Inventiones mathematicae (2024) 236:1–16
https://doi.org/10.1007/s00222-024-01236-6

Fano 4-folds with $b_2 > 12$ are products of surfaces

C. Casagrande[1]

Dedicated to Lorenzo, Sabrina, and Fabrizio

Received: 4 February 2023 / Accepted: 12 January 2024 / Published online: 5 February 2024

Abstract
Let X be a smooth, complex Fano 4-fold, and ρ_X its Picard number. We show that if $\rho_X > 12$, then X is a product of del Pezzo surfaces. The proof relies on a careful study of divisorial elementary contractions $f: X \to Y$ such that $\dim f(\mathrm{Exc}(f)) = 2$, together with the author's previous work on Fano 4-folds. In particular, given $f: X \to Y$ as above, under suitable assumptions we show that $S := f(\mathrm{Exc}(f))$ is a smooth del Pezzo surface with $-K_S = (-K_Y)_{|S}$.

Mathematics Subject Classification 14J45 · 14J35 · 14E30

1 Introduction

Smooth, complex Fano varieties have been classically intensively studied, and have attracted a lot of attention also in the last decades, due to their role in the framework of the Minimal Model Program. The Fano condition is a natural positivity condition of the tangent bundle, and it ensures a rich geometry, from both the points of view of birational geometry and of families of rational curves.

It has been known since the 90's that Fano varieties form a bounded family in each dimension. Del Pezzo surfaces are known classically, and the classification of Fano 3-folds have been in achieved in the 80's, there are 105 families.

Starting from dimension 4, there are probably too many families to get a complete classification; still we aim to better understand and describe the behavior and properties of these varieties. In this paper we focus on Fano 4-folds X with "large" Picard number ρ_X; let us recall that since X is Fano, ρ_X is equal to the second Betti number $b_2(X)$. We show the following result.

Theorem 1.1 *Let X be a smooth Fano 4-fold with $\rho_X > 12$. Then $X \cong S_1 \times S_2$, where S_i are del Pezzo surfaces.*

✉ C. Casagrande
cinzia.casagrande@unito.it

[1] Università di Torino, Dipartimento di Matematica, via Carlo Alberto 10, 10123 Torino, Italy

To the author's knowledge, all known examples of Fano 4-folds which are not products of surfaces have $\rho \leq 9$, so that we do not know whether the condition $\rho > 12$ in Theorem 1.1 is sharp. We refer the reader to [7, §6] for an overview of known Fano 4-folds with $\rho \geq 6$; there are few examples and it is an interesting problem to construct new ones.

As $\rho_{S_1 \times S_2} = \rho_{S_1} + \rho_{S_2}$, and del Pezzo surfaces have $\rho \leq 9$, Theorem 1.1 implies the following.

Corollary 1.2 *Let X be a smooth Fano 4-fold. Then $\rho_X \leq 18$.*

Let us note that Theorem 1.1 and Corollary 1.2 generalize to dimension 4 the analogous result for Fano 3-folds, established by Mori and Mukai in the 80's:

Theorem 1.3 ([15], Theorem 1.2) *Let X be a smooth Fano 3-fold with $\rho_X > 5$. Then $X \cong S \times \mathbb{P}^1$ where S is a del Pezzo surface. In particular $\rho_X \leq 10$.*

The proof of Theorem 1.1 relies on a careful study of *elementary contractions of X of type* $(3, 2)$, together with the author's previous work on Fano 4-folds. To explain this, let us introduce some notation.

Let X be a Fano 4-fold. A *contraction* is a surjective morphism $f : X \to Y$, with connected fibers, where Y is normal and projective; f is *elementary* if $\rho_X - \rho_Y = 1$. As usual, an elementary contraction can be of fiber type, divisorial, or small.

We say that an elementary contraction $f : X \to Y$ is *of type* $(3, 2)$ if it is divisorial with $\dim S = 2$, where $E := \mathrm{Exc}(f)$ and $S := f(E) \subset Y$. Such f can have at most finitely many 2-dimensional fibers; outside the images of these fibers, Y and S are smooth, and f is just the blow-up of the surface S. If $y_0 \in S$ is the image of a two-dimensional fiber, then either Y or S are singular at y_0; these singularities have been described by Andreatta and Wiśniewski, see Theorem 2.1. In any case, Y has at most isolated locally factorial and terminal singularities, while S can be not normal.

We denote by $\mathcal{N}_1(X)$ the real vector space of one-cycles with real coefficients, modulo numerical equivalence; we have $\dim \mathcal{N}_1(X) = \rho_X$. For any closed subset $Z \subset X$, we set

$$\mathcal{N}_1(Z, X) := \iota_*(\mathcal{N}_1(Z)) \subset \mathcal{N}_1(X)$$

where $\iota : Z \hookrightarrow X$ is the inclusion, so that $\mathcal{N}_1(Z, X)$ is the subspace of $\mathcal{N}_1(X)$ spanned by classes of curves in Z, and $\dim \mathcal{N}_1(Z, X) \leq \rho_Z$.

We study an elementary contraction $f : X \to Y$ of type $(3, 2)$ under the hypothesis that:

$$\dim \mathcal{N}_1(E, X) \geq 4.$$

In particular this implies that Y is Fano too (Lemma 2.3).

We would like to compare $(-K_Y)_{|S}$ to $-K_S$, but since S may be singular, we consider the minimal resolution of singularities $\mu : S' \to S$ and set $L := \mu^*((-K_Y)_{|S})$, a nef and big divisor class on S'. We show that $K_{S'} + L$ is semiample (Proposition 3.1). Then our strategy is to look for curves in S' on which $K_{S'} + L$ is trivial, using other elementary contractions of X of type $(3, 2)$ whose exceptional divisor intersects E in a suitable way.

Hence let us assume that X has another elementary contraction g_1 of type $(3, 2)$ whose exceptional divisor E_1 intersects E, and such that $E \cdot \Gamma_1 = 0$ for a curve Γ_1 contracted by g_1. Set $D := f(E_1) \subset Y$. We show that an irreducible component C_1 of $D \cap S$ is a (-1)-curve contained in the smooth locus S_{reg}, and such that $-K_Y \cdot C_1 = 1$ (Proposition 3.4, see Figure 1). If $C'_1 \subset S'$ is the transform of C_1, we have $(K_{S'} + L) \cdot C'_1 = 0$.

Finally let us assume that X has three elementary contractions g_1, g_2, g_3, all of type $(3, 2)$, satisfying the same assumptions as g_1 above. We also assume that $E_1 \cdot \Gamma_2 > 0$ and $E_1 \cdot \Gamma_3 > 0$, where $E_1 = \mathrm{Exc}(g_1)$ and Γ_2, Γ_3 are curves contracted by g_2, g_3 respectively. Then we show that S is a smooth del Pezzo surface with $-K_S = (-K_Y)_{|S}$ (Propositions 3.7 and 3.9); let us give an overview of the proof.

The previous construction yields three distinct (-1)-curves $C'_1, C'_2, C'_3 \subset S'$ such that $(K_{S'} + L) \cdot C'_i = 0$ and C'_1 intersects both C'_2 and C'_3. This shows that the contraction of S' given by $K_{S'} + L$ cannot be birational, namely $K_{S'} + L$ is not big. We also rule out the possibility of a contraction onto a curve, and conclude that $K_{S'} + L \equiv 0$. Finally we show that $\omega_S \cong \mathcal{O}_Y(K_Y)_{|S}$, where ω_S is the dualizing sheaf of S, and conclude that S is smooth and del Pezzo.

We believe that these results can be useful in the study of Fano 4-folds besides their use in the present work. It would be interesting to generalize this technique to higher dimensions.

