Deforming the ODE/IM correspondence with $T\overline{T}$

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Abstract: The ODE/IM correspondence is an exact link between classical and quantum integrable models. The primary purpose of this work is to show that it remains valid after $T\overline{T}$ perturbation on both sides of the correspondence. In particular, we prove that the deformed Lax pair of the sinh-Gordon model, obtained from the unperturbed one through a dynamical change of coordinates, leads to the same Burgers-type equation governing the quantum spectral flow induced by $T\overline{T}$. Our main conclusions have general validity, as the analysis may be easily adapted to all the known ODE/IM examples involving integrable quantum field theories.

Contents

1 Introduction

The first instance of the ODE/IM correspondence dates back to 1998 [1] when a surprising connection between works on spectral determinants for specific Sturm-Liouville problems [2, 3] and the functional approach to conformal field theories (CFTs) [4–10] was first established. The correspondence holds its roots in the fact that seemingly different quantities in the two contexts fulfil the same set of functional relations [1] with identical analytic and asymptotic properties.

Though initially referring only to the CFT vacuum states, the ODE/IM correspondence was later generalised to encompass excited states [11, 12] and more recently extended to massive integrable quantum field theories based on the $\mathfrak{sl}_2$ algebra [13, 14]. The correspondence was also extended to systems based on higher-rank algebras, both critical [8, 15–17] and off-critical [18–20]. In this work, we focus on systems based on the $\mathfrak{sl}_2$ algebra. In the off-critical extension of the original results [14], the Sturm-Liouville equation is replaced by the Lax equations of the modified sinh-Gordon model, written as a pair of second-order differential equations coupled through a field-dependent generalised potential (cf. equation (3.7), below). In this perspective, we are dealing with two, a priori unrelated systems: a

classical integrable equation — the modified sinh-Gordon one — and a quantum integrable model — such as the sine- and sinh-Gordon models. The correspondence can then be viewed as an equality between the conserved quantities of these two systems. [14, 21, 22].

The second main ingredient, relevant to the current purposes, is the recent discovery that specific irrelevant perturbations of quantum field theories can be studied using integrable models tools and hydrodynamics-type flow equations [23, 24]. The observations made in [23, 24], triggered a considerable amount of research activity, with applications ranging from simple systems in quantum mechanics to AdS/CFT and nonlinear electrodynamics [25–31]. The perturbation involving Zamolodchikov's $T\overline{T}$ operator [32] is arguably the most interesting representative of an infinite tower of irrelevant deformations [33–35] (see also [36] for some early results on irrelevant integrable perturbations). It is related to the Nambu-Goto string [37–39], quantum JT gravity [40–45], and it possesses compelling interpretation within the AdS/CFT [46–51] and supersymmetry [52–57] frameworks. While at the quantum level, an inviscid Burgers' equation governs the evolution of the $T\overline{T}$-deformed spectrum, at the classical level, the perturbation turns out to be equivalent to a dynamical change of the space-time coordinates [58–61].

The main objective of this work is to unify these two research strands by proving the validity of the ODE/IM after $T\overline{T}$ perturbations of both the classical and the quantum sides of the correspondence. In order to do so, we will prove that the conserved energy and momentum of the theory on the $T\overline{T}$-deformed classical side of the correspondence satisfy the same Burgers' equation as the finite-size spectrum on the $T\overline{T}$-deformed quantum side. This fact ensures that if the ODE/IM is valid for the undeformed systems, it will hold for an arbitrary $T\overline{T}$ deformation. We will proceed in steps, first recalling in section 2 how a $T\overline{T}$ deformation alters the finite-size spectrum of a quantum integrable field theory and subsequently reviewing in section 3 the elements of the ODE/IM correspondence relevant for this work. Section 4 contains the key arguments and result: a proof that the energy and momentum of a $T\overline{T}$-deformed classical field theory satisfy the inviscid Burgers' equation.

In this work we choose to focus on the specific instance of the ODE/IM correspondence analysed in [14], involving the quantum sine-Gordon and the classical modified sinh-Gordon models. Consequently, the arguments of section 4 are adapted to the specificities of the latter. However, the final result has a much broader validity. In particular, our derivation can be used — with minimal modifications — to produce a proof of the Burgers' equation for the energy and momentum of a generic, not necessarily integrable, $T\overline{T}$-deformed classical field theory. To our knowledge, the only previous appearance of the Burgers' equation in a classical theory concerns a very specific class of zero-momentum solutions to the sine-Gordon model [59].

Finally, our results should be considered as a first step toward the study of irrelevant deformations using the ODE/IM correspondence as a powerful quantisation tool.

2 The quantum sine-Gordon model at finite volume

Integrable models and their properties have been extensively studied during past decades, and powerful methods developed to determine the finite-size spectrum of an integrable quantum

field theory. One of these is the non-linear integral equation (NLIE) approach [5, 62–68], which we briefly review in this section before discussing the effects of the $T\overline{T}$ deformation.

2.1 The non-linear integral equation

Consider the sine-Gordon quantum field theory defined on a cylinder of radius $R/2\pi$. The complete information on its spectrum can be extracted from the *counting function* $f_\nu(\theta)$, solution to the following NLIE [63–65, 67, 68]

$$
\begin{aligned}
f_\nu(\theta) = \nu(R, k \,|\, \theta) - \int_{\mathcal{C}_1} d\theta' \, \mathcal{K}(\theta - \theta') \log\Big(1 + e^{-f_\nu(\theta')}\Big) \\
+ \int_{\mathcal{C}_2} d\theta' \, \mathcal{K}(\theta - \theta') \log\Big(1 + e^{f_\nu(\theta')}\Big),
\end{aligned}
\tag{2.1}
$$

where

$$
\nu(R, k \,|\, \theta) = 2\pi \imath \, k - \imath \, mR \sinh(\theta),
\tag{2.2}
$$

is the so-called driving term. In equations (2.1) and (2.2), m denotes the sine-Gordon soliton mass and $k \in [-1/2, 1/2]$ is the quasi-momentum, also known as the *twist*, whose role is to select the vacuum [14, 69]. In (2.1), the convolution kernel $\mathcal{K}(\theta)$ is

$$
\mathcal{K}(\theta) = \int_{\mathbb{R}} \frac{dp}{2\pi} \cos(p\,\theta) \frac{\sinh\Big(\pi p \frac{1-\alpha}{2\alpha}\Big)}{2\cosh\big(\pi \frac{p}{2}\big) \sinh\big(\pi \frac{p}{2\alpha}\big)},
\tag{2.3}
$$

where $\alpha = \beta^{-2} - 1$ and $\beta^2 < 1$ is the sine-Gordon coupling. For technical reasons, as in [14], in the following we shall limit our attention to the range $\beta^2 < 1/2$, which corresponds to the region $\alpha > 1$.

The information on the specific energy eigenstate state under consideration is encoded in the choice of the integration contours $\mathcal{C}_1$ and $\mathcal{C}_2$. For the ground state $\mathcal{C}_1 = \mathbb{R} + \imath 0^+$ and $\mathcal{C}_2 = \mathbb{R} - \imath 0^+ = \mathcal{C}_1^*$. For the excited states the contours $\mathcal{C}_1$ and $\mathcal{C}_2$ encircle a number of singularities $\{\theta_i\}$ of $\log\Big(1 + e^{f_\nu(\theta_i)}\Big)$. See [67, 68, 70, 71] for more details.

