

1/2 BPS Wilson loops in non-conformal $\mathcal{N} = 2$ gauge theories and localization: a three-loop analysis

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ABSTRACT: We study the 1/2 BPS circular Wilson loop in four-dimensional $SU(N)$ $\mathcal{N} = 2$ SYM theories with massless hypermultiplets and non-vanishing β -function. Using supersymmetric localization on S^4 , we map the path-integral associated with this observable onto an interacting matrix model. Despite the breaking of conformal symmetry at the quantum level, we show that, within a specific regime, the matrix model predictions remain consistent with the perturbative results in flat space up to order g^6 . At this order, our analysis reveals that the reorganization of Feynman diagrams based on the matrix model interaction potential, widely tested in (super)conformal models, also applies to these non-conformal set-ups and is realized by interference mechanisms.

KEYWORDS: Extended Supersymmetry, Wilson, 't Hooft and Polyakov loops, Supersymmetric Gauge Theory, Renormalization and Regularization

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1 Introduction and main results

Supersymmetric gauge theories provide a powerful theoretical laboratory for controlling the dynamics of fields at the quantum level. In four dimensions, these models exhibit interesting dynamics, including confinement without chiral symmetry breaking and the emergence of gapless gauge bosons in the infrared [1, 2]. Moreover, through advanced techniques, such as dualities [2–4] and gauge-gravity correspondences [5, 6], it has been possible to probe the non-perturbative properties of these models, confirming the presence of mechanisms that also are expected in physical theories like QCD [7].

Recently, extended supersymmetry has allowed to develop new analytical approaches, such as supersymmetric localization [8, 9]. Unlike integrability [10, 11], resurgence [12] and bootstrap approaches [13, 14], supersymmetric localization provides a direct technique for computing path integrals. Under suitable conditions, partition functions and classes of local and non-local observables for the theory defined on a compact space-time manifold, such as \mathbb{S}^4 , can be calculated exactly in terms of matrix models. These are typically characterized by complex interaction potentials that encode both the conventional perturbative series and non-perturbative contributions. The latter are often associated with semiclassical configurations, such as instantons [15], monopoles [16] and fluxes [17, 18]. Localization thus offers an alternative technique for testing methods that provide informations only in particular regimes and for refining techniques that require external inputs or data.¹ Furthermore, the matrix models generated by supersymmetric localization also offer new insights on the perturbative techniques, suggesting a convenient reorganization of Feynman diagrams and predicting their large-order behaviours. In four dimensions, these features have been extensively studied in (super)conformal models, where the computations on compact spaces naturally extend to the Euclidean configurations. Less attention has been given to non-conformal cases.

In this paper, we continue the analysis initiated in [20] regarding the localization approaches in non-conformal four-dimensional $\mathcal{N} = 2$ supersymmetric theories. More precisely, we will consider $SU(N)$ $\mathcal{N} = 2$ super-Yang-Mills theories (SYM) with massless hypermultiplets in an arbitrary representation \mathcal{R} . In these set-ups, classical conformal symmetry is broken at the quantum level by the (one-loop exact) β -function [21, 22]

$$\beta(g) = -\epsilon g + \beta_0 g^3, \quad \text{where} \quad \beta_0 = \frac{i_{\mathcal{R}} - N}{8\pi^2}. \quad (1.1)$$

In the previous expression, the first term is the classical contribution in

$$d = 4 - 2\epsilon \quad (1.2)$$

dimensions, while $i_{\mathcal{R}}$ denotes the Dynkin index of the representation \mathcal{R} . In the following, we will focus on asymptotically free theories, where $i_{\mathcal{R}} < N$ and we fix $i_F = 1/2$ for the fundamental representation. Compactifying these theories on the four-sphere \mathbb{S}^4 , we can employ supersymmetric localization [8] to reduce the path-integral associated with the partition function and with the expectation values of protected operators into a matrix model.

¹Localization data have been often used in superconformal bootstrap to refine bounds on anomalous dimensions and OPE coefficients, see for example [19].

When the theory remains conformal at the quantum level, i.e. when $i_{\mathcal{R}} = N$ and the β -function vanishes, localization results on \mathbb{S}^4 naturally extend to flat-space observables. For instance, in $\mathcal{N} = 4$ SYM theories, supersymmetric localization was employed to derive the analytical expression of the 1/2 BPS Wilson loop [8], originally conjectured in [23, 24]. Moreover, the same technique also applies to supersymmetric Wilson loops which preserve fewer global supercharges than the circular configuration [25] and families of BPS local operators [26]. In these cases, the matrix model generated by localization is connected to Yang-Mills theories in two-dimensions [27–29] and successfully captures the perturbative results based on standard Feynman diagrams.

Unlike the $\mathcal{N} = 4$ theory, where the matrix model generated by localization on \mathbb{S}^4 is purely Gaussian, $\mathcal{N} = 2$ theories involve non-trivial interaction potentials. Standard perturbative techniques in flat Euclidean space perfectly reproduce the localization predictions for several protected observables, including supersymmetric Wilson loops [30–34], chiral operators [35–39] and Bremsstrahlung functions [40–44]. These results show that the perturbative computations in flat space are encoded by a one-loop effective action on \mathbb{S}^4 [8], which provides an elegant reorganization of Feynman diagrams.

However, when the theory involves dimensionful parameters, such as a mass term in the $\mathcal{N} = 2^*$ theories or a scale generated by dimensional transmutation, the short and long distance properties of the model are different and it is expected that the flat-space calculations do not coincide with those on the sphere. In particular, when a mass term is present, observables on \mathbb{S}^4 naturally depend on the mass scale and on the radius of the sphere by their product. The dependence on this dimensionless parameter of the observables on \mathbb{S}^4 usually differs from the flat-space counterpart. This scenario was analysed in [45], where the authors studied the 1/2 BPS Wilson loop in $\mathcal{N} = 2^*$ SYM and showed that the two-loop perturbative computations of the observable on \mathbb{S}^4 coincide with the matrix model predictions, while the analogous flat-space calculation exhibits a different behaviour.

While a mass deformation breaks explicitly conformal symmetry, in theories with massless matter and a non-vanishing β -function the violation of conformal symmetry occurs at the quantum level. Compactifying these set-ups on the four-dimensional sphere, we can still apply supersymmetric localization to map the expectation value of specific protected operator into matrix models. However, when the matter representation \mathcal{R} is associated with a non-vanishing β -function, the one-loop determinants generated by localization requires a regularization based on additional massive supermultiplets of mass M (see in particular section 4 of [8] and section 2 of [20] for more details). These are properly introduced in order to make the β -function vanish and the one-loop determinants expressible via well-defined products of H -functions² (see eq. (2.7)). In the limit $M \rightarrow \infty$, the massive degrees of freedom decouple and we remain with a well-defined matrix model for $\mathcal{N} = 2$ SYM with massless hypermultiplets in an arbitrary representation \mathcal{R} . This regularization leads to a matrix model which depends on the one-loop exact running coupling

$$\frac{1}{g^2} = \frac{1}{g_*^2} + \beta_0 \log M^2 R^2, \quad (1.3)$$

²Ref. [8] discusses the case of pure $\mathcal{N} = 2$ SYM, while in section 2 of [20] the authors describe in detail non-conformal $\mathcal{N} = 2$ SQCD and generalize the procedure to the theories under examination.

where g_* is the renormalized coupling evaluated at the scale M which, from the perspective of the massless theory, plays the role of a UV cut-off, while R is the radius of the sphere; it is also the radius of the BPS Wilson loop on \mathbb{S}^4 . Eq. (1.3) also describes the running coupling constant of the flat-space theory evaluated at the energy scale $1/R$, with R being the radius of the circular Wilson loop.

The dependence of the matrix model on the running coupling g is obviously expected and analogous to the flat-space computations. It is therefore important to investigate whether the conventional perturbative series in Euclidean space, when expressed in terms of the running coupling, is encoded in the localization effective action or to understand which part of this series (if any) is univocally determined by the localization approach. This question was addressed in [46] for the correlators of chiral primary operators. The analysis revealed that the flat-space calculation matches the localization prediction at order g^4 , while at order g^6 the agreement occurs only for dimensionless ratios of correlators. A similar analysis is presented in [20], where it was showed that the calculation of the 1/2 BPS Wilson loop in flat space matches the localization predictions up to order g^4 .

In the present work, which is a detailed version of a recent short letter [47], we extend the results presented in [20] up to order g^6 . In an asymptotically free $\mathcal{N} = 2$ theory with massless hypermultiplets in an representation \mathcal{R} of $SU(N)$, the perturbative prediction of the matrix model for the 1/2 BPS Wilson loops takes the following form

$$W(g) = W_0(g) + g^6 \frac{3\zeta(3)}{28\pi^4 N} \mathcal{K}'_4 + g^6 \frac{\zeta(3) C_F N \beta_0}{16\pi^2} + \mathcal{O}(g^8). \tag{1.4}$$

In the previous expression, $W_0(g)$ is the expectation value of the operator in the Gaussian matrix model, while \mathcal{K}'_4 is a colour factor which depends on the representation \mathcal{R} (see eq. (2.19)). The previous expression is *valid* only in the range of scales

$$\frac{1}{\Lambda} \gg R \gg \frac{1}{M}, \quad \text{where} \quad \Lambda = M e^{\frac{1}{2\beta_0 g_*^2}} \tag{1.5}$$

is the infrared strong coupling scale generated by dimensional transmutation. In this work, we will show that perturbation theory in flat space exactly reproduces eq. (2.20) within the regime (1.5) where the running coupling g , defined in eq. (1.3), is small. Conversely, for $\Lambda R \sim 1$ the running coupling g grows so that a resummation of the perturbative series would be needed in order to include in the observables non-perturbative power-like corrections³ of the form $C_n(R\Lambda)^n$.

On general grounds, we expect that the functional dependence of the observable on RA suffers from a conformal anomaly and differs between the sphere and flat space. Similarly, when $MR \sim 1$, the massive degrees of freedom become relevant and the nature of the theory changes. As a result, the observables acquire a further dependence on RM which is not purely logarithmic.

In the following, we will show that standard perturbation in flat Euclidean space perfectly reproduces eq. (1.4) within the range of validity (1.5). In particular, the two $\zeta(3)$ -like

³In special multicolour models, such $\mathcal{N} = 2^*$ SYM or the massive deformation of superconformal $\mathcal{N} = 2$ SQCD, the coefficients C_n can be calculated on the four-sphere by matrix model generated via supersymmetric localization [48]. Moreover, also instantons, which we neglected in our analysis, could contribute to the calculation of the observables with power-like corrections.

corrections in eq. (1.4) have a different origin: the contribution proportional to \mathcal{K}'_4 is also present in superconformal set-ups [30, 31] and arises from a Feynman integral which retains the same form in flat space and on the sphere, while that involving the coefficient β_0 , emerges by interference effects between evanescent terms and the UV divergence of the bare coupling constant. Our analysis highlights how the localization matrix model organizes in a compact and elegant way different and complicated diagrammatic contributions, encoding efficiently ultraviolet cancellations and subtle effects resulting from regularizing and renormalizing the flat-space perturbative series.

Field theory set-up. In flat space, we consider $SU(N)$ $\mathcal{N} = 2$ SYM theories with massless hypermultiplets in an arbitrary representation \mathcal{R} such that the β -function is non-vanishing. The explicit expression of the actions is reported in appendix A.1.

