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Letter

Four-loop splitting functions in QCD – The quark-to-gluon case

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ABSTRACT

We present the even-N moments $N \leq 20$ of the fourth-order (N^{3}LO) contribution $P_{\mathrm{gq}}^{(3)}(x)$ to the quark-to-gluon splitting function in perturbative QCD. These moments, obtained by analytically computing off-shell operator matrix elements for a general gauge group, agree with all known results, in particular with the moments $N \leq 10$ derived before from structure functions in deep-inelastic scattering. Using the new moments and the available endpoint constraints, we construct approximations for $P_{\mathrm{gq}}^{(3)}(x)$ which improve upon those obtained from the lowest five even moments. The remaining uncertainties of this function are now practically irrelevant at momentum fractions $x > 0.1$. The resulting errors of the convolution of P_{gq} at N^{3}LO with a typical quark distribution are small at $x \gtrsim 10^{-3}$ and exceed 1% only at $x \lesssim 10^{-4}$ for a strong coupling $\alpha_s = 0.2$. The present results for $P_{\mathrm{gq}}^{(3)}(x)$ should thus be sufficient for most collider-physics applications.

Next-to-next-to-next-to-leading order (N^{3}LO) radiative corrections in QCD form an essential ingredient for precision physics at the Large Hadron Collider (LHC) [1] and the future Electron-Ion Collider [2]. Almost a decade after the pioneering computation of the total cross section for Higgs-boson production via gluon-gluon fusion [3], N^{3}LO partonic cross sections are becoming available for an increasing number of LHC processes. Besides the partonic cross sections, fully consistent N^{3}LO calculations of collider-physics observables require the fourth-order contributions to the splitting functions for the scale dependence (evolution) of the parton distribution functions (PDFs). These N^{3}LO splitting functions are not fully known yet, but substantial progress towards their determination has been made over the past years.

The N^{3}LO flavour non-singlet quark splitting functions $P_{\mathrm{ns}}^{(3)}(x)$ are fully known in the limit of a large number of colours n_c [4]. Analytic results for their x-dependence in full QCD are known for the leading [5] and next-to-leading [6] contributions in the limit of a large number of flavours n_f; recently the QED-like $C_F^3 n_f$ contribution has also been completed [7]. For all other colour factors, only the first eight even- or odd-N moments have been published [4]; in the meantime those calculations have reached $N = 22$ [8].

So far, the situation is less favourable for the evolution of the flavour-singlet PDFs,

$$\frac{d}{d\ln\mu^2}\begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix} \quad \text{with} \quad P_{\mathrm{ik}}(x,\alpha_s) = \sum_{n=0} a_s^{n+1} P_{\mathrm{ik}}^{(n)}(x)\,, \tag{1}$$

where $a_s = \alpha_s/(4\pi) = g_s^2/(4\pi)^2$ denote the strong coupling, $q_s = \sum_{i=1}^{n_f}(q_i + \bar{q}_i)$ and g are the singlet quark and gluon PDFs, and $\otimes$ represents the Mellin convolution in the momentum variable x. Only the n_f^3 leading large-n_f part of the splitting-function matrix in eq. (1) is fully known at N^{3}LO [6]; recently the n_f^2 contributions to the initial-quark quantities $P_{\mathrm{ps}}^{(3)}(x) = P_{\mathrm{qq}}^{(3)}(x) - P_{\mathrm{ns}}^{(3)+}(x)$ and $P_{\mathrm{gq}}^{(3)}(x)$ have been derived in refs. [9,10]. The even moments of $P_{\mathrm{ik}}^{(3)}(x)$ have been computed to $N = 12$ for ik = qq and to $N = 10$ for all other entries in eq. (1) using structure functions in deep-inelastic scattering (DIS) [11,12].

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Up to a conventional sign, these even-N moments are identical to the anomalous dimensions of the twist-2 operators of spin N,

$$\gamma_{\rm ik}(N) = -\int_0^1 dx\, x^{N-1}\, P_{\rm ik}(x)\,, \quad \gamma_{\rm ik}(N,\alpha_{\rm s}) = \sum_{n=0} a_{\rm s}^{n+1}\, \gamma_{\rm ik}^{(n)}(N)\,, \tag{2}$$

which can be computed efficiently by renormalising off-shell operator matrix elements (OMEs) $A_{\rm ik} = \langle\, {\rm k}(p)|O_{\rm i}|\,{\rm k}(p)\rangle$, where $O_{\rm i}$ indicates the quark (i = q) or gluon (i = g) twist-2 operators. $\gamma_{\rm ps}^{(3)}$ and $\gamma_{\rm qg}^{(3)}$ were recently determined in this manner up to $N = 20$ in refs. [13,14]. Here we report on the corresponding results for $\gamma_{\rm gq}^{(3)}$.

In the non-abelian flavour-singlet cases, the renormalisation of the OMEs involves a mixing between gauge invariant and unphysical operators, also known as *aliens* [15–20]. Following ref. [19], the aliens entering the calculation of $\gamma_{\rm gq}$ were organised as follows. The first class is given by

$$O_{\rm A}^{I} \;=\; \eta(N)\,\left(D.F^a + g_{\rm s}\overline{\psi}\,\Delta\!\!\!/\, t^a\psi\right)\,\left(\partial^{N-2}A_a\right)\,, \quad O_{\rm c}^{I} \;=\; -\eta(N)\,(\partial\,\bar{c}^a)\,\left(\partial^{N-1}c_a\right)\,, \tag{3}$$

where Δ is a lightlike vector, $A_a = \Delta_\mu A_a^\mu$, $\partial = \Delta_\mu \partial^\mu$ and $D.F^a = \Delta_\nu(\partial_\mu\delta^{ab} + g_{\rm s} f^{acb}A_\mu^c)\,F^{\mu\nu,b}$. We have determined $\eta(N \leq 20)$ to order $\alpha_{\rm s}^3$ and found agreement with ref. [20]. Next we need

$$O_{\rm A}^{II} = g_{\rm s}\, f^{\,aa_1a_2}\left(D.F^a + g_{\rm s}\overline{\psi}\,\Delta\!\!\!/\, t^a\psi\right) \sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=N-3}} \kappa^{(1)}_{n_1n_2}\left(\partial^{n_1}A_{a_1}\right)\left(\partial^{n_2}A_{a_2}\right), \tag{4}$$

$$O_{\rm c}^{II} = -g_{\rm s}\, f^{\,aa_1a_2}\,(\partial\,\bar{c}^a) \sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=N-3}} \eta^{(1)}_{n_1n_2}\left(\partial^{n_1}A_{a_1}\right)\left(\partial^{n_2+1}c_{a_2}\right). \tag{5}$$

