal{M}^2 \times S^1$, where the R-symmetry background produces Landau levels for quantum mechanics on S^1 [14,23,42,43].

For $r \neq 1$, the partition functions $Z_{\rm chi}$ in (1.5) reproduces three-dimensional holomorphic blocks [44,45], also obtained by performing supersymmetric localization on $D^2 \times S^1$ [33]. In light of (1.5), including non-normalizable contributions in the partition function calculation amounts to trivially gluing together $Z_{\rm chi}(r<1)$ and $Z_{\rm chi}(r>1)$. This procedure yields the partition function on $S^2 \times S^1$, explaining the accidental coincidence between the 3d superconformal index of a chiral multiplet with arbitrary R-charge [46] and the (a priori different) topologically twisted index (1.2).

As we have already mentioned, the value $r = 1$ is special from the viewpoint of boundary conditions as well. Indeed, an R-charge $r > 1$ implies Dirichlet boundary conditions on the fields contributing to $Z_{\rm chi}$; namely, the scalars $\phi, \widetilde{\phi}$ are supposed to vanish at the (conformal) boundary. On the other hand, $r < 1$ requires Robin boundary conditions, meaning that derivatives of $\phi, \widetilde{\phi}$ go to zero at the boundary. The case $r = 1$ does not correspond to any set of BPS boundary conditions and, in fact, there are no fields contributing to $Z_{\rm chi}$ non-trivially for $r = 1$.

1.1. Outlook

In this paper we studied an $\mathcal{N} = 2$ chiral multiplet on topologically twisted $H^2 \times S^1$ coupled to a background vector multiplet incorporating a real mass deformation. It would be very interesting to generalize the results of the present work by including dynamical vector multiplets, Chern-Simons terms as well as general BPS observables. This would provide a complete study of gauge theories on $H^2 \times S^1$, helping out to clarify universal features of supersymmetric theories on non-compact manifolds, also unveiling possible dualities intertwining them.

Furthermore, it would be intriguing to apply a similar analysis to gauge theories defined on higher dimensional non-compact manifolds. This not only would be fascinating per se, but should also shed a new light on our findings concerning matter multiplets on $H^2 \times S^1$.

Finally, it would be compelling to explore the link between partition functions on $H^2 \times S^1$, the half-index on $D^2 \times S^1$ and 3d holomorphic blocks. In particular, a rigorous interpretation of (1.5) in terms of quantum mechanics for states on H^2 would be desirable.[1]

1.2. Outline

In Section 2 we describe the geometry of topologically twisted $H^2 \times S^1$, constructing the corresponding Killing spinors. In Section 3 we write down the supersymmetry transformations and action for an $\mathcal{N} = 2$ chiral multiplet coupled to a background vector multiplet. We shall also introduce twisted fields, which simplify the localization computation, and discuss the asymptotic boundary conditions. Eventually, Section 4 contains the computation of the one-loop determinant for the chiral multiplet on twisted $H^2 \times S^1$.

1.3. Acknowledgements

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[1] We thank Pietro Longhi for raising this point.

2. Background geometry

2.1. Metric and killing spinors

We use the conventions of [39], which are the same as the conventions of [9] apart from a sign in the definition of the spin connection. Let us consider $H^2 \times S^1$ with line element

$$\begin{aligned} ds^2 &= L^2 \left[d\eta^2 + \sinh^2\eta\, (d\chi + \alpha\, d\varphi)^2 \right] + L^2\beta^2 d\varphi^2 , \\ \eta &\in \mathbb{R}^+ , \quad \chi, \varphi \in [0, 2\pi] , \end{aligned} \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}$. Consequently, the orthonormal frame e^a is

$$e^1 = L\, d\eta , \qquad e^2{}_\chi = L \sinh\eta\, d\chi + L\,\alpha \sinh\eta\, d\varphi , \qquad e^3 = L\,\beta\, d\varphi . \tag{2.2}$$