Energy and momentum can be obtained from the counting function from the following expression

$$
\begin{aligned}
E &= m\bigg(\int_{\mathcal{C}_1} \frac{dy}{2\pi\imath} \sinh(y) \log\Big(1 + e^{-f_\nu(y)}\Big) - \int_{\mathcal{C}_2} \frac{dy}{2\pi\imath} \sinh(y) \log\Big(1 + e^{f_\nu(y)}\Big)\bigg), \\
P &= m\bigg(\int_{\mathcal{C}_1} \frac{dy}{2\pi\imath} \cosh(y) \log\Big(1 + e^{-f_\nu(y)}\Big) - \int_{\mathcal{C}_2} \frac{dy}{2\pi\imath} \cosh(y) \log\Big(1 + e^{f_\nu(y)}\Big)\bigg).
\end{aligned}
\tag{2.4}
$$

In particular, the momentum can be computed exactly via the so-called *dilogarithm trick*. Using this, one easily checks that $P(R) = 2\pi k/R$ with $k \in \mathbb{Z}$ (see for example the Lemma in section 7 of [66]).

Finally, in the following we shall adopt the following specific parametrization [14]

$$
mR = 2\sqrt{\pi}\mathbf{s}^{\alpha+1} \frac{\Gamma\Big(1 + \frac{1}{2\alpha}\Big)}{\Gamma\Big(\frac{3}{2} + \frac{1}{2\alpha}\Big)},
\tag{2.5}
$$

where $\mathbf{s}$ is a dimensionless scaling constant.

2.2 The $T\overline{T}$ deformation

At the level of the non-linear integral equation (2.1), the $T\overline{T}$ perturbation is introduced by implementing the following modification of the convolution kernel [24]

$$\mathcal{K}(\theta) \xrightarrow{T\overline{T}} \mathcal{K}(\theta) - \tau \frac{m^2}{2\pi} \cosh(\theta) \,. \tag{2.6}$$

Inserting (2.6) into equation (2.1), after simple manipulations, one finds that $f_\nu^{(\tau)}(\theta)$, the counting function of the deformed theory, still satisfies (2.1), up to a redefinition of the driving term

$$\nu^{(\tau)} = \nu(R_0, k \,|\, \theta - \theta_0) \,, \tag{2.7}$$

with

$$R_0^2 = (R + \tau E(R, \tau))^2 - \tau^2 P^2(R, \tau) \,, \tag{2.8}$$

$$\tanh \theta_0 = \tau \frac{P(R)}{R + \tau E(R, \tau)} \,. \tag{2.9}$$

The redefinition (2.7) implies the famous inviscid Burgers' flow equation for the finite-volume quantum spectrum of $T\overline{T}$-deformed field theories [23, 24]:

$$\frac{\partial}{\partial \tau} E(R, \tau) = E(R, \tau) \frac{\partial}{\partial R} E(R, \tau) + \frac{P(R)^2}{R} \,. \tag{2.10}$$

3 The modified classical sinh-Gordon model

In this section, we will follow very closely [14] in the definition of the quantities of interest. Let us consider the modified sinh-Gordon (mShG) model, with equation of motion (EoM)

$$\partial_z \partial_{\bar{z}} \eta(\mathbf{z}) - e^{2\eta(\mathbf{z})} + p(z, s) p(\bar{z}, s) e^{-2\eta(\mathbf{z})} = 0 \,, \tag{3.1}$$

where the complex coordinates $\mathbf{z} = (z, \bar{z})$ are dimensionless. The function

$$p(z, s) = z^{2\alpha} - s^{2\alpha} \,, \tag{3.2}$$

is characterized by the pair of parameters α and $s \in \mathbb{R}_{>0}$. Introducing polar coordinates,

$$z = \rho \, e^{\iota\varphi} \,, \quad \bar{z} = \rho \, e^{-\iota\varphi} \,, \tag{3.3}$$

we can describe the field configurations $\eta(\rho, \varphi)$ relevant to the ODE/IM correspondence as those that satisfy the following requirements

- $e^{-\eta(\rho,\varphi)}$ is a single-valued, non-zero complex function on the cone $C_{\pi/\alpha}$ with apex angle π/α and $L + \bar{L}$ punctures;

- the $\rho \to \infty$ asymptotic behaviour is $e^{-\eta(\rho,\varphi)} \sim \rho^{-\alpha}$;

- the $\rho \to 0$ asymptotic behaviour is $e^{-\eta(\rho,\varphi)} \sim \rho^{-l}$, with $l \in [-1/2, 1/2]$.

Through the ODE/IM correspondence, the conserved quantities of the *classical* mShG model, evaluated on the above field configurations, are identified with the *quantum* ones of the sine-Gordon model, such as its energy and momentum (2.4). The parameter l is then related to the quasi-momentum in (2.2) as $l = 2|k| - 1/2$. The number and positions of the $L + \bar{L}$ punctures determine the specific energy eigenstate on the quantum side of the correspondence and are constrained by a system of algebraic equations, the *monodromy-free conditions* of [12, 72], also described in [22]. In this section we will describe more in detail the construction of the classical conserved charges of the mShG model. These charges can be understood geometrically as integrals along specific contours. We will leverage this perspective in the following section to apply the $T\bar{T}$ deformation on the classical side of the ODE/IM correspondence.

3.1 Second order linear differential equations

As it is well known, the mShG equation (3.1) can be interpreted as the compatibility condition of the linear problem

$$\partial_z \Psi = L_1 \Psi, \quad \partial_{\bar{z}} \Psi = L_2 \Psi, \tag{3.4}$$

which involves the Lax pair

$$
\begin{aligned}
L_1 &= -\frac{1}{2}\partial_z \eta(\mathbf{z})\, \sigma^3 + e^\vartheta \Big(\sigma^+ e^{\eta(\mathbf{z})} - \sigma^- p(z,s) e^{-\eta(\mathbf{z})} \Big), \\
L_2 &= \frac{1}{2}\partial_{\bar{z}} \eta(\mathbf{z})\, \sigma^3 + e^{-\vartheta} \Big(\sigma^- e^{\eta(\mathbf{z})} - \sigma^+ p(\bar{z},s) e^{-\eta(\mathbf{z})} \Big).
\end{aligned}
\tag{3.5}
$$

Here σ^3 and $\sigma^\pm$ are the Pauli matrices and ϑ is the spectral parameter while Ψ in (3.4) is a two-dimensional vector.

One can write the general solution of the linear problem (3.4) as

$$
\Psi = \begin{pmatrix} e^{\frac{\vartheta}{2}} e^{\frac{\eta}{2}} \psi \\ e^{-\frac{\vartheta}{2}} e^{-\frac{\eta}{2}} (\partial_z + \partial_z \eta)\psi \end{pmatrix} = \begin{pmatrix} e^{\frac{\vartheta}{2}} e^{-\frac{\eta}{2}} (\partial_{\bar{z}} + \partial_{\bar{z}}\eta)\bar{\psi} \\ e^{-\frac{\vartheta}{2}} e^{\frac{\eta}{2}} \bar{\psi} \end{pmatrix}, \tag{3.6}
$$

where the auxiliary fields ψ and $\bar{\psi}$ solve the Schrödinger-type equations

$$
\begin{aligned}
\Big(\partial_z^2 - u(\mathbf{z}) - e^{2\vartheta} p(z,s) \Big)\psi(\mathbf{z}) &= 0, \\
\Big(\partial_{\bar{z}}^2 - \bar{u}(\mathbf{z}) - e^{-2\vartheta} p(\bar{z},s) \Big)\bar{\psi}(\mathbf{z}) &= 0.
\end{aligned}
\tag{3.7}
$$