The 1/2 BPS Wilson loop operator in the fundamental representation is defined by

$$\widehat{\mathcal{W}} = \frac{1}{N} \text{tr} \mathcal{P} \exp \left\{ g_B \int_C d\tau \left[iA^\mu(x(\tau)) \dot{x}_\mu(\tau) + \frac{R}{\sqrt{2}} (\bar{\phi}(x(\tau)) + \phi(x(\tau))) \right] \right\}, \quad (1.6)$$

where g_B is the bare coupling constant, while \mathcal{P} denotes the path-ordering operator. In the previous expression, the gauge field $A_\mu(x(\tau))$ and the vector-multiplet scalar $\phi(x(\tau))$ are integrated over a circle C of radius R and canonically parametrized by

$$x^\mu(\tau) = R(\cos \tau, \sin \tau, 0, 0), \quad \text{with } 0 \leq \tau < 2\pi. \quad (1.7)$$

The vacuum expectation value of (1.6) contains ultraviolet divergent diagrams. To regularize the singular corrections and preserve the extended supersymmetry we dimensionally reduce the theory from four to $d = 4 - 2\epsilon$ dimensions [23]. In this scheme, the gauge field A_μ is a d -dimensional vector, while the real scalars generated by the reduction are denoted with A_i , with $i = 1, \dots, 2\epsilon$. Since the bare coupling is dimensionless only when $d = 4$, this regularization scheme breaks classical conformal symmetry. As a result, the dimensionally regularized observable can only depend on the combination $\hat{g}_B = R^\epsilon g_B$. Perturbatively, we expand the expectation value as follows

$$\langle \widehat{\mathcal{W}} \rangle \equiv \mathcal{W} = 1 + \mathcal{W}_2 + \mathcal{W}_4 + \mathcal{W}_6 + \mathcal{O}(\hat{g}_B^8), \quad (1.8)$$

where the quantities \mathcal{W}_{2k} are proportional to \hat{g}_B^{2k} . Throughout this work, unless stated otherwise, the Feynman gauge will be always understood.

Structure of the paper. This paper is organized as follows. In section 2, we present the structure of Pestun's matrix model in general massless $\mathcal{N} = 2$ theories with matter representation associated with a non-vanishing β -function. Subsequently, we consider the insertion of the 1/2 BPS Wilson loops and derive the explicit prediction of localization for its perturbative expansion up to order g^6 . In section 3, we present the field theory analysis in flat space. We will first review the two-loop results obtained in [20] and explain the non-trivial role of additional evanescent terms which result from the integration over the Wilson loop contour. Upon renormalization, these contributions produce finite three-loop corrections which combine with the diagrams presented in subsection 3.3. Finally, in section 4, we

discuss the renormalization of the Wilson loop operator. We show that the structure of the divergences respects the usual renormalization properties expected for this operator and that within the specific range of energy scales (1.5), the perturbative series in flat space coincides with the prediction of the matrix model. Finally, in section 5, we draw our conclusions and present some possible future directions. Calculation details of the three-loop diagrams involves several intermediate steps, mainly related to intricate path-ordered integration over the Wilson loop contour which, as far as we know, have not been performed in the current literature. These computations are presented in detail in five different appendices.

2 Predictions from localization

In this work, we consider $\mathcal{N} = 2$ theories with $SU(N)$ gauge group and massless hypermultiplets in an arbitrary representation \mathcal{R} such that the β -function is non-vanishing. When these theories are compactified on S^4 , supersymmetric localization enables to reduce the path integral to an interacting matrix model. However, the one-loop fluctuation determinants require a regularization which involves additional degrees of freedom of mass M [8, 20]. The purpose of this section is to introduce the (regularized) matrix model⁴ which describes the vacuum expectation value of the 1/2 BPS Wilson loop on S^4 for this class of theories and present the three-loop prediction for this observable.

2.1 The S^4 partition function

Compactifying a generic $SU(N)$ $\mathcal{N} = 2$ SYM theory on a four-sphere S^4 of radius R , localization [8] maps the partition function into a matrix model, i.e.

$$\mathcal{Z} = \int \mathcal{D}a \ |Z(ia, g, R)|^2. \tag{2.1}$$

In the previous expression, a is an $N \times N$ Hermitian traceless matrix whose eigenvalues a_u parametrize the Coulomb moduli space and the integration measure is given by

$$\mathcal{D}a = \prod_{u=1}^N da_u \Delta(a) \delta\left(\sum_{v=1}^N a_v\right), \quad \text{with} \quad \Delta(a) = \prod_{u < v}^N (a_u - a_v)^2, \tag{2.2}$$

denoting the Vandermonde determinant. This quantity represents the Jacobian of the transformation which connects the integration over a Lie algebra \mathfrak{g} to its Cartan subalgebra \mathfrak{h} . This means that $\mathcal{D}a$ is equivalent to the flat integration measure

$$da = \prod_{b=1}^{N^2-1} da_b, \quad \text{where} \quad a = a_b t^b. \tag{2.3}$$

In the previous expression, we denoted with t_n the n -th hermitian traceless generator of $\mathfrak{su}(n)$ in the fundamental representation where⁵

$$\text{tr } t_a t_b = \frac{\delta_{ab}}{2}. \tag{2.4}$$

⁴In particular, we refer to section 1 of [20] for the technical details.

⁵The normalization of eq. (2.4) fixes the Dynkin index of the fundamental representation to $i_F = 1/2$.

In the localized partition function (2.1), the integrand consists of three different factors

$$Z = Z_{1\text{-loop}}^{\mathcal{R}} Z_{\text{inst}} Z_{\text{cl}}. \tag{2.5}$$

In the previous expression, Z_{inst} describes the instanton contribution, which can be discarded since we will primarily work in perturbation theory, while Z_{cl} and $Z_{1\text{-loop}}^{\mathcal{R}}$ denote, respectively, the classical term of the matrix model and its interaction potential, which depends on the representation \mathcal{R} . These quantities are defined as follows [20]

$$|Z_{\text{cl}}(ia, g)|^2 = e^{-\frac{8\pi^2 R^2}{g^2} \text{tr} a^2}, \quad |Z_{1\text{-loop}}^{\mathcal{R}}|^2 = \frac{\prod_{\mathbf{w}_{\text{Adj}}} H(R \mathbf{w}_{\text{Adj}} \cdot \mathbf{a})}{\prod_{\mathbf{w}_{\mathcal{R}}} H(R \mathbf{w}_{\mathcal{R}} \cdot \mathbf{a})}. \tag{2.6}$$

In the previous expression, g is the running coupling defined in eq. (1.3), \mathbf{a} denotes an N -dimensional vector containing the eigenvalues of the matrix a , while $\mathbf{w}_{\mathcal{R}}$ and \mathbf{w}_{Adj} are the weight-vectors of the representation \mathcal{R} and of the adjoint one respectively. Moreover, $H(x)$ is defined through the product of Barnes' G-function as follows [48]

$$H(x) = G(1 + ix)G(1 - ix) e^{-(1+\gamma)x^2} = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}, \tag{2.7}$$

where γ is the Euler's constant. Using the properties of the G -function, it is straightforward to show that for small values of the argument we have

$$\log H(z) = - \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(2m-1) z^{2m}}{m}. \tag{2.8}$$

The contribution of the one-loop determinants in eq. (2.6) can be exponentiated and interpreted as an interaction potential for the matrix model, i.e.

$$S_{\text{int}}(a) \equiv - \log |Z_{1\text{-loop}}^{\mathcal{R}}|^2 = (\text{Tr}_{\mathcal{R}} - \text{Tr}_{\text{Adj}}) H(Ra). \tag{2.9}$$

Combining together the relations of this subsection and rescaling the integration variable according to $a \rightarrow \left(\frac{g^2}{8\pi^2 R^2}\right)^{\frac{1}{2}} a$, we can write the localized partition function with the contribution of the instanton suppressed as follows⁶ [20, 31]

$$\mathcal{Z} = \int da e^{-\text{tr} a^2 - S_{\text{int}}(a, g)}. \tag{2.10}$$

In the previous expression, the measure da is defined in eq. (2.3) and is normalized in such a way that $\int da e^{-\text{tr} a^2} = 1$, while the interaction potential of eq. (2.9) acquires a dependence on g and can be expanded as a power series by eq. (2.8), i.e.

$$S_{\text{int}}(a, g) = - \sum_{m=2}^{\infty} \left(-\frac{g^2}{8\pi^2}\right)^m \frac{\zeta(2m-1)}{m} \text{Tr}'_{\mathcal{R}} a^{2m}, \tag{2.11}$$

⁶In eq. (2.10), we did not include the Jacobian of the transformation $a \rightarrow \left(\frac{g^2}{8\pi^2 R^2}\right)^{\frac{1}{2}} a$ since it introduces a multiplicative constant which disappears in properly normalized expectation values.

The first term on the right-hand side of the previous expression denotes the expectation value of the BPS Wilson loop in the Gaussian matrix model, i.e. [20, 31]

$$\begin{aligned}
 W_0 &= \frac{1}{N} L_{N-1}^1 \left(-\frac{g^2}{4} \right) \exp \left(\frac{g^2}{8} \left(1 - \frac{1}{N} \right) \right) \\
 &= 1 + \frac{g^2 C_F}{4} + \frac{g^4 C_F (2N^2 - 3)}{192N} + \frac{g^6 C_F (N^4 - 3N^2 + 3)}{4608N^2} + \dots, \tag{2.16}
 \end{aligned}$$

where $C_F = (N^2 - 2)/2N$ is the fundamental Casimir, while $L_m^n(x)$ denotes the n -th generalized Laguerre polynomial of degree m . In $\mathcal{N} = 4$ SYM, where the matrix model is Gaussian and g is a pure parameter, the observable is precisely given by the previous expression which, from a diagrammatic point of view, encodes the resummation of the ladder-like corrections [23, 24].

Turning our attention to the effects of the interaction action (2.15), we note that these become evident only at three-loop accuracy. In particular, expanding the Wilson loop operator via eq. (2.14), we find that the lowest order contribution takes the form

$$\left(\frac{g^2}{8\pi^2} \right)^2 \frac{\zeta(3)}{2} \langle \mathcal{W}(a, g) \text{Tr}'_{\mathcal{R}} a^4 \rangle_{0,c} = \left(\frac{g^2}{8\pi^2} \right)^2 \frac{\zeta(3)}{2} \frac{g^2}{4N} \langle \text{tr} a^2 \text{Tr}'_{\mathcal{R}} a^4 \rangle_{0,c} + \mathcal{O}(g^8). \tag{2.17}$$

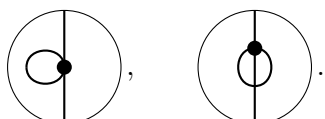
To evaluate the connected correlator for an arbitrary \mathcal{R} we can introduce the free contraction $\langle a^a a^b \rangle_0 = \delta^{ab}$ and apply Wick theorem. By considering the legitimate contractions, it is straightforward to show that

$$\left(\frac{g^2}{8\pi^2} \right)^2 \frac{\zeta(3)}{2} \frac{g^2}{4N} \langle \text{tr} a^2 \text{Tr}'_{\mathcal{R}} a^4 \rangle_{0,c} = \frac{g^6 3\zeta(3)}{2^8 \pi^4 N} \mathcal{K}'_4 + \frac{g^6 \zeta(3) C_F N \beta_0}{16\pi^2}. \tag{2.18}$$

In the previous expression, β_0 is the one-loop coefficient of the β -function, defined in eq. (1.1), and we introduced the $SU(N)$ -invariant quantity

$$\mathcal{K}'_4 = \text{Tr}'_{\mathcal{R}} T_a T_e T^a T^e = 2N C_F \left(C_{\mathcal{R}} i_{\mathcal{R}} - \frac{N i_{\mathcal{R}}}{2} - \frac{N^2}{2} \right). \tag{2.19}$$

The two interaction contributions in eq. (2.18) correspond to the two inequivalent contractions of matrix model quartic vertex


(2.20)

The correspondence between the matrix model vertices and matter loops (2.12) suggests that these interaction contributions proportional to $\zeta(3)$ should emerge in perturbation theory from two inequivalent single-exchange diagrams. As we already stressed in the previous section, this correspondence was originally tested in [30, 31] for generic superconformal set-ups, where only the correction proportional to \mathcal{K}'_4 is present. In non-conformal models, the prediction of the matrix model also includes an additional term proportional to β_0 . In the following sections, we will show that this novel contribution emerges in perturbative field

theory by interference effects between the (UV) poles of the bare coupling and evanescent factors associated with special parts of diagrams which behave as single exchange correction.

Finally, combining together the relations we derived in this subsection, we obtain a simple expression for the three-loop prediction, i.e.

$$W(g) = W_0(g) + g^6 \frac{3\zeta(3)}{2^8 \pi^4 N} \mathcal{K}'_4 + g^6 \frac{\zeta(3) C_F N \beta_0}{16\pi^2} + \mathcal{O}(g^8), \quad (2.21)$$

where we recall that $W_0(g)$ is given by (2.16). Let us stress again that the previous expression is valid within the range (1.5). Relaxing this condition, we expect that the observable receives non-perturbative *infrared* corrections (see comments after eq. (1.5)) which make the result on the sphere different from the flat-space counterpart.