Let us now explain how we use these results to prove Theorem 1.1. We define the *Lefschetz defect* of X as:

$$\delta_X := \max\{\operatorname{codim} \mathcal{N}_1(D, X) \mid D \subset X \text{ a prime divisor}\}.$$

This invariant, introduced in [3], measures the difference between the Picard number of X and that of its prime divisors; we refer the reader to [7] for a survey on δ_X.

Fano 4-folds with $\delta_X \geq 3$ are classified, as follows.

Theorem 1.4 ([3], Theorem 3.3) *Let X be a smooth Fano 4-fold. If $\delta_X \geq 4$, then $X \cong S_1 \times S_2$ where S_i are del Pezzo surfaces, and $\delta_X = \max_i \rho_{S_i} - 1$.*

Theorem 1.5 ([8], Proposition 1.5) *Smooth Fano 4-folds with $\delta_X = 3$ are classified. They have $5 \leq \rho_X \leq 8$, and if $\rho_X \in \{7, 8\}$ then X is a product of surfaces.*

Therefore in our study of Fano 4-folds we can assume that $\delta_X \leq 2$, that is, $\operatorname{codim} \mathcal{N}_1(D, X) \leq 2$ for every prime divisor $D \subset X$. To prove that $\rho_X \leq 12$, we look for a prime divisor $D \subset X$ with $\dim \mathcal{N}_1(D, X) \leq 10$.

To produce such a divisor, we look at contractions of X. If X has an elementary contraction of fiber type, or a divisorial elementary contraction $f : X \to Y$ with $\dim f(\mathrm{Exc}(f)) \leq 1$, it is not difficult to find a prime divisor $D \subset X$ such that $\dim \mathcal{N}_1(D, X) \leq 3$, hence $\rho_X \leq 5$ (Lemmas 2.6 and 2.7).

The case where X has a small elementary contraction is much harder and is treated in [6], where the following result is proven.

Theorem 1.6 ([6], Theorem 1.1) *Let X be a smooth Fano 4-fold. If X has a small elementary contraction, then $\rho_X \leq 12$.*

We are left with the case where every elementary contraction $f: X \to Y$ is of type $(3, 2)$. In this case we show (Theorem 4.1) that, if $\rho_X \geq 8$, we can apply our previous study of elementary contractions of type $(3, 2)$, so that if $E := \mathrm{Exc}(f)$ and $S := f(E) \subset Y$, then S is a smooth del Pezzo surface. This implies that $\dim \mathcal{N}_1(S, Y) \leq \rho_S \leq 9$, $\dim \mathcal{N}_1(E, X) = \dim \mathcal{N}_1(S, Y) + 1 \leq 10$, and finally that $\rho_X \leq 12$, proving Theorem 1.1.

The structure of the paper is as follows. In §2 we gather some preliminary results. Then in §3 we develop our study of elementary contractions of type $(3, 2)$, while in §4 we prove Theorem 1.1.

1.1 Notation

We work over the field of complex numbers.

We will frequently use the definitions and apply the techniques of birational geometry and the Minimal Model Program, without explicit references. We refer the reader to [9, 13, 14] for background and details.

Let X be a projective variety.

We denote by $\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$) the real vector space of one-cycles (respectively, Cartier divisors) with real coefficients, modulo numerical equivalence; $\dim \mathcal{N}_1(X) = \dim \mathcal{N}^1(X) = \rho_X$ is the Picard number of X.

For any closed subset $Z \subset X$, we denote by $\mathcal{N}_1(Z, X)$ the subspace of $\mathcal{N}_1(X)$ spanned by classes of curves in Z.

Let C be a one-cycle of X, and D a Cartier divisor. We denote by $[C]$ (respectively, $[D]$) the numerical equivalence class in $\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$). We also denote by $D^\perp \subset \mathcal{N}_1(X)$ the orthogonal hyperplane to the class $[D]$.

The symbol $\equiv$ stands for numerical equivalence (for both one-cycles and divisors), and $\sim$ stands for linear equivalence of divisors.

$\mathrm{NE}(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves, and $\overline{\mathrm{NE}}(X)$ is its closure. An *extremal ray* R is a one-dimensional face of $\overline{\mathrm{NE}}(X)$. If D is a Cartier divisor in X, we write $D \cdot R > 0$, $D \cdot R = 0$, and so on, if $D \cdot \gamma > 0$, $D \cdot \gamma = 0$, and so on, for a non-zero class $\gamma \in R$. We say that R is K-negative if $K_X \cdot R < 0$.

A *contraction* is a surjective morphism, with connected fibers, between normal projective varieties.

Suppose that X has terminal and locally factorial singularities, and is Fano. Then $\mathrm{NE}(X)$ is a convex polyhedral cone. Given a contraction $f: X \to Y$, we denote by $\mathrm{NE}(f)$ the convex subcone of $\mathrm{NE}(X)$ generated by classes of curves contracted by f; we recall that there is a bijection between contractions of X and faces of $\mathrm{NE}(X)$, given by $f \mapsto \mathrm{NE}(f)$. Moreover $\dim \mathrm{NE}(f) = \rho_X - \rho_Y$, in particular f is *elementary* (that is, $\rho_X - \rho_Y = 1$) if and only if $\mathrm{NE}(f)$ is an extremal ray.

When $\dim X = 4$, we say that an extremal ray R is of type $(3, 2)$ if the associated elementary contraction f is of type $(3, 2)$, namely if f is divisorial with $\dim f(\mathrm{Exc}(f)) = 2$. We also set $E_R := \mathrm{Exc}(f)$ and denote by $C_R \subset E_R$ a general fiber of $f_{|E_R}$; note that $E_R \cdot C_R = -1$ and $-K_X \cdot C_R = 1$.

We will also consider the cones $\mathrm{Eff}(X) \subset \mathcal{N}^1(X)$ of classes of effective divisors, and $\mathrm{mov}(X) \subset \mathcal{N}_1(X)$ of classes of curves moving in a family covering X. Since X is Fano, both cones are polyhedral; we have the duality relation $\mathrm{Eff}(X) = \mathrm{mov}(X)^\vee$.

If $\mathcal{N}$ is a real vector space and $S \subset \mathcal{N}$ is a subset, we denote by $\mathbb{R}S$ the linear span of S.

2 Preliminaries

In this section we gather some preliminary results that will be used in the sequel.

Andreatta and Wiśniewski have classified the possible 2-dimensional fibers of an elementary contraction of type $(3, 2)$ of a smooth Fano 4-fold. In doing this, they also describe precisely the singularities both of the target, and of the image of the exceptional divisor, as follows.

Theorem 2.1 ([1], Theorem on p. 256) *Let X be a smooth Fano 4-fold and $f: X \to Y$ an elementary contraction of type $(3, 2)$. Set $S := f(\mathrm{Exc}(f))$.*

Then f can have at most finitely many 2-dimensional fibers. Outside the images of these fibers, Y and S are smooth, and f is the blow-up of S.

Let $y_0 \in S \subset Y$ be the image of a 2-dimensional fiber; then one of the following holds:

(i) S is smooth at y_0, while Y has an ordinary double point at y_0, locally factorial and terminal;
(ii) Y is smooth at y_0, while S is singular at y_0. More precisely either S is not normal at y_0, or it has a singularity of type $\frac{1}{3}(1, 1)$ at y_0 (as the cone over a twisted cubic).

In particular the singularities of Y are at most isolated, locally factorial, and terminal.

We will need the following elementary estimates on $\dim \mathcal{N}_1(Z, X)$ in terms of a contraction $f: X \to Y$ and of $f(Z) \subset Y$.

Remark 2.2 Let $f: X \to Y$ be a contraction between normal projective varieties, and $Z \subset X$ an irreducible closed subset. Consider the pushforward of one-cycles $f_*: \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$. We have the following:

(a) $f_*(\mathcal{N}_1(Z, X)) = \mathcal{N}_1(f(Z), Y)$;
(b) $\dim \mathcal{N}_1(Z, X) \leq \rho_X - \rho_Y + \dim \mathcal{N}_1(f(Z), Y)$;
(c) if $\dim f(Z) \leq 1$, then $\dim \mathcal{N}_1(Z, X) \leq \rho_X - \rho_Y + 1$.