The function $u(\mathbf{z})$ depends on the field η appearing in (3.1) as:

$$u(\mathbf{z}) = (\partial_z \eta(\mathbf{z}))^2 - \partial_z^2 \eta(\mathbf{z}), \quad \bar{u}(\mathbf{z}) = (\partial_{\bar{z}} \eta(\mathbf{z}))^2 - \partial_{\bar{z}}^2 \eta(\mathbf{z}). \tag{3.8}$$

3.2 Integrals of motion

The modified sinh-Gordon equation (3.1) is related to the unmodified one by a simple change of variables. In terms of the new coordinates $\mathbf{w} = (w, \bar{w})$,

$$\begin{pmatrix} dw \\ d\bar{w} \end{pmatrix} = \begin{pmatrix} \sqrt{p(z,s)}dz \\ \sqrt{p(\bar{z},s)}d\bar{z} \end{pmatrix}, \tag{3.9}$$

the field

$$\hat{\eta}(\mathbf{w}) = \eta(\mathbf{z}(\mathbf{w})) - \frac{1}{4} \ln p(z(w), s) p(\bar{z}(\bar{w}), s) \,, \tag{3.10}$$

satisfies the sinh-Gordon equation

$$\partial_w \partial_{\bar{w}} \hat{\eta} - e^{2\hat{\eta}} + e^{-2\hat{\eta}} = 0 \,. \tag{3.11}$$

From a WKB-type analysis of (3.7) [73, 74], one can find the expressions for the integrals of motion (**IM**s) $\mathcal{J}_{2n-1}$ and $\bar{\mathcal{J}}_{2n-1}$ with $n \in \mathbb{N}_{>0}$. Explicitly:

$$\begin{aligned}
\mathcal{J}_{2n-1} &= \frac{1}{2(2n-1)\sin\left(\frac{\pi(2n-1)}{2\alpha}\right)} \int_{\Gamma_1} \left(dw \hat{P}_{2n} + d\bar{w} \hat{R}_{2n-2} \right), \\
\bar{\mathcal{J}}_{2n-1} &= \frac{1}{2(2n-1)\sin\left(\frac{\pi(2n-1)}{2\alpha}\right)} \int_{\bar{\Gamma}_1} \left(d\bar{w} \hat{\bar{P}}_{2n} + dw \hat{\bar{R}}_{2n-2} \right).
\end{aligned} \tag{3.12}$$

The specification of the integration contours Γ_1 and $\bar{\Gamma}_1$ is of primary importance and will be discussed in a moment. The integrands appearing in (3.12) are closed 1-forms, meaning that they satisfy a continuity equation

$$\partial_{\bar{w}} \hat{P}_{2n} = \partial_w \hat{R}_{2n-2} \,, \quad \partial_w \hat{\bar{P}}_{2n} = \partial_{\bar{w}} \hat{\bar{R}}_{2n-2} \,. \tag{3.13}$$

The expressions for $\hat{P}_{2n}$ and $\hat{R}_{2n-2}$, in terms of $\hat{\eta}$, are known for generic n and can be found in [14], however the following analysis will only involve $\hat{P}_2$ and $\hat{R}_0$:

$$\begin{aligned}
\hat{P}_2 &= \frac{\hat{u}}{2} \,, \quad \hat{R}_0 = e^{-2\hat{\eta}} - 1 \,, \\
\hat{\bar{P}}_2 &= \frac{\hat{\bar{u}}}{2} \,, \quad \hat{\bar{R}}_0 = e^{-2\hat{\eta}} - 1 \,.
\end{aligned} \tag{3.14}$$

The potentials $\hat{u}$ and $\hat{\bar{u}}$ are defined as in (3.8) with η substituted for $\hat{\eta}$ and $\mathbf{z}$ for $\mathbf{w}$.

As a consequence of the continuity equation (3.13), these densities may be written in terms of the stress-energy tensor components T_2, $\bar{T}_2$ and Θ_0 of the sinh-Gordon model in dimensionless coordinates:

$$\begin{aligned}
T_2(\mathbf{w}) &= \hat{P}_2(\mathbf{w}) \,, \quad \bar{T}_2(\mathbf{w}) = \hat{\bar{P}}_2(\mathbf{w}) \,, \\
\Theta_0(\mathbf{w}) &= \hat{R}_0(\mathbf{w}) = \hat{\bar{R}}_0(\mathbf{w}) \,.
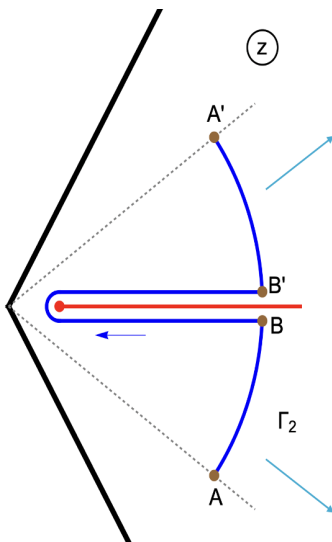\end{aligned} \tag{3.15}$$

Finally, the total energy E and the total momentum P can be expressed as

$$E = \mathcal{E} + \bar{\mathcal{E}} \,, \quad P = \mathcal{E} - \bar{\mathcal{E}} \,, \tag{3.16}$$

where

$$\begin{aligned}
\mathcal{E} &= \frac{\mathcal{M}}{4} \int_{\Gamma_1} \left(dw\, T_2(\mathbf{w}) + d\bar{w}\, \Theta_0(\mathbf{w}) \right), \\
\bar{\mathcal{E}} &= \frac{\mathcal{M}}{4} \int_{\bar{\Gamma}_1} \left(dw\, \Theta_0(\mathbf{w}) + d\bar{w}\, \bar{T}_2(\mathbf{w}) \right).
\end{aligned} \tag{3.17}$$

The dimensionful constant $\mathcal{M}$ is the equivalent of the soliton mass m, appearing on the ODE/IM correspondence's quantum side.

Figure 1. The z-plane with Γ_2 in blue. The solid rays $\varphi = \pm\pi/(2\alpha)$ in black and the dashed rays $\varphi = \pm\pi/(2\alpha + 2)$ in gray.

3.3 The integration countours Γ_1 and Γ_2

While the analysis of the modified sinh-Gordon equation (3.1) is more easily performed in the coordinates $\mathbf{w}$, we need to keep in mind that the properties of the relevant field configurations are established in the coordinates $\mathbf{z}$. Hence, we need to consider Γ_1 as the image, through the map $\mathbf{w}(\mathbf{z})$ of a fundamental contour Γ_2. Roughly speaking, this is a regularization of the contour γ_2 introduced in [14] by a pair of arches around $z = \infty$ (in the following denoted as $c_2^\pm$) starting just above and just below the positive real axis and ending on the anti-Stokes lines with directions $\varphi = \pi/(2\alpha + 2)$ and $\varphi = -\pi/(2\alpha + 2)$. The original contour γ_2 in [14] started from $+\infty$ just below the real z axis, wound around the turning point $z = s$ and went back to $+\infty$ just above the real axis, thus encircling the branch cut of $\sqrt{p(z,s)}$ clockwise. As we will mention momentarily, the regularization provided by Γ_2 is necessary to ensure the convergence of certain quantities relevant for the ODE/IM correspondence. Notice that, this simple contour-type regularization works well only in the regime $\alpha > 1$ considered in this paper. We shall return to this issue in section 3.4 below.