3 Field theory in flat space

Let us begin with observing that at any perturbative order \hat{g}_B^{2k} , we can organize the quantities \mathcal{W}_{2k} of eq. (1.8) as follows:

$$\mathcal{W}_{2k} = \mathcal{W}_{2k}^{\text{ladder}} + \mathcal{W}_{2k}^{\text{v.m.}} + \mathcal{W}_{2k}^{\mathcal{R}}. \quad (3.1)$$

The first two contributions capture, respectively, the ladder-like diagrams, in which the gauge field A_μ and the scalar field ϕ are exchanged at tree-level, and the interaction corrections with internal vertices and lines of the vector multiplet only. These contributions are in common with the $\mathcal{N} = 4$ theory. By $\mathcal{W}_{2k}^{\mathcal{R}}$ we denote, instead, the diagrams that contain internal lines associated with the matter hypermultiplets in the representation \mathcal{R} .

It is well known that in the $\mathcal{N} = 4$ theory, where matter transforms in the adjoint representation, only the ladder-like diagrams contribute to the expectation value of the Wilson loop in the limit $d \rightarrow 4$. This means that, in general, we can write

$$\mathcal{W}_{2k}^{\text{v.m.}} = -\mathcal{W}_{2k}^{\text{Adj}} + \delta\mathcal{W}_{2k}^{\text{v.m.}}, \quad (3.2)$$

where $\delta\mathcal{W}_{2k}^{\text{v.m.}}$ is an evanescent corrections: it vanishes for $d = 4$ and can be expanded in power series of $\epsilon = (4 - d)/2$. As we will discuss in section 4, upon renormalization, the ultraviolet poles of the bare coupling constant \hat{g}_B interfere with the evanescent terms and produce finite corrections at higher orders in perturbation theory. This means that the renormalized expectation value at three loops, also receives non-trivial contributions from the two-loop evanescent corrections $\delta\mathcal{W}_4^{\text{v.m.}}$ which we will compute explicitly in the following subsection.

Substituting eq. (3.2) into eq. (3.1), we have

$$\mathcal{W}_{2k} = \mathcal{W}_{2k}^{\text{ladder}} + \mathcal{W}'_{2k} + \delta\mathcal{W}_{2k}^{\text{v.m.}}, \quad \text{where} \quad \mathcal{W}'_{2k} \equiv \mathcal{W}_{2k}^{\mathcal{R}} - \mathcal{W}_{2k}^{\text{Adj}}. \quad (3.3)$$

Thus, besides the ladder-like diagram and the corrections $\delta\mathcal{W}_{2k}$, at any perturbative order the interaction contributions are constructed by subtracting from $\mathcal{W}_{2k}^{\mathcal{R}}$ exactly the same diagrams in which the internal matter lines are in the adjoint representation. This combination of contributions, denoted by \mathcal{W}'_{2k} , precisely encodes the *difference theory* diagrams predicted by the interaction action of the matrix model, see eq. (2.9).

For $d = 4$ the ladder-like contributions $\mathcal{W}_{2k}^{\text{ladder}}$ are known for every k and are captured by a Gaussian matrix model through eq. (2.16). Thus, for $d \rightarrow 4$, we can write

$$\mathcal{W}_{2k}^{\text{ladder}} = \mathcal{W}_{2k}^{\text{ladder}} \Big|_{d=4} + \delta\mathcal{W}_{2k}^{\text{ladder}}. \tag{3.4}$$

The evanescent corrections $\delta\mathcal{W}_{2k}^{\text{ladder}}$ can contribute, upon renormalization, to higher perturbative orders. For our purposes, we will have to compute $\delta\mathcal{W}_4^{\text{ladder}}$.

3.1 One-loop corrections

At order \hat{g}_B^2 , the Wilson loop expectation value receives contributions from a single class of ladder-like diagrams, i.e.

$$\mathcal{W}_2 = \text{Diagram 1} + \text{Diagram 2} \equiv \text{Diagram 3}. \tag{3.5}$$

In the previous expression, we employed the double straight/wiggly line of eq. (2.12) to depict the tree level propagators of the adjoint scalar and of the gauge-field. In the d dimensional Euclidean space, their expression is given by

$$\begin{aligned} \langle \phi^a(x_1) \bar{\phi}^b(x_2) \rangle_0 &= \delta^{ab} \Delta(x_{12}), \\ \langle A_\mu^a(x_1) A_\nu^b(x_2) \rangle_0 &= \delta_{\mu\nu} \delta^{ab} \Delta(x_{12}), \end{aligned} \tag{3.6}$$

where we introduced the notation $x_{12} \equiv x_1 - x_2$, while the function $\Delta(x_{12})$ is given by⁹

$$\Delta(x_{12}) = \mathcal{D}(x_{12}, 1) = \frac{\Gamma(1 - \epsilon)}{4\pi^{2-\epsilon} (x_{12}^2)^{1-\epsilon}}. \tag{3.7}$$

Expanding the Wilson loop (1.6) at order g_B^2 , and employing the free Wick contractions (3.6), we obtain the following representation for the diagrams in eq. (3.5):

$$\mathcal{W}_2 = \text{Diagram 3} = \frac{g_B^2 C_F}{2} \oint d^2\tau (R^2 - \dot{x}_1 \cdot \dot{x}_2) \Delta(x_{12}). \tag{3.8}$$

The two terms above are, respectively, associated with the propagation of the adjoint scalar and of the gauge field inside the Wilson loop. In particular, as will see in the following sections, this combination enters all the diagrams contributing to the BPS Wilson loop (1.6). Consequently, it is convenient to introduce the following *effective (tree-level) propagator* on the Wilson loop:

$$\hat{\Delta}(x_{12}) = (R^2 - \dot{x}_1 \cdot \dot{x}_2) \Delta(x_{12}) = \frac{\Gamma(1 - \epsilon)}{8\pi^{2-\epsilon}} \left(4R^2 \sin^2\left(\frac{\tau_{12}}{2}\right) \right)^\epsilon, \tag{3.9}$$

⁹This corresponds to the case $s = 1$ in eq. (B.7), since the in momentum space the tree-level propagator is simply $1/p^2$.

where in the second step we used the parametrization (1.7). Substituting the previous expression in eq. (3.8), we observe that the integration over the contour reduces to a single integral of the form considered in eq. (F.17), namely

$$a_0(\alpha) \equiv \frac{1}{\pi} \oint d\tau \frac{1}{(4 \sin^2(\frac{\tau}{2}))^\alpha} = \frac{\sec(\pi\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)\Gamma(2\alpha)}. \tag{3.10}$$

As a result, it is straightforward to express the one-loop correction $\mathcal{W}_2 = \mathcal{W}_2^{\text{ladder}}$ in a closed form which is valid for any d :

$$\mathcal{W}_2^{\text{ladder}} = \hat{g}_B^2 C_F \frac{\Gamma(1-\epsilon)}{8\pi^\epsilon} a_0(-\epsilon) \equiv \hat{g}_B^2 C_F B_1(\epsilon). \tag{3.11}$$

In the previous expression, we introduced, for future convenience, the set of functions

$$B_n(\epsilon) = \frac{\Gamma^n(1-\epsilon)}{8\pi^{n\epsilon}} a_0(-n\epsilon), \tag{3.12}$$

which are regular and independent of n for $\epsilon \rightarrow 0$. As we will see in the following, single-exchange contributions, dressed with the $(n-1)$ -th loop corrections to the propagators, are expressed in terms of the function $B_n(\epsilon)$.

Expanding eq. (3.11) about $\epsilon \rightarrow 0$ we can construct explicitly the two terms of eq. (3.4) at one loop. To do this properly we have, however, to re-express the bare coupling in terms of the renormalized one; we will do this in section 4.

3.2 Two loop corrections

The two-loop corrections to the expectation value of Wilson loop were analysed in great details in [20]. We devote this subsection to review the results at order \hat{g}_B^2 and determine the relevant evanescent corrections we will employ for the three-loop analysis. According to eq. (3.1), we organize the different families of two-loop diagrams in terms of three distinct classes of terms, i.e. $\mathcal{W}_4^{\text{ladder}}$, $\mathcal{W}_4^{\text{v.m.}}$ and $\mathcal{W}_4^{\mathcal{R}}$.

Let us begin with discussing the two-loop ladder-like diagrams. Expanding the Wilson loop operator (1.6) at order g_B^4 and employing the tree-level propagators of the adjoint scalar and gauge field (3.6), we find the ladder corrections

$$\begin{aligned} \text{Diagram} &= \frac{g_B^4}{N} \oint_{\mathcal{D}} d^4\tau \left\{ C^{aabb} \left(\hat{\Delta}(x_{12})\hat{\Delta}(x_{34}) + \hat{\Delta}(x_{14})\hat{\Delta}(x_{23}) \right) + C^{abab} \hat{\Delta}(x_{13})\hat{\Delta}(x_{24}) \right\} \\ &= \mathcal{W}_4^{\text{ladder}}. \end{aligned} \tag{3.13}$$

In the previous expression, the domain of integration \mathcal{D} denotes the ordered region $\tau_1 > \tau_2 > \tau_3 > \tau_4$, the propagator $\hat{\Delta}(x)$ is defined in eq. (3.9) and we introduced the $SU(N)$ tensor

$$C^{abcd} = \text{tr} T^a T^b T^c T^d. \tag{3.14}$$

Using the properties of the non-Abelian exponentiation of the Wilson loop [49, 50], we can reduce eq. (3.13) to the following expression

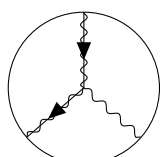
$$\mathcal{W}_4^{\text{ladder}} = \frac{1}{2} \left(\mathcal{W}_2^{\text{ladder}} \right)^2 + \frac{\hat{g}_B^4}{2N} \text{tr} \left([T^b, T^a] \right)^2 \oint_{\mathcal{D}} d^4\tau \hat{\Delta}(x_{13})\hat{\Delta}(x_{24}), \tag{3.15}$$

where $\mathcal{W}_2^{\text{ladder}}$ is the ladder-like contribution of eq. (3.11), while the second term defines the so-called *maximally non-Abelian part* of the diagrams. The nested integration in this last term is treated in detail in appendix F by Fourier representations. Employing the parametrization (1.7) and eq. (F.20), we finally find

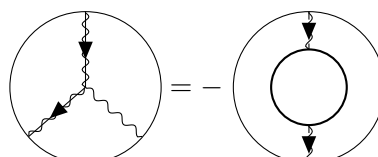
$$\mathcal{W}_4^{\text{ladder}} = \hat{g}_B^4 \frac{C_F(2N^2 - 3)}{12N} B_1^2(\epsilon) - \epsilon \hat{g}_B^4 \frac{C_F N \zeta(3)}{16\pi^2} + \mathcal{O}(\epsilon)^2. \quad (3.16)$$

Note that the term proportional to $\zeta(3)$ arises from the maximally non-Abelian part of the diagram. Further expanding the function $B_1(\epsilon)$, by employing eq. (3.12), we can determine the complete expression of the evanescent correction $\delta\mathcal{W}_4^{\text{ladder}}$. For convenience, however, we will present this calculation in section 4, where we will discuss the renormalization of the Wilson loop.