In our conventions, the Ricci scalar of $H^2 \times S^1$ is $R = -2/L^2$. The Killing spinor equations for a three-dimensional manifold with $\mathcal{N} = 2$ supersymmetry read

$$\begin{aligned} \nabla_\mu \zeta - \mathrm{i} A_\mu \zeta &= -\frac{H}{2}\gamma_\mu \zeta - \mathrm{i} V_\mu \zeta - \frac{1}{2}\epsilon_{\mu\nu\rho} V^\nu \gamma^\rho \zeta , \\ \nabla_\mu \widetilde{\zeta} + \mathrm{i} A_\mu \widetilde{\zeta} &= -\frac{H}{2}\gamma_\mu \widetilde{\zeta} + \mathrm{i} V_\mu \widetilde{\zeta} + \frac{1}{2}\epsilon_{\mu\nu\rho} V^\nu \gamma^\rho \widetilde{\zeta} . \end{aligned} \tag{2.3}$$

If we choose the background fields

$$A = \frac{1}{2}\cosh\eta\, (d\chi + \alpha\, d\varphi) , \qquad H = 0 , \qquad V = 0 , \tag{2.4}$$

we find that the spinors

$$\zeta_\alpha = \frac{1}{\sqrt{2}}(1, \mathrm{i})_\alpha , \qquad \widetilde{\zeta}_\alpha = -\frac{1}{\sqrt{2}}(\mathrm{i}, 1)_\alpha , \tag{2.5}$$

solve (2.3). The Killing spinors $\zeta, \widetilde{\zeta}$ in (2.5) have R-charges $1, -1$ respectively. They satisfy $\zeta\widetilde{\zeta} = -\widetilde{\zeta}\zeta = +1$ as well as $\zeta^\dagger = -\widetilde{\zeta}$, implying $|\zeta|^2 = \zeta^\dagger\zeta = |\widetilde{\zeta}|^2 = \widetilde{\zeta}^\dagger\widetilde{\zeta} = +1$.

2.2. Three-dimensional frame

The Killing spinors $\zeta, \widetilde{\zeta}$ allow for constructing the bilinears

$$K^\mu = \zeta\gamma^\mu\widetilde{\zeta} , \qquad P^\mu = \zeta\gamma^\mu\zeta , \qquad \widetilde{P}^\mu = \widetilde{\zeta}\gamma^\mu\widetilde{\zeta} . \tag{2.6}$$

The vectors $K^\mu, P^\mu, \widetilde{P}^\mu$ have R-charges $0, 2, -2$ and fulfil

$$\begin{aligned} & g^{\mu\nu} = K^\mu K^\nu - P^{(\mu}\widetilde{P}^{\nu)} , \quad K_\mu K^\mu = 1 , \quad \widetilde{P}_\mu P^\mu = -2 , \\ & (K_\mu)^* = K_\mu , \quad (P_\mu)^* = -\widetilde{P}_\mu . \end{aligned} \tag{2.7}$$

By contracting (2.6) with ∂_μ, we obtain a representation in terms of Lie derivatives $\mathcal{L}_K = K^\mu\partial_\mu$, $\mathcal{L}_P = P^\mu\partial_\mu$ and $\mathcal{L}_{\widetilde{P}} = \widetilde{P}^\mu\partial_\mu$:

$$\begin{aligned} & \mathcal{L}_K = -\frac{1}{L\beta}\left(\alpha\,\partial_\chi - \partial_\varphi\right) , \qquad \mathcal{L}_P = \frac{1}{L}\left(\mathrm{i}\,\partial_\eta + \frac{1}{\sinh\eta}\partial_\chi\right) , \\ & \mathcal{L}_{\widetilde{P}} = \frac{1}{L}\left(\mathrm{i}\,\partial_\eta - \frac{1}{\sinh\eta}\partial_\chi\right) . \end{aligned} \tag{2.8}$$

Especially, the parameter α deforms $\mathcal{L}_K$ by a term proportional to ∂_χ, where the latter is the angular momentum operator on H^2.