After picking a basis of independent constants $\eta^{(1)}_{n_1n_2}$ and $\kappa^{(1)}_{n_1n_2}$ under the relations imposed by BRST and antiBRST symmetry [19], we have determined all the required two-loop mixing constants up to $N = 20$. The calculation of the renormalisation constants has been performed using a specifically adapted implementation of R^* [21–23] which will be described in detail in a future publication. We found complete agreement for those renormalisation constants which were given in ref. [20]. In contrast to refs. [13,14], we now also require the class-*III* aliens

$$O_{\rm A_1}^{III} = g_{\rm s}^2 f^{\,aa_1x} f^{\,a_2a_3x}\,\left(D.F^a + g_{\rm s}\bar{\psi}\Delta\!\!\!/ T^a\psi\right) \sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=N-4}} \kappa^{(1)}_{n_1n_2n_3}\left(\partial^{n_1}A_{a_1}\right)\left(\partial^{n_2}A_{a_2}\right)\left(\partial^{n_3}A_{a_3}\right), \tag{6}$$

$$O_{\rm A_2}^{III} = g_{\rm s}^2 d^{\,aa_1a_2a_3}\,\left(D.F^a + g_{\rm s}\bar{\psi}\Delta\!\!\!/ T^a\psi\right) \sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=N-4}} \kappa^{(2)}_{n_1n_2n_3}\left(\partial^{n_1}A_{a_1}\right)\left(\partial^{n_2}A_{a_2}\right)\left(\partial^{n_3}A_{a_3}\right), \tag{7}$$

$$O_{\rm c}^{III} = -g_{\rm s}^2 f^{\,aa_1x} f^{\,a_2a_3x}\,(\partial\bar{c}^a) \sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=N-4}} \eta^{(1)}_{n_1n_2n_3}\left(\partial^{n_1}A_{a_1}\right)\left(\partial^{n_2}A_{a_2}\right)\left(\partial^{n_3+1}c_{a_3}\right). \tag{8}$$

We have computed a basis of independent mixing constants $\kappa^{(1)}_{ijk}$ at one-loop order, finding agreement with ref. [20]. While the gluonic and ghost parts were already presented in ref. [19], the quark contributions to the operators in eq. (6) are presented here for the first time in this form. Additional operators will be required for the determination of the gluon-gluon splitting function.

The calculation of $\gamma_{\rm gq}^{(3)}$ requires the OMEs $A_{\rm gq}$ to four loops and $A_{\rm qq}$ to three loops. In addition, the OMEs $A_{\rm iq}$ for all alien operators i are needed at the third order. The Feynman diagrams for the OMEs have been generated using QGRAF [24] and processed, see ref. [25] using a FORM [26–28] program that classifies them according to their colour factors [29] and topologies. The calculation of the two-point functions has been performed by an optimised in-house version of FORCER [30]. By renormalising the OMEs computed in $4 - 2\varepsilon$ dimensions, we have obtained the anomalous dimension $\gamma_{\rm gq}^{(3)}$. For QCD, i.e., the gauge group SU($n_c = 3$), this results in the numerical values

$$\begin{aligned}
\gamma_{\rm gq}^{(3)}(N=2) &= -16663.2255 + 4439.14375\,n_f - 202.555479\,n_f^2 - 6.37539072\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=4) &= -6565.75315 + 1291.00675\,n_f - 16.1461902\,n_f^2 - 0.83976340\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=6) &= -3937.47937 + 679.718506\,n_f - 1.37207753\,n_f^2 - 0.13979433\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=8) &= -2803.64411 + 436.393057\,n_f + 1.81494625\,n_f^2 + 0.07358858\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=10) &= -2179.48761 + 310.063163\,n_f + 2.65636842\,n_f^2 + 0.15719522\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=12) &= -1786.31231 + 234.383019\,n_f + 2.82817592\,n_f^2 + 0.19211953\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=14) &= -1516.59810 + 184.745296\,n_f + 2.78076831\,n_f^2 + 0.20536518\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=16) &= -1320.36106 + 150.076970\,n_f + 2.66194730\,n_f^2 + 0.20798493\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=18) &= -1171.29329 + 124.717778\,n_f + 2.52563073\,n_f^2 + 0.20512226\,n_f^3\,,\\
\gamma_{\rm gq}^{(3)}(N=20) &= -1054.26140 + 105.497994\,n_f + 2.39223358\,n_f^2 + 0.19938504\,n_f^3\,.
\end{aligned} \tag{9}$$

The n_f^2 and n_f^3 parts of eqs. (9) agree with the all-N expressions of refs. [6,10]. The values at $N \leq 10$ agree with those obtained in refs. [11,12] from structure functions in DIS – a conceptionally much simpler but computationally much more involved approach. Our new exact results at $N \geq 12$, expressed in terms of rational numbers and values of Riemann's ζ-function, can be found in eqs. (A.3) – (A.7) in the appendix for a general compact simple gauge group.

The all-N expressions of the N^{n}LO anomalous dimensions in the $\overline{\text{MS}}$ scheme include ζ_m up to $m_{\max} = 2n-1$ (but not ζ_2), powers of simple denominators D_a and harmonic sums $S_{\vec{w}}(N)$ [31], see also ref. [32]. Up to three loops (N^{2}LO) [33,34] the anomalous dimensions (2) have the form

$$\gamma^{(n)}_{\rm ik}(N)\Big|_{\zeta_m} = \sum_a \sum_{p=p_0}^{2n+1-m} \sum_{w=0}^{2n+1-m-p} c^{(n)}_{mapw}\, D_a^p\, S_w(N) \quad \text{where} \quad D_a = \frac{1}{(N+a)}\,, \tag{10}$$

$$S_{\pm m_1}(N) = \sum_{j=1}^{N} (\pm 1)^j \frac{1}{j^{m_1}}\,, \quad S_{\pm m_1, m_2}(N) = \sum_{j=1}^{N} (\pm 1)^j \frac{1}{j^{m_1}} S_{m_2}(j)\,, \quad \ldots \tag{11}$$

with $\zeta_0 \equiv 1$, offsets $a = -1, 0, 1, 2$ and minimal powers $p_0 = 0$ for i = k and $p_0 = 1$ for i $\neq$ k. $S_w(N)$ in eq. (10) is a shorthand for all $2^w - 1$ harmonic sums of a given weight w – defined as the sum of the absolute values of the indices m_i ($m_i = -1$ does not occur) – with $S_0(N) \equiv 1$. The coefficients $c^{(n)}_{mapw}$ are integer, modulo small-prime denominators that can be readily removed. Thus Diophantine equations can be used for their determination from a limited number of N-values.

On top of the above contributions, terms with D_{-2} (in special combinations with sums $S_{\vec{w}}$, as there is no pole at $N = 2$) occur in the coefficient functions for inclusive DIS already at two loops [35–37]. At the third order in α_s, positive powers of N enter the DIS coefficient functions with a special weight-5 combination of sums and Riemann-ζ values [38],

$$f(N) = 5\zeta_5 - 2S_{-5} + 4\zeta_3 S_{-2} - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5\,. \tag{12}$$

If such additional contributions are present in $\gamma_{\rm ik}(N)$ beyond three loops, then they are visible already in the terms with $\zeta_{m\geq 3}$, in the case of D_{-2} by the appearance of delta-function contributions $\delta(N-2)$. The comparatively simple $\zeta_{m\geq 3}$ terms can thus provide useful information about the most complex parts of the functions, the non-ζ contributions.

