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Stationary rotating and axially symmetric dust systems as peculiar General Relativistic objects

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Abstract: We study an exact solution of Einstein's equations describing a self-gravitating system, made of dust, distributed with axial symmetry and in stationary rotation, and we prove that this type of system has no Newtonian analogue. In a low-energy limit, its existence depends on the solution of a Grad-Shafranov equation in vacuum which can be interpreted as a Laplace equation for the toroidal component of the gravitomagnetic potential; in particular, in this system the relativistic rotational effects are of the order of magnitude of Newtonian ones. We therefore argue that this exact solution should contain singularities and discuss the possible consequences of using such a system as simplified model for galactic dynamics.

Keywords: Exact solutions, black holes and black hole thermodynamics in GR and beyond, gravity

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Contents

1 Introduction

General Relativity (GR) is the best model that we have to describe gravitational interactions: one century after its birth, we know that it passed with great success numerous tests and helped to greatly improve our knowledge of the near and far Universe [1]. The Einsteinian picture drastically changed our understanding of the structure of spacetime, however GR effects can be often considered small corrections with respect to the Newtonian theory of gravitation, at least in regions where the gravitational field is weak and the speeds are small if compared to the speed of light, which is reasonably true in the terrestrial environment and in the Solar System. Nonetheless, extreme astrophysical events exist in which spacetime is greatly deformed by the presence of very compact objects that are fast moving or rotating. In these cases, new phenomena arise which do not possess at all a Newtonian analogue: just to mention few of them, we can think of the existence of neutron stars, black holes and of the emission of gravitational waves.

Actually, also in condition of weak-gravitational field there are GR effects without Newtonian analogue: this is the case of the so-called gravitomagnetic effects [2] which, roughly speaking, are determined by mass currents. As a matter of fact, it is not true that GR effects are always (much) smaller than the corresponding Newtonian ones, since the latter at times simply do not exist; but even if they do exist, the situation is not always straightforward. In fact, if we consider the bending of a light a ray by a source like the Sun, we know that it can be calculated using a Newtonian approach, but the result differs by a factor 2 from the general relativistic one [3]: hence, both the Newtonian and the GR effect are of the same order of magnitude.

The purpose of this paper is to discuss another situation where, surprisingly enough, GR and Newtonian effects are expected to be of the same order of magnitude and, in addition, the very existence of the system under consideration could not be possible in a classical, i.e. Newtonian, framework. The motivations derive from this simple question: "Do there exist general relativistic self-gravitating systems, made of dust, in stationary and axially symmetric rotation?". After analyzing the question in the context of the exact solutions of Einstein's equations, we suggest that if such systems can be used as a model for a galaxy, its dynamics, i.e. the rotation curves, are also influenced by peculiar relativistic effects.

2 The exact solution

We use cylindrical coordinates (ct, ϕ, r, z) and the signature is $(-1, 1, 1, 1)$; due to the symmetry of the system, we know that matter is allowed to flow along the Killing vectors ∂_t and ∂_ϕ: as a consequence, all functions considered will depend on the coordinates (r, z) only. Accordingly, we may write the energy momentum tensor $T^{\mu\nu} = \rho u^\mu u^\nu$, where ρ is the energy density ($\rho = \rho_m c^2$, where ρ_m is the matter density) and u^μ is the fluid four-velocity. Given these symmetries and matter distribution, Einstein's equations can be integrated up to quadratures, using techniques that can be traced to the work of Geroch [4], Geroch [5], Hansen and Winicour [6], Hansen and Winicour [7], Winicour [8]: a summary of the approaches to these kinds of exact solutions can be found in the textbook by Stephani et al. [9], where it is shown that the solution of Einstein's equations is completely determined by the choice of a negative function $H(\eta)$, on which the physical properties depend: the meaning of η will be clarified below. Accordingly, the fluid velocity can be written as $u^\mu = \frac{1}{\sqrt{-H}}(1, \Omega, 0, 0)$, where $\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t}$ is the angular velocity of the fluid as seen by observers at rest with respect to the given set of coordinates. The function $H(\eta)$ depends on the existence of the auxiliary function $\mathcal{F}(\eta)$,[1]

$$\mathcal{F} = 2\eta + r^2 \int \frac{H'}{H} \frac{d\eta}{\eta} - \int \frac{H'}{H} \eta d\eta, \tag{2.1}$$

which needs to identically satisfy the equation

$$\mathcal{F}_{,rr} - \frac{1}{r}\mathcal{F}_{,r} + \mathcal{F}_{,zz} = 0. \tag{2.2}$$