3. Chiral multiplet on $H^2 \times S^1$

3.1. Supersymmetry transformations and action

The supersymmetric transformations for a chiral multiplet (ϕ, ψ, F) of R-charge r on $H^2 \times S^1$ with respect to the supercharge $\delta = \delta_\zeta + \delta_{\widetilde{\zeta}}$ are [9]

$$\begin{aligned} \delta\phi &= \sqrt{2}\,\zeta\psi , \\ \delta\psi &= \sqrt{2}\,\zeta F + \mathrm{i}\sqrt{2}\,\sigma\,\phi\widetilde{\zeta} - \mathrm{i}\sqrt{2}\,\gamma^\mu\widetilde{\zeta}\mathcal{D}_\mu\phi , \\ \delta F &= -\mathrm{i}\sqrt{2}\,\sigma\,\widetilde{\zeta}\psi - \mathrm{i}\sqrt{2}\,\mathcal{D}_\mu\left(\widetilde{\zeta}\gamma^\mu\psi\right) , \end{aligned} \tag{3.1}$$

where we introduced the covariant derivative $\mathcal{D}_\mu = \nabla_\mu - \mathrm{i}\, q_R \left(A_\mu - \frac{1}{2}V_\mu\right) - \mathrm{i}\, v_\mu$. Here, q_R is the R-charge, σ a constant scalar encoding a real mass deformation and v_μ a background gauge field corresponding to a flavor symmetry $\mathrm{U}(1)_F$. Similarly, we can write down the supersymmetry transformations for an anti-chiral multiplet $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{F})$ of R-charge $-r$:

$$\begin{aligned} \delta\widetilde{\phi} &= -\sqrt{2}\,\widetilde{\zeta}\widetilde{\psi} , \\ \delta\widetilde{\psi} &= \sqrt{2}\,\widetilde{\zeta}\widetilde{F} - \mathrm{i}\sqrt{2}\,\sigma\,\widetilde{\phi}\zeta + \mathrm{i}\sqrt{2}\,\gamma^\mu\zeta\mathcal{D}_\mu\widetilde{\phi} , \\ \delta\widetilde{F} &= -\mathrm{i}\sqrt{2}\,\sigma\,\zeta\widetilde{\psi} - \mathrm{i}\sqrt{2}\,\mathcal{D}_\mu\left(\zeta\gamma^\mu\widetilde{\psi}\right) . \end{aligned} \tag{3.2}$$

The supersymmetric variations $\delta_\zeta, \delta_{\widetilde{\zeta}}$ are nilpotent, while δ squares to an isometry of the background $\mathcal{L}_K$ plus a central charge given by the background fields:

$$\delta^2 = \left\{\delta_\zeta, \delta_{\widetilde{\zeta}}\right\} = -2\mathrm{i}\,\mathcal{L}_K + 2\mathrm{i}\left(\sigma + \mathrm{i}\,K^\mu\, v_\mu\right) . \tag{3.3}$$

The action for the above chiral multiplet is given by integrating over $H^2 \times S^1$ the following Lagrangian:

$$\mathcal{L}_{\text{chi}} = \mathcal{D}^\mu\widetilde{\phi}\mathcal{D}_\mu\phi + \left(\sigma^2 - \frac{r}{2L^2}\right)\widetilde{\phi}\phi - \widetilde{F}F - \mathrm{i}\widetilde{\psi}\gamma^\mu\mathcal{D}_\mu\psi - \mathrm{i}\sigma\widetilde{\psi}\psi . \tag{3.4}$$

3.2. Twisted fields

Let us introduce the twisted fields $B, C, \widetilde{B}, \widetilde{C}$, which are Graßmann-odd scalars of R-charge $(r-2, r, 2-r, -r)$ defined as [39]

$$\begin{aligned} & B = \widetilde{\zeta}\psi , \qquad C = \zeta\psi , \qquad \widetilde{B} = \zeta\widetilde{\psi} , \qquad C = -\widetilde{\zeta}\widetilde{\psi} , \\ & \psi = \zeta B + \widetilde{\zeta} C , \qquad \widetilde{\psi} = \widetilde{\zeta}\widetilde{B} + \zeta\widetilde{C} . \end{aligned} \tag{3.5}$$