Up to terms in addition to eq. (10), we expect the ζ_5 contributions to $\gamma^{(3)}_{\rm gq}$ to resemble the ζ_3 parts of $\gamma^{(2)}_{\rm gq}$. In particular, $S_1 \equiv S_1(N)$ should enter with the leading-order combination of $1/(N+a)$ denominators, $p_{\rm gq} = 2/(N-1) - 2/N + 1/(N+1)$. The available 10 moments are then sufficient for a direct determination of the coefficients $c^{(3)}_{5apw}$ by systems of linear equations, resulting in

$$\begin{aligned}
\gamma^{(3)}_{\rm gq}(N)\Big|_{\zeta_5} &= 160\, C_F^4 \Big((N+2)/12 - 28\, D_{-1} - 239/3\, D_0 + 473/6\, D_1 + 52\, D_0^2 + 26\, D_1^2 + 14\, p_{\rm gq}\, S_1 \Big) \\
&+ 160\, C_F^3 C_A \Big(-(N+2)/6 + 44\, D_{-1} + 442/3\, D_0 - 428/3\, D_1 - 92\, D_0^2 - 46\, D_1^2 - 22\, p_{\rm gq}\, S_1 \Big) \\
&+ 80\, C_F^2 C_A^2 \Big((N+2)/6 - 44\, D_{-1} - 548/3\, D_0 + 1007/6\, D_1 - 8/3\, D_2 + 102\, D_0^2 + 51\, D_1^2 \\
&- 8\, D_{-1}^2 + 25\, p_{\rm gq}\, S_1 \Big) + 80/3\, C_F C_A^3 \Big(26\, D_{-1} + 224/3\, D_0 - 166/3\, D_1 + 16/3\, D_2 - 24\, D_0^2 \\
&- 12\, D_1^2 + 16\, D_{-1}^2 - 19\, p_{\rm gq}\, S_1 \Big) + 320\, d_R^{abcd} d_A^{abcd}/n_c \Big(-10\, D_{-1} - 202/3\, D_0 + 391/6\, D_1 \\
&- 8/3\, D_2 - 8\, D_{-1}^2 + 42\, D_0^2 + 21\, D_1^2 + 8\, p_{\rm gq}\, S_1 \Big) + 320\, n_f\, C_F^3\, p_{\rm gq} - 160/3\, n_f\, C_F^2 C_A\, p_{\rm gq} \\
&+ 160/9\, n_f\, C_F C_A^2 \Big(-26\, D_{-1} + 62\, D_0 - 49\, D_1 - 24\, D_0^2 - 12\, D_1^2 \Big) \\
&+ 640/3\, n_f\, d_R^{abcd} d_R^{abcd}/n_c \Big(64\, D_0 - 68\, D_1 + 8\, D_{-1} - 48\, D_0^2 - 24\, D_1^2 \Big)\,.
\end{aligned} \tag{13}$$

The contributions of the two quartic colour factors, see eqs. (A.1) and (A.2), have been obtained before [39], the remaining terms are new. The $C_F^k C_A^{4-k}$ contributions at $k \geq 2$ indeed include terms beyond eq. (10); these numerator-$(N+2)$ terms hint at the appearance of the function $(N+2)f(N)$ in the n_f^0-part of $\gamma^{(3)}_{\rm gq}$. This situation is completely analogous to $\gamma^{(3)}_{\rm qg}$ in ref. [14], where these terms are of the form $(N-1)\zeta_5$, cf. the relation between the off-diagonal $\gamma^{(3)}_{\rm ik}$ in ref. [39]. In both cases these extra contributions vanish for $C_F = C_A$, which is part of the choice of the colour factors that leads to a $\mathcal{N} = 1$ supersymmetric theory, for lower-order discussions see refs. [40,41].

The ζ_4 parts of $\gamma_{\rm ik}$ are special, as all π^2 terms – see ref. [42] for first five-loop results that include also $\zeta_6 \propto \pi^6$. Using the three-loop DIS coefficient functions of refs. [38,43], the ζ_4 part of $\gamma^{(3)}_{\rm gq}$ can be predicted from the no-π^2 theorem [44–46]; the result is given by eq. (11) of ref. [47].

Ten N-values are not sufficient for all-N determinations of the more complex $n_f^0\, \zeta_3$ and $n_f^1\, \zeta_3$ contributions to $\gamma^{(3)}_{\rm gq}$ – except for the quartic colour factors, which are very closely related [39] to those occurring in $\gamma^{(3)}_{\rm qg}$ [14]. For one colour factor, $n_f\, C_F^3$, the diagrams are simple enough to allow the extension of our FORCER calculations to $N = 30$. This was sufficient to obtain and check the corresponding coefficients $c^{(3)}_{3apw}$ in eq. (10) by a system of Diophantine equations. We hence find

$$\begin{aligned}
\gamma_{\rm gq}(N)\Big|_{\zeta_3} &= 128\, d_R^{abcd} d_A^{abcd}/n_c \Big(\big\{ 7/12 + 33/2\,\eta - 96\, S_1\, \eta + 32\, S_1\, \nu - 24\, S_{-2} - 6\, S_1\, \eta^2 \\
&+ 10\, S_{-2}\, \eta - 16\, S_{-2}\, \nu + 43\, S_1^2\, \eta - 16\, S_1^2\, \nu - 8\,(S_{-2,1} - S_{1,-2}) + 8\, S_1\, S_{-2} - S_3 \big\}\, p_{\rm gq} \\
&- \big\{ 40 + 24\, S_1 - 72\, S_{-2} - 10\, S_1^2 + 72\,(S_{-2,1} - S_{1,-2}) + 72\, S_1\, S_{-2} + 36\, S_3 \big\}\, D_{-1}\, D_0\, D_1 \Big) \\
&+ 32/9\, n_f\, C_F^3 \Big(-10/3\, \delta(N-2) - 1372/3\, D_{-1} - 2233/2\, D_0 + 6049/4\, D_1 - 20/3\, D_2 \\
&- 16\, D_{-1}^2 + 1477\, D_0^2 + 305\, D_1^2 + 32\, D_2^2 - 186\, D_0^3 - 87\, D_1^3 + 216\,(D_0^4 - D_1^4) - \big\{ 30\, D_{-1} \\
&- 856\, D_0 + 725\, D_1 - 32\, D_2 - 96\, D_{-1}^2 + 288\, D_0^2 + 300\, D_1^2 - 108\,(2\, D_0^3 - D_1^3) \big\}\, S_1 \\
&+ \big\{ 60\, S_{-2} - 78\, S_{1,1} + 72\, S_2 \big\}\, p_{\rm gq} \Big) + 256\, n_f\, d_R^{abcd} d_R^{abcd}/n_c \Big(\big\{ -7/12 - 8\, S_1\, \eta^2 + 8\, S_{-2}\, \eta
\end{aligned}$$

$$- 3\,\eta \big\}\, p_{\rm gq} + \big\{ 4 - 16\, S_{-2} \big\}\, D_{-1}\, D_0\, D_1 \Big) + 64/27\, n_f^2\, C_F^2 \Big(52\, D_{-1} - 89\, D_0 + 76\, D_1 + 4\, D_2$$
$$+ 12\, D_{-1}^2 + 42\, D_0^2 + 9\, D_1^2 - 6\, S_1\, p_{\rm gq} \Big) + 32/9\, n_f^2\, C_F C_A \Big(\delta(N-2)/3 - 103/3\, D_{-1} + 63\, D_0$$
$$- 54\, D_1 - 8/3\, D_2 - 8\, D_{-1}^2 - 28\, D_0^2 - 12\, D_1^2 + 6\, S_1\, p_{\rm gq} \Big) - 128/27\, n_f^3\, C_F\, p_{\rm gq} + \ldots \tag{14}$$

with

$$p_{\rm gq} \;=\; 2D_{-1} - 2D_0 + D_1\,, \quad \eta \;=\; D_0 - D_1\,, \quad \nu \;=\; D_{-1} - D_2\,. \tag{15}$$

As above, we have omitted also the argument N of the sums for brevity. A very considerable effort would be required to find the contributions, indicated by '$+\ldots$', that are still missing in eq. (14).