Indeed (a) follows from the definitions and the surjectivity of f, and (b) follows from (a) because f_* is a surjective linear map. For (c), we have $\mathcal{N}_1(f(Z), Y) = \{0\}$ if $f(Z) = \{pt\}$, and $\mathcal{N}_1(f(Z), Y) = \mathbb{R}[f(Z)]$ if $f(Z)$ is a curve; in any case $\dim \mathcal{N}_1(f(Z), Y) \leq 1$, and we apply (b).

Now we give some simple preliminary results on extremal rays of type $(3, 2)$.

Lemma 2.3 *Let X be a smooth Fano 4-fold and $f: X \to Y$ an elementary contraction of type* $(3,2)$; *set* $E := \mathrm{Exc}(f)$. *If* $\dim \mathcal{N}_1(E,X) \geq 4$, *then* $E \cdot R \geq 0$ *for every extremal ray R of X different from* $\mathrm{NE}(f)$, *and Y is Fano*.

Proof It follows from [5, Lemma 2.16 and Remark 2.17] that $\mathrm{NE}(f)$ is the unique extremal ray of X having negative intersection with E, $-K_X + E = f^*(-K_Y)$ is nef, and $(-K_X+E)^{\perp} \cap \mathrm{NE}(X) = \mathrm{NE}(f)$, so that $-K_Y$ is ample. □

Lemma 2.4 *Let X be a smooth Fano 4-fold and R_1, R_2 extremal rays of X of type* $(3,2)$ *such that* $\dim \mathcal{N}_1(E_{R_1},X) \geq 4$ *and* $E_{R_1} \cdot R_2 = 0$.

Then $E_{R_2} \cdot R_1 = 0$ and $R_1 + R_2$ is a face of $\mathrm{NE}(X)$ *whose associated contraction is birational, with exceptional locus $E_{R_1} \cup E_{R_2}$.*

Proof Let H be a nef divisor on X such that $H^{\perp} \cap \mathrm{NE}(X) = R_2$, and set $H' := H + (H \cdot C_{R_1})E_{R_1}$. Then $H' \cdot C_{R_1} = H' \cdot C_{R_2} = 0$, and if R_3 is an extremal ray of $\mathrm{NE}(X)$ different from R_1 and R_2, we have $E_{R_1} \cdot R_3 \geq 0$ by Lemma 2.3, hence $H' \cdot R_3 > 0$. Therefore H' is nef and $(H')^{\perp} \cap \mathrm{NE}(X) = R_1 + R_2$ is a face of $\mathrm{NE}(X)$.

If $\Gamma \subset X$ is an irreducible curve with $[\Gamma] \in R_1 + R_2$, then $H' \cdot \Gamma = 0$, so that either $E_{R_1} \cdot \Gamma < 0$ and $\Gamma \subset E_{R_1}$, or $H \cdot \Gamma = 0$, $[\Gamma] \in R_2$ and $\Gamma \subset E_{R_2}$. This shows that the contraction of $R_1 + R_2$ is birational with exceptional locus $E_{R_1} \cup E_{R_2}$.

We show that $E_{R_2} \cdot R_1 = 0$. By contradiction, suppose that $E_{R_2} \cdot R_1 \neq 0$. If $E_{R_2} \cdot R_1 < 0$, then $E_{R_1} = E_{R_2}$, thus $\dim \mathcal{N}_1(E_{R_2},X) \geq 4$, contradicting Lemma 2.3.

Suppose that $E_{R_2} \cdot R_1 > 0$, and let f_i be the contraction of R_i, $i = 1,2$. Since $E_{R_2} \cdot R_1 > 0$, E_{R_2} meets every non-trivial fiber of f_1, and $f_1(E_{R_1} \cap E_{R_2}) = f_1(E_{R_1})$; let Z be an irreducible component of $E_{R_1} \cap E_{R_2}$ such that $f_1(Z) = f_1(E_{R_1})$.

On the other hand $E_{R_1} \cdot R_2 = 0$, thus $E_{R_1} \cap E_{R_2}$ is a union of fibers of f_2, and $\dim f_2(Z) \leq 1$. This yields $\dim \mathcal{N}_1(Z,X) \leq 2$ by Remark 2.2(*c*).

We also have $f_1(Z) = f_1(E_{R_1})$, thus $(f_1)_*(\mathcal{N}_1(E_{R_1},X)) = (f_1)_*(\mathcal{N}_1(Z,X))$ by Remark 2.2(*a*), and $\dim (f_1)_*(\mathcal{N}_1(E_{R_1},X)) \leq \dim \mathcal{N}_1(Z,X) \leq 2$. We deduce that $\dim \mathcal{N}_1(E_{R_1},X) \leq 3$ by Remark 2.2(*b*), against our assumptions. □

Lemma 2.5 *Let X be a smooth Fano 4-fold and R_1, R_2 distinct extremal rays of X of type* $(3,2)$ *with* $\dim \mathcal{N}_1(E_{R_i},X) \geq 4$ *for $i = 1,2$. If there exists a birational contraction $g: X \to Z$ with $R_1, R_2 \subset$* $\mathrm{NE}(g)$, *then* $E_{R_1} \cdot R_2 = E_{R_2} \cdot R_1 = 0$.

Proof We note first of all that $E_{R_i} \cdot R_j \geq 0$ for $i \neq j$ by Lemma 2.3. Suppose that $E_{R_1} \cdot R_2 > 0$. Then $E_{R_1} \cdot (C_{R_1} + C_{R_2}) = E_{R_1} \cdot C_{R_2} - 1 \geq 0$. Moreover $E_{R_2} \cdot R_1 > 0$ by Lemma 2.4, so that $E_{R_2} \cdot (C_{R_1} + C_{R_2}) \geq 0$. On the other hand for every prime divisor D different from E_{R_1}, E_{R_2} we have $D \cdot (C_{R_1} + C_{R_2}) \geq 0$, therefore $[C_{R_1} + C_{R_2}] \in \mathrm{Eff}(X)^{\vee} = \mathrm{mov}(X)$. Since $[C_{R_1} + C_{R_2}] \in \mathrm{NE}(g)$, g should be of fiber type, a contradiction. □

Lemma 2.6 *Let X be a smooth Fano 4-fold with $\delta_X \leq 2$, and $g: X \to Z$ a contraction of fiber type. Then $\rho_Z \leq 4$.*

Proof This follows from [3]; for the reader's convenience we report the proof.

If $\dim Z \leq 1$, then $\rho_Z \leq 1$. If Z is a surface, take any prime divisor $D \subset X$ such that $g(D) \subsetneq Z$, namely $\dim g(D) \leq 1$. We have $\dim \mathcal{N}_1(D, X) \leq \rho_X - \rho_Z + 1$ by Remark 2.2(*c*), thus $\operatorname{codim} \mathcal{N}_1(D, X) \geq \rho_Z - 1$. Therefore $\delta_X \leq 2$ yields $\rho_Z \leq 3$.

Suppose now that $\dim Z = 3$. By [2, Lemma 2.6] we know that Z has some elementary contraction $h \colon Z \to W$. If $\dim W \leq 2$, by applying the first part of the proof to $h \circ g \colon X \to W$, we get $\rho_W \leq 3$ and hence $\rho_Z \leq 4$.