The non-trivial supersymmetric variations of (ϕ, B, C, F) read

$$\begin{aligned} & \delta_\zeta\phi = \sqrt{2}C , \qquad \delta_{\widetilde{\zeta}}C = -\mathrm{i}\sqrt{2}\hat{\mathcal{L}}_K\phi + \mathrm{i}\sqrt{2}\sigma\phi , \\ & \delta_\zeta B = \sqrt{2}F , \qquad \delta_{\widetilde{\zeta}}B = \mathrm{i}\sqrt{2}\hat{\mathcal{L}}_{\widetilde{P}}\phi , \\ & \delta_{\widetilde{\zeta}}F = -\mathrm{i}\sqrt{2}\hat{\mathcal{L}}_K B + \mathrm{i}\sqrt{2}\sigma B - \mathrm{i}\sqrt{2}\hat{\mathcal{L}}_{\widetilde{P}}C , \end{aligned} \tag{3.6}$$

while those of $(\widetilde{\phi}, \widetilde{B}, \widetilde{C}, \widetilde{F})$ are

$$\begin{aligned} & \delta_{\widetilde{\zeta}}\widetilde{\phi} = \sqrt{2}\widetilde{C} , \qquad \delta_\zeta\widetilde{C} = -\mathrm{i}\sqrt{2}\hat{\mathcal{L}}_K\widetilde{\phi} - \mathrm{i}\sqrt{2}\sigma\widetilde{\phi} , \\ & \delta_{\widetilde{\zeta}}\widetilde{B} = \sqrt{2}\widetilde{F} , \qquad \delta_\zeta\widetilde{B} = \mathrm{i}\sqrt{2}\hat{\mathcal{L}}_P\widetilde{\phi} , \\ & \delta_\zeta\widetilde{F} = -\mathrm{i}\sqrt{2}\hat{\mathcal{L}}_K\widetilde{B} - \mathrm{i}\sqrt{2}\sigma\widetilde{B} - \mathrm{i}\sqrt{2}\hat{\mathcal{L}}_P\widetilde{C} . \end{aligned} \tag{3.7}$$

Here, *hatted* Lie derivatives are covariant, for example $\hat{\mathcal{L}}_X = X^\mu\mathcal{D}_\mu$. Via twisted fields we can write down the deformation term

$$\begin{aligned} \mathcal{V}_{\text{chi}} &= \frac{1}{2}\left[(\delta_\zeta B)^\ddagger B + (\delta_\zeta\widetilde{B})^\ddagger\widetilde{B} + (\delta_\zeta\widetilde{C})^\ddagger\widetilde{C}\right] \\ &= \frac{1}{\sqrt{2}}\left[\mathrm{i}\,\widetilde{B}\hat{\mathcal{L}}_{\widetilde{P}}\phi + \mathrm{i}\,\widetilde{C}\left(\hat{\mathcal{L}}_K\phi + \sigma\phi\right) - \widetilde{F}B\right] , \end{aligned} \tag{3.8}$$

where we used the reality conditions $\phi^\ddagger = \widetilde{\phi}$ and $F^\ddagger = -\widetilde{F}$, the involution $\ddagger$ acting as complex conjugation upon c-numbers. The variation of $\mathcal{V}_{\text{chi}}$ with respect to the supercharge δ_ζ yields the Lagrangian

$$\mathcal{L}'_{\text{chi}} = \mathcal{L}_K \widetilde{\phi} \mathcal{L}_K \phi - \mathcal{L}_P \widetilde{\phi} \mathcal{L}_{\widetilde{P}} \phi + \sigma^2 \widetilde{\phi} \phi - \widetilde{F} F + \\ - \mathrm{i}\, B \mathcal{L}_K \widetilde{B} - \mathrm{i}\, \widetilde{C} \mathcal{L}_K C - \mathrm{i}\, B \mathcal{L}_P \widetilde{C} - \mathrm{i}\, \widetilde{B} \mathcal{L}_{\widetilde{P}} C + \mathrm{i}\, \sigma \left(\widetilde{B} B + C \widetilde{C} \right) , \tag{3.9}$$

coinciding with (3.4) up to total derivatives. By construction, (3.9) is supersymmetric under both δ_ζ and $\delta_{\widetilde{\zeta}}$ without imposing any boundary condition.[2]