Also eq. (14) includes a structure that did not occur in the anomalous dimensions up to the third order in α_s but was mentioned above: Both the $n_f C_F^3$ and $n_f^2 C_F C_A$ terms receive a $\delta(N-2)$ contribution. In the latter case, which is fully known [10], this arises from the non-ζ contribution $(N-2)^{-1}\, S_{-2}(N-2)$, corresponding to $x^{-2}\, {\rm H}_{-1,0}(x) = x^{-2}\,[\,\ln(x)\ln(1+x) + {\rm Li}_2(-x)]$ [37,48]. Exactly the same function occurs in the two-loop coefficient functions for inclusive DIS [35–37] and in the $n_f^2 C_F C_A$ contribution to $\gamma_{\rm ps}^{(3)}(N)$ recently determined in ref. [9].

The above partial all-N results are not relevant to N^{3}LO analyses of LHC processes. Until the complete functions $P_{\rm ik}^{(3)}(x)$ become known, such analyses will have to rely on approximations based on the available moments and information about the large-x (threshold) and small-x (high-energy) limits. With eqs. (9) we are now in the position to improve upon the $N \le 10$ based approximations of ref. [12], thus putting $P_{\rm gq}^{(3)}$ on the same footing as $P_{\rm ps}^{(3)}$ and $P_{\rm qg}^{(3)}$ in refs. [13,14].

Up to the presence of a leading-logarithmic BFKL small-x contribution [49,50], the endpoint structure of $P_{\rm gq}^{(3)}(x)$ is the same as that of $P_{\rm qg}^{(3)}(x)$ given in eqs. (10) and (11) of ref. [14]. So far the next-to-leading logarithmic (NLL) small-x correction has been calculated [51,52] and transformed to the $\overline{\rm MS}$ scheme [53,54] only for $P_{\rm gg}$. At order α_s^3, the NLL BFKL contributions to $P_{\rm gq}$ and $P_{\rm gg}$ are related by Casimir scaling in the large-n_c limit. In QCD the breaking of this relation is small, amounting to less than 0.5% at $n_f = 3,\ldots,6$ [34]. We have checked that, due to the very large uncertainties of the unknown NNLL $x^{-1}\ln x$ terms, no relevant bias is generated by assuming Casimir scaling for the $x^{-1}\ln^2 x$ term of $P_{\rm gq}^{(3)}$. Under this assumption the known endpoint contributions analogous to eq. (15) of ref. [14] are, to eight significant figures, given by [43,49–57]

$$\begin{aligned} p_{\rm gq,0}^{(n_f)}(x) \;=\;& -3692.7188\, L_0^3/x - (47516.440 + 442.83691\, n_f)\, L_0^2/x \\ &+ (52.235940 - 7.3744856\, n_f)\, L_0^6 - (292.21399 - 1.8436214\, n_f)\, L_0^5 \\ &+ (7310.6077 - 378.87135\, n_f - 32.438957\, n_f^2)\, L_0^4 + (13.443073 - 0.54869684\, n_f)\, L_1^5 \\ &+ (375.39831 - 34.494742\, n_f + 0.87791495\, n_f^2)\, L_1^4 + (22.222222 - 0.54869684\, n_f)\, x_1 L_1^5 \\ &+ (662.42163 - 47.992684\, n_f + 0.87791495\, n_f^2)\, x_1 L_1^4\,, \end{aligned} \tag{16}$$

where we have used the abbreviations $x_1 = 1-x$, $L_1 = \ln(1-x)$ and $L_0 = \ln x$. The coefficient of L_0^2/x in the first line of eq. (16) will be referred to as ${\rm bfkl}_1$ below.

Our procedure for the construction of approximate expressions for $P_{\rm gq}^{(3)}(x)$ is identical to that used for its counterpart $P_{\rm qg}^{(3)}(x)$ in ref. [14]. Also here it is not possible, even with ten moments, to make a 'direct fit' for the coefficient of $x^{-1}\ln x$. In fact, knowing five additional moments on top of those employed in ref. [12] leads to considerable improvements elsewhere, see below, but does not enable us to assign a smaller uncertainty to this important coefficient.

The coefficients of $\ln^{1,2,3}(1-x)$, $\ln^{1,2,3} x$, $(1-x)/x$ are determined from the ten moments for 10 choices of a two-parameter polynomial in x and 8 (di-)logarithmic 'interpolating functions'. The approximations selected from the resulting 80 functions to represent our error bands for the physically relevant numbers $n_f = 3, 4, 5$ of light flavours read (with ${\rm bfkl}_1$ as defined below eq. (16))

$$\begin{aligned} P_{\rm gq,\,A}^{(3)}(n_f=3,x) =\;& p_{\rm gq,0}^{(n_f=3)}(x) + 6\,{\rm bfkl}_1\, L_0/x - 744384\, x_1/x + 2453640 - 1540404\, x(2+x) \\ &+ 1933026\, L_0 + 1142069\, L_0^2 + 162196\, L_0^3 - 2172.1\, L_1^3 - 93264.1\, L_1^2 - 786973\, L_1 + 875383\, x_1 L_1^2\,, \\ P_{\rm gq,\,B}^{(3)}(n_f=3,x) =\;& p_{\rm gq,0}^{(n_f=3)}(x) + 3\,{\rm bfkl}_1\, L_0/x + 142414\, x_1/x - 326525 + 2159787\, x(2-x) \\ &- 289064\, L_0 - 176358\, L_0^2 + 156541\, L_0^3 + 9016.5\, L_1^3 + 136063\, L_1^2 + 829482\, L_1 - 2359050\, L_0 L_1\,, \end{aligned} \tag{17}$$

$$\begin{aligned} P_{\rm gq,\,A}^{(3)}(n_f=4,x) =\;& p_{\rm gq,0}^{(n_f=4)}(x) + 6\,{\rm bfkl}_1\, L_0/x - 743535\, x_1/x + 2125286 - 1332472\, x(2+x) \\ &+ 1631173\, L_0 + 1015255\, L_0^2 + 42612\, L_0^3 - 1910.4\, L_1^3 - 80851\, L_1^2 - 680219\, L_1 + 752733\, x_1 L_1^2\,, \\ P_{\rm gq,\,B}^{(3)}(n_f=4,x) =\;& p_{\rm gq,0}^{(n_f=4)}(x) + 3\,{\rm bfkl}_1\, L_0/x + 160568\, x_1/x - 361207 + 2048948\, x(2-x) \\ &- 245963\, L_0 - 171312\, L_0^2 + 163099\, L_0^3 + 8132.2\, L_1^3 + 124425\, L_1^2 + 762435\, L_1 - 2193335\, L_0 L_1\,, \end{aligned} \tag{18}$$