Once that $H(\eta), \mathcal{F}(\eta)$ have been determined, it is possible to obtain the fluid angular velocity

$$\Omega = \frac{1}{2} \int H' \frac{d\eta}{\eta}. \tag{2.3}$$

In summary, the metric components read

$$g_{tt} = \frac{(H - \eta\Omega)^2 - r^2\Omega^2}{H}, \tag{2.4}$$

$$g_{t\phi} = \frac{\eta^2 - r^2}{(-H)}\Omega + \eta, \tag{2.5}$$

$$g_{\phi\phi} = \frac{r^2 - \eta^2}{(-H)}, \tag{2.6}$$

and the remaining metric components $g_{zz} = g_{rr} =: e^\Psi$ can be obtained using the following equations

$$\Psi_{,r} = \frac{1}{2r}\left[(g_{tt})_{,r}(g_{\phi\phi})_{,r} - (g_{tt})_{,z}(g_{\phi\phi})_{,z} - ((g_{t\phi})_{,r})^2 + ((g_{t\phi})_{,z})^2\right], \tag{2.7}$$

$$\Psi_{,z} = \frac{1}{2r}\left[(g_{tt})_{,z}(g_{\phi\phi})_{,r} + (g_{tt})_{,r}(g_{\phi\phi})_{,z} - 2(g_{t\phi})_{,r}(g_{t\phi})_{,z}\right]. \tag{2.8}$$

[1]We use the following notation: for any function of one argument, like $H(\eta)$, with a prime we mean the derivative with respect to its argument; in addition, we use a comma to indicate partial derivative with respect to a given coordinate.

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Moreover, the energy density is given by

$$8\pi G\rho = \frac{\eta^2 r^{-2}(2-\eta l)^2 - r^2 l^2}{4g_{rr}} \frac{\eta_{,r}^2 + \eta_{,z}^2}{\eta^2}, \tag{2.9}$$

where $l = \dfrac{H'}{H}$.

We can learn more about the meaning of this solution if we consider the Zero Angular Momentum Observers (ZAMO) [10]: as we discussed in [11], the metric can be written in the form

$$ds^2 = H\gamma^2 c^2 dt^2 - r^2 \frac{1}{H\gamma^2} (d\phi - \chi dt)^2 + e^{\Psi} \left(dr^2 + dz^2\right), \tag{2.10}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, being v the velocity of the dust fluid as measured by the ZAMO, and $\chi \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{H\eta}{(r^2-\eta^2)} + \Omega$ is the angular velocity of the ZAMO as seen by asymptotic inertial observers at infinity. In addition, $\eta = vr$, so this function is related to the angular momentum per unit mass of a dust element. It is possible to get the following relation

$$r\Omega = r\chi - v\gamma^2 H \tag{2.11}$$

between the coordinate velocity $r\Omega$ of the dust, its corresponding ZAMO expression v and the ZAMO's velocity $r\chi$.

The metric (2.10) is *non time-orthogonal*, because $g_{0i} \neq 0$: this is an expected feature, since these off-diagonal terms are generally related to the rotational features of the reference frame and to the rotation of the sources of the gravitational field [2]. In particular, from g_{0i} it possible to formally introduce a gravitomagnetic potential

$$A_i = -c^2 \frac{g_{0i}}{2} \tag{2.12}$$

which, in the weak-field and slow-motion approximation, enables to describe the motion of free test particles in terms of the action of a Lorentz-like force equation, exploiting the gravitoelectromagnetic analogy [2, 12, 13]. In our case the gravitomagnetic effects are related to the function χ, since $g_{0\phi} = \frac{r^2\chi}{H\gamma^2}$ and $\mathbf{A} = A_\phi \mathbf{e}_\phi$.