If h is birational and divisorial, then $\dim h(\operatorname{Exc}(h)) \leq 1$, and Remark 2.2(*c*) yields $\dim \mathcal{N}_1(\operatorname{Exc}(h), Z) \leq 2$. Moreover we can take a prime divisor $D \subset X$ such that $g(D) \subseteq \operatorname{Exc}(h)$, thus $\dim \mathcal{N}_1(g(D), Z) \leq 2$. Reasoning as above we conclude that $\operatorname{codim} \mathcal{N}_1(D, X) \geq \rho_Z - 2$ and $\rho_Z \leq 4$.

Finally we assume that h is birational and small. Then $\operatorname{Exc}(h)$ is a curve in Z, and $\mathcal{N}_1(\operatorname{Exc}(h), Z) = \mathbb{R}\operatorname{NE}(h)$ is one-dimensional. We show that there exists a prime divisor $D \subset X$ such that $g(D) \subseteq \operatorname{Exc}(h)$. We consider the lifting of h in X (see [2, §2.5]), which is an elementary contraction $h' \colon X \to W'$ fitting into a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{h'} & W' \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \xrightarrow{h} & W \end{array}$$

and such that $g_*(\operatorname{NE}(h')) = \operatorname{NE}(h)$. If $F \subset X$ is a non-trivial fiber of h', then g must be finite on F and $g(F) \subseteq \operatorname{Exc}(h)$. This implies that h' is a K-negative birational elementary contraction with fibers of dimension ≤ 1, therefore it must be of type $(3, 2)$ (see [16, Theorem 1.2]); let D be its exceptional divisor. Then $g(D) \subseteq \operatorname{Exc}(h)$.

Hence $\dim \mathcal{N}_1(g(D), Z) = 1$, and reasoning as above we get $\operatorname{codim} \mathcal{N}_1(D, X) \geq \rho_Z - 1$ and $\rho_Z \leq 3$. □

Lemma 2.7 ([5], Remark 2.17(1)) *Let X be a smooth Fano 4-fold. If X has a divisorial elementary contraction not of type $(3, 2)$, then $\rho_X \leq 5$.*

3 Showing that S is a del Pezzo surface

In this section we study elementary contractions of type $(3, 2)$ of a Fano 4-fold. We focus on the surface S which is the image of the exceptional divisor; as explained in the Introduction, our goal is to show that under suitable assumptions, S is a smooth del Pezzo surface.

Recall that S has isolated singularities by Theorem 2.1.

Proposition 3.1 *Let X be a smooth Fano 4-fold and $f \colon X \to Y$ an elementary contraction of type $(3, 2)$. Set $E := \operatorname{Exc}(f)$ and $S := f(E)$, and assume that $\dim \mathcal{N}_1(E, X) \geq 4$.*

Let $\mu \colon S' \to S$ be the minimal resolution of singularities, and set $L := \mu^*((-K_Y)_{|S})$. Then $K_{S'} + L$ is semiample.

If moreover $K_{S'} + L \equiv 0$*, then* S *is a smooth del Pezzo surface, and* $-K_S = (-K_Y)_{|S}$.

Proof Note that $-K_Y$ is Cartier by Theorem 2.1, and ample by Lemma 2.3, so that L is nef and big on S', and for every irreducible curve $\Gamma \subset S'$, we have $L \cdot \Gamma = 0$ if and only if Γ is μ-exceptional.

Consider the pushforward of one-cycles $f_*\colon \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$. Then $f_*(\mathcal{N}_1(E, X)) = \mathcal{N}_1(S, Y)$, therefore $\rho_{S'} \geq \rho_S \geq \dim \mathcal{N}_1(S, Y) \geq 3$.

Recall that by the Cone Theorem we have:

$$\overline{\mathrm{NE}}(S') = \overline{\mathrm{NE}}(S')_{K_{S'} \geq 0} + \sum_i R_i$$

where R_i are the $K_{S'}$-negative extremal rays of $\overline{\mathrm{NE}}(S')$ (and they are at most countably many). We show that $K_{S'} + L$ is nef; for this it is enough to show that it is non-negative on each summand.

Since L is nef, if $\gamma \in \overline{\mathrm{NE}}(S')_{K_{S'} \geq 0}$, we have $(K_{S'} + L) \cdot \gamma = K_{S'} \cdot \gamma + L \cdot \gamma \geq 0$.

Suppose now that $\overline{\mathrm{NE}}(S')$ has a $K_{S'}$-negative extremal ray R. The contraction associated to R can be onto a point (if $S' \cong \mathbb{P}^2$), onto a curve (so that $\rho_{S'} = 2$), or the blow-up of a smooth point (see for instance [14, Theorem 1-4-8]). Since $\rho_{S'} > 2$, R is generated by the class of a (-1)-curve Γ, that cannot be μ-exceptional, because μ is minimal. Then $L \cdot \Gamma > 0$ and $(K_{S'} + L) \cdot \Gamma = L \cdot \Gamma - 1 \geq 0$.

We conclude that $K_{S'} + L$ is nef on S', and also semiample by the Base-Point-Free Theorem.

We assume now that $K_{S'} + L \equiv 0$. In particular $-K_{S'}$ is nef and big, namely S' is a weak del Pezzo surface.

Set for simplicity $\mathcal{F} := \mathcal{O}_Y(K_Y)_{|S}$, invertible sheaf on S, and let ω_S be the dualizing sheaf of S. We have $K_{S'} \equiv \mu^*(\mathcal{F})$, and since S' is rational, we also have $\mathcal{O}_{S'}(K_{S'}) \cong \mu^*(\mathcal{F})$. By restricting to the open subset $\mu^{-1}(S_{reg})$, we conclude that $(\omega_S)_{|S_{reg}} \cong \mathcal{F}_{|S_{reg}}$. Now we use the following.

Lemma 3.2 *Let* S *be a reduced and irreducible projective surface with isolated singularities, and* ω_S *its dualizing sheaf. If there exists an invertible sheaf* $\mathcal{F}$ *on* S *such that* $(\omega_S)_{|S_{reg}} \cong \mathcal{F}_{|S_{reg}}$*, then* S *is normal and* $\omega_S \cong \mathcal{F}$.

This should be well-known to experts, we include a proof for lack of references. We postpone the proof of Lemma 3.2 and carry on with the proof of Proposition 3.1.

By Lemma 3.2 we have that S is normal and $\omega_S \cong \mathcal{F}$, in particular ω_S is locally free. If y_0 is a singular point of S, then by Theorem 2.1 y_0 is a singularity of type $\frac{1}{3}(1, 1)$, but this contradicts the fact that ω_S is locally free. We conclude that S is smooth, and finally that $-K_S = (-K_Y)_{|S}$ is ample, so that S is a del Pezzo surface. □

Remark 3.3 In the setting of Proposition 3.1, when $K_{S'} + L \equiv 0$ we cannot conclude that Y is smooth. A priori Y could have isolated singularities at some $y_0 \in S$; by [1] in this case $f^{-1}(y_0) \cong \mathbb{P}^2$.

Proof of Lemma 3.2 Recall that S has isolated singularities. The surface S is reduced, thus it satisfies condition (S_1), namely

$$\operatorname{depth} \mathcal{O}_{S,y} \geq 1 \quad \text{for every } y \in S.$$

Then by [12, Lemma 1.3] the dualizing sheaf ω_S satisfies condition (S_2):

$$\operatorname{depth} \omega_{S,y} \geq 2 \quad \text{for every } y \in S,$$

where $\operatorname{depth} \omega_{S,y}$ is the depth of the stalk $\omega_{S,y}$ as an $\mathcal{O}_{S,y}$-module.

Then, for every open subset $U \subset S$ such that $S \smallsetminus U$ is finite, we have $\omega_S = j_*((\omega_S)_{|U})$, where $j\colon U \hookrightarrow S$ is the inclusion. This is analogous to the properties of reflexive sheaves on normal varieties, see [11, Propositions 1.3 and 1.6] and [12, Remark 1.8]; for the reader's convenience, we recall the proof using local cohomology [10].