Secondly, we analyse the quantity $\mathcal{W}_4^{\text{v.m.}}$, which encodes all the two-loop diagrams uniquely characterized by internal vertices and lines associated with the vector-multiplet. The only non-trivial contributions result from the *Mercedes-like* diagrams:¹⁰

$$\mathcal{W}_4^{\text{v.m.}} = \text{Diagram} \quad (3.17)$$


This class of corrections were originally discussed in [23], where the authors studied the supersymmetric Wilson loop in $\mathcal{N} = 4$ SYM and showed that

$$\text{Diagram} = - \text{Diagram} + \delta\mathcal{W}_4^{\text{v.m.}} = -\mathcal{W}_4^{\text{Adj}} + \delta\mathcal{W}_4^{\text{v.m.}}. \quad (3.18)$$


This expression provides a concrete realization of eq. (3.2) at two loops. In particular, the bubble-like contribution denotes the one-loop correction to the adjoint scalar and gauge field propagator in $\mathcal{N} = 4$ SYM, where the hypermultiplets are in the adjoint representation, while the evanescent correction $\delta\mathcal{W}_4^{\text{v.m.}}$ is given by

$$\delta\mathcal{W}_4^{\text{v.m.}} = \epsilon \frac{\hat{g}_B^4 C_F N \Gamma(1 - 2\epsilon)}{(2\pi)^{-2\epsilon} 128\pi^4} \int_0^1 dF (\alpha\beta\gamma)^{-\epsilon} \oint d^3\tau \varepsilon(\tau) \frac{\sin \tau_{13}}{Q^{1-2\epsilon}} + \mathcal{O}(\epsilon)^2. \quad (3.19)$$

In the previous expression, we introduced the quantities

$$Q = \alpha\beta(1 - s\tau_{12}) + \beta\gamma(1 - s\tau_{23}) + \gamma\alpha(1 - \cos \tau_{13}), \quad (3.20)$$

$$dF = d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma), \quad (3.21)$$

$$\varepsilon(\tau) = \theta(\tau_{12})\theta(\tau_{23}) - \theta(\tau_{13})\theta(\tau_{32}) + \text{permutations}. \quad (3.22)$$

¹⁰In principle, one could also expect single-exchange diagrams dressed with the one-loop corrections to the adjoint scalar and gauge field propagators resulting from the vector-multiplet interaction. However, it follows from eqs. (B.11) and (B.12) that these specific contributions are not present.

The path-ordered integral in eq. (3.19) is completely regular in the limit $\epsilon \rightarrow 0$ and is evaluated in appendix F. In particular, using eq. (F.15), we find that

$$\int_0^1 dF (\alpha\beta\gamma)^{-\epsilon} \oint d^3\tau \varepsilon(\tau) \frac{\sin \tau_{13}}{Q^{1-2\epsilon}} = -16\pi^2 \zeta(3) + \mathcal{O}(\epsilon). \quad (3.23)$$

Substituting this expression in eq. (3.19) and expanding the prefactor about $\epsilon \rightarrow 0$, we finally arrive at the following result:

$$\delta\mathcal{W}_4^{\text{v.m.}} = -\epsilon \frac{\hat{g}_B^4 C_F N \zeta(3)}{8\pi^2} + \mathcal{O}(\epsilon)^2. \quad (3.24)$$

The last quantity we have to determine is the correction $\mathcal{W}_4^{\mathcal{R}}$, which encodes all the diagrams characterized by internal lines associated with the matter hypermultiplets in the representation \mathcal{R} . At two loops, we find

$$\mathcal{W}_4^{\mathcal{R}} = \left(\text{Diagram with dashed loop} \right), \quad (3.25)$$

where the dashed virtual loop denote the one-loop corrections to the adjoint scalar and gauge field propagator resulting from matter field in the representation \mathcal{R} . We can now combine the previous expression with eq. (3.18) to construct the difference theory diagrams at two-loop, i.e.

$$\mathcal{W}'_4 = \mathcal{W}_4^{\mathcal{R}} - \mathcal{W}_4^{\text{Adj}} = \left(\text{Diagram with dashed loop} \right) - \left(\text{Diagram with solid loop} \right) \equiv \left(\text{Diagram with dashed loop} \right). \quad (3.26)$$

Thus, we remain with a single-exchange contribution dressed with the one-loop correction to the adjoint scalar and gauge field propagator in the difference theory approach. The expression of these propagators in configuration space are given by eqs. (B.16), (B.17). Note that the correction to the gluon propagator involves the gauge-like term $\partial_{1,\mu} \partial_{2,\nu} \Delta^{(1),g}(x_{12})$ which, when contracted with the tangent vectors $\dot{x}_1^\mu \dot{x}_2^\nu$, gives rise to total derivatives integrated over a closed path. These obviously vanish and we remain with

$$\mathcal{W}'_4 = \frac{g_B^2 C_F}{2} \oint d^2\tau \hat{\Delta}^{(1)}(x_{12}). \quad (3.27)$$

In analogy to the ladder-like correction (3.8), we introduced an effective one-loop propagator on the Wilson loop contour

$$\begin{aligned} \hat{\Delta}^{(1)}(x_{12}) &= (R^2 - \dot{x}_1 \cdot \dot{x}_2) \Delta^{(1)}(x_{12}) \\ &= \frac{g_B^2 f^{(1)}(\epsilon) \Gamma(1-2\epsilon)}{2^{3+2\epsilon} \pi^{2-\epsilon} \Gamma(1+\epsilon)} \left(4R^2 \sin^2\left(\frac{\tau_{12}}{2}\right) \right)^{2\epsilon}, \end{aligned} \quad (3.28)$$

where, to obtain the second equality, we used the explicit definition of the function $\Delta^{(1)}(x_{12})$, given by eq. (B.16), and the parametrization (1.7). Performing the integration over the

contour by eq. (3.10) and by employing the definition of $f^{(1)}(\epsilon)$ in eq. (B.15), we produce a factor $2\pi^2 a_0(-2\epsilon)$ and arrive at the following result:

$$\mathcal{W}'_4 = \hat{g}_B^4 C_F P_2(\epsilon) B_2(\epsilon), \quad \text{where} \quad P_2(\epsilon) = -\frac{\beta_0}{\epsilon(1-2\epsilon)} \quad (3.29)$$

and we recall that the function $B_2(\epsilon)$ was defined in eq. (3.12). Combining together the relations we derived in this subsection, we find that the two-loop corrections to Wilson loop v.e.v can be written as follows:

$$\mathcal{W}_4 = \mathcal{W}_4^{\text{ladder}} + \mathcal{W}'_4 + \delta\mathcal{W}_4^{\text{v.m.}}. \quad (3.30)$$

3.3 Three-loop corrections

The calculation of the three-loop diagrams is significantly more involved and technical than the two-loop one. However, the logical steps are identical except for the fact that we do not have to calculate the evanescent corrections since, upon renormalization, they contribute to four loops. This means that the three-loop corrections take the following form

$$\mathcal{W}_6 = \mathcal{W}_6^{\text{ladder}} \Big|_{d=4} + \mathcal{W}'_6 + \mathcal{O}(\epsilon). \quad (3.31)$$

Let us begin with analysing the ladder diagrams. In $d = 4$ dimensions, their expression is captured by eq. (2.16). We find that

$$\mathcal{W}_6^{\text{ladder}} = \left(\text{Diagram: Circle with three vertical wavy lines and arrows pointing down} \right) = \frac{\hat{g}_B^6 C_F (N^4 - 3N^2 + 3)}{4608N^2} + \mathcal{O}(\epsilon). \quad (3.32)$$

The three-loop interaction contributions are encoded in the difference-theory term $\mathcal{W}'_6 = \mathcal{W}_6^{\mathcal{R}} - \mathcal{W}_6^{\text{Adj}}$. Unlike its two-loop counterpart (3.26), \mathcal{W}'_6 consists of three different classes of Feynman diagrams which can be organized according to the number of insertions in the Wilson loop contour. We use the notation

$$\mathcal{W}'_6 = \mathcal{W}'_{6(2)} + \mathcal{W}'_{6(3)} + \mathcal{W}'_{6(4)}, \quad (3.33)$$

to distinguish each contribution which we will discuss in turn.

3.3.1 Diagrams with two insertions

At order g_B^6 , we can insert in the Wilson loop contour a single scalar/gauge-field propagator dressed with the two-loop corrections in the difference theory approach. The explicit expressions of these corrections in configuration space is computed in appendix B.2, see eqs. (B.33), (B.34). Expanding the Wilson loop at order g_B^2 and employing these relations, we find, using the usual difference-theory notation, the following expression

$$\mathcal{W}'_{6(2)} = \left(\text{Diagram: Circle with a dashed inner circle labeled '2-loop' and two vertical wavy lines with arrows pointing down} \right) = \frac{g_B^2 C_F}{2} \oint d^2\tau \hat{\Delta}^{(2)}(x_{12}). \quad (3.34)$$

In analogy to the one/two-loop corrections (3.8) and (3.26), we defined the two-loop effective propagator on the Wilson loop contour as follows

$$\begin{aligned} \hat{\Delta}^{(2)}(x_{12}) &= (R^2 - \dot{x}_1 \cdot \dot{x}_2) \Delta^{(2)}(x_{12}) \\ &= f^{(2)}(\epsilon) \frac{g_B^4 \Gamma(1 - 3\epsilon)}{2^{3+4\epsilon} \pi^{2-\epsilon} \Gamma(1 + 2\epsilon)} \frac{1}{(4R^2 \sin^2 \frac{\tau_{12}}{2})^{-3\epsilon}}, \end{aligned} \quad (3.35)$$

where to obtain the second equality we employed eq. (B.33) and the parametrization (1.7).

Substituting eq. (3.35) in eq. (3.34), we can easily integrate over the Wilson loop contour by means of eq. (3.10). Moreover, recalling that $f^{(2)}(\epsilon)$, given by eqs. (B.31), (B.32), contains four different terms, we finally find

$$\mathcal{W}'_{6(2)} = \sum_{i=1}^4 F_i^{(2)}, \quad \text{where} \quad F_i^{(2)} = f_i^{(2)}(\epsilon) \frac{\hat{g}_B^6 C_F \Gamma(1 - 3\epsilon)}{2^{3+4\epsilon} \pi^{-\epsilon} \Gamma(1 + 2\epsilon)} a_0(-3\epsilon). \quad (3.36)$$

Using the explicit form (B.32) of the functions $f_i^{(2)}(\epsilon)$ and simple manipulations, we can recast these contributions as follows:

$$\begin{aligned} F_1^{(2)} &= -\hat{g}_B^6 \frac{C_F i_{\mathcal{R}} P_2(\epsilon) B_3(\epsilon)}{8\pi^2 \epsilon(1 - 2\epsilon)} + \mathcal{O}(\epsilon), \\ F_2^{(2)} &= -\hat{g}_B^6 \frac{C_F N P_2(\epsilon) B_3(\epsilon)}{16\pi^2 \epsilon(1 - 3\epsilon)}, \\ F_3^{(2)} &= \hat{g}_B^6 \frac{C_F N P_2(\epsilon) B_3(\epsilon)}{32\pi^2 \epsilon(1 + \epsilon)}, \\ F_4^{(2)} &= \hat{g}_B^6 \frac{\mathcal{K}'_4 3\zeta(3)}{N (4\pi)^4} + \mathcal{O}(\epsilon). \end{aligned} \quad (3.37)$$

Note that only the last contribute is regular in the limit $\epsilon \rightarrow 0$, while the others exhibit single and double UV poles. Note also that the contribution $F_3^{(2)}$ arises from the gauge-like part of the gluon self-energy in the second diagram of eq. (B.25). By gauge invariance, we expect that it should eventually cancel against similar contributions resulting from other diagrams.

3.3.2 Diagrams with three insertions

The three-loop diagrams with three insertions on the Wilson loop contour fall in two distinct classes, corresponding to one-loop reducible and irreducible corrections to the gauge-scalar and pure gauge vertex in the difference theory approach. These diagrams are computed in appendix C and appendix D. The complexity of the calculation lies on the *path-ordered* integration over the contour which we have to perform in arbitrary dimension d due to the presence of UV singularities. Although the computations are extremely technical, the final result is quite simple and follows from eqs. (C.20) and (D.60). We find

$$\begin{aligned} \mathcal{W}'_{6(3)} &= \text{[Diagram 1]} + \text{[Diagram 2]} \\ &= \frac{N}{i_{\mathcal{R}}} F_1^{(2)} - F_2^{(2)} - 2F_3^{(2)} + \hat{g}_B^6 \frac{C_F N \beta_0}{4\pi^2} \zeta(3) + \mathcal{O}(\epsilon). \end{aligned} \quad (3.38)$$

Thus, up to a finite term proportional to $\zeta(3)$, these diagrams with internal vertices are expressible as linear combinations of the bubble-like contributions $F_i^{(2)}$ that emerge from the single-exchange corrections of the same order, see eqs. (3.36), (3.37). As it occurred in eq. (3.36), the $F_3^{(2)}$ contribution above results from diagrams involving the gauge-like part of the gluon self-energy at one-loop.