3.3. Boundary conditions

If we use the Lagrangian (3.9), we find that the equations of motion of $B, F, \widetilde{B}, \widetilde{F}$ generate bulk terms only. Instead, the equations of motion of $\phi, C, \widetilde{\phi}, \widetilde{C}$ give[3]

$$\begin{aligned} \delta_{\text{eom}} S'_{\text{chi}} &= \delta_{\text{eom}} \int_M d^3x \sqrt{g}\, \mathcal{L}'_{\text{chi}} \\ &= (\text{bulk}) - \int_M d^3x \sqrt{g} \left[\mathcal{L}_P (\delta \widetilde{\phi} \mathcal{L}_{\widetilde{P}} \phi) + \mathcal{L}_{\widetilde{P}} (\delta \phi \mathcal{L}_P \widetilde{\phi}) \right. \\ &\quad \left. + \mathrm{i}\, \mathcal{L}_P (B \delta \widetilde{C}) + \mathrm{i}\, \mathcal{L}_{\widetilde{P}} (\widetilde{B} \delta C) \right] , \end{aligned} \tag{3.10}$$

where $M = H^2 \times S^1$. The (bulk) terms vanish by the equations of motion, while the boundary terms disappear if we impose at the conformal boundary either Dirichlet boundary conditions, $\phi = \widetilde{\phi} = C = \widetilde{C} = 0$, or Robin boundary conditions $\mathcal{L}_{\widetilde{P}} \phi = \mathcal{L}_P \widetilde{\phi} = B = \widetilde{B} = 0$. If we choose to leave the field variations $\delta\phi, \delta C, \delta\widetilde{\phi}, \delta\widetilde{C}$ free to oscillate at the conformal boundary, the action S'_{chi} forces us to impose Robin. As we shall see in the next section, asymptotic boundary conditions will constrain the R-symmetry of the modes contributing to the one-loop determinant of the partition function.

4. Localization

4.1. BPS locus

The deformation term (3.8) leads to the Lagrangian (3.9), whose bosonic part is positive definite. The saddle point configurations of the path integral are then obtained by solving the BPS equations

$$\delta_\zeta B = \delta_\zeta \widetilde{B} = \delta_\zeta C = \delta_\zeta \widetilde{C} = 0 . \tag{4.1}$$

These constraints immediately imply $F = \widetilde{F} = 0$. Furthermore, periodicity along (φ, χ) directions yield $\phi = \widetilde{\phi} = 0$. We then find the trivial locus $\phi = \widetilde{\phi} = F = \widetilde{F} = 0$.

4.2. One loop determinant

We compute the one loop determinant by means of the unpaired eigenvalues method, see e.g. [26,39,40,48]. This exploits two main facts: first, $\hat{\mathcal{L}}_P, \hat{\mathcal{L}}_{\widetilde{P}}$ commute with the operator δ^2, whose functional determinant provides the chiral multiplet partition function Z_{chi}. Second, $\hat{\mathcal{L}}_P, \hat{\mathcal{L}}_{\widetilde{P}}$ map to each other bosonic and fermionic modes. As a result, the neat contribution to Z_{chi} is given by modes belonging to the kernels of $\hat{\mathcal{L}}_P, \hat{\mathcal{L}}_{\widetilde{P}}$:

$$Z_{\text{chi}} = \frac{\det_{\text{Ker}\, \hat{\mathcal{L}}_P} \delta^2}{\det_{\text{Ker}\, \hat{\mathcal{L}}_{\widetilde{P}}} \delta^2} . \tag{4.2}$$

In our setup, such modes have the form

$$\begin{aligned} \text{Ker}\, \hat{\mathcal{L}}_{\widetilde{P}} &: \phi_{m_\varphi, m_\chi} = e^{\mathrm{i}\, m_\varphi \varphi + \mathrm{i}\, m_\chi \chi} \left(\tanh \frac{\eta}{2} \right)^{m_\chi} (\sinh \eta)^{-\frac{r}{2}} , \\ \text{Ker}\, \hat{\mathcal{L}}_P &: B_{n_\varphi, n_\chi} = e^{\mathrm{i}\, n_\varphi \varphi + \mathrm{i}\, n_\chi \chi} \left(\coth \frac{\eta}{2} \right)^{n_\chi} (\sinh \eta)^{\frac{r-2}{2}} , \end{aligned} \tag{4.3}$$