The contour Γ_2 is represented in figure 1, where the limit $\rho \to +\infty$ for the arches' radius is implicitly assumed. Points on the contour Γ_1 are defined as

$$w(P) - w(A) = \int_A^{P \in \Gamma_2} dz \sqrt{p(z,s)}. \tag{3.18}$$

The qualitative form of the contour Γ_1, obtained through the map (3.18), is represented in figure 2. Note that, $\bar{\Gamma}_2 = \Gamma_2^*$ and $\bar{\Gamma}_1 = \Gamma_1^*$. It is important to notice that, in the absence of singularities in the upper part of the w-complex plane, Γ_1 can be straightened to a segment joining the point $w(A)$ to $w(A')$.

Finally, for integrands with a sufficiently fast vanishing asymptotics for $\rho \to +\infty$ in the sector $|\arg(z)| \leq \pi/(2\alpha + 2)$, the contours Γ_2 $(\bar{\Gamma}_2)$ and γ_2 $(\bar{\gamma}_2)$ become totally equivalent.
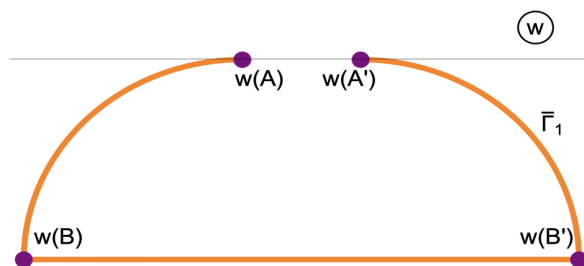
Figure 2. The complex w-plane with (pictorial) $\bar{\Gamma}_1$ in orange. $w(A') - w(A) = \bar{w}(A) - \bar{w}(A') > 0$ is fixed at $\rho \to +\infty$ (the points $w(B)$ and $w(B')$ are pushed to infinity).

In particular, given the asymptotic requirements for the field $\eta(z, \bar{z})$, in the definition of the **IM**s $\mathcal{J}_{2n-1}$, $\bar{\mathcal{J}}_{2n-1}$ one can safely trade the contour Γ_1 with the image of γ_2 under the map $\mathbf{w(z)}$.

3.4 The circumference of the cylinder

We shall now introduce the classical equivalent $\mathcal{MR}$ of the dimensionless quantity mR appearing in (2.2). The former is written in terms of simple integrals over Γ_2 and $\bar{\Gamma}_2$

$$\mathcal{MR} = \mathcal{MR}(s) = \frac{2}{\tan\left(\frac{\pi}{2\alpha}\right)} \int_{\bar{\Gamma}_2} dz \sqrt{p(z, s)} = \frac{2}{\tan\left(\frac{\pi}{2\alpha}\right)} \int_{\Gamma_2} d\bar{z} \sqrt{p(\bar{z}, s)}, \qquad (3.19)$$

where $\Gamma_2 = c_2^- \cup \gamma_2 \cup c_2^+$ is represented in figure 1. In (3.19) we purposefully defined $\mathcal{MR}$ as *either* an integral of $\sqrt{p(z, s)}$ over $\bar{\Gamma}_2$ or of $\sqrt{p(\bar{z}, s)}$ over Γ_2. At this stage this distinction might seem moot, since the two integrals are identical. However, as we are going to see later, they will not transform in the same way under a $T\bar{T}$ deformation, so it is in our interest to keep the two quantities in (3.19) separate from the start. These objects can be explicitly computed. First, let us consider the integral appearing in [14]

$$\begin{aligned}
\int_{\bar{\gamma}_2} dz \sqrt{p(z, s)} &= 2 \lim_{\rho \to +\infty} \int_\rho^s dt \sqrt{p(t, s)} \\
&= 2 \lim_{\rho \to +\infty} \left(\int_\rho^s dt \left(\sqrt{p(t, s)} - t^\alpha \right) + \frac{s^{\alpha+1}}{\alpha + 1} - \frac{\rho^{\alpha+1}}{\alpha + 1} \right) \\
&= \tan\left(\frac{\pi}{2\alpha}\right) \sqrt{\pi} s^{\alpha+1} \frac{\Gamma\left(1 + \frac{1}{2\alpha}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{2\alpha}\right)} - 2 \lim_{\rho \to +\infty} \frac{\rho^{\alpha+1}}{\alpha + 1},
\end{aligned} \qquad (3.20)$$

where the factor two is due to the discontinuity on the square root type branch cut. This integral is obviously divergent. We choose to regularize it by altering the integration contour. In fact, we calculate the limiting behaviour

$$\lim_{\rho \to +\infty} \int_{\bar{c}_2^+(\rho) \cup \bar{c}_2^-(\rho)} dz \sqrt{p(z, s)} = \left(2 - 2\cos(\Omega(\alpha + 1))\right) \lim_{\rho \to +\infty} \frac{\rho^{\alpha+1}}{\alpha + 1}. \qquad (3.21)$$

Now, if we set $\Omega = \pi/(2\alpha + 2)$, the above quantity exactly cancels the divergent term in (3.20) and we obtain the following finite expressions

$$\mathcal{M}\mathcal{R}(s) = 2\sqrt{\pi}s^{\alpha+1}\frac{\Gamma\left(1 + \frac{1}{2\alpha}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{2\alpha}\right)}, \tag{3.22}$$

establishing a match with the quantity defined in (2.5) on the quantum IM-side of the correspondence: $\mathcal{M}\mathcal{R} = mR$ or, equivalently, $s = \mathbf{s}$.

This same result could be obtained by more standard regularization schemes, for example by subtracting the $\rho \to +\infty$ divergent part appearing on the right-hand side of (3.20), or by replacing

$$\sqrt{p(z,s)} \to (p(z,s))^a, \tag{3.23}$$

with $\text{Re}(a) < -1/(2\alpha)$ and analytically continuing the outcome to $a = 1/2$. However, our contour type regularization is more natural, especially in view of the introduction of the T$\overline{\text{T}}$ perturbation as a field dependent change of coordinates.

The same computation in the $\mathbf{w}$ complex plane yields the following simple expression

$$\mathcal{M}\mathcal{R}\tan\left(\frac{\pi}{2\alpha}\right) = 2(\bar{w}(A) - \bar{w}(A')) = 2(w(A') - w(A)). \tag{3.24}$$

According to the previous definitions (3.17) and to the above expression, the total energy and momentum associated to a given field configuration should depend on both $\mathcal{R}$ and $\mathcal{M}$. However, an explicit calculation shows that the momentum is independent of $\mathcal{M}$: $P = P(\mathcal{R})$ (see also the discussion in appendix A).

4 Adding the T$\overline{\text{T}}$

At the classical level, the simplest way to introduce the T$\overline{\text{T}}$ perturbation is through a dynamical change of coordinates that involves the components of the stress-energy tensor [60]. Since we will repeatedly switch from one set of coordinates to the other, we take a moment to fix the notations. The set of coordinates $\mathbf{w}$ will always be that in which the EoMs (3.11) and, consequently the current densities, look like the unperturbed ones. In these coordinates, the effect of the T$\overline{\text{T}}$ perturbation is implicitly hidden in the integration contour $\Gamma_1^{(t)}$, which is (equivalent to) the image of Γ_1, under a dynamical change of coordinates [60]. On the other hand, in the set of coordinates $\mathbf{x}$, the effect of the T$\overline{\text{T}}$ perturbation will be visible as a modification of both the EoMs and the current densities. One of the main results of [60] is that these two alternative descriptions of a T$\overline{\text{T}}$-perturbed theory are in fact equivalent. This will allow us to prove that the energy of the T$\overline{\text{T}}$-deformed, classical mShG model satisfies the Burgers' equation (2.10), thus proving the validity of the ODE/IM correspondence for T$\overline{\text{T}}$-deformed integrable quantum field theories.