3.3.3 Diagrams with four insertions

This class of corrections arises when dressing the internal lines of the two-loop ladder-like corrections (3.13) with the one-loop correction to the adjoint scalar and gauge field propagator in the difference approach. The intermediate steps of the calculation are reported in appendix E. In particular, by employing eqs. (E.7) and (E.11), we find that

$$\begin{aligned}
 \mathcal{W}'_{6(4)} &= \text{Diagram} \tag{3.39} \\
 &= F_3^{(2)} + \hat{g}_B^6 \frac{C_F(2N^2 - 3)}{6N} B_1(\epsilon) B_2(\epsilon) P_2(\epsilon) + \hat{g}_B^6 C_F N \beta_0 \frac{3\zeta(3)}{16\pi^2} + \mathcal{O}(\epsilon),
 \end{aligned}$$

where we recall that $F_3^{(2)}$ is the three-loop bubble-like contributions defined in eq. (3.37) and, again, it results from diagrams involving the gauge-like part of the gluon self-energy.

3.4 Summary of the three-loop results

Let us summarise our findings at three-loop accuracy for the difference-theory interaction correction defined in eq. (3.33). Using the results (3.36), (3.38), (3.39), we obtain

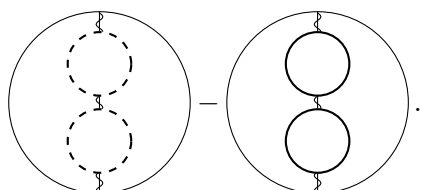
$$\mathcal{W}'_6 = \frac{i_{\mathcal{R}} - N}{i_{\mathcal{R}}} F_1^{(2)} + \hat{g}_B^6 C_F \left(\frac{2N^2 - 3}{6N} B_1(\epsilon) B_2(\epsilon) P_2(\epsilon) + N \beta_0 \frac{7\zeta(3)}{16\pi^2} + \frac{\mathcal{K}'_4}{N} \frac{3\zeta(3)}{2^8 \pi^4 C_F} \right) + \mathcal{O}(\epsilon), \tag{3.40}$$

where we recall that the functions $B_n(\epsilon)$ and $P_2(\epsilon)$ are defined, respectively, in eqs. (3.12) and (3.29). As anticipated, the final result does not include any $F_3^{(2)}$ contributions as a consequence due to gauge invariance. Actually, an analogous cancellation also occurs for the $F_2^{(2)}$ contributions and, as we will shortly see, this is essential to ensure the correct renormalization properties of the Wilson loop observable.

The first contribution in the previous expression can be further simplified by using the explicit definition of the bubble-like contribution $F_1^{(2)}$ given by eq. (3.37). We find that it accounts for a double insertion in the single-exchange diagram (3.8) of the one-loop correction to the adjoint scalar and gauge field in the difference theory:

$$\frac{i_{\mathcal{R}} - N}{i_{\mathcal{R}}} F_1^{(2)} = \hat{g}_B^6 C_F P_2^2(\epsilon) B_3(\epsilon) + \mathcal{O}(\epsilon) = \text{Diagram} + \mathcal{O}(\epsilon). \tag{3.41}$$

Let us note that the internal exchange in the previous expression *does not* represent the reducible component of the internal correction associated with the contribution $\mathcal{W}'_{6(2)}$ (3.34) which, instead, is given by



$$(3.42)$$

In fact, eq. (3.41) arises when adding to the previous diagrams the first term in eq. (3.38), resulting from the diagrams with internal vertices $\mathcal{W}'_{6(3)}$ (3.41). This additional correction introduces the “cross terms” characterized by one of the two internal bubbles in the representation \mathcal{R} and the second one in the adjoint.

Altogether, taking into account all the results described above, we get the following expression of the Wilson loop v.e.v. up to three loops:

$$\begin{aligned} \mathcal{W} = & 1 + \hat{g}_B^2 C_F B_1(\epsilon) + \hat{g}_B^4 C_F \left(\frac{(2N^2 - 3)}{12N} B_1^2(\epsilon) + P_2(\epsilon) B_2(\epsilon) - \epsilon N \frac{3\zeta(3)}{16\pi^2} \right) \\ & + \hat{g}_B^6 C_F \left(\frac{N^4 - 3N^2 + 3}{46098N^2} + \frac{2N^2 - 3}{6N} B_1(\epsilon) B_2(\epsilon) P_2(\epsilon) + P_2^2(\epsilon) B_3(\epsilon) + \beta_0 N \frac{7\zeta(3)}{16\pi^2} \right) \\ & + \hat{g}_B^6 \frac{3\zeta(3)\mathcal{K}'_4}{28\pi^4 N} + \dots, \end{aligned} \tag{3.43}$$

where the dots stand for $\mathcal{O}(\epsilon)$ terms which only contribute at four loops.

4 Renormalization

The vacuum expectation value of the 1/2 BPS Wilson loop (1.6) is (UV) divergent and we have to renormalize it in order to obtain a finite result. The divergences are encoded in the function $P_2(\epsilon)$, defined in eq. (3.29), which is singular in the limit $\epsilon \rightarrow 0$. Since the circular Wilson loop operator is defined over a smooth curve, the singularities are reabsorbed by the charge renormalization [51–53] which, in terms of $\hat{g}_B = g_B R^\epsilon$, reads

$$\hat{g}_B = g_* (RM)^\epsilon Z_{g_*}(\epsilon). \tag{4.1}$$

In the previous expression, g_* is the renormalized coupling evaluated at the renormalization scale M , while $Z_{g_*}(\epsilon)$ encodes the so-called subtraction terms. These can be easily calculated by the explicit expression of the β -function (1.1). In particular, acting on eq. (4.1) with the logarithmic derivative with respect to M and requiring that g_B does not depend on M we find, in the MS scheme, that

$$\begin{aligned} Z_{g_*}(\epsilon) &= \exp\left(-\int_0^{g_*} \frac{dt (\epsilon t + \beta(t))}{t \beta(t)}\right) \\ &= \left(1 - \frac{\beta_0 g_*^2}{\epsilon}\right)^{-\frac{1}{2}} = 1 + \frac{\beta_0 g_*^2}{2\epsilon} + \frac{3(\beta_0)^2 g_*^4}{8\epsilon^2} + \dots \end{aligned} \tag{4.2}$$

The renormalized Wilson loop average is obtained by replacing the bare coupling \hat{g}_B with the renormalized one g_* in the dimensionally regularized observable (3.43) and taking the limit $\epsilon \rightarrow 0$, i.e.

$$W_* = \lim_{\epsilon \rightarrow 0} \mathcal{W}(g_*). \tag{4.3}$$

Note that when $\epsilon \rightarrow 0$, the overall dependence on the renormalization scale M must vanish. This means that W_* satisfies a Callan-Symanzik equation [20] which constrains the dependence of the renormalized Wilson loop average on M , g_* and R , making them to appear in the running coupling constant $g(R)$, defined in eq. (1.3).

If we consider the three-loop results in eq. (3.43), we can verify that all the divergences cancel out upon introducing the renormalized coupling and taking the limit $\epsilon \rightarrow 0$. Moreover, the final result can be expressed in terms of the running coupling. For instance, let us examine the terms

$$\hat{g}_B^2 C_F B_1(\epsilon) + \hat{g}_B^4 C_F P_2(\epsilon) B_2(\epsilon) + \hat{g}_B^6 C_F P_2^2(\epsilon) B_3(\epsilon), \tag{4.4}$$

which correspond, respectively, to a single-exchange diagrams dressed with zero, one or two corrections to the adjoint scalar and gauge field propagator at one-loop in the difference theory. To proceed with the computation, we use eqs. (3.12) and (3.29) to expand the functions $B_n(\epsilon)$ and $P_2(\epsilon)$ about $\epsilon \rightarrow 0$, i.e.

$$\begin{aligned} P_2(\epsilon) &= -\beta_0 \left(\frac{1}{\epsilon} + 2 + 4\epsilon + \mathcal{O}(\epsilon^2) \right), \\ B_1(\epsilon) &= \frac{1}{4} + \frac{1}{4} (\gamma + \log \pi) \epsilon + \frac{1}{16} (\pi^2 + (\gamma + \log \pi)^2) \epsilon^2 + \mathcal{O}(\epsilon^3), \\ B_2(\epsilon) &= \frac{1}{4} + \frac{1}{2} (\gamma + \log \pi) \epsilon + \mathcal{O}(\epsilon^2), \\ B_3(\epsilon) &= \frac{1}{4} + \mathcal{O}(\epsilon), \end{aligned} \tag{4.5}$$

and we replace the bare coupling in eq. (4.4) with the renormalized one (4.1). By employing the subtraction terms (4.2) and the expansions (4.5), it is straightforward to verify that the final result is divergence free. Analogously, it is also straightforward to show that, up to four-loop terms, the finite term takes the following form:

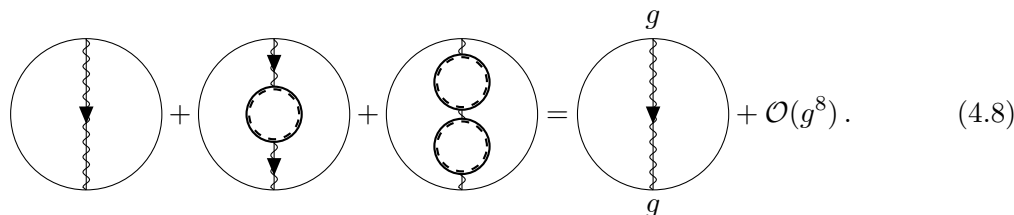
$$\frac{g_*^2}{4} \left(1 - \beta_0 g_*^2 (\log M^2 R^2 + 2 + \gamma + \log \pi) + \beta_0^2 g_*^4 \left((\log M^2 R^2 + 2 + \gamma + \log \pi)^2 + \frac{\pi^2}{3} \right) \right). \tag{4.6}$$

Let us focus on the regime (1.5) in which we derived the matrix model on \mathbb{S}^4 . Within this range, $\log RM \gg 0$ so that the logarithmic terms, associated with the short-distance properties of theory, dominate over $\mathcal{O}(M^0)$ ones. Thus, we can write¹¹

$$\frac{g_*^2}{4} \left(1 - \beta_0 g_*^2 \log M^2 R^2 + \beta_0^2 g_*^4 (\log M^2 R^2)^2 \right) + \mathcal{O}(g_*^8) = \frac{g^2}{4} + \mathcal{O}(g^8), \tag{4.7}$$

¹¹These (scheme-dependent) finite terms are not completely captured by the matrix model, even if we could reabsorb many of them by using as a renormalization scale the quantity \tilde{M} such that $\log \tilde{M}^2 R^2 = \log M^2 R^2 + 2 + \gamma + \log \pi$.

where we recognized the expansion up to order g_*^6 of the running coupling constant defined in eq. (1.3). It is interesting to observe that the previous expression admits a graphical description in terms of a resummation of single-exchange:



$$\text{Tree} + \text{1-loop} + \text{2-loop} = \text{Tree}(g) + \mathcal{O}(g^8). \quad (4.8)$$

The right-hand side of the previous expression highlights that the final result can be obtained by the usual ladder-like contribution by replacing the bare coupling with the running one, defined in eq. (1.3).

Going back to eq. (3.43), we repeat the same analysis for the terms proportional to the colour factor $(2N^2 - 3)$, characterizing the double-exchange diagrams. Exploiting analogous manipulations, we find, within the regime (1.5), that

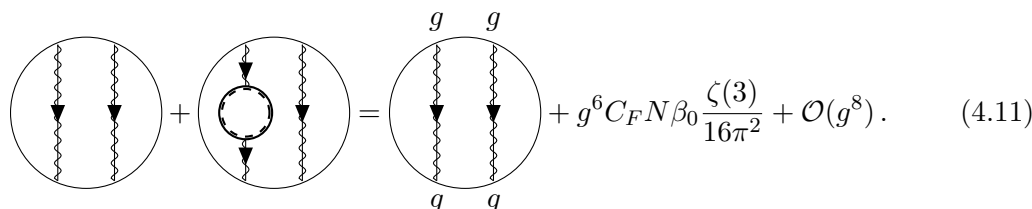
$$\hat{g}_B^4 C_F \frac{2N^2 - 3}{12N} \left(B_1^2(\epsilon) + 2\hat{g}_B^2 B_1(\epsilon) B_2(\epsilon) P_2(\epsilon) \right) = g^4 C_F \frac{2N^2 - 3}{192N} + \mathcal{O}(g^8). \quad (4.9)$$

Let us now consider the terms in (3.43) proportional to $\zeta(3)$ and characterized by the colour factor $C_F N$, for which we have

$$\hat{g}_B^4 C_F N \frac{\zeta(3)}{16\pi^2} \left(-3\epsilon + 7\beta_0 \hat{g}_B^2 \right) = g^6 C_F N \beta_0 \frac{\zeta(3)}{16\pi^2} + \mathcal{O}(g^8). \quad (4.10)$$

Note that each coefficient on the l.h.s. represents the sum of two types of contributions: $-3\epsilon \hat{g}_B^4 = (-2\epsilon - \epsilon) \hat{g}_B^4$ and $7\beta_0 \hat{g}_B^6 = (4 + 3)\beta_0 \hat{g}_B^6$. More specifically, the $(-2\epsilon \hat{g}_B^4)$ term results from the evanescent correction $\delta\mathcal{W}_4^{\text{v.m.}}$ of the two-loop diagrams with internal vertices, defined in eq. (3.18) and explicitly given by eq. (3.24). Upon renormalization, this evanescence interferes with the UV poles of the bare coupling and *precisely remove* the $4\beta_0 \hat{g}_B^6$ term, resulting from the same family of diagrams at three-loop, i.e. the Mercedes and lifesaver corrections we presented in eq. (3.38). This means that all the terms proportional to $\beta_0 \zeta(3)$ only originate from the *ladder-like* diagrams depicted in eqs. (3.16) and (3.39), which are responsible, respectively, for the contributions $-\epsilon \hat{g}_B^4$ and $3\beta_0 \hat{g}_B^6$.