3 The equilibrium conditions

The exact solution considered describes the motion of a dust fluid; from the conservation law of the energy-momentum tensor $T^{\mu\nu}{}_{;\nu} = 0$, we deduce that the dust elements are in geodesic motions, which *by construction* are circular trajectories in planes at constant z. Let us see a first consequence of this hypothesis. For simplicity, we define the function $a = -H\gamma^2$ in the metric (2.10), and then we write the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[\left(-a + \frac{r^2}{a}\chi^2\right)c^2 - \frac{2r^2\chi}{a}\dot{\phi} + \frac{r^2}{a}\dot{\phi}^2 + e^{\Psi}\left(\dot{r}^2 + \dot{z}^2\right)\right], \tag{3.1}$$

where dot means derivative with respect to the coordinate time. Now, we are interested in the components of the geodesics in the z direction: we get $\frac{\partial \mathcal{L}}{\partial \dot{z}} = e^{\Psi}\dot{z}$ and, on setting $z = \text{const}$, from the Euler-Lagrange equation we get $\frac{\partial \mathcal{L}}{\partial z} = 0$, or

$$\frac{\partial}{\partial z}\left(-a + \frac{r^2}{a}\chi^2\right)c^2 + \frac{\partial}{\partial z}\left(-\frac{2\chi}{a}\right)r^2\dot{\phi} + \frac{\partial}{\partial z}\left(\frac{1}{a}\right)r^2\dot{\phi}^2 = 0. \tag{3.2}$$

If we suppose that $\chi = 0$, we get

$$\frac{\partial a}{\partial z}\left(c^2 + \frac{r^2\dot{\phi}^2}{a^2}\right) = 0. \tag{3.3}$$

So, in this case circular geodesics at $z = \text{const}$ are realizable only if the system has *cylindrical symmetry*, which means that it is not possible to obtain a compact structure. Actually, this is what happens in Newtonian gravity, where no compact or finite dust object can exist, as Bonnor [14] pointed out.

Until now, we made no assumptions on the nature of the system we are considering. If we suppose that we refer to an actual physical system, it is reasonable to expect that this solution can be used to describe some low-energy limits and, in this condition, the exact metric (2.10) can be expanded in negative powers of c, as it is customary in the post-Newtonian development [11]. Accordingly, we may write $a = 1 - \frac{2U}{c^2} + O(c^{-4})$, where U is the gravitoelectric or Newtonian potential;[2] to simplify the results, we introduce the function $\psi = \chi r^2$. Now, we consider the Euler-Lagrange equations for the coordinates r, z and, by hypothesis, we set $z = \text{const}, r = \text{const}$ to describe the geodesic motions of the dust fluid. We get

$$0 = \frac{\partial U}{\partial r} + \frac{\psi}{r}\frac{\partial}{\partial r}\left(\frac{\psi}{r}\right) - \frac{\partial \psi}{\partial r}\dot{\phi} + r\dot{\phi}^2 \tag{3.4}$$

$$0 = \frac{\partial U}{\partial z} + \frac{\psi}{r^2}\frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial z}\dot{\phi} \tag{3.5}$$

We see from the above equations that, in order to get the equilibrium for the geodesic motions, the effects determined by ψ needs to be of the same order as the Newtonian ones.

In addition, using relations (2.3) and (2.11), we can calculate the expression of the function $\mathcal{F}$ from (2.1), and we get $\mathcal{F} = -2\psi$. Accordingly, the function ψ satisfies the equation

$$\psi_{,rr} - \frac{1}{r}\psi_{,r} + \psi_{,zz} = 0, \tag{3.6}$$

which is in the form of the homogenous Grad-Shafranov equation [15, 16]. The latter equation is often used to describe the equilibrium of a two dimensional plasma, in magnetohydrodynamics; in particular, it is easy to show that using the definition of the gravitomagnetic potential (2.12), the above equation (3.6) can be regarded as a Laplace equation for the gravitomagnetic vector potential $\mathbf{A} = A_\phi \mathbf{e}_\phi$:

$$\nabla^2 \mathbf{A} = 0 \tag{3.7}$$

[2] Notice that U is defined in analogy with electromagnetism and differs by a minus sign from the standard definition of the Newtonian potential.