$$\begin{aligned} P_{\rm gq,\,A}^{(3)}(n_f=5,x) =\;& p_{\rm gq,0}^{(n_f=5)}(x) + 6\,{\rm bfkl}_1\, L_0/x - 785864\, x_1/x + 285034 - 131648\, x(2+x) \\ &- 162840\, L_0 + 321220\, L_0^2 + 12688\, L_0^3 + 1423.4\, L_1^3 + 1278.9\, L_1^2 - 30919.9\, L_1 + 47588\, x_1 L_1^2\,, \\ P_{\rm gq,\,B}^{(3)}(n_f=5,x) =\;& p_{\rm gq,0}^{(n_f=5)}(x) + 3\,{\rm bfkl}_1\, L_0/x + 177094\, x_1/x - 470694 + 1348823\, x(2-x) \\ &- 52985\, L_0 - 87354\, L_0^2 + 176885\, L_0^3 + 4748.8\, L_1^3 + 65811.9\, L_1^2 + 396390\, L_1 - 1190212\, L_0 L_1\,. \end{aligned} \tag{19}$$

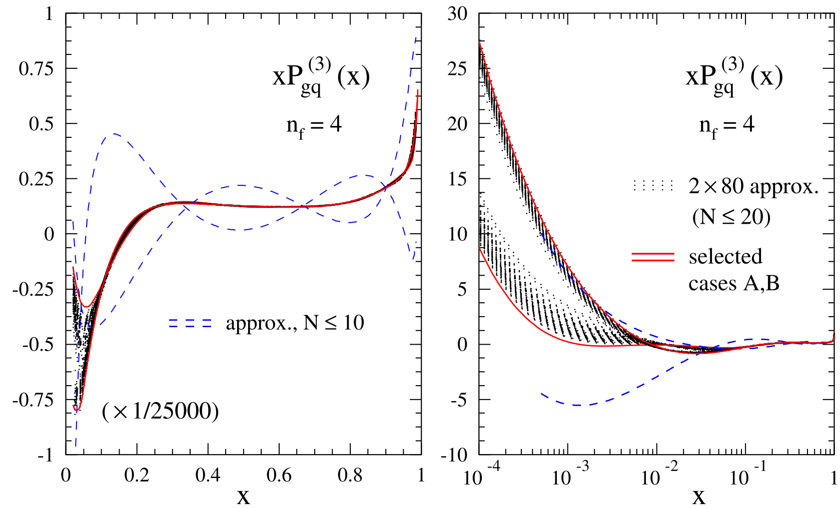

Fig. 1. Two sets of 80 trial functions for the four-loop (N^{3}LO) contribution to the quark-to-gluon splitting function at $n_f = 4$. The two cases selected for eq. (18) are shown by the solid (red) lines. Also shown, by the dashed (blue) lines, are the selected approximations of ref. [12] based on only the first 5 even moments.

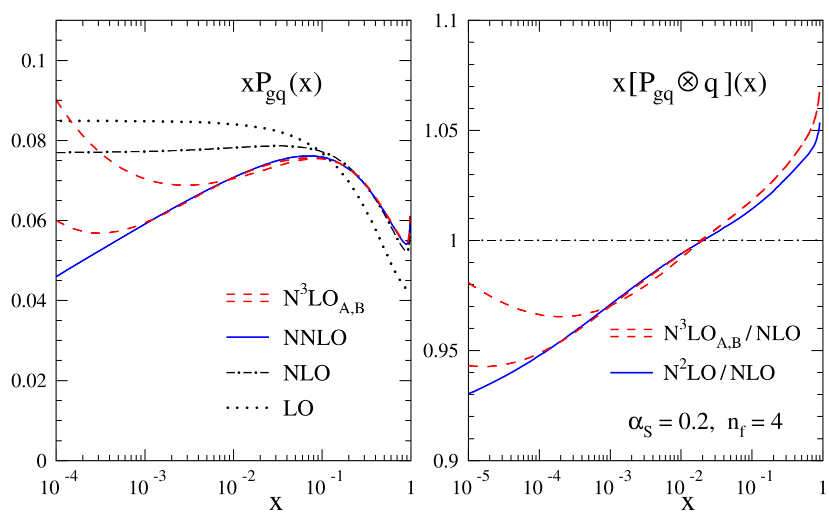

Fig. 2. Left: the perturbative expansion of the splitting functions P_{gq} to N^{3}LO for $n_f = 4$ and $\alpha_s = 0.2$, using eq. (18) for the four-loop contribution. Right: the resulting N^{2}LO and N^{3}LO convolutions with the reference quark distribution (21) normalised to the NLO results.

At $N = 22$ we predict, with the brackets indicating a conservative uncertainty of the last digit:

$$-\gamma^{(3)}_{gq}(N=22) = 662.7981(3),\ 549.2876(3),\ 426.6266(2) \quad \text{for} \quad n_f = 3,4,5\,. \tag{20}$$

The selection procedure and its results are illustrated in Fig. 1 for $n_f = 4$; the two functions in eq. (19) are shown by the red (solid) lines. A comparison with the corresponding $N \leq 10$ results of ref. [12], shown by the blue (dashed) lines, reveals an impressive improvement down to $x \approx 10^{-2}$. $P^{(3)}_{gq}(x)$ can now be considered as sufficiently well constrained at x-values above 0.1.

The uncertainties of $P^{(3)}_{gq}(x)$ 'as such' are still rather large at $10^{-2} \lesssim x < 10^{-1}$. However, as shown in the left panel of Fig. 2, the resulting uncertainties of the expansion of the splitting function P_{gq} to N^{3}LO appear to be perfectly tolerable also in this range, down to (for LHC purposes) rather low scales. Of course, the splitting functions enter physical quantities only via the convolution with the respective quark or gluon PDF. This convolution is illustrated in the right panel of Fig. 2 using a sufficiently realistic order-independent model input [34] for the singlet quark PDF in eq. (1),

$$xq_s(x,\mu_0^2) = 0.6\,x^{-0.3}(1-x)^{3.5}\left(1+5.0\,x^{0.8}\right)\,, \tag{21}$$

together with a strong coupling of $\alpha_s(\mu_0^2) = 0.2$ corresponding to a scale $\mu_0^2 \approx 30\ldots 50$ GeV^2. Here the uncertainties of $P^{(3)}_{gq}(x)$ do not appear to have a relevant impact at $x \gtrsim 10^{-3}$. An overall uncertainty of $\pm 1\%$ of the NLO result for $P_{gq} \otimes q_s$ is reached only at $x \approx 10^{-4}$.

To summarise, we have computed the even-N moments $N \leq 20$ of the fourth-order (N^{3}LO) quark-to-gluon splitting function $P^{(3)}_{gq}(x)$ in the framework of the operator-product expansion. These results facilitate the determination of the all-N form of the ζ_5 contribution and of parts of the corresponding ζ_3 terms. These expressions show deviations from the functional form of the anomalous dimensions up to three loops which should be of interest to future research.