4.1 The dynamical change of coordinates

The dynamical change of space-time coordinates [60] is defined as the following differential map:

$$\begin{pmatrix} dw \\ d\bar{w} \end{pmatrix} = J^T \begin{pmatrix} dx \\ d\bar{x} \end{pmatrix}. \tag{4.1}$$

The Jacobian and its inverse are

$$J^T(\mathbf{x}) = \frac{1}{\Delta(\mathbf{w}(\mathbf{x}),0)} \begin{pmatrix} 1 + 2t\Theta_0(\mathbf{w}(\mathbf{x}),0) & -2t\bar{T}_2(\mathbf{w}(\mathbf{x}),0) \\ -2tT_2(\mathbf{w}(\mathbf{x}),0) & 1 + 2t\Theta_0(\mathbf{w}(\mathbf{x}),0) \end{pmatrix},$$

$$\left(J^T\right)^{-1}(\mathbf{w},0) = \begin{pmatrix} 1 + 2t\Theta_0(\mathbf{w},0) & 2t\bar{T}_2(\mathbf{w},0) \\ 2tT_2(\mathbf{w},0) & 1 + 2t\Theta_0(\mathbf{w},0) \end{pmatrix}, \tag{4.2}$$

where t is a dimensionless coupling trivially related to τ in (2.6), as we will see shortly, while

$$\Delta(\mathbf{w},0) = (1 + 2t\Theta_0(\mathbf{w}),0)^2 - 4t^2 T_2(\mathbf{w},0)\bar{T}_2(\mathbf{w},0). \tag{4.3}$$

The explicit form of (4.1) is

$$dw = \frac{1}{\Delta(\mathbf{w}(\mathbf{x}),0)} \Big(dx + 2t\Theta_0(\mathbf{w}(\mathbf{x}),0)dx - 2t\bar{T}_2(\mathbf{w}(\mathbf{x}),0)d\bar{x} \Big),$$

$$d\bar{w} = \frac{1}{\Delta(\mathbf{w}(\mathbf{x}),0)} (d\bar{x} + 2t\Theta_0(\mathbf{w}(\mathbf{x}),0)d\bar{x} - 2tT_2(\mathbf{w}(\mathbf{x}),0)dx). \tag{4.4}$$

Now, after a $T\bar{T}$ deformation, all the quantities introduced in the previous sections will depend on the two independent parameters $\mathcal{R}$ and t. So for a generic value of t, and in agreement with the previous discussions, we can write:

$$\mathcal{E}(\mathcal{R},t) = \frac{\mathcal{M}}{4} \int_{\Gamma_1^{(t)}} (dw\, T_2(\mathbf{w},0) + d\bar{w}\, \Theta_0(\mathbf{w},0)),$$

$$\bar{\mathcal{E}}(\mathcal{R},t) = \frac{\mathcal{M}}{4} \int_{\bar{\Gamma}_1^{(t)}} \Big(dw\, \Theta_0(\mathbf{w},0) + d\bar{w}\, \bar{T}_2(\mathbf{w},0) \Big), \tag{4.5}$$

where, according to (3.16):

$$E(\mathcal{R},t) = \mathcal{E}(\mathcal{R},t) + \bar{\mathcal{E}}(\mathcal{R},t), \quad P(\mathcal{R},t) = \mathcal{E}(\mathcal{R},t) - \bar{\mathcal{E}}(\mathcal{R},t). \tag{4.6}$$

Let us stress again that in equations (4.5) it is the deformation of the integration contours which drives the evolution in t of the conserved charges. Notice also how the map (4.1) implies that the contours $\bar{\Gamma}_1^{(t)}$ and $\Gamma_1^{(t)*}$ are not necessarily equivalent, since in general

$$(T_2(\mathbf{w}),0)^* - \bar{T}_2(\mathbf{w},0) \neq 0. \tag{4.7}$$

We shall now shift to the alternative point of view, from which the current densities evolve in t while the contours remain untouched. Using the invariance of 1-forms under coordinate transformations, we can write

$$T_2(\mathbf{w},0)\, dw + \Theta_0(\mathbf{w},0)\, d\bar{w} = T_2(\mathbf{x},t)\, dx + \Theta_0(\mathbf{x},t)\, d\bar{x},$$

$$\Theta_0(\mathbf{w},0)\, dw + \bar{T}_2(\mathbf{w},0)\, d\bar{w} = \Theta_0(\mathbf{x},t)\, dx + \bar{T}_2(\mathbf{x},t)\, d\bar{x}, \tag{4.8}$$

so that

$$\mathcal{E}(\mathcal{R},t) = \frac{\mathcal{M}}{4} \int_{\Gamma_1} (dx\, T_2(\mathbf{x},t) + d\bar{x}\, \Theta_0(\mathbf{x},t)),$$

$$\bar{\mathcal{E}}(\mathcal{R},t) = \frac{\mathcal{M}}{4} \int_{\bar{\Gamma}_1} \Big(dx\, \Theta_0(\mathbf{x},t) + d\bar{x}\, \bar{T}_2(\mathbf{x},t) \Big). \tag{4.9}$$

Using the explicit form of the map (4.4) we obtain

$$T_2(\mathbf{x}, t) = \frac{T_2(\mathbf{w}(\mathbf{x}), 0)}{\Delta(\mathbf{w}(\mathbf{x}), 0)}, \quad \bar{T}_2(\mathbf{x}, t) = \frac{\bar{T}_2(\mathbf{w}(\mathbf{x}), 0)}{\Delta(\mathbf{w}(\mathbf{x}), 0)},$$

$$\Theta_0(\mathbf{x}, t) = \frac{\Theta_0(\mathbf{w}(\mathbf{x}), 0) + 2t(\Theta_0(\mathbf{w}(\mathbf{x}), 0)^2 - T_2(\mathbf{w}(\mathbf{x}), 0)\bar{T}_2(\mathbf{w}(\mathbf{x}), 0))}{\Delta(\mathbf{w}(\mathbf{x}), 0)},$$

(4.10)

from which we conclude that

$$dw = dx - 2t\Big(\Theta_0(\mathbf{x}, t)\, dx + \bar{T}_2(\mathbf{x}, t)\, d\bar{x}\Big),$$
$$d\bar{w} = d\bar{x} - 2t(\Theta_0(\mathbf{x}, t)\, d\bar{x} + T_2(\mathbf{x}, t)\, dx).$$

(4.11)

The inverse map is

$$dx = dw + 2t\Big(\Theta_0(\mathbf{w}, 0)\, dw + \bar{T}_2(\mathbf{w}, 0)\, d\bar{w}\Big),$$
$$d\bar{x} = d\bar{w} + 2t(\Theta_0(\mathbf{w}, 0)\, d\bar{w} + T_2(\mathbf{w}, 0)\, dw).$$

(4.12)

Next, we are going to consider the effect of the $T\bar{T}$ perturbation on the 'circumference' $\mathcal{R}$ (3.24). From the explicit form of the coordinate map (4.11), it is clear that the two expressions in (3.24) change in different ways. It turns out to be useful to introduce the following parametrization

$$\mathcal{M}\mathcal{R}_0\, e^{-\theta_0} = \frac{2}{\tan\left(\frac{\pi}{2\alpha}\right)} \int\limits_{\bar{\Gamma}_1^{(t)}} dw, \quad \mathcal{M}\mathcal{R}_0\, e^{+\theta_0} = \frac{2}{\tan\left(\frac{\pi}{2\alpha}\right)} \int\limits_{\Gamma_1^{(t)}} d\bar{w},$$