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If we compare the above equation with the corresponding one obtained when we consider the weak-field and slow-motion approximation of Einstein's equations [17]

$$\nabla^2 \mathbf{A} = -\frac{8\pi G}{c}\mathbf{j}, \tag{3.8}$$

where $\mathbf{j}$ is the mass-energy currents, we see that the gravitomagnetic potential determined by ψ *is not originated by the local mass distribution*, rather its sources should be located elsewhere and, as we are going to show below, they have a singular behaviour at infinity or along the siymmetry axis.

To avoid misunderstandings, it is important to give the right meaning to words. Even if there are various gravitoelectromagnetic analogies that arise in GR [2], gravitomagnetic effects are generally understood as the solutions of eq. (3.8), while the gravitoelectric ones are the solutions of the corresponding equation for the gravitoelectric or Newtoninan potential. Properly, in the Newtonian limit, U reduces to the Newtonian gravitational potential, while $A_i = O(c^{-1})$ [13].

But what we are focusing on here is different: in fact the solutions of eqs. (3.6) or (3.7) by no means go to zero as far as $c \to \infty$. From now on, we will call them *rotation effects* (or *homogenous solutions* as done previous works [11, 17, 18]) to distinguish them from the popular gravitomagnetic ones.

The fact that the Grad-Shafranov equation in vacuum coincides with the Laplace equation for the toroidal component of the vector potential [19, 20] suggests that the solutions can be found in analogy with electromagnetism. For instance, since $\mathbf{A} = \dfrac{\mathbf{m} \wedge \mathbf{x}}{|\mathbf{x}|^3}$ is the solution of the Laplace equation describing the vector potential of a magnetic-dipole $\mathbf{m}$, we see that

$$\psi = \frac{mr^2}{(r^2 + z^2)^{3/2}} \tag{3.9}$$

is a solution of the above equation (3.6) and corresponds to the Bonnor's solution [14]. More generally speaking, it is possible to obtain the solution of the above equation (3.6) as a multipole expansion (see e.g. Soběhart [21], Reusch and Neilson [22]): using spherical coordinates R, θ, φ, the solutions are in the form

$$\psi(R,\theta) = \sum_{n=2}^{\infty} \left(\alpha_n R^n + \beta_n R^{1-n}\right) \sin\theta P^1_{n-1}(\cos\theta) \tag{3.10}$$

where $P^1_{n-1}(\cos\theta)$ are the Legendre functions, and α_n, β_n are arbitrary constants. Notice that the solutions which multiply α_n are regular along the symmetry axis, while the others are regular at infinity. In particular, the solution (3.9) corresponds to $\alpha_2 = 0$ and $\beta_2 = m$ and the other terms are null.

We remark that if suppose that our system has a finite extension, the solutions that are regular at the origin do not necessarily give singularities at infinity, because it is expected that the internal solution described by (2.10) should be matched to an external solution that extends to infinity.

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4 Discussion and conclusions

We considered a self-gravitating system, made of an axially symmetric dust fluid in stationary rotation: the metric elements of the corresponding exact solution of Einstein's equations are given by eqs. (2.4)–(2.8). These elements are completely determined by the choice of the negative function $H(\eta)$, taking into account the auxiliary function $\mathcal{F}(\eta)$ which satisfies the condition expressed by eq. (2.2). Since, by hypothesis, the system is made of dust particles, their motion is geodesic: accordingly, the solution of the geodesic equations must give circular spatial trajectories at constant z coordinate.

A first point that needs to be stressed is that the very existence of this system rests on the presence of the rotation effects determined by the solution of eq. (3.6) in the low-energy limit, or of eq. (2.2) in the exact solution: in fact, if they were absent, the system would be cylindrical symmetric, i.e. with infinite extension along the symmetry axis. This is what happens in Newtonian gravity, where it is impossible to build a limited system, stationary rotating with axial symmetry: so, the system that we are considering is peculiar since it has no Newtonian analogue.