Set $\{y_1, \dots, y_m\} := S \smallsetminus U$. We have $\operatorname{depth}_{\{y_1,\dots,y_m\}} \omega_S := \min_i \operatorname{depth} \omega_{S,y_i} \geq 2$ [10, p. 43]. By [10, Theorem 3.8] this is equivalent to $\underline{H}^i_{\{y_1,\dots,y_m\}}(\omega_S) = 0$ for $i = 0, 1$, where $\underline{H}^i_{\{y_1,\dots,y_m\}}(\omega_S)$ is the ith local cohomology sheaf of S with coefficients in ω_S and supports in $\{y_1, \dots, y_m\}$ [10, §1], in particular $\underline{H}^0_{\{y_1,\dots,y_m\}}(\omega_S)$ is the subsheaf of ω_S of sections with support contained in $\{y_1, \dots, y_m\}$. There is an exact sequence of sheaves:

$$0 \longrightarrow \underline{H}^0_{\{y_1,\dots,y_m\}}(\omega_S) \longrightarrow \omega_S \longrightarrow j_*\big((\omega_S)_{|U}\big) \longrightarrow \underline{H}^1_{\{y_1,\dots,y_m\}}(\omega_S) \longrightarrow 0$$

[10, Corollary 1.9], hence $\underline{H}^i_{\{y_1,\dots,y_m\}}(\omega_S) = 0$ for $i = 0, 1$ is in turn equivalent to $\omega_S = j_*((\omega_S)_{|U})$.

For $U = S_{reg}$ we have $\omega_S = j_*((\omega_S)_{|S_{reg}})$. Since $\mathcal{F}$ is locally free, we get

$$\omega_S = j_*\big((\omega_S)_{|S_{reg}}\big) \cong j_*(\mathcal{F}_{|S_{reg}}) = \mathcal{F},$$

in particular ω_S is an invertible sheaf and for every $y \in Y$ we have $\omega_{S,y} \cong \mathcal{O}_{S,y}$ as an $\mathcal{O}_{S,y}$-module, thus $\operatorname{depth} \mathcal{O}_{S,y} = 2$. Therefore S has property (S_2), and it is normal by Serre's criterion. □

Proposition 3.4 *Let X be a smooth Fano 4-fold and $f\colon X \to Y$ an elementary contraction of type $(3,2)$. Set $E := \operatorname{Exc}(f)$ and $S := f(E)$, and assume that $\dim \mathcal{N}_1(E, X) \geq 4$. Let $\mu\colon S' \to S$ be the minimal resolution of singularities, and set $L := \mu^*((-K_Y)_{|S})$.*

Suppose that X has an extremal ray R_1 of type $(3,2)$ such that:

$$E \cdot R_1 = 0 \quad \text{and} \quad E \cap E_{R_1} \neq \emptyset.$$

Set $D := f(E_{R_1}) \subset Y$.

Then $D_{|S} = C_1 + \cdots + C_r$ where C_i are pairwise disjoint (-1)-curves contained in S_{reg}, $E_{R_1} = f^*(D)$, and $f_*(C_{R_1}) \equiv_Y C_i$. Moreover if $C_i' \subset S'$ is the transform of C_i, we have $(K_{S'} + L) \cdot C_i' = 0$ for every $i = 1, \dots, r$.

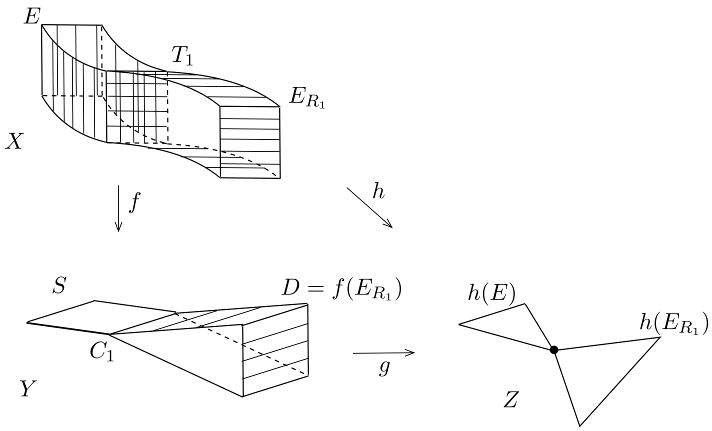

Fig. 1 The varieties in Proposition 3.4.

Proof By Lemma 2.4 we have $E_{R_1} \cdot \mathrm{NE}(f) = 0$ and $\mathrm{NE}(f) + R_1$ is a face of $\mathrm{NE}(X)$, whose associated contraction $h\colon X \to Z$ is birational with $\mathrm{Exc}(h) = E \cup E_{R_1}$. We have a diagram (see Figure 1):

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \searrow^{h} & \\ Y & \xrightarrow[g]{} & Z \end{array} \tag{3.5}$$

where g is an elementary, K-negative, divisorial contraction, with $\mathrm{Exc}(g) = D$ (recall that Y is locally factorial by Theorem 2.1, and Fano by Lemma 2.3).

Since $E_{R_1} \cdot \mathrm{NE}(f) = E \cdot R_1 = 0$, $E \cap E_{R_1}$ is both a union of fibers of f and of fibers of the contraction of R_1. This implies that $\dim f(E \cap E_{R_1}) \leq 1$, that $\mathcal{N}_1(E \cap E_{R_1}) = \ker f_* \oplus \mathbb{R}R_1 = \ker h_*$, and that $\dim h(E \cap E_{R_1}) = 0$.

We also note that both E and E_{R_1} have non-positive intersection with every irreducible curve contracted by h, thus they are both unions of fibers of h, and

$$E \cap E_{R_1} = h^{-1}\big(h(E)\big) \cap h^{-1}\big(h(E_{R_1})\big) = h^{-1}\big(h(E) \cap h(E_{R_1})\big).$$

Both $h(E)$ and $h(E_{R_1})$ are surfaces in Z, and the general fiber of h over these surfaces is one-dimensional. Moreover $h(E) \cap h(E_{R_1}) = h(E \cap E_{R_1})$ is finite, and the connected components of $E \cap E_{R_1}$ are 2-dimensional fibers of h over these points.

Using the classification of the possible 2-dimensional fibers of h in [1], as in [6, Lemma 4.15] we see that every connected component T_i of $E \cap E_{R_1}$ (which is non-empty by assumption) is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1)$, for $i = 1, \dots, r$. Set $C_i := f(T_i)$, so that $D \cap S = f(E \cap E_{R_1}) = f(\cup_i T_i) = \cup_i C_i$. Then $C_i \cong \mathbb{P}^1$, $C_i \cap C_j = \emptyset$ if $i \neq j$, and f has fibers of dimension one over C_i, therefore $C_i \subset S_{reg}$ and $C_i \subset Y_{reg}$ by Theorem 2.1.

Moreover $g(D) = h(E_{R_1})$ is a surface, namely g is of type $(3, 2)$, and C_i is a one-dimensional fiber of g contained in Y_{reg}, hence $K_Y \cdot C_i = D \cdot C_i = -1$. We also have $E_{R_1} = f^*(D)$ and $f_*(C_{R_1}) \equiv_Y C_i$.

Since $C_i \subset S_{reg}$, it is a Cartier divisor in S, and we can write $D_{|S} = m_1 C_1 + \cdots + m_r C_r$ with $m_i \in \mathbb{Z}_{>0}$ for every $i = 1, \ldots, r$. In S we have $C_i \cdot C_j = 0$ for $i \neq j$, hence for $i \in \{1, \ldots, r\}$ we get

$$-1 = D \cdot C_i = (m_1 C_1 + \cdots + m_r C_r) \cdot C_i = m_i C_i^2$$

and we conclude that $m_i = 1$ and $C_i^2 = -1$, so that C_i is a (-1)-curve in S.