Together with (approximately) known endpoint constraints at $x \to 1$ and $x \to 0$, our additional moments – only the results up to $N = 10$ had been obtained before [11,12] – have been employed to construct improved approximations for $P^{(3)}_{\rm gq}(x)$ that should be sufficiently accurate for a wide range of phenomenological applications in collider physics. Their uncertainties are still large at $x \lesssim 10^{-3}$. A drastic improvement in this region would be obtained if the next-to-leading and next-to-next-leading small-x contributions, $x^{-1}\ln^2 x$ and $x^{-1}\ln x$, became known.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Mellin moments of $P^{(3)}_{\rm gq}$

The exact results for the four-loop anomalous dimensions $\gamma^{(3)}_{\rm gq}(N)$ at even $2 \le N \le 10$ have already been presented, for a general compact simple gauge group, in eqs. (11) - (13) of ref. [11] and eqs. (9) and (10) of ref. [12]. We agree with the results given in those articles, which were obtained in a different theoretical framework. Here we report the corresponding results at $12 \le N \le 20$. The numerical values for QCD, i.e., SU($n_c = 3$), have been given in eqs. (9) above.

The quadratic Casimir invariants are $C_A = n_c$ and $C_F = (n_c^2 - 1)/(2n_c)$ in SU(n_c). The quartic group invariants are products of two symmetrised traces of four generators T_r^a of the fundamental (R) or adjoint (A) representation,

$$d_r^{\,abcd} \;=\; \frac{1}{6}\,{\rm Tr}\,(\,T_r^a T_r^b T_r^c T_r^d \;+\; \text{five } bcd \text{ permutations}\,)\,, \tag{A.1}$$

which leads to

$$\frac{d^{(4)}_{RA}}{n_c} \;=\; \frac{(n_c^2-1)\,(n_c^2+6)}{48}\;, \qquad \frac{d^{(4)}_{RR}}{n_c} \;=\; \frac{(n_c^2-1)\,(n_c^4-6\,n_c^2+18)}{96\,n_c^3}\;. \tag{A.2}$$

Their values in QCD are $\frac{d^{(4)}_{RA}}{n_c} \equiv d_R^{\,abcd} d_A^{\,abcd}/n_c \;=\; 5/2$ and $\frac{d^{(4)}_{RR}}{n_c} \equiv d_R^{\,abcd} d_R^{\,abcd}/n_c \;=\; 5/36$.

$$\begin{aligned}
\gamma^{(3)}_{\rm gq}(N=12) = {} & C_F^4 \Big(- \frac{256825576112753878132420570039956677}{1359167659377896124397761120000000} - \frac{788732866421532776 27}{313680161535741000}\,\zeta_3 \\
& + \frac{12335483}{193050}\,\zeta_4 + \frac{17406716}{42471}\,\zeta_5 \Big) + C_A\,C_F^3 \Big(\frac{4226991801435309744806542280206048 13}{11438190172426969981944794880000000} \\
& + \frac{1000644815142900639 61}{313680161535741000}\,\zeta_3 - \frac{760772056049}{9591882300}\,\zeta_4 - \frac{237624788}{351351}\,\zeta_5 \Big) \\
& + C_A^2\,C_F^2 \Big(- \frac{224250321575347263949905289161744199}{145576965830888708861115571200000 00} - \frac{2010183037743051616}{39210020191967625}\,\zeta_3 \\
& + \frac{3810092749}{532882350}\,\zeta_4 + \frac{1211419205}{3864861}\,\zeta_5 \Big) + C_A^3\,C_F \Big(- \frac{60579732448689350058530653522 1627}{18906099458556975176768256000000} \\
& - \frac{1052537460264167687}{25347891841272000}\,\zeta_3 + \frac{79289578229}{9591882300}\,\zeta_4 - \frac{23156863}{1054053}\,\zeta_5 \Big) \\
& + \frac{d^{(4)}_{RA}}{n_c} \Big(- \frac{1135272951753220436 99}{2082683928124800000} - \frac{66259676094769}{321401840760}\,\zeta_3 + \frac{1159790344}{3864861}\,\zeta_5 \Big) \\
& + n_f\,C_F^3 \Big(- \frac{3320434820843043641839672741241 53}{190636502873782833032413248000 00} - \frac{393343213522261}{33948069430275}\,\zeta_3 - \frac{6070628}{495495}\,\zeta_4 \\
& + \frac{12640}{429}\,\zeta_5 \Big) + n_f\,C_A\,C_F^2 \Big(\frac{29361179497371363836008643846 11}{1575508288213081264730688000 00} - \frac{345775231864711 9}{67896138860550}\,\zeta_3 \\
& + \frac{8619754063}{251215965}\,\zeta_4 - \frac{6320}{1287}\,\zeta_5 \Big) + n_f\,C_A^2\,C_F \Big(- \frac{3409052330654655340645389731}{1469005396935273906508800000} \\
& + \frac{25514383657902523}{407376833163300}\,\zeta_3 - \frac{5541945667}{251215965}\,\zeta_4 - \frac{119360}{5577}\,\zeta_5 \Big) \\
& + n_f\,\frac{d^{(4)}_{RR}}{n_c} \Big(- \frac{1432266432400637 2519}{260335491015600000} - \frac{33040603052}{2608781175}\,\zeta_3 + \frac{421760}{5577}\,\zeta_5 \Big)
\end{aligned}$$