In graphical terms, we can summarize the content of eqs. (4.9) and (4.10) as follows:¹²



$$\text{Tree} + \text{1-loop} = \text{Tree}(g) + g^6 C_F N \beta_0 \frac{\zeta(3)}{16\pi^2} + \mathcal{O}(g^8). \quad (4.11)$$

¹²This is actually not precise: the right hand side includes also, as reported in eq. (3.39), the term $F_3^{(2)}$. However, as we already pointed out, this contribution does not contribute since it is exactly removed by analogous contributions resulting from the correction $\mathcal{W}'_{6(3)}$, see eq. (3.38), and $\mathcal{W}'_{6(2)}$, given by eq. (3.41).

Since our analysis regards the three-loop correction, the renormalization of the triple-exchange terms (3.32) is trivial and provides us with the following contribution

$$g^6 \frac{C_F(N^4 - 3N^2 + 3)}{4608N^2} + \mathcal{O}(g^8). \tag{4.12}$$

The last term in eq. (3.43), proportional to the colour factor \mathcal{K}'_4 , results from the irreducible part of the internal correction in the single-exchange diagrams (3.34), namely from the $F_4^{(2)}$ function in eq. (3.34). We find that

$$\hat{g}_B^6 \frac{\mathcal{K}'_4}{N} \frac{3\zeta(3)}{2^8\pi^4} = g^6 \frac{\mathcal{K}'_4}{N} \frac{3\zeta(3)}{2^8\pi^4} + \mathcal{O}(g^8). \tag{4.13}$$

Collecting all the results we derived in this subsection, we can write the renormalized Wilson loop vev W_* in terms of the running coupling constant g as follows:

$$W_* = W_0 + g^6 \frac{\mathcal{K}'_4}{N} \frac{3\zeta(3)}{2^8\pi^4} + g^6 C_F N \beta_0 \frac{\zeta(3)}{16\pi^2} + \mathcal{O}(g^8), \tag{4.14}$$

where W_0 was introduced in eq. (2.16) and contains the ladder diagrams computed with the running coupling constant g , while the two terms proportional to $\zeta(3)$ coincide exactly with the prediction of the localization matrix model, as follows from eqs. (2.15) and (2.18). Let us stress that this agreement holds within the regime (1.5). From the field theory point of view, the final result, when expressed in terms of the running coupling, is purely due to ladder-like diagrams, see eqs. (4.8), (4.11) and (4.12). Moreover, the final outcome also ties perfectly in with the matrix model diagrams (2.20), which suggest that the two terms proportional to $\zeta(3)$ have to be associated with single-exchange diagrams. Indeed, as we previously explained, the correction involving the coefficient \mathcal{K}'_4 results from the diagram (3.34), while the term $\beta_0\zeta(3)$ is proportional to the fundamental Casimir C_F , which is the expected colour coefficient of the as single-exchange diagrams (3.5).

5 Conclusions and outlook

In this paper, we investigated the relation between supersymmetric localization on \mathbb{S}^4 and standard perturbative techniques in flat space for a generic $\mathcal{N} = 2$ SYM theory with non-vanishing β -function. The analysis has been performed by studying the vacuum expectation value of the 1/2 BPS Wilson loop, for which localization provides an explicit result in term of an interacting matrix model. Although conformal invariance is broken at quantum level, preventing a direct connection between the sphere and the Euclidean space, we found a precise agreement in the specific regime described in eq. (1.5). Within this range of validity, the contribution of instantons and power-like corrections are suppressed and we showed that the matrix model predictions match standard perturbation theory based on Feynman diagrams techniques in flat space up to order g^6 . At this perturbative order the matrix model produces two non-trivial $\zeta(3)$ -like terms, that have a different origin: one is already present in the conformal case [30, 31], while the other is peculiar of the models with non-vanishing β -function. We successfully compared the effective matrix diagrams associated with these contributions with the flat-space perturbative expansion, finding crucial interference effects

between evanescent terms and the UV divergences of the bare coupling constant. Our results not only provide a non-trivial test of the localization approach for generic $\mathcal{N} = 2$ SYM theories, but also make manifest the subtle reorganization of the conventional Feynman diagrams into the matrix-model average. On the technical side, the perturbative computations of the three-loop contributions involved multiple ordered integrations of position-space Green functions along a circular domain. As far as our knowledge is concerned, this type of calculations have never been considered before at such precision level: we have devoted a series of appendices to illustrate the procedure and the actual emergence of the evanescent terms and finite contributions relevant for the final result.

Clearly, there are some possible improvements and extensions of our work. It would be interesting to expand our analysis to the next perturbative order and try to generalize the understanding at all loops. This would imply a more systematic approach to the calculation of Feynman diagrams for circular Wilson loops involving complicated path-ordered trigonometric integrations. In the case of cusped Wilson loops, the path-integration is performed over straight lines by techniques involving heavy quark effective theory. These have provided beautiful results for the cusp anomalous dimension [53] at high-loop order, both in supersymmetric and non-supersymmetric theories (see [54] for status review). It would be nice to develop an analogous tool to face circular contours. Another natural investigation would be to examine correlators of local operators in this non-conformal set-up: supersymmetric localization still gives exact results for classes of two-point functions that can be compared with flat-space perturbation theory [46]. Studying these local observables in light of the present computations could further improve our understanding of the effects associated with a non-trivial β -function. We plan to explore these two-point functions in the near future. A more speculative direction concerns the study the large-order behaviour of the perturbative series in presence of a running coupling constant. Exact all-orders expressions on \mathbb{S}^4 have been already used to explore asymptotic properties of the matrix-model perturbative expansion, in connection with resurgent techniques [55]. The analysis has been performed for different $\mathcal{N} = 2$ SYM theories, obtaining explicit results in the conformal and massive cases. It would be interesting to reconsider the non-conformal case and its relation with a flat-space set-up to shed light on the convergence properties of the perturbative series and, possibly, on some gauge-invariant resummations.

Acknowledgments

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A Field theory set-ups and conventions

Our conventions follow those of [31, 35, 36]. In Euclidean space the spin group is $\text{Spin}(4) \simeq \text{SU}(2)_\alpha \otimes \text{SU}(2)_{\dot{\alpha}}$. Chiral spinors carry undotted indices α, β, \dots , while anti-chiral spinors carry dotted indices $\dot{\alpha}, \dot{\beta}, \dots$, which are contracted as follows

$$\psi_\chi \equiv \psi^\alpha \chi_\alpha, \quad \bar{\psi}_{\bar{\chi}} \equiv \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}. \quad (\text{A.1})$$

In the following, we raise and lower indices as follows

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \quad (\text{A.2})$$

where $\epsilon^{12} = \epsilon_{21} = \epsilon^{i\dot{2}} = \epsilon_{\dot{2}i} = 1$. Let us note in Euclidean spacetime spinors satisfy *pseudoreality* conditions, i.e.

$$(\psi_\alpha)^\dagger = \psi^\alpha. \quad (\text{A.3})$$

The matrices $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$ and $(\sigma^\mu)_{\alpha\dot{\beta}}$ are defined as follows

$$\sigma^\mu = (\vec{\tau}, -i\mathbb{I}), \quad \bar{\sigma}^\mu = (-\vec{\tau}, -i\mathbb{I}), \quad (\text{A.4})$$

where $\vec{\tau}$ are the ordinary Pauli matrices. Furthermore, these matrices are such that

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} (\sigma^\mu)_{\beta\dot{\beta}} \quad (\text{A.5})$$

and satisfy the Clifford algebra

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = -2\delta^{\mu\nu} \mathbb{I}, \quad (\text{A.6})$$

$$\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = -2\delta^{\mu\nu} \mathbb{I}. \quad (\text{A.7})$$

The previous expressions obviously implies that

$$\text{Tr} \sigma^\mu \bar{\sigma}^\nu = -2\delta^{\mu\nu}. \quad (\text{A.8})$$

It also is straightforward to show that the following set of relations hold

$$\begin{aligned} \text{tr}(\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho \sigma^\sigma) &= 2(\delta^{\mu\nu} \delta^{\rho\sigma} - \delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\nu\rho} + \epsilon^{\mu\nu\rho\sigma}), \\ \text{tr}(\sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\sigma) &= 2(\delta^{\mu\nu} \delta^{\rho\sigma} - \delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\nu\rho} - \epsilon^{\mu\nu\rho\sigma}), \\ \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho &= -\delta^{\mu\nu} \bar{\sigma}^\rho + \delta^{\mu\rho} \bar{\sigma}^\nu - \delta^{\nu\rho} \bar{\sigma}^\mu - \epsilon^{\mu\nu\rho\alpha} \bar{\sigma}_\alpha, \end{aligned} \quad (\text{A.9})$$

where we normalize $\epsilon^{1234} = \epsilon_{1234} = 1$.

A.1 Euclidean actions in flat space

We consider $\mathcal{N} = 2$ super-Yang-Mills theories with gauge group $\text{SU}(N)$ and with massless hypermultiplets in an arbitrary representation \mathcal{R} . The Lie algebra of the gauge group is $\mathfrak{su}(n)$ and spanned by hermitian traceless generators T^a , with $a = 1, \dots, N^2 - 1$, satisfying

$$[T^a, T^b] = i f^{abc} T^c. \quad (\text{A.10})$$

On the other hand, the tree-level propagators of the two *gauginos* λ and ψ exhibits a more complicated structure. Here we consider in detail the relevant expressions for the Weyl spinor λ but analogous results hold for ψ . We have two relevant Wick contractions, i.e.

$$\langle \lambda_\alpha^a(x) \bar{\lambda}_{\dot{\alpha}}^b(y) \rangle_0, \quad \langle \bar{\lambda}_{\dot{\alpha}}^b(y) \lambda_\alpha^a(x) \rangle_0. \quad (\text{A.17})$$

In our conventions, the arrow associated with the particle flow always goes from the dotted index to the undotted one. As a result, in momentum space we represent the first contraction as follows

$$\langle \lambda_\alpha^a(x) \bar{\lambda}_{\dot{\alpha}}^b(y) \rangle_0 \leftrightarrow \begin{array}{c} \xrightarrow{p} \\ \text{---} \xrightarrow{\quad} \text{---} \\ \dot{\alpha}, b \qquad \qquad \alpha, a \end{array} = \frac{\delta^{ab} \sigma_{\alpha\dot{\alpha}} \cdot p}{p^2}, \quad (\text{A.18})$$

where $\sigma_{\alpha\dot{\alpha}} \cdot p = \sigma_{\alpha\dot{\alpha}}^\mu p_\mu$, with $\sigma_{\alpha\dot{\alpha}}^\mu$ defined in eq. (A.4). The tree-level propagator with raised indices in eq. (A.17) is obtained from the previous expression by employing the ϵ -tensor as explained in eq. (A.4). We find

$$\langle \bar{\lambda}_{\dot{\alpha}}^b(y) \lambda_\alpha^a(x) \rangle_0 \leftrightarrow \begin{array}{c} \xleftarrow{p} \\ \text{---} \xrightarrow{\quad} \text{---} \\ \dot{\alpha}, b \qquad \qquad \alpha, a \end{array} = \frac{\delta^{ab} \bar{\sigma}^{\dot{\alpha}\alpha} \cdot p}{p^2}. \quad (\text{A.19})$$