Finally $-K_S \cdot C_i = -K_Y \cdot C_i = 1$, hence if $C_i' \subset S'$ is the transform of C_i, we have $(K_{S'} + L) \cdot C_i' = 0$. □

Corollary 3.6 *Let X be a smooth Fano 4-fold and $f : X \to Y$ an elementary contraction of type $(3, 2)$. Set $E := \mathrm{Exc}(f)$, and assume that $\dim \mathcal{N}_1(E, X) \geq 4$. Suppose that X has an extremal ray R_1 of type $(3, 2)$ such that $E \cdot R_1 = 0$.*

Then $R_1' := f_*(R_1)$ is an extremal ray of Y of type $(3, 2)$, and $E_{R_1} = f^*(E_{R_1'})$.

Proof If $E \cap E_{R_1} \neq \emptyset$, we are in the setting of Proposition 3.4; consider the elementary contraction $g : Y \to Z$ as in (3.5). Then $\mathrm{NE}(g) = f_*(R_1) = R_1'$ is an extremal ray of Y of type $(3, 2)$, and $f^*(E_{R_1'}) = E_{R_1}$.

If $E \cap E_{R_1} = \emptyset$, then we still have a diagram as (3.5), where g is locally isomorphic to the contraction of R_1 in X, and the statement is clear. □

Proposition 3.7 *Let X be a smooth Fano 4-fold and $f : X \to Y$ an elementary contraction of type $(3, 2)$. Set $E := \mathrm{Exc}(f)$ and $S := f(E)$, and assume that $\dim \mathcal{N}_1(E, X) \geq 4$. Let $\mu : S' \to S$ be the minimal resolution of singularities, and set $L := \mu^*((-K_Y)_{|S})$.*

Suppose that X has two extremal rays R_1, R_2 of type $(3, 2)$ such that:

$$E_{R_1} \cdot R_2 > 0 \text{ and } E \cdot R_i = 0, \ E \cap E_{R_i} \neq \emptyset \text{ for } i = 1, 2.$$

Then one of the following holds:

(i) $K_{S'} + L \equiv 0$;

(ii) *there is a contraction $g : S' \to B$ with $\dim B = 1$ such that $\mathrm{NE}(g) = (K_{S'} + L)^{\perp} \cap \overline{\mathrm{NE}}(S')$, and $E_{R_1} \cdot C_{R_2} = E_{R_2} \cdot C_{R_1} = 1$.*

Proof We apply Proposition 3.4 to f, R_1 and to f, R_2. Write $f(E_{R_1})_{|S} = C_1 + \cdots + C_r$, and let Γ_2 be an irreducible component of $f(E_{R_2})_{|S}$, so that $C_1, \ldots, C_r, \Gamma_2$ are (-1)-curves contained in S_{reg}, and $\Gamma_2 \equiv f_*(C_{R_2})$. Then

$$0 < E_{R_1} \cdot C_{R_2} = f^*(f(E_{R_1})) \cdot C_{R_2} = f(E_{R_1}) \cdot \Gamma_2 = (C_1 + \cdots + C_r) \cdot \Gamma_2, \quad (3.8)$$

hence $C_i \cdot \Gamma_2 > 0$ for some i, say $i = 1$. Since C_1 cannot be a component of $f(E_{R_2})_{|S}$, we also get $E_{R_2} \cdot C_{R_1} = f(E_{R_2})_{|S} \cdot C_1 \geq \Gamma_2 \cdot C_1 > 0$.

Let Γ_2' and C_1' in S' be the transforms of Γ_2 and C_1 respectively; then Γ_2' and C_1' are disjoint from the μ-exceptional locus, are (-1)-curves in S', $C_1' \cdot \Gamma_2' > 0$, and still by Proposition 3.4 we have $(K_{S'} + L) \cdot C_1' = (K_{S'} + L) \cdot \Gamma_2' = 0$.

Recall that $K_{S'} + L$ is semiample by Proposition 3.1. In particular, the face $(K_{S'} + L)^{\perp} \cap \overline{\mathrm{NE}}(S')$ contains the classes of two distinct (-1)-curves which meet. This means that the associated contraction cannot be birational, and we have two possibilities: either $K_{S'} + L \equiv 0$, or $K_{S'} + L$ yields a contraction $g\colon S' \to B$ onto a smooth curve. We show that this second case yields (ii).

Let $F \subset S'$ be a general fiber F of g, so that $-K_{S'} \cdot F = L \cdot F$. Since F is not μ-exceptional, we have $L \cdot F > 0$ and hence $-K_{S'} \cdot F > 0$. Thus there is a non-empty open subset $B_0 \subseteq B$ such that $(-K_{S'})_{|g^{-1}(B_0)}$ is g-ample, therefore $g_{|g^{-1}(B_0)}\colon g^{-1}(B_0) \to B_0$ is a conic bundle, $F \cong \mathbb{P}^1$, and $-K_{S'} \cdot F = 2$.

The curves C'_1 and Γ'_2 are components of the same fiber F_0 of g, and $-K_{S'} \cdot F_0 = 2 = -K_{S'} \cdot (C'_1 + \Gamma'_2)$. For any irreducible curve C_0 contained in F_0 we have $-K_{S'} \cdot C_0 = L \cdot C_0 \geq 0$, so that if C_0 is different from C'_1 and Γ'_2, we must have $-K_{S'} \cdot C_0 = L \cdot C_0 = 0$ and C_0 is μ-exceptional. But C'_1 and Γ'_2 are disjoint from the μ-exceptional locus, thus $C_0 \cap (C'_1 \cup \Gamma'_2) = \emptyset$. Since F_0 is connected, we conclude that $F_0 = C'_1 + \Gamma'_2$ and $F_0 \subset g^{-1}(B_0)$, hence F_0 is isomorphic to a reducible conic.

This also shows that C'_i for $i > 1$ are contained in different fibers of g, so that

$$C_1 \cdot \Gamma_2 = \Gamma_2 \cdot C_1 = 1 \quad \text{and} \quad C_i \cdot \Gamma_2 = 0 \quad \text{for every } i = 2, \dots, r,$$

and finally using (3.8)

$$E_{R_1} \cdot C_{R_2} = (C_1 + \cdots + C_r) \cdot \Gamma_2 = 1.$$

Similarly we conclude that $E_{R_2} \cdot C_{R_1} = 1$. □

Proposition 3.9 *Let X be a smooth Fano 4-fold and $f\colon X \to Y$ an elementary contraction of type $(3, 2)$. Set $E := \mathrm{Exc}(f)$ and $S := f(E)$, and assume that* $\dim \mathcal{N}_1(E, X) \geq 4$.

Suppose that X has three distinct extremal rays R_1, R_2, R_3 of type $(3, 2)$ such that:

$$E \cdot R_i = 0, \ E \cap E_{R_i} \neq \emptyset \textit{ for } i = 1, 2, 3, \textit{ and } E_{R_1} \cdot R_j > 0 \textit{ for } j = 2, 3.$$

Then S is a smooth del Pezzo surface and $-K_S = (-K_Y)_{|S}$.

Proof We apply Proposition 3.7 to f, R_1, R_2 and to f, R_1, R_3; we show that we are in case (i), which yields the statement by Proposition 3.1.

By contradiction, suppose that we are in case (ii); we keep the same notation as in the proof of Proposition 3.7. Then $K_{S'} + L$ yields a contraction $g\colon S' \to B$ onto a curve, $E_{R_2} \cdot R_1 > 0$, and $E_{R_3} \cdot R_1 > 0$.

Let $C_1 \subset S$ be an irreducible component of $f(E_{R_1})_{|S}$, and $C'_1 \subset S'$ its transform. For $j \in \{2, 3\}$ write $f(E_{R_j})_{|S} = \Gamma_{j1} + \cdots + \Gamma_{jr_j}$, and let $\Gamma'_{ji} \subset S'$ be the transform of Γ_{ji}.