$$
\begin{aligned}
&+n_f^2\,C_F^2\Big(\frac{1690332943553209185963140 39}{793525236737357779855200 00}+\frac{81408784}{15810795}\,\zeta_3-\frac{5056}{1287}\,\zeta_4\Big)\\
&+n_f^2\,C_A\,C_F\Big(-\frac{199054683271297463698 0331}{1109825505926374517280000}-\frac{171649172}{57972915}\,\zeta_3+\frac{5056}{1287}\,\zeta_4\Big)\\
&+n_f^3\,C_F\Big(\frac{2265689177355445577}{3387745744586002800}-\frac{5056}{11583}\,\zeta_3\Big)\,,
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\gamma_{\rm gq}^{(3)}(N=14) = {}& C_F^4\Big(-\frac{11758585141839781791810882123296 22671}{583824290050959926161765572 0000000}\\
&-\frac{32053546295279752556}{124759155156260625}\,\zeta_3+\frac{10403997892}{165540375}\,\zeta_4+\frac{5234629744}{12297285}\,\zeta_5\Big)\\
&+C_A\,C_F^3\Big(\frac{435029240281577148104394100536180709}{1077829458555618325221721056000000}+\frac{172973892630128742931}{499036620625042500}\,\zeta_3\\
&-\frac{5900838170593}{71016820875}\,\zeta_4-\frac{160919504}{223587}\,\zeta_5\Big)\\
&+C_A^2\,C_F^2\Big(-\frac{626848860230295572619233480440154 57}{331639833401728715452837248000000}-\frac{173609406082076933}{2268348275568375}\,\zeta_3\\
&+\frac{231212408549}{15781515750}\,\zeta_4+\frac{4199718508}{12297285}\,\zeta_5\Big)+C_A^3\,C_F\Big(-\frac{427280796934450930964335754478824 3}{255107564155175934963720960000000}\\
&-\frac{926544367976770183}{25591621570515000}\,\zeta_3+\frac{794134472909}{142033641750}\,\zeta_4-\frac{177252716}{7378371}\,\zeta_5\Big)\\
&+\frac{d_{RA}^{(4)}}{n_c}\Big(-\frac{213352131545679397444 19}{4145842694423430000 00}-\frac{613323018929398364}{3198952696314375}\,\zeta_3+\frac{3460252064}{12297285}\,\zeta_5\Big)\\
&+n_f\,C_F^3\Big(-\frac{490379446041031040879323234108051}{408268734301370577735500400000 00}-\frac{33894392681638}{2403420508125}\,\zeta_3-\frac{46632848}{4099095}\,\zeta_4\\
&+\frac{6784}{273}\,\zeta_5\Big)+n_f\,C_A\,C_F^2\Big(\frac{407591207989083358411810474974 91}{314052872539515829027308000000 0}-\frac{992731197367267}{26437625589375}\,\zeta_3\\
&+\frac{65246478712}{2152024875}\,\zeta_4-\frac{3392}{819}\,\zeta_5\Big)+n_f\,C_A^2\,C_F\Big(-\frac{958430136662854610670607569547}{322105510296939311822880000000}\\
&+\frac{9023658476762282}{174488328889875}\,\zeta_3-\frac{40764233512}{2152024875}\,\zeta_4-\frac{4416}{245}\,\zeta_5\Big)\\
&+n_f\,\frac{d_{RR}^{(4)}}{n_c}\Big(-\frac{8670331171595460795 7}{1850822631439031250}-\frac{244011856707712}{22370298575625}\,\zeta_3+\frac{615424}{9555}\,\zeta_5\Big)\\
&+n_f^2\,C_F^2\Big(\frac{6497249939571360052326071 41}{3398840611899521959170000 00}+\frac{178574416}{42567525}\,\zeta_3-\frac{13568}{4095}\,\zeta_4\Big)\\
&+n_f^2\,C_A\,C_F\Big(-\frac{249311118093524342947660 3}{149399587336242723480000 0}-\frac{83451856}{36891855}\,\zeta_3+\frac{13568}{4095}\,\zeta_4\Big)\\
&+n_f^3\,C_F\Big(\frac{321519364624645823}{538959550275045900}-\frac{13568}{36855}\,\zeta_3\Big)\,,
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
\gamma_{\rm gq}^{(3)}(N=16) = {}& C_F^4\Big(-\frac{913337946814886058526420486334866085908776073 51}{431323535597978497478425306351127101440000000}\\
&-\frac{1546180599641264856584891 9}{5862686606238125664000 0}\,\zeta_3+\frac{105621100681}{1715313600}\,\zeta_4+\frac{2466482078}{5579145}\,\zeta_5\Big)\\
&+C_A\,C_F^3\Big(\frac{697091037207664897235959976597124105262457068 81}{161746325849241936554409489881672663040000000}\\
&+\frac{561806270069913957735019}{150325297595849376 0000}\,\zeta_3-\frac{181207395103216571}{2126662954416000}\,\zeta_4-\frac{2706171203}{3550365}\,\zeta_5\Big)\\
&+C_A^2\,C_F^2\Big(-\frac{290588766399196445429954480260091203851994445 9}{134788604874368280462007908234727219200000 00}\\
&-\frac{140622850159330174818 79}{142125735908803046400}\,\zeta_3+\frac{14216141064343763}{708887651472000}\,\zeta_4+\frac{93724168}{255255}\,\zeta_5\Big)\\
&+C_A^3\,C_F\Big(-\frac{317835545673384782764570229718975673012928297}{539154419497473121848031632938908876800000 00}\\
&-\frac{1262927249130586239359921}{3908457737492083776 0000}\,\zeta_3+\frac{475554692179667}{132916434651000}\,\zeta_4-\frac{23407688153}{937296360}\,\zeta_5\Big)\\
&+\frac{d_{RA}^{(4)}}{n_c}\Big(-\frac{1177370014231117991570212 1}{240974431125652224000000}-\frac{1011690819926689807 37}{563503134009816000}\,\zeta_3+\frac{4137449651}{15621606}\,\zeta_5\Big)\\
&+n_f\,C_F^3\Big(-\frac{5729833912344913671847440196223042451336 1}{718154404925039123340701475776102400000 0}-\frac{87782813700795439}{5546240318619000}\,\zeta_3\\
&-\frac{16614041449}{1562160600}\,\zeta_4+\frac{1096}{51}\,\zeta_5\Big)+n_f\,C_A\,C_F^2\Big(\frac{933443441649638575905699644076117008927 11}{107723160738755868501105221366415360000 00}\\
&-\frac{247511909262822012 7}{88739845097904000}\,\zeta_3+\frac{643872057559}{23605982400}\,\zeta_4-\frac{548}{153}\,\zeta_5\Big)\\
&+n_f\,C_A^2\,C_F\Big(-\frac{203330836821302717393891806247985487319 3}{633665651404446285300618949214208000000}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{15529936726131895757}{354959380391616000}\,\zeta_3-\frac{3535338880967}{212453841600}\,\zeta_4-\frac{121513}{7803}\,\zeta_5\Big)\\
&+n_f\,\frac{d_{RR}^{(4)}}{n_c}\Big(-\frac{40522338694523474080858 7}{99401952839331542400000}-\frac{14956670451523}{1563722760600}\,\zeta_3+\frac{145816}{2601}\,\zeta_5\Big)\\
&+n_f^2\,C_F^2\Big(\frac{9044520444067665064750987439226131}{5275271823213838538966191718400000}+\frac{4119326989}{1171620450}\,\zeta_3-\frac{2192}{765}\,\zeta_4\Big)\\
&+n_f^2\,C_A\,C_F\Big(-\frac{6050468506198910363093667967243 9}{3878876340598410690416317440000 0}-\frac{3125731858}{1757430675}\,\zeta_3+\frac{2192}{765}\,\zeta_4\Big)\\
&+n_f^3\,C_F\Big(\frac{34108337373135323374301}{63317015347371757171200}-\frac{2192}{6885}\,\zeta_3\Big),
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
\gamma_{\rm gq}^{(3)}(N=18) = {}& C_F^4\Big(-\frac{199441346676346613221100951050417553229794351172058063}{903627934773265885372610492741460297127036416000000}\\
&-\frac{9734182701413671124997104021}{3581396161929181756210050 0}\,\zeta_3+\frac{55191569546}{916620705}\,\zeta_4+\frac{555009226736}{1208442807}\,\zeta_5\Big)\\