Finally, we consider the propagators associated with the spacetime fields of the massless hypermultiplets in the representation \mathcal{R} . For the complex scalars q and \tilde{q} we have

$$\begin{array}{c} \text{---} \xrightarrow{\quad} \text{---} \\ \bar{q}^u \qquad \qquad q_v \end{array} = \frac{\delta_v^u}{p^2} \quad (\text{A.20})$$

$$\begin{array}{c} \text{---} \xrightarrow{\quad} \text{---} \\ \bar{\tilde{q}}_u \qquad \qquad \tilde{q}^v \end{array} = \frac{\delta_u^v}{p^2},$$

where $u, v = 1, \dots, \dim \mathcal{R}$. Finally, we consider the fermionic propagators associated with the fermions η and $\tilde{\eta}$. For simplicity, we only depict the contractions with lowered indices i.e.

$$\begin{array}{c} \xrightarrow{p} \\ \text{---} \xrightarrow{\quad} \text{---} \\ \bar{\eta}_{\dot{\alpha}}^v \qquad \qquad \eta_{\alpha,u} \end{array} = \frac{\delta_u^v \sigma_{\alpha\dot{\alpha}} \cdot p}{p^2} \quad (\text{A.21})$$

$$\begin{array}{c} \xrightarrow{p} \\ \text{---} \xrightarrow{\quad} \text{---} \\ \bar{\tilde{\eta}}_{\dot{\alpha},v} \qquad \qquad \tilde{\eta}_{\alpha}^u \end{array} = \frac{\delta_v^u \sigma_{\alpha\dot{\alpha}} \cdot p}{p^2}. \quad (\text{A.22})$$

The relevant expressions with raised indices are analogous to the propagators presented in eq. (A.19).

B Perturbative corrections to propagators

In this section, we introduce our notations and conventions for the calculation of the Feynman integrals entering the perturbative corrections to the propagators at one/two-loop accuracy.

We will primarily work in momentum space and will follow the formalism presented in [56]. At one-loop accuracy, we consider the basis integral

$$G(n_1, n_2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^{n_1} ((k+p)^2)^{n_2}} = (p^2)^{d/2-n_1-n_2} \tilde{G}(n_1, n_2), \quad (\text{B.1})$$

where the overall dependence on external momentum p^2 follows from dimensionality, while $\tilde{G}(n_1, n_2)$ is a function of the dimension d and of the integers n_1 and n_2 .¹³ Employing usual Feynman parameters for the different propagators, it is straightforward to show that

$$\tilde{G}(n_1, n_2) = \frac{\Gamma(n_1 + n_2 - d/2)}{(4\pi)^{d/2} \Gamma(n_1) \Gamma(n_2)} \frac{\Gamma(d/2 - n_1) \Gamma(d/2 - n_2)}{\Gamma(d - n_1 - n_2)}, \quad (\text{B.2})$$

where $\Gamma(x)$ is the Euler gamma function. At two-loop accuracy, the basis integral we consider is [30, 56]

$$\begin{aligned} G(n_1, n_2, n_3, n_4, n_5) &= \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{((k+p)^2)^{n_1} ((l+p)^2)^{n_2} (k^2)^{n_3} (l^2)^{n_4} ((l-k)^2)^{n_5}} \\ &= (p^2)^{d-\sum n_i} \tilde{G}(n_1, n_2, n_3, n_4, n_5) \end{aligned}$$

where n_i are integers. Note that the previous expression is symmetric under the interchanges $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ and $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$. When one of the parameters n_i vanishes, eq. (B) reduces to a product of the one-loop integrals we introduced (B.1). In particular, we will use the identities

$$\tilde{G}(n_1, n_2, n_3, n_4, 0) = \tilde{G}(n_1, n_3) \tilde{G}(n_2, n_4), \quad (\text{B.3})$$

$$\tilde{G}(0, n_2, n_3, n_4, n_5) = \tilde{G}(n_3, n_5) \tilde{G}(n_2, n_3 + n_4 + n_5 - d/2), \quad (\text{B.4})$$

which can be derived by repeated applications of eq. (B.1). When all the indices n_i in eq. (B) are equal to one, it is possible to employ integration by parts (see section 5.1 of [56] for the technical details) to derive the following relation:

$$G(d) \equiv G(1, 1, 1, 1, 1) = \frac{2G(1, 1)}{d-4} \left(G(2, 1) - (p^2)^{2-d/2} G(2, 3-d/2) \right). \quad (\text{B.5})$$

Using eq. (B.1), it is straightforward to prove that the previous expression is regular in the limit $d \rightarrow 4$ and yields the well-known result proportional to $\zeta(3)$, i.e.

$$G(d) = (p^2)^{d-5} \tilde{G}(d) = \frac{6\zeta(3)}{(4\pi)^4 p^2} + \mathcal{O}(d-4). \quad (\text{B.6})$$

Finally, to Fourier transform in configuration space, we will employ the formula

$$\mathcal{D}(x, s) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip \cdot x}}{(p^2)^s} = \frac{\Gamma(d/2 - s)}{4^s \pi^{d/2} \Gamma(s)} \frac{1}{(x^2)^{d/2-s}}. \quad (\text{B.7})$$

The tree-level propagators in configuration space are proportional to $\Delta(x) = \mathcal{D}(x, 1)$.

¹³Let us note that when $n_1 \leq 0$ or $n_2 \leq 0$ eq. (B.1) vanishes.

B.1 One-loop corrections

In this subsection, we examine in detail the one-loop corrections to the propagators which enter the calculation of the Wilson loop.

We begin with considering the gauge field and the adjoint scalar propagators. By gauge invariance, we can deduce that

$$\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\rangle\text{---} = \frac{\delta^{ab}g_B^2}{(p^2)^2}\pi_S^{(1)}(p^2), \quad (\text{B.8})$$

$$\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\rangle\text{---} = \frac{\delta^{ab}g_B^2}{(p^2)^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\pi_G^{(1)}(p^2), \quad (\text{B.9})$$

where $\pi_G^{(1)}$ and $\pi_S^{(1)}$ are the gluon and scalar polarization operator, respectively. For the theories under examination, these quantities were computed in appendix C of [20], where it is explicitly showed that they coincide in the Feynman gauge, as expected by supersymmetry. For future reference, we report the relevant Feynman diagrams that contribute to eq. (B.8). Using the conventions of appendix A.1, we find that

$$\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\rangle\text{---} = \text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} + \text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} + \text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} \quad (\text{B.10})$$

where ψ and λ denote the two gauginos of the vector multiplet, while η and $\tilde{\eta}$ are the Weyl fermions associated with the massless hypermultiplets in the representation \mathcal{R} . Going through the calculation of eq. (B.10), it is possible to show that the first two diagrams cancel each other out and consequently, we remain with [20]

$$\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\text{---} = \text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} \equiv \text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} = -2\frac{\delta_{ab}g_B^2}{p^2}i_{\mathcal{R}}G(1,1), \quad (\text{B.11})$$

where $G(1,1)$ is defined in (B.1) and we recall $i_{\mathcal{R}}$ is the Dynkin index of the representation \mathcal{R} . Since $\pi_G^{(1)}(p^2) = \pi_S^{(1)}(p^2)$ in the Feynman gauge, we deduce that [20]

$$\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\text{---} = \text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} = -2i_{\mathcal{R}}\frac{\delta_{ab}g_B^2}{(p^2)}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)G(1,1). \quad (\text{B.12})$$

Using these results, we can easily derive the one-loop corrections to the propagators in the difference theory method. Subtracting off the contributions of $\mathcal{N} = 4$ SYM, where the hypermultiplets transform in the adjoint representation, we find that

$$\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} = \delta_{ab}\frac{g_B^2\Pi^{(1)}(p^2)}{(p^2)^2}, \quad (\text{B.13})$$

$$\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\text{---}\langle\!\!\langle\bigcirc\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\!\!\!\rangle\text{---} = \delta_{ab}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{g_B^2\Pi^{(1)}(p^2)}{(p^2)^2}, \quad (\text{B.14})$$

with the one-loop polarization operator in the difference theory being given by

$$\Pi^{(1)}(p^2) = f^{(1)}(d)(p^2)^{d/2-1}, \quad \text{where } f^{(1)}(d) = -16\pi^2\beta_0\tilde{G}(1, 1). \quad (\text{B.15})$$

We recall that the dimensionless function $\tilde{G}(1, 1)$ is given by (B.1), while β_0 is the one-loop coefficient of the β -function (1.1). In configuration space, using eq. (B.7) to perform the Fourier transform, we find the following result

$$x_1 \rightarrow \text{loop} \rightarrow x_2 = f^{(1)}(d)g_B^2 \mathcal{D}(x_{12}, 3 - d/2) \equiv \Delta^{(1)}(x_{12}), \quad (\text{B.16})$$

for the scalar propagator. Repeating the same calculation for the gluon, we have

$$\begin{aligned} x_1 \text{ wavy} \text{ loop} \text{ wavy} x_2 &= g_B^2 f^{(1)}(d) (\delta_{\mu\nu} \mathcal{D}(x_{12}, 3 - d/2) - \partial_{1,\mu} \partial_{2,\nu} \mathcal{D}(x_{12}, 4 - d/2)) \\ &\equiv \delta_{\mu\nu} \Delta^{(1)}(x_{12}) - \partial_{1,\mu} \partial_{2,\nu} \Delta^{(1);g}(x_{12}). \end{aligned} \quad (\text{B.17})$$

By gauge invariance, we expect that all the Wilson loop diagrams which involves the gauge-like term $\partial_{1,\mu} \partial_{2,\nu} \Delta^{(1);g}(x_{12})$ do not contribute to the final results and in the following, we will verify this property explicitly.

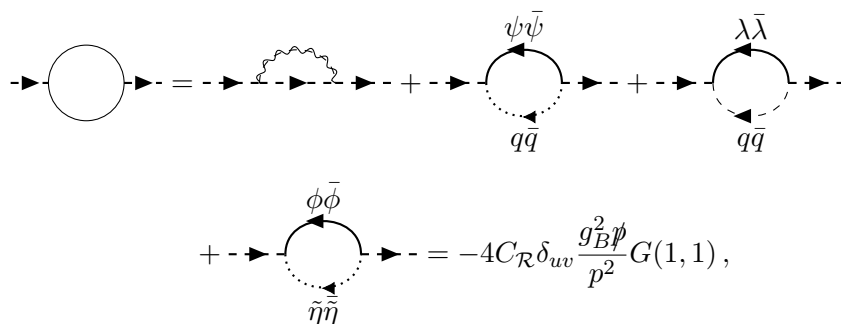
Finally, we consider the fermionic propagators at one-loop accuracy. These will enter the calculation of the two-loop corrections to the adjoint scalar propagator we will examine in the following section. Specifically, we begin with considering the vector multiplet fermions, i.e. the gauginos ψ and λ . For the Weyl fermion ψ , we find

$$\begin{aligned} \text{loop} &= \text{loop} + \text{loop} + \text{loop} \\ &+ \text{loop} = -2(N + i_{\mathcal{R}}) \delta_{ab} \frac{g_B^2 \not{p}}{p^2} G(1, 1), \end{aligned} \quad (\text{B.18})$$

where $\not{p} \equiv p_\mu \sigma^\mu$, with σ^μ given by (A.4). In the previous expression, q and \tilde{q} are the complex scalars associated with hyperts in the representation \mathcal{R} , while the first diagram results from the interaction of the fermion ψ with the gauge field A_μ and with the real scalars A_i , where $i = 1, \dots, 4 - d$, which emerge from dimensional reduction. We verified that the one-loop corrections to the propagator of the gaugino λ give us the same result, as expected from supersymmetry. From eq. (B.18), we can easily deduce the one-loop correction to the fermion propagator in the difference method, i.e.

$$\text{loop} = 2(N - i_{\mathcal{R}}) \delta_{ab} \frac{g_B^2 \not{p}}{p^2} G(1, 1). \quad (\text{B.19})$$

Finally, we consider the corrections to the propagators of the spinors η and $\tilde{\eta}$. For the fermion η we find



$$\begin{aligned}
 & \text{Tree-level propagator} = \text{One-loop correction} + \text{Feynman diagram with } \psi\bar{\psi} \text{ and } q\bar{q} \\
 & \text{Feynman diagram with } \phi\bar{\phi} \text{ and } \tilde{\eta}\bar{\tilde{\eta}} = -4C_{\mathcal{R}}\delta_{uv} \frac{g_B^2 \not{p}}{p^2} G(1,1),
 \end{aligned}
 \tag{B.20}$$

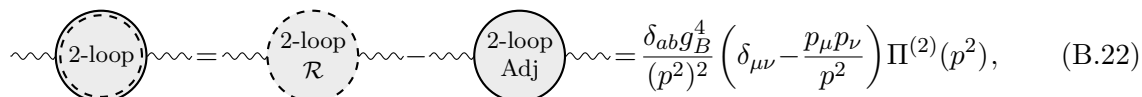
where $u, v = 1, \dots, \dim \mathcal{R}$ and we recall that $C_{\mathcal{R}}$ is the quadratic Casimir¹⁴ of the representation \mathcal{R} . We find an identical result for fermion $\tilde{\eta}$ as expected from supersymmetry.