A second important point is that an inspection of the geodesic equation (3.5) reveals that to have an equilibrium along the symmetry axis, the rotation effects determined by ψ and deriving from the off-diagonals terms in the spacetime metric, cannot be negligible with respect to the Newtonian ones, represented by U.

The rotation effects stem from the solution of the vacumm Grad-Shafranov equation which can be interpreted as a Laplace equation for the toroidal component of the gravitomagnetic potential. Consequently, what we have shown suggests that, if they exist, these exact solutions of Einstein's equations should have singularities. This is not surprising: in fact, a particular case of this class is represented by the Balasin and Grumiller solution [23], which describes a rigidly rotating (i.e. $\Omega = \text{const}$) dust [24]. A recent analysis by Costa et al. [25] shows that this solution contains singularities along the axis, namely a pair of NUT rods and a cosmic string; we remember that a Newman-Unti-Tamburino (NUT) spacetime is a solution of Einstein's equations that generlises the Schwarzschild solution since, in addition to the mass parameter, it contains a second parameter, the so-called NUT charge, that can be interpreted as gravitomagnetic monopole (see e.g. Jefremov and Perlick [26] and references therein).

Seemingly, the solution of Einstein's equation describing a self-gravitating system, made of an axially symmetric dust fluid in stationary rotation, requires a vacuum solution for the rotation term ψ. Notice that, as discussed by Astesiano and Ruggiero [17], in a low-energy limit, these vacuum solutions become sources of the Poisson equation for the Newtonian potential:

$$\left[\nabla^2 U + \frac{(\partial_z \psi)^2 + \left(\partial_r \psi - 2\frac{\psi}{r}\right)^2}{2r^2}\right] = -4\pi G \rho_m \tag{4.1}$$

Actually, this is not surprising since the same happens for the exact axially symmetric solutions of Einstein equations in vacuum (see e.g. Reina and Treves [27], Bonnor [28]). Differently speaking, our approach naturally suggests that spacetime curvature, through the rotation term ψ, modifies the interplay between the sources of the gravitational field and the Newtonian potential U: the key point is that these additional sources are not necessary

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small. The ψ term contributes with an effective matter density in the form

$$\rho_\psi = \frac{1}{4\pi G}\left(\frac{(\partial_z\psi)^2 + \left(\partial_r\psi - 2\frac{\psi}{r}\right)^2}{2r^2}\right). \tag{4.2}$$

In particular, we get for the dipole solution (3.9), $\rho_\psi = \frac{1}{8\pi G}\left(\frac{9mr^2}{(r^2+z^2)^4}\right)$, which is rapidly increasing at the origin. On the other hand, if we consider a solution in the form $\psi = \alpha_3 zr^2$, we get $\rho_\psi = \frac{1}{8\pi G}\left(\alpha_3 r^2\right)$, which is smooth at the origin and has cylindrical symmetry.

Eq. (4.1) can be interpreted in a Machian sense, since the state of the system, i.e. its rotation with respect to asymptotical inertial observers, determines the local effective mass distribution which is the source of the Poisson equation.

Our analysis is quite general and does not depend on the choice of a specific system, which can be defined only when a given mass distribution is taken into account. Conversely, it shows that the existence of such a system is determined by rotation effects which are of the same order of magnitude of Newtonian ones: in other words, this is a purely relativistic system, which cannot be studied in analogy with the Newtonian case, but only using the framework of General Relativity.

The simple question: "are there any self-gravitating systems, made of dust, stationary rotating with axial symmetry?" seemingly leads us to the key role of rotation effects, that are naturally incorporated in GR, but absent in pre-relativistic gravity. We also emphasize that the system under consideration is by no means exotic, but it is made of the simplest kind of matter: dust. We argue that the existence of this type of systems can be intended as a new test of GR and we suggest that astrophysics is a natural scenario to look for possible candidates.