&+C_A\,C_F^3\Big(\frac{170307903917822151841266753834768510267119818033212 7}{375208831047722858978246571380537147596001280000 0}\\
&+\frac{5815195693143541304951632 9}{1449957960295215285915 00}\,\zeta_3-\frac{20821174767688520 6}{2413254243364965}\,\zeta_4-\frac{618848170736}{769009059}\,\zeta_5\Big)\\
&+C_A^2\,C_F^2\Big(-\frac{12331962659659773256391355766903032266820234425500 7}{52112337645517063746978690469519048277222400000 0}\\
&-\frac{97421341297611056710882606}{813953673165723126411375}\,\zeta_3+\frac{7737412238766461}{321767232448662}\,\zeta_4+\frac{1104257588788}{2819699883}\,\zeta_5\Big)\\
&+C_A^3\,C_F\Big(\frac{115832786415748476638280610626833293796768491507}{551777692717239498497421428500789922935296000 00}\\
&-\frac{3955588702413341460682019 03}{134829031978510371998496 00}\,\zeta_3+\frac{9748563515611381}{4826508486729930}\,\zeta_4-\frac{214104770108}{8459099649}\,\zeta_5\Big)\\
&+\frac{d_{RA}^{(4)}}{n_c}\Big(-\frac{15342548557364634582858604875 97}{32941608367048794488275200000}-\frac{2630633360777559874070 89}{155619843003820835640 0}\,\zeta_3\\
&+\frac{705610821248}{2819699883}\,\zeta_5\Big)+n_f\,C_F^3\Big(-\frac{1548713485085741602882737839658973739768394 91}{3186088974930713587727368488246319435130880 0}\\
&-\frac{181004516970751808 17}{106292880082756867 5}\,\zeta_3-\frac{84572797384}{8459099649}\,\zeta_4+\frac{55040}{2907}\,\zeta_5\Big)\\
&+n_f\,C_A\,C_F^2\Big(\frac{494213149462187941188004051358795571999087887}{937084992626680466978637790660682186803200 00}\\
&-\frac{1975854809531024826 19}{956635920744811807 5}\,\zeta_3+\frac{59943194135296}{2410843399965}\,\zeta_4-\frac{27520}{8721}\,\zeta_5\Big)\\
&+n_f\,C_A^2\,C_F\Big(-\frac{936013096859225678199723846435281783145737}{290119192763678163151281049740149283840000}\\
&+\frac{1220577866492603197 79}{323784465482859381 0}\,\zeta_3-\frac{35839946880856}{2410843399965}\,\zeta_4-\frac{20446720}{1491291}\,\zeta_5\Big)\\
&+n_f\,\frac{d_{RR}^{(4)}}{n_c}\Big(-\frac{2376495270782792811304983137}{658832167340975889765504 00}-\frac{9726278907372152}{1143120431067615}\,\zeta_3+\frac{24663040}{497097}\,\zeta_5\Big)\\
&+n_f^2\,C_F^2\Big(\frac{20769881139161155063875496211825892 19}{13506302570736633907681621683093600 00}+\frac{13478091680}{4478346873}\,\zeta_3-\frac{22016}{8721}\,\zeta_4\Big)\\
&+n_f^2\,C_A\,C_F\Big(-\frac{5160355411866549202545627806335534 7}{3531059495617420629459247498848000 0}-\frac{543557988224}{380659484205}\,\zeta_3+\frac{22016}{8721}\,\zeta_4\Big)\\
&+n_f^3\,C_F\Big(\frac{25322735830563737715837587}{5157210473178021728942472 0}-\frac{22016}{78489}\,\zeta_3\Big),
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\gamma_{\rm gq}^{(3)}(N=20) = {}& C_F^4\Big(-\frac{18011842890292937504270512411100571563413211395232599}{7874753244211467410654557670949545073002496000000 0}\\
&-\frac{27623613597125033101695488051}{98312835817663812915570000}\,\zeta_3+\frac{57192203098}{972173475}\,\zeta_4+\frac{923526447854}{1935088155}\,\zeta_5\Big)\\
&+C_A\,C_F^3\Big(\frac{742257701799954265313171119485669995895179080771700597}{15666614349010182532775909471678568618920755200000 00}\\
&+\frac{30463838310709426493320 7}{710296405760118870000}\,\zeta_3-\frac{570786749229593761}{6585879026727000}\,\zeta_4-\frac{149116082954}{175917105}\,\zeta_5\Big)\\
&+C_A^2\,C_F^2\Big(-\frac{135951719798984873981123660606505036199246133659 49}{535427694771366457032669496639732351979520000 00}\\
&-\frac{137066032120612148849243108 47}{983128358176638129155700 00}\,\zeta_3+\frac{1068938712970319969}{39515274160362000}\,\zeta_4+\frac{178657350101}{430019590}\,\zeta_5\Big)\\
&+C_A^3\,C_F\Big(\frac{8128052126051322869747930125993834197558195023}{9962769440190231789549937470702090547200000 00}\\
&-\frac{44686861440755693439865115 69}{16557951295606536912096000 0}\,\zeta_3+\frac{494114186999099}{627226573974000}\,\zeta_4-\frac{10876666809}{430019590}\,\zeta_5\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d_{RA}^{(4)}}{n_c}\Big(-\frac{993424778726313579388174412147119 3}{2230446355616176557715215360000 00}-\frac{200558188401722575067052 13}{125439024966716188728000}\zeta_3 \\
&+\frac{153050440696}{645029385}\zeta_5\Big)+n_f\, C_F^3\Big(-\frac{9412487805761017126474048385436243207670819 07}{39488445681722861095591031770186985856000000 0} \\
&-\frac{633979491609139435645 1}{354748373774649855000}\zeta_3-\frac{91361957041}{9675440775}\zeta_4+\frac{6752}{399}\zeta_5\Big) \\
&+n_f\, C_A\, C_F^2\Big(\frac{102664753783997698300409838737001288852130553}{403791625016113466842133858702663915520000 00} \\
&-\frac{534739354474288369879 7}{354748373774649855000}\zeta_3+\frac{2584501408063}{112880142375}\zeta_4-\frac{3376}{1197}\zeta_5\Big) \\
&+n_f\, C_A^2\, C_F\Big(-\frac{465634445017831591325444801806925052804292331}{148765335532252329889207211100981442560000000} \\
&+\frac{3171458341939781449}{96255155006010000}\zeta_3-\frac{4555835727754}{338640427125}\zeta_4-\frac{1539424}{125685}\zeta_5\Big) \\
&+n_f\,\frac{d_{RR}^{(4)}}{n_c}\Big(-\frac{2704920902087866223629472523102433}{836417383356066209143205760000 00}-\frac{108318373596246439}{14144672000584125}\zeta_3 \\
&+\frac{1863808}{41895}\zeta_5\Big)+n_f^2\, C_F^2\Big(\frac{4773708126683882394527205560488250927}{3460438305705072216870062548871040000}+\frac{4000788628}{1527701175}\zeta_3 \\
&-\frac{13504}{5985}\zeta_4\Big)+n_f^2\, C_A\, C_F\Big(-\frac{1132553467354958443895482249966696 7}{8259788293841919601074263155200000}-\frac{11266215274}{9675440775}\zeta_3 \\
&+\frac{13504}{5985}\zeta_4\Big)+n_f^3\, C_F\Big(\frac{236420317660870976851379 41}{524335124360873668883040 00}-\frac{13504}{53865}\zeta_3\Big)\,.
\end{aligned}
\tag{A.7}
$$

FORM files with the results for $\gamma_{\rm gq}(N)$ at even $N \leq 20$ and the partial all-N expressions in the main text have been deposited at the preprint server http://arXiv.org together with a FORTRAN subroutine of our approximations for the splitting function $P_{\rm gq}^{(3)}(x)$ in eqs. (16) - (19). These files are also available from the authors upon request.

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