(4.13)

where $\theta_0(t = 0) = 0$ and $\mathcal{R}_0(t = 0) = \mathcal{R}$. These expressions will be useful to make contact with the $T\bar{T}$-deformed theory on quantum side of the correspondence [24]. Using the map (4.11) we write

$$\int\limits_{\bar{\Gamma}_1^{(t)}} dw = \int\limits_{\bar{\Gamma}_1} dx - 2t \int\limits_{\bar{\Gamma}_1} \Big(dx\,\Theta_0(\mathbf{x}, t) + d\bar{x}\,\bar{T}_2(\mathbf{x}, t)\Big)$$

$$= \frac{1}{2}\tan\left(\frac{\pi}{2\alpha}\right)\mathcal{M}\mathcal{R} - \frac{8}{\mathcal{M}}t\bar{\mathcal{E}}(\mathcal{R}, t),$$

(4.14)

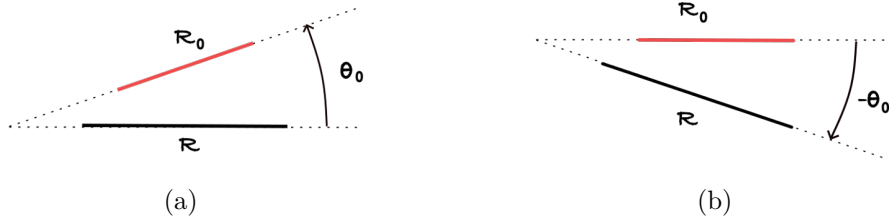where relations (4.3) and (4.10) were used. Similarly, we obtain

$$\int\limits_{\Gamma_1^{(t)}} d\bar{w} = \frac{1}{2}\tan\left(\frac{\pi}{2\alpha}\right)\mathcal{M}\mathcal{R} - \frac{8}{\mathcal{M}}t\mathcal{E}(\mathcal{R}, t).$$

(4.15)

Setting

$$\tau = -\frac{8}{\mathcal{M}^2 \tan\left(\frac{\pi}{2\alpha}\right)}t,$$

(4.16)

and using relations (4.6) we arrive at

$$\begin{cases} \mathcal{R}_0\, e^{-\theta_0} = \mathcal{R} + \tau(E(\mathcal{R}, \tau) - P(\mathcal{R}, \tau)) \\ \mathcal{R}_0\, e^{+\theta_0} = \mathcal{R} + \tau(E(\mathcal{R}, \tau) + P(\mathcal{R}, \tau)) \end{cases}.$$

(4.17)

Figure 3. Pictorial representation of the net effect of the $\mathrm{T\overline{T}}$ perturbation, as a rescaling of the 'volume' plus a Minkowski rotation. Figure 3(a) corresponds to equation (4.18), while figure 3(b) to the inverse relation (4.22).

The previous analysis and the resulting equations (4.17) have the following simple geometric interpretation: the net effect of the $\mathrm{T\overline{T}}$ perturbation is equivalent to a rescaling of $\mathcal{R}$ plus a Minkowski rotation

$$(\mathcal{R}, \mathcal{R}) \xrightarrow{\mathrm{T\overline{T}}} \left(\mathcal{R}_0 e^{-\theta_0}, \mathcal{R}_0 \, e^{+\theta_0}\right), \tag{4.18}$$

with rapidity

$$\theta_0 = \frac{1}{2} \log\left(\frac{\mathcal{R} + \tau(E(\mathcal{R}, \tau) + P(\mathcal{R}, \tau))}{\mathcal{R} + \tau(E(\mathcal{R}, \tau) - P(\mathcal{R}, \tau))}\right), \tag{4.19}$$

and

$$\mathcal{R}_0^2 = (\mathcal{R} + \tau E(\mathcal{R}, \tau))^2 - \tau^2 P^2(\mathcal{R}, \tau), \tag{4.20}$$

as pictorially represented in figure 3(a). On the other hand, the dynamical interpretation is encoded in the dependence of θ_0 and $\mathcal{R}_0$ on the energy and momentum of the state. Therefore, the net effect of the $\mathrm{T\overline{T}}$ perturbation corresponds to a doubling/redefinition of the scaling parameter $s \to (s^-, s^+)$ of the previous sections. For example, at fixed 'mass-scale' $\mathcal{M}$, it corresponds to

$$\left(\mathcal{R}_0 \, e^{-\theta_0}, \mathcal{R}_0 \, e^{+\theta_0}\right) = \left(\mathcal{R}(s^-), \mathcal{R}(s^+)\right). \tag{4.21}$$

Notice also that, one might find the interpretation of equations (4.17) more intuitively after switching to the Euclidean version of the theory, obtained through the formal replacements: $P \to \imath P$ and $\theta_0 \to \imath\theta_0$. The interpretation using complex variables is also helpful to guide the intuition in the following final steps of our analysis. We now need further, independent knowledge to close our system of equations and make it equivalent to the Burgers' equation. Extra constraints can be obtained by demanding the consistency between (4.12) and (4.11), encoding exact information on the fundamental invertibility property of the Jacobian matrix (4.1). Concretely, the idea is to derive the analogue of (4.17), for the inverse transformation

$$(\mathcal{R}_0, \mathcal{R}_0) \mapsto \left(\mathcal{R} e^{+\theta_0}, \mathcal{R} \, e^{-\theta_0}\right), \tag{4.22}$$

as depicted in figure 3(b), by integrating the r.h.s. of (4.12) along appropriate contours C_1 and $\bar{C}_1 = C_1^*$, of a form similar to that of Γ_1 and $\bar{\Gamma}_1$ (see figure 2). The endpoints of $\bar{C}_1$ are $w(\tilde{A})$ and $w(\tilde{A}')$, with relative distance

$$w(\tilde{A}') - w(\tilde{A}) = \frac{1}{2} \mathcal{M} \tan\left(\frac{\pi}{2\alpha}\right) \mathcal{R}_0. \tag{4.23}$$

Under the coordinate transformation $\mathbf{w} \to \mathbf{x}$, C_1 and $\bar{C}_1$ are mapped to $C_1^{(t)}$ and $\bar{C}_1^{(t)}$, respectively, with

$$
\begin{aligned}
\bar{C}_1^{(t)} : \qquad & x(w(\tilde{A}')) - x(w(\tilde{A})) = \frac{1}{2}\,\mathcal{M}\tan\!\left(\frac{\pi}{2\alpha}\right)\mathcal{R}e^{+\theta_0}\,, \\
C_1^{(t)} : \qquad & \bar{x}(\bar{w}(\tilde{A})) - \bar{x}(\bar{w}(\tilde{A}')) = \frac{1}{2}\,\mathcal{M}\tan\!\left(\frac{\pi}{2\alpha}\right)\mathcal{R}e^{-\theta_0}\,.
\end{aligned}
\tag{4.24}
$$

Proceeding as above, we arrive at the following equations

$$
\begin{cases}
\mathcal{R}\,e^{+\theta_0} = \mathcal{R}_0 - \tau(E(\mathcal{R}_0,0) - P(\mathcal{R}_0,0)) \\
\mathcal{R}\,e^{-\theta_0} = \mathcal{R}_0 - \tau(E(\mathcal{R}_0,0) + P(\mathcal{R}_0,0))
\end{cases}.
\tag{4.25}
$$

4.2 The Burgers' equation

As a final step in our analysis, we shall derive the Burgers' equation involving only classical physical quantities.