However, to ascertain the existence of such systems, it is crucial to note that, as of today, there is no known global solution to Einstein's equations that describes a rotating isolated matter distribution: the solution considered here is valid *within* the source. This is primarily due to the absence of a systematic procedure for matching the internal solution to the external one along an unknown surface. Consequently, global solutions remain elusive, with only limiting cases, such as rotating disks, being attainable [29, 30]. Nevertheless, it has been proposed that this system could be regarded as a model for a rotating cloud of dark matter, as dust gravitates without interacting with other particles: in particular, Ilyas et al. [31] explored Bonnor's solution, as expressed in eq. (3.9), given its asymptotically flatness and the positivity of its energy density throughout. This renders it physically plausible, with the exception of a singularity at the center attributed to the diverging vorticity field of the dust fluid there [32].

Another possible application of such systems was considered in previous works [11, 17, 18], where the relevance in the study of galactic dynamics was focused on. In doing so, it was assumed that a galaxy can be modelled as a rotating dust system: the present analysis suggests that for such a model to exist it would have to contain singularities.

At this point, a clarification is important: when a galaxy is considered as a collisionless system, the distribution functions (DF) that are solutions of the Vlasov equation are considered (see e.g. Binney and Tremaine [33]). This equation can be coupled to the Einstein equations by specifying the energy-momentum tensor in terms of the distribution function. However, it is possible to show (see [34] and references therein) that the solutions of the Vlasov-Einstein

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system where the DF has a distributional form are in one-to-one correspondence with dust solutions of the Einstein equations, where dust is meant to be a perfect fluid with zero pressure, which is what we consider.

It is significant to emphasize that the application of the solution considered in this and our previous papers to the galactic dynamic problem is different from the approach proposed by Ludwig [35], who considered the gravitomagnetic effects originating from mass currents into the solution of Einstein equations in weak-field and slow-motion approximation. In fact, in this regime mass currents produce post-Newtonian effects and their impact is negligibly small with respect to the dominant Newtonian ones [36, 37].

A recent work by Govaerts [38] focuses on an exact vacuum solution of GR equations, describing two rotating massive black holes of equal masses carrying opposite NUT charges along the symmetry axis; in this work, the possibility is suggested that the flattening of the galactic rotation curves [39–41] could be a consequence of these singular energy-momentum distributions, positioned along the rotation axis at a distance much larger than the visible spatial extent of the galaxy. In particular, the rotation effects are in the form of a dipole contribution like (3.9). The author points out that his results are just preliminary but, in view of our analysis, they appear intriguing.

We are aware that there is no guarantee that a galaxy can be described as a rotating dust fluid: however, if this can be done, at least for a very simplified model, our analysis suggests that singularities can play a role on its dynamics. It is relevant to point out that there are suggestions that collapsed objects could be described by a Kerr-Taub-NUT [42] spacetime, instead of a Kerr spacetime (see Chakraborty and Bhattacharyya [43], Chakraborty and Bhattacharyya [44] and references therein): in other words, the debate about the nature of the singularity hosted by a galaxy is indeed open. As a matter of fact, after one century of relativity, we learned that exact solutions of Einstein's equations must be taken seriously, even if they denote a strange behaviour: in fact, we have now a concrete evidence that a black hole exists at the center of a galaxy [45] and, accordingly, the Schwarzschild or, more generally, the Kerr metric can be a faithful description of natural phenomena.

In conclusion, we have shown that the presence of ψ introduces a richer geometric structure, whose impact on the effective sources of the Newtonian potential U is not trivial; furthermore the geodesic equations are greatly influenced by the presence of the rotation terms ψ. As a result, we expect rotational effects can have a twofold impact on the system dynamics. In this regard, we emphasize once again that we do not want to claim that GR solutions can explain rotation curves without dark matter: rather, we suggest that there are hints that if a galaxy (or at least a limited region) can be modeled in this way, the curvature of spacetime may play a role on its dynamics. This fact should be studied to better understand the geometric structure and, if applicable, the impact of dark matter. Or, alternatively, viewed from a different perspective, this dust fluid could be a model for dark matter itself, as previously mentioned.

In any case, we believe that these systems of self-gravitating dust in stationary rotation, due to their peculiar relativistic nature, deserve further attention to understand if they can be considered a model of real astrophysical objects.

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