with $m_\varphi, m_\chi, n_\varphi, n_\chi \in \mathbb{Z}$. Regularity of the modes ϕ_{m_φ, m_χ} and B_{n_φ, n_χ} at $\eta = 0$ requires $m_\chi \geq r/2$ and $n_\chi \leq (r-2)/2$. The Lagrangian $\mathcal{L}'_{\text{chi}}$ that we use as a δ-exact deformation term encodes Robin boundary conditions, meaning that $\hat{\mathcal{L}}_{\widetilde{P}} \phi$, $\hat{\mathcal{L}}_P \widetilde{\phi}$, B and $\widetilde{B}$ have to vanish at $\eta \to \infty$. The bosonic modes contributing to Z_{chi} satisfy Robin conditions already in the bulk of $H^2 \times S^1$; thus, they are left unconstrained. On the other hand, the fermionic modes are supposed to vanish at infinity. This leads us to consider normalizable modes for B, forcing $r < 1$. Conversely, Dirichlet conditions at infinity leave B unconstrained and fix $r > 1$. To infer regularity and normalizability of the fields we employed the norm induced by the inner product

$$\langle X_1, X_2 \rangle = \int_M d^3x \sqrt{g}\, (X_1)^\ddagger X_2 . \tag{4.4}$$

Consequently, the one-loop determinant (4.2) with Robin boundary conditions is

$$\begin{aligned} Z_{\text{chi}} &= \prod_{n_\varphi \in \mathbb{Z}} \prod_{n_\chi \geq 0} \frac{n_\varphi + \alpha\, (n_\chi - \frac{r-2}{2}) + \mathrm{i}\, u}{n_\varphi + \alpha\, (n_\chi + \frac{r}{2}) - \mathrm{i}\, u} , \\ &= \frac{\Gamma_2(\frac{\alpha r}{2} - \mathrm{i}u | 1, \alpha) \Gamma_2(1 - \frac{\alpha r}{2} + \mathrm{i}u | 1, -\alpha)}{\Gamma_2(\alpha - \frac{\alpha r}{2} + \mathrm{i}u | 1, \alpha) \Gamma_2(1 - \alpha + \frac{\alpha r}{2} - \mathrm{i}u | 1, -\alpha)} \\ &= e^{\mathrm{i}\pi\, \mathcal{A}_{\text{chi}}} \frac{(t\, q^{1-\frac{r}{2}}; q)}{(t^{-1}\, q^{\frac{r}{2}}; q)} , \end{aligned} \tag{4.5}$$

with $r > 1$, $u = L\, \beta\, (\sigma + \mathrm{i}\, K \cdot v)$ as well as $t = e^{2\pi u}$ and $q = e^{2\pi \mathrm{i}\alpha}$. The phase factor $\mathcal{A}_{\text{chi}}$ is

$$\mathcal{A}_{\text{chi}} = \zeta_2(0, \alpha - \frac{\alpha r}{2} + \mathrm{i}u | 1, \alpha) - \mathrm{i}\pi\, \zeta_2(0, \frac{\alpha r}{2} - \mathrm{i}u | 1, \alpha), \tag{4.6}$$

proving (1.2). In computing Z_{chi} we regularized the infinite products via Shintani-Barnes multiple Zeta and Gamma functions. If we require all fields to be normalizable according to (4.4), we see that ϕ and B cannot contribute to Z_{chi} at the same time. In particular, ϕ modes will generate a non-trivial Z_ϕ for $r > 1$, whereas B-modes will produce Z_B for $r < 1$. This shows (1.5).

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[2] Indeed, $\delta_\zeta \mathcal{L}'_{\text{chi}} = 0$, while $\delta_{\widetilde{\zeta}} \mathcal{L}'_{\text{chi}} = -2\mathrm{i}\, \mathcal{L}_K \mathcal{V}_{\text{chi}} \to \delta_{\widetilde{\zeta}} S'_{\text{chi}} = 0$ as $\mathcal{L}_K$ is parallel to the boundary.

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