From equations (4.17) and (4.25), we have

$$
\sinh\theta_0 = \tau\frac{P(\mathcal{R},\tau)}{\mathcal{R}_0} = \tau\frac{P(\mathcal{R}_0,0)}{\mathcal{R}} \quad\Longrightarrow\quad \mathcal{R}P(\mathcal{R},\tau) = \mathcal{R}_0 P(\mathcal{R}_0,0)\,,
\tag{4.26}
$$

implying that $P(\mathcal{R},\tau) = P(\mathcal{R}) \propto \mathcal{R}^{-1}$. Alternatively, the exactly-quantised expression

$$
P(\mathcal{R}) = \frac{2\pi k}{\mathcal{R}}\,, \quad k \in \mathbb{Z}\,,
\tag{4.27}
$$

can be obtained through the WKB type analysis reported in appendix A. Putting (4.17) and (4.25) together, one get

$$
\begin{cases}
e^{+\theta_0}\left(E(\mathcal{R},\tau) - P(\mathcal{R},\tau)\right) = E(\mathcal{R}_0,0) - P(\mathcal{R}_0,0) \\
e^{-\theta_0}\left(E(\mathcal{R},\tau) + P(\mathcal{R},\tau)\right) = E(\mathcal{R}_0,0) + P(\mathcal{R}_0,0)
\end{cases},
\tag{4.28}
$$

that is

$$
E^2(\mathcal{R},\tau) - P^2(\mathcal{R},\tau) = E^2(\mathcal{R}_0,0) - P^2(\mathcal{R}_0,0)\,.
\tag{4.29}
$$

Moreover, (4.17) and (4.25) imply the additional constraint [60]

$$
\frac{\partial}{\partial\tau}\mathcal{R} = -E(\mathcal{R},\tau)\,,
\tag{4.30}
$$

at fixed $\mathcal{R}_0$. This result together with (4.29) is the solution, in implicit form, of the Burgers' equation [58]:

$$
\frac{\partial}{\partial\tau}E(\mathcal{R},\tau) = E(\mathcal{R},\tau)\frac{\partial}{\partial\mathcal{R}}E(\mathcal{R},\tau) + \frac{P(\mathcal{R})^2}{\mathcal{R}}\,.
\tag{4.31}
$$

In conclusion, assuming the validity of the ODE/IM at $\tau = 0$, i.e.

$$
E(R,0)_{\text{quantum}} = E(R,0)_{\text{classical}}\,, \qquad P(R)_{\text{quantum}} = P(R)_{\text{classical}}\,,
\tag{4.32}
$$

the fact that the quantum and the classical $T\overline{T}$-evolution equations coincide, implies

$$
E(R,\tau)_{\text{quantum}} = E(R,\tau)_{\text{classical}}\,.
\tag{4.33}
$$

This equality constitutes the key result of the present work.

5 Conclusions

The main result of this paper is the extension to $T\overline{T}$ perturbed models of the classical/quantum duality associated with the off-critical variant of the ODE/IM correspondence. Our work links properties of the $T\overline{T}$ perturbation on quantum and classical levels that previously stood on separate grounds. The analysis highlights a deep connection between the classical coordinate transformation, its invertibility, and the factorization property of the operator $T\overline{T}$, which plays an essential role at the quantum level. In this work, attention was restricted to the modified sinh-Gordon case of [14], however, our result has a much wider validity. First, we notice that the explicit form of the mShG potential (3.2) does not appear anywhere in section 4 — except for a few constants. Consequently, the validity of the result (4.33) can be immediately extended to ODE/IM correspondence based on more general potentials, e.g. the polynomial ones used in the context of AdS/CFT correspondence for the computation of polygonal Wilson loops [75, 76]. Similarly, it is not difficult to see how the arguments of section 4 remain valid for $T\overline{T}$ deformations of other known instances of the ODE/IM correspondence [18, 19, 21, 22, 77, 78] and are also easily adaptable to the study of the Lorentz breaking deformations introduced in [60].

We wish to stress that the Burgers' equation (4.31) is actually a general property of $T\overline{T}$-deformed classical field theories. In fact, the argument used in section 4 teaches us that everything we need to prove (4.31) are the differential maps (4.11), (4.12) and an appropriate choice of Cauchy cycles for both the **w** and the **x** coordinates, over which we integrate the closed 1-forms (4.8) in order to obtain the energy and the momentum. Integrating (4.11), (4.12) over these cycles we arrive at the equivalent of (4.17), (4.25), from which the Burgers' equation (4.31) immediately follows.

Another significant point emerging from our analysis is the link between the dynamical change of coordinates and the factorization property of $T\overline{T}$ [32]. This fact suggests the existence of coordinate transformations also for deformations driven by operators built using higher spin currents. However, preliminary inspections indicate that spacetime maps exist, besides $T\overline{T}$, only for $J\overline{T}$ [79, 80] and the $T\overline{T}_s$ Lorentz-breaking models of [60], not for the generic $T_s\overline{T}_{s'}$ deformations. Nevertheless, it may be worth investigating the relation between the latter and maps between the spacetime and more complicated surfaces embedded in higher dimensional spaces. The additional dimensions would correspond to the higher times associated to the commuting Hamiltonian flows generated by the higher conserved charges. A further, natural follow-up would be the extension of the ODE/IM to the recently discovered $\sqrt{T\overline{T}}$ deformations [61, 81–86].

Finally, it would be nice to use the current framework to understand better the geometric meaning of the Hagedorn singularity appearing in the deformed quantum spectrum and explore at a deeper level the consequences of the $T\overline{T}$ perturbation in the context of minimal surfaces and polygonal Wilson loops [75, 76] (see also the recent review [87]), and in the framework of quasi-normal modes of black holes [88, 89].

Acknowledgments

A The conformal field theory limit

At the classical level, the ultraviolet limit yielding the conformal field theory behaviour corresponds to the following (left-moving) scaling limit:

$$z = y\,e^{-\frac{\vartheta}{1+\alpha}}\,, \quad \vartheta \to +\infty\,, \quad |z| \sim s \to 0^+\,, \tag{A.1}$$

with

$$y = e^{\frac{\vartheta}{1+\alpha}}z \quad \mathbf{E} = e^{2\alpha\vartheta/(1+\alpha)}s^{2\alpha}\,, \tag{A.2}$$

kept finite. In this limit, the first equation in (3.7) reduced to the Schrödinger equation of [1], with the addition of a finite number of regular singularities $\{y_a\}$ [11, 12]

$$\left(\frac{d^2}{dy^2} - V\!\left(y, \{y_a\}_{a=1}^L\right) - y^{2\alpha} + \mathbf{E}\right)\psi_{\mathrm{CFT}}(y) = 0\,. \tag{A.3}$$

Here V is the so-called 'monster potential' [11],

$$V\!\left(y, \{y_a\}_{a=1}^L\right) = \frac{l(l+1)}{y} + 2\frac{d^2}{dy^2}\sum_{a=1}^L \log\!\left(y^{2\alpha+2} - y_a\right), \tag{A.4}$$

and the additional singularities are defined on the cone $y_a \in C_{\pi/\alpha} \setminus [\mathbf{E}^{1/(2\alpha)}, +\infty)$, away from the apex: $y_a \neq 0$.