Using (3.8) as in the proof of Proposition 3.7, we see that $(\Gamma_{j1} + \cdots + \Gamma_{jr_j}) \cdot C_1 > 0$, hence $\Gamma_{ja_j} \cdot C_1 > 0$ for some $a_j \in \{1, \dots, r_j\}$, and $\Gamma'_{ja_j} \cdot C'_1 > 0$ in S'. Then the proof of Proposition 3.7 shows that $C'_1 + \Gamma'_{2a_2}$ and $C'_1 + \Gamma'_{3a_3}$ are both fibers of g, so they should coincide, but $\Gamma'_{2a_2} \neq \Gamma'_{3a_3}$ because $R_2 \neq R_3$, and we get a contradiction. □

Corollary 3.10 *Let X be a smooth Fano 4-fold with $\delta_X \leq 2$. Suppose that X has four distinct extremal rays R_0, R_1, R_2, R_3 of type $(3, 2)$ such that:*

$$E_{R_0} \cdot R_i = 0 \text{ for } i = 1, 2, 3, \text{ and } E_{R_1} \cdot R_j > 0 \text{ for } j = 2, 3.$$

Then one of the following holds:

(i) $\dim \mathcal{N}_1(E_{R_i}, X) \leq 3$ for some $i \in \{0, 1, 2, 3\}$, in particular $\rho_X \leq 5$;
(ii) $\dim \mathcal{N}_1(E_{R_0}, X) \leq 10$, in particular $\rho_X \leq 12$. Moreover if $f\colon X \to Y$ is the contraction of R_0 and $S := f(E_{R_0})$, then S is a smooth del Pezzo surface and $-K_S = (-K_Y)_{|S}$.

Proof We assume that $\dim \mathcal{N}_1(E_{R_i}, X) \geq 4$ for every $i = 0, 1, 2, 3$, and prove (ii).

We show that $E_{R_0} \cap E_{R_i} \neq \emptyset$ for every $i = 1, 2, 3$. If $E_{R_0} \cap E_{R_i} = \emptyset$ for some $i \in \{1, 2, 3\}$, then for every curve $C \subset E_{R_0}$ we have $E_{R_i} \cdot C = 0$, so that $[C] \in (E_{R_i})^{\perp}$, and $\mathcal{N}_1(E_{R_0}, X) \subset (E_{R_i})^{\perp}$.

Since the classes $[E_{R_1}], [E_{R_2}], [E_{R_3}] \in \mathcal{N}^1(X)$ generate distinct one dimensional faces of $\mathrm{Eff}(X)$ (see [4, Remark 2.19]), they are linearly independent, hence in $\mathcal{N}_1(X)$ we have

$$\operatorname{codim}\big((E_{R_1})^{\perp} \cap (E_{R_2})^{\perp} \cap (E_{R_3})^{\perp}\big) = 3.$$

On the other hand $\operatorname{codim} \mathcal{N}_1(E_{R_0}, X) \leq \delta_X \leq 2$, thus $\mathcal{N}_1(E_{R_0}, X)$ cannot be contained in the above intersection. Then $\mathcal{N}_1(E_{R_0}, X) \not\subset (E_{R_h})^{\perp}$ for some $h \in \{1, 2, 3\}$, hence $E_{R_0} \cap E_{R_h} \neq \emptyset$. In particular, since $E_{R_0} \cdot R_h = 0$, there exists an irreducible curve $C \subset E_{R_0}$ with $[C] \in R_h$.

For $j = 2, 3$ we have $E_{R_1} \cdot R_j > 0$, and by Lemma 2.4 also $E_{R_j} \cdot R_1 > 0$. This implies that $E_{R_0} \cap E_{R_i} \neq \emptyset$ for every $i = 1, 2, 3$. For instance say $h = 3$: then $E_{R_1} \cdot R_3 > 0$ yields $E_{R_1} \cap C \neq \emptyset$, hence $E_{R_0} \cap E_{R_1} \neq \emptyset$. Then there exists an irreducible curve $C' \subset E_{R_0}$ with $[C'] \in R_1$, and $E_{R_2} \cdot R_1 > 0$ yields $E_{R_0} \cap E_{R_2} \neq \emptyset$.

Finally we apply Proposition 3.9 to get that S is a smooth del Pezzo surface and $-K_S = (-K_Y)_{|S}$. Therefore $\dim \mathcal{N}_1(S, Y) \leq \rho_S \leq 9$ and $\dim \mathcal{N}_1(E_{R_0}, X) = \dim \mathcal{N}_1(S, X) + 1 \leq 10$, so we get (ii). □

4 Proof of Theorem 1.1

In this section we show how to apply the results of §3 to bound ρ_X; the following is our main result.

Theorem 4.1 *Let X be a smooth Fano 4-fold with $\delta_X \leq 2$ and $\rho_X \geq 8$, and with no small elementary contraction.*

Then $\rho_X \leq \delta_X + 10 \leq 12$. Moreover every elementary contraction $f\colon X \to Y$ is of type $(3, 2)$, and $S := f(\mathrm{Exc}(f)) \subset Y$ is a smooth del Pezzo surface with $-K_S = (-K_Y)_{|S}$.

In the proof we will use the following terminology: if R_1, R_2 are distinct one-dimensional faces of a convex polyhedral cone $\mathcal{C}$, we say that R_1 and R_2 are *adjacent* if $R_1 + R_2$ is a face of $\mathcal{C}$. A *facet* of $\mathcal{C}$ is a face of codimension one. We will also need the following elementary fact.

Lemma 4.2 *Let $\mathcal{C}$ be a convex polyhedral cone not containing non-zero linear subspaces, and R_0 a one-dimensional face of $\mathcal{C}$. Let $R_1, \ldots, R_m$ be the one-dimensional faces of $\mathcal{C}$ that are adjacent to R_0.*

Then the linear span of $R_0, R_1, \ldots, R_m$ is $\mathbb{R}\mathcal{C}$.

Proof We can assume that $\mathcal{C} \subset \mathbb{R}^n$ with $n = \dim \mathcal{C}$. Since $\mathcal{C}$ does not contain non-zero linear subspaces, there exists an affine hyperplane $H \subset \mathbb{R}^n$ such that $P := H \cap \mathcal{C}$ is an $(n-1)$-dimensional convex polytope, and $\mathcal{C}$ is the cone over P. Then $v_i := R_i \cap H$ is a vertex of P for $i = 0, 1, \ldots, m$, and $v_1, \ldots, v_m$ are the vertices of P that are adjacent to v_0. The claim is that the affine span of $v_0, v_1, \ldots, v_m$ is H.

Up to translation we can assume that $v_0 = 0$ in $H = \mathbb{R}^{n-1}$. Let $\mathcal{D} \subset H$ be the convex cone generated by $v_1, \ldots, v_m$, with vertex v_0. Since P is convex, we have $P \subset \mathcal{D}$, and $\dim \mathcal{D} = \dim P = n - 1$. Thus the affine span of $v_0, v_1, \ldots, v_m$ has dimension $n - 1$, and coincides with H. □

Proof of Theorem 4.1 Let $f : X \to Y$ be an elementary contraction; note that $\rho_Y = \rho_X - 1 \geq 7$. Then f is not of fiber type by Lemma 2.6, and not small by assumption, so that f is divisorial. Moreover f is of type $(3, 2)$ by Lemma 2.7.

Set $E := \mathrm{Exc}(f)$ and $S := f(E) \subset Y$; we have $\dim \mathcal{N}_1(E, X) \geq \rho_X - \delta_X \geq 6$, and if $R' \neq \mathrm{NE}(f)$ is another extremal ray of X, we have $E \cdot R' \geq 0$ by Lemma 2.3. Moreover, if R' is adjacent to $\mathrm{NE}(f)$, then $E \cdot R' = 0$. Indeed the contraction $g : X \to Z$ of the face $R' + \mathrm{NE}(f)$ cannot be of fiber type by Lemma 2.6, thus it is birational and we apply Lemma 2.5.