In the right-moving sector, the opposite scaling limit

$$\bar{z} = y\,e^{-\frac{\vartheta}{1+\alpha}}\,, \quad \vartheta \to -\infty\,, \quad |\bar{z}| \sim s \to 0^+, \tag{A.5}$$

should be implemented. In this sector, a different number of singularities, $\bar{L}$, will mark the monster potential.

A.1 WKB analysis and the monster potential

A naive, straightforward WKB analysis of (A.3) runs into problems, due to the presence of two competing scales $\mathbf{E}$ and α. A more refined analysis involves the following change of variables [90]:

$$y = \mathbf{E}^{\frac{1}{2\alpha}}\hat{y}^{\frac{1}{2l+1}}\,, \quad \psi_{\mathrm{CFT}}(y) = \mathbf{E}^{-\frac{l}{2\alpha}}\hat{y}^{-\frac{l}{2l+1}}\hat{\psi}_{\mathrm{CFT}}(\hat{y})\,. \tag{A.6}$$

In this new set of coordinates, equation (A.3) reads

$$\left(-\epsilon^2\frac{d^2}{d\hat{y}^2} + q(\hat{y}) + \epsilon^2\xi_L(\hat{y}, \epsilon)\right)\hat{\psi}_{\mathrm{CFT}}(\hat{y}) = 0\,, \tag{A.7}$$

with

$$q(\hat{y}) = \frac{1}{4\lambda^2}\hat{y}^{\frac{1}{\lambda}}\left(\hat{y}^{\frac{\alpha}{\lambda}} - 1\right).\tag{A.8}$$

The last term in the parenthesis, in equation (A.7), is

$$\xi_L(\hat{y}, \epsilon) = -\frac{1}{\lambda}\frac{d^2}{d\hat{y}^2}\sum_{a=1}^{L}\log\left(\hat{y}^{\frac{\alpha+1}{\lambda}} - \epsilon^2 y_a\right)$$

$$+ \frac{2\lambda - 1}{\lambda^3}(\alpha+1)^2\hat{y}^{\frac{\alpha+1}{\lambda}-2}\sum_{a=1}^{L}\frac{\epsilon^2 y_a}{\left(\hat{y}^{\frac{\alpha+1}{\lambda}} - \epsilon^2 y_a\right)^2},\tag{A.9}$$

with

$$\lambda = l + \frac{1}{2}, \quad \epsilon = \mathbf{E}^{-\frac{\alpha+1}{2\alpha}}.\tag{A.10}$$

Now the WKB expansion of $\hat{\psi}_{\text{CFT}}(\hat{y})$ is naturally organized by the single parameter ϵ, which combines the two scales $\mathbf{E}$ and α

$$\hat{\psi}_{\text{CFT}}(\hat{y}) = \exp\left(\frac{1}{\epsilon}\sum_{b\geq 0}\epsilon^b S_b(\hat{y})\right),\tag{A.11}$$

while

$$\xi_L(\hat{y}, \epsilon) = \xi_L(\hat{y}, 0) + \sum_{c\geq 1}\epsilon^c\xi_c^{(L)}(\hat{x}).\tag{A.12}$$

The first few terms of the solution are easily computed by recursion,

$$S_0'(\hat{y}) = -\sqrt{q(\hat{y})}, \quad S_1'(\hat{y}) = -\frac{q'(\hat{y})}{4q(\hat{y})},\tag{A.13}$$

$$S_2^{(L)'}(\hat{y}) = -\frac{1}{48}\left(\frac{q''(\hat{y})}{q(\hat{y})^{\frac{3}{2}}} + 5\frac{d}{d\hat{y}}\left(\frac{q''(\hat{y})}{q(\hat{y})^{\frac{3}{2}}}\right)\right) - \frac{\xi_L(\hat{y}, 0)}{2\sqrt{q(\hat{y})}}.\tag{A.14}$$

From $S_2^{(L)}(\hat{y})$ we can extract the first **IM** [87] as

$$\mathcal{E} = \frac{M}{4}\int_{\gamma_{\text{WKB}}}d\hat{y}\,S_2^{(L)'}(\hat{y}).\tag{A.15}$$

Here the WKB integration contour γ_{WKB} starts from $+\infty$ just above the real axis, encircles the turning point $\hat{y} = 1$ and goes back to $+\infty$, just below the real axis. The mass M sets the energy scale. Similarly, for the right-moving sector

$$\bar{\mathcal{E}} = \frac{M}{4}\int_{\gamma_{\text{WKB}}}d\hat{y}\,S_2^{(\bar{L})'}(\hat{y}).\tag{A.16}$$

Now, we can find the exact expression for the total momentum:

$$P = \mathcal{E} - \bar{\mathcal{E}} = -\frac{M}{4}\int_{\gamma_{\text{WKB}}}d\hat{y}\,\frac{\xi_L(\hat{y}, 0)}{2\sqrt{q(\hat{y})}} + \frac{M}{4}\int_{\gamma_{\text{WKB}}}d\hat{y}\,\frac{\xi_{\bar{L}}(\hat{y}, 0)}{2\sqrt{q(\hat{y})}}$$

$$= \frac{M}{\epsilon}L(\alpha+1)\int_{\mathbf{E}^{\frac{1}{2\alpha}}}^{+\infty}\frac{dy}{y^2}\frac{1}{\sqrt{y^{2\alpha} - \mathbf{E}}} - \frac{M}{\epsilon}\bar{L}(\alpha+1)\int_{\mathbf{E}^{\frac{1}{2\alpha}}}^{+\infty}\frac{dy}{y^2}\frac{1}{\sqrt{y^{2\alpha} - \mathbf{E}}}$$

$$= \frac{2\pi(L - \bar{L})}{\mathcal{R}},\tag{A.17}$$

where, in the last equality in (A.17), we used the explicit form of $\mathcal{R}$

$$\mathcal{R} = \frac{2\sqrt{\pi}}{M} \frac{\Gamma\left(1 + \frac{1}{2\alpha}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{2\alpha}\right)}. \tag{A.18}$$

Therefore, we arrived at the quantization rule for the total momentum

$$P = \frac{2\pi k}{\mathcal{R}}, \quad k \in \mathbb{Z}. \tag{A.19}$$

Similarly we can compute the total energy, arriving at the following expression

$$E = \mathcal{E} + \bar{\mathcal{E}} = -\frac{\pi}{6\mathcal{R}} c_{\text{eff}}, \tag{A.20}$$

where c_{eff} is the effective central charge:

$$c_{\text{eff}} = 1 - \frac{6(l + \frac{1}{2})^2}{\alpha + 1} - 12(L + \bar{L}). \tag{A.21}$$

The results (A.19) and (A.20) can also be derived following other routes. As an example, we have checked the scaling limit of [14] (in which $s^{\alpha+1}/\mathcal{M} = M^{-1}$ is kept constant). From the general assumptions in section 3, the WKB analysis reassures us that (A.19) is a rather general result. See also [91] on recent interesting developments in the study of excited-states in the ODE/IM framework.

Finally, it is worth stressing that the total contribution to the momentum P comes from the region $(z, \bar{z}) \sim (0, 0)$, therefore the result (A.19) is correct also for the full massive version, and after the perturbation with $\mathrm{T}\overline{\mathrm{T}}$. This is an expected result as, at both the classical and the quantum levels, the additions of the sin(h)-Gordon potential and $\mathrm{T}\overline{\mathrm{T}}$ do not modify the original translational properties along the space direction.

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