We are going to show that there exists three extremal rays R'_1, R'_2, R'_3 adjacent to $\mathrm{NE}(f)$ such that $E_{R'_1} \cdot R'_j > 0$ for $j = 2, 3$, and then apply Corollary 3.10.

Let us consider the cone $\mathrm{NE}(Y)$. It is a convex polyhedral cone whose extremal rays R are in bijection with the extremal rays R' of X adjacent to $\mathrm{NE}(f)$, via $R = f_*(R')$, see [2, §2.5].

By Corollary 3.6, R is still of type $(3, 2)$, and $f^*(E_R) = E_{R'}$. Thus for every pair R_1, R_2 of distinct extremal rays of Y, with $R_i = f_*(R'_i)$ for $i = 1, 2$, we have $E_{R_1} \cdot R_2 = E_{R'_1} \cdot R'_2 \geq 0$.

If R_1 and R_2 are adjacent, we show that $E_{R_1} \cdot R_2 = E_{R_2} \cdot R_1 = 0$. Indeed consider the contraction $Y \to Z$ of the face $R_1 + R_2$ and the composition $g : X \to Z$, which contracts R'_1 and R'_2. Again g cannot be of fiber type by Lemma 2.6, thus it is birational and we apply Lemma 2.5 to get $E_{R'_1} \cdot R'_2 = E_{R'_2} \cdot R'_1 = 0$, thus $E_{R_1} \cdot R_2 = E_{R_2} \cdot R_1 = 0$.

Fix an extremal ray R_1 of Y. We show that there exist two distinct extremal rays R_2, R_3 of Y with $E_{R_1} \cdot R_j > 0$ for $j = 2, 3$.

Indeed since E_{R_1} is an effective divisor, there exists some curve $C \subset Y$ with $E_{R_1} \cdot C > 0$, hence there exists some extremal ray R_2 with $E_{R_1} \cdot R_2 > 0$.

By contradiction, let us assume that $E_{R_1} \cdot R = 0$ for every extremal ray R of Y different from R_1, R_2. This means that the cone $\mathrm{NE}(Y)$ has the extremal ray R_1 in the halfspace $\mathcal{N}_1(Y)_{E_{R_1} < 0}$, the extremal ray R_2 in the halfspace $\mathcal{N}_1(Y)_{E_{R_1} > 0}$, and all other extremal rays in the hyperplane $(E_{R_1})^{\perp}$.

Fix $R \neq R_1, R_2$, and let τ be a facet of $\mathrm{NE}(Y)$ containing R and not R_1. Note that $\mathbb{R}\tau \neq (E_{R_1})^{\perp}$, as E_{R_1} and $-E_{R_1}$ are not nef. By Lemma 4.2 the rays adjacent to R in τ cannot be all contained in $(E_{R_1})^{\perp}$. We conclude that R_2 is adjacent to R, therefore $E_{R_2} \cdot R = 0$, namely $R \subset (E_{R_2})^{\perp}$.

Summing up, we have shown that every extremal ray $R \neq R_1, R_2$ of Y is contained in both $(E_{R_1})^{\perp}$ and $(E_{R_2})^{\perp}$. On the other hand these rays include all the rays adjacent to R_1, so by Lemma 4.2 their linear span must be at least a hyperplane. Therefore $(E_{R_1})^{\perp} = (E_{R_2})^{\perp}$ and the classes $[E_{R_1}], [E_{R_2}] \in \mathcal{N}^1(Y)$ are proportional, which is impossible, because they generate distinct one dimensional faces of the cone $\mathrm{Eff}(Y)$ (see [4, Remark 2.19]).

We conclude that there exist two distinct extremal rays R_2, R_3 of Y with $E_{R_1} \cdot R_j > 0$ for $j = 2, 3$.

For $i = 1, 2, 3$ we have $R_i = f_*(R_i')$ where R_i' is an extremal ray of X adjacent to $\mathrm{NE}(f)$, so that $E \cdot R_i' = 0$. Moreover for $j = 2, 3$ we have $E_{R_1'} \cdot R_j' = E_{R_1} \cdot R_j > 0$.

We apply Corollary 3.10 to $\mathrm{NE}(f)$, R_1', R_2', R_3'. We have already excluded (i), and (ii) yields the statement. □

We can finally prove the following more detailed version of Theorem 1.1.

Theorem 4.3 *Let X be a smooth Fano 4-fold which is not a product of surfaces.*

Then $\rho_X \leq 12$, and if $\rho_X = 12$, then there exist $X \overset{\varphi}{\dashrightarrow} X' \overset{g}{\to} Z$ where φ is a finite sequence of flips, X' is smooth, g is a contraction, and $\dim Z = 3$.

Proof Since X is not a product of surfaces, we have $\delta_X \leq 3$ by Theorem 1.4. Moreover $\delta_X = 3$ yields $\rho_X \leq 6$ by Theorem 1.5, while $\delta_X \leq 2$ yields $\rho_X \leq 12$ by Theorems 1.6 and 4.1.

If $\rho_X = 12$, the statement follows from [6, Theorems 2.7 and 9.1]. □

Acknowledgements I thank the referee for comments that helped to improve the readability of the paper.

Funding Open access funding provided by Università degli Studi di Torino within the CRUI-CARE Agreement.

References

1. Andreatta, M., Wiśniewski, J.A.: On contractions of smooth varieties. J. Algebraic Geom. **7**, 253–312 (1998)
2. Casagrande, C.: Quasi-elementary contractions of Fano manifolds. Compos. Math. **144**, 1429–1460 (2008)

3. Casagrande, C.: On the Picard number of divisors in Fano manifolds. Ann. Sci. Éc. Norm. Supér. **45**, 363–403 (2012)
4. Casagrande, C.: On the birational geometry of Fano 4-folds. Math. Ann. **355**, 585–628 (2013)
5. Casagrande, C.: Fano 4-folds, flips, and blow-ups of points. J. Algebra **483**, 362–414 (2017)
6. Casagrande, C.: Fano 4-folds with a small contraction. Adv. Math. **405**, 108492 (2022)
7. Casagrande, C.: The Lefschetz defect of Fano varieties. Rend. Circ. Mat. Palermo (2) **72**, 3061–3075 (2023). special issue on Fano varieties
8. Casagrande, C., Romano, E.A., Secci, S.A.: Fano manifolds with Lefschetz defect 3. J. Math. Pures Appl. **163**, 625–653 (2022). Corrigendum: **168**, 108–109 (2022)
9. Debarre, O.: Higher-Dimensional Algebraic Geometry. Universitext. Springer, Berlin (2001)
10. Hartshorne, R.: Local Cohomology. a Seminar Given by A. Groethendieck, Harvard University. Fall, 1961. Lecture Notes in Math., vol. 41. Springer, Berlin (1961)
11. Hartshorne, R.: Stable reflexive sheaves. Math. Ann. **254**, 121–176 (1980)
12. Hartshorne, R.: Generalized divisors and biliaison. Illinois J. Math. **51**, 83–98 (2007)
13. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties. Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998)
14. Matsuki, K.: Introduction to the Mori Program. Universitext. Springer, Berlin (2002)
15. Mori, S., Mukai, S.: Classification of Fano 3-folds with $b_2 \geq 2$, I. In: Algebraic and Topological Theories – to the Memory of Dr. Takehiko Miyata, Kinosaki, 1984, pp. 496–545. Kinokuniya, Tokyo (1986)
16. Wiśniewski, J.A.: On contractions of extremal rays of Fano manifolds. J. Reine Angew. Math. **417**, 141–157 (1991)