B.2 Two-loop corrections to the propagators

The three-loop analysis of the 1/2 BPS Wilson loop involves diagrams characterized by the two-loop corrections to the adjoint scalar and gauge field propagator in the *difference theory approach*. In the Feynman gauge, the expectation based on supersymmetry is that these quantities coincide up to spacetime indices,¹⁵ as it occurs at one-loop accuracy (see eqs. (B.13) and (B.14)). Therefore, in the following, we will assume that



$$\text{2-loop} \text{ (solid)} \rightarrow \text{2-loop } \mathcal{R} \text{ (dashed)} \rightarrow \text{2-loop Adj} \text{ (solid)} = \frac{\delta_{ab} g_B^4}{(p^2)^2} \Pi^{(2)}(p^2),
 \tag{B.21}$$



$$\text{2-loop} \text{ (wavy)} \rightarrow \text{2-loop } \mathcal{R} \text{ (dashed)} \rightarrow \text{2-loop Adj} \text{ (wavy)} = \frac{\delta_{ab} g_B^4}{(p^2)^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Pi^{(2)}(p^2),
 \tag{B.22}$$

and we will calculate the two-loop polarization operator $\Pi^{(2)}(p^2)$ by considering the scalar propagator. In the previous expression, the contribution labelled by \mathcal{R} encodes all the two-loop diagrams in $\mathcal{N} = 2$ SYM in which the scalar ϕ (or the gluon) interacts with matter fields in representation \mathcal{R} , while the other contribution denotes the corrections resulting from $\mathcal{N} = 4$ SYM, where matter transforms in the adjoint representation, i.e. $\mathcal{R} = \text{Adj}$. By dimensional reasons, the polarization operators can be written as

$$\Pi^{(2)}(p^2) = (p^2)^{d-3} f^{(2)}(d),
 \tag{B.23}$$

where $f^{(2)}(d)$ is a dimensionless function of d and includes colour factors. To avoid cumbersome expressions, we find convenient to express every diagram by the basis integrals (B.1) and (B) and directly provide their contributions to $f^{(2)}(d)$, omitting the overall prefactor $g_B^4 \delta_{ab} / (p^2)^{5-d}$.

¹⁴The quadratic Casimir is defined via the relation $T_{\mathcal{R}}^a T_{\mathcal{R}}^a = C_{\mathcal{R}} \mathbb{I}$.

¹⁵An explicit test of this property at two-loop accuracy can be found in [30], where the authors studied the 1/2 BPS Wilson loop in superconformal $\mathcal{N} = 2$ QCD.

On the one-hand, we find that the reducible corrections are simply given by

$$\begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} = 4(i_{\mathcal{R}}^2 - N^2) \tilde{G}(1, 1)^2, \quad (\text{B.24})$$

as it follows from eq. (B.11). On the other hand, the irreducible contributions can be organized in two classes of diagrams.

The first one arises when decorating the internal lines of the diagrams depicted in eq. (B.10) with the one-loop self-energies associated with the fields of the virtual loops. In the difference theory approach, we find the following classes of diagrams

$$\begin{array}{c} \text{---} \circ \text{---} \text{---} + \text{---} \circ \text{---} \text{---} + \text{---} \circ \text{---} \text{---} \\ \text{---} \circ \text{---} \text{---} \end{array} = 2N(N - i_{\mathcal{R}}) \left(5\tilde{G}(0, 1, 1, 0, 1) - \tilde{G}(0, 1, 1, 2, 1) \right), \quad (\text{B.25})$$

$$\begin{array}{c} \text{---} \circ \text{---} \end{array} = 16 \left(C_{\mathcal{R}} i_{\mathcal{R}} - N^2 \right) \left(\tilde{G}(0, 1, 1, 1, 1) - \tilde{G}(0, 1, 1, 0, 1) \right). \quad (\text{B.26})$$

In eq. (B.25), the internal bubbles in the double dashed/continuous line notation denote, respectively, the one-loop correction to the adjoint scalar, gauge field and gaugino propagators in the difference method (see, respectively, eqs. (B.13), (B.14) and (B.19)). Similarly, eq. (B.26) arises when we decorate the matter loop in eq. (B.11) with the one-loop correction (B.20) and we subtract the contribution of $\mathcal{N} = 4$ SYM.

The second class of irreducible corrections emerges from pure two-loop diagrams, which we organize in terms of three fermionic loops and one sunset-like correction, i.e.

$$\begin{array}{c} \text{---} \circ \text{---} \end{array} = \frac{2\mathcal{K}'_4}{NC_F} \left(\tilde{G}(d) - \tilde{G}(1, 0, 1, 1, 1) + 2\tilde{G}(0, 1, 1, 0, 1) \right), \quad (\text{B.27})$$

$$\begin{array}{c} \text{---} \circ \text{---} \end{array} = -4N(i_{\mathcal{R}} - N) \left(\tilde{G}(0, 1, 1, 1, 1) - \tilde{G}(1, 1)^2 \right), \quad (\text{B.28})$$

$$\begin{array}{c} \text{---} \circ \text{---} \end{array} = 4N(i_{\mathcal{R}} - N) \left(4\tilde{G}(0, 1, 1, 0, 1) - 2\tilde{G}(1, 1)^2 \right), \quad (\text{B.29})$$

$$\begin{array}{c} \text{---} \circ \text{---} \end{array} = \left(8(C_{\mathcal{R}} i_{\mathcal{R}} - N^2) - 2N(i_{\mathcal{R}} - N) \right) \tilde{G}(0, 1, 1, 0, 1). \quad (\text{B.30})$$

The colour factor \mathcal{K}'_4 in eq. (B.27) was defined in eq. (2.19), while the double wiggly/continuous line denotes the propagation of the gauge field A_{μ} and of the $4 - d$ real scalars resulting from dimensional reduction inside the fermion loop. Similarly, the diagrams in eq. (B.29) arise from the Yukawa-like vertices in which the adjoint scalar ϕ interacts with the matter fermions, with

the two gauginos and with the matter scalars. Note that the external continuous line with which we depicted the internal bubble in eq. (B.29) has the meaning as in eqs. (B.21) and (B.22).

Combining together the results we derived in this subsection, we can express the dimensionless function $f^{(2)}(d)$, that determines the scalar polarization $\Pi^{(2)}$ through eq. (B.23), as the sum of four different terms

$$f^{(2)}(d) = f_1^{(2)}(d) + f_2^{(2)}(d) + f_3^{(2)}(d) + f_4^{(2)}(d). \tag{B.31}$$

Recalling the explicit definition of the coefficient β_0 , given by eq. (1.1), we finally obtain

$$\begin{aligned} f_1^{(2)}(d) &= 32\pi^2 \beta_0 i_{\mathcal{R}} \tilde{G}(1, 1)^2, \\ f_2^{(2)}(d) &= 32\pi^2 \beta_0 N \tilde{G}(0, 1, 1, 1, 1), \\ f_3^{(2)}(d) &= 16\pi^2 \beta_0 N \tilde{G}(0, 1, 1, 2, 1), \\ f_4^{(2)}(d) &= \frac{2\mathcal{K}'_4}{NC_F} \tilde{G}(d). \end{aligned} \tag{B.32}$$

Finally, it is straightforward to obtain the expressions of these propagators in configuration space. By employing eq. (B.7), we find

$$x_1 \text{---} \textcircled{\text{2-loop}} \text{---} x_2 = g_B^4 f^{(2)}(d) \mathcal{D}(x_{12}, 5-d) \equiv \Delta^{(2)}(x_{12}), \tag{B.33}$$

for the adjoint scalar field. Conversely, for the gluon propagator, we get two terms:

$$\begin{aligned} x_1 \text{---} \textcircled{\text{2-loop}} \text{---} x_2 &= g_B^4 f^{(2)}(d) (\delta_{\mu\nu} \mathcal{D}(x_{12}, 5-d) - \partial_{1,\mu} \partial_{2,\nu} \mathcal{D}(x_{12}, 6-d)) \\ &\equiv \delta_{\mu\nu} \Delta^{(2)}(x_{12}) - \partial_{1,\mu} \partial_{2,\nu} \Delta^{(2),g}(x_{12}). \end{aligned} \tag{B.34}$$

Note that the gauge-like term $\partial_{1,\mu} \partial_{2,\nu} \Delta^{(2),g}(x_{12})$ is completely irrelevant when inserted in the single-exchange correction (3.34) since, when contracted with the tangent vectors $\hat{x}_1^\mu \hat{x}_2^\nu$, it provides total derivatives integrated over a closed path.

C Mercedes-like diagrams

In this section, we provide the technical details regarding the calculation of the *Mercedes-like* correction

$$M \equiv \textcircled{\text{Mercedes-like diagram}}, \tag{C.1}$$

where we used the notation introduced in eqs. (B.16) and (B.17).

Expanding the Wilson loop operator (1.6) at order g_B^3 , we obtain the following representation for the Mercedes-like corrections

$$M = \oint d^3\tau \left(\frac{(ig_B)^3}{3!N} \langle \text{tr } \mathcal{P} \mathcal{A}(\tau_1) \mathcal{A}(\tau_2) \mathcal{A}(\tau_2) \rangle_M + \frac{ig_B^3 R^2}{2N} \langle \text{tr } \mathcal{P} \mathcal{A}(\tau_1) \Phi(\tau_2) \bar{\Phi}(\tau_3) \rangle_M \right), \tag{C.2}$$

where we recall that \mathcal{P} denotes the path-ordering operator, we introduced the notation $\mathcal{A}_i \equiv \dot{x}^\mu(\tau_i)A_\mu^a(x(\tau_i))T^a$ and $\Phi_i \equiv \phi^a(x(\tau_i))T^a$ and we used the subscript M to restrict the Wick contractions inside the correlators to the internal diagrams depicted in eq. (C.1).

Before entering the calculation of eq. (C.2), it is convenient to recall that the one-loop correction to the gauge-field propagator, defined in eq. (B.17), involves the gauge-like term $\partial_{1,\mu}\partial_{2,\nu}\Delta^{(2),g}(x_{12})$. By gauge invariance, we expect that the sum of all three-loop corrections to the expectation value of the Wilson loop involving this gauge-like term vanishes. To check this fact explicitly, it is convenient to introduce the following diagrammatic notation for eq. (B.17):

$$\begin{aligned}
 \text{Diagram} &= \delta_{ab}\delta_{\mu\nu}\Delta^{(1)}(x_{12}) + \delta_{ab}\partial_{1,\mu}\partial_{1,\nu}\Delta_{\mu\nu}^{(1),g}(x_{12}) \\
 &\equiv \text{Diagram } \delta + \text{Diagram } \partial .
 \end{aligned}
 \tag{C.3}$$

We use the symbols δ and ∂ inside the dashed/continuous bubbles, to distinguish the tensor structures of the two terms and we recall that $\Delta^{(1)}(x)$ and $\Delta^{(1),g}(x)$ are defined in eqs. (B.16) and (B.17) respectively, from which it follows that

$$\Delta^{(1)}(x_{12}) = \frac{g_B^2\beta_0\Gamma^2(d/2-1)}{4\pi^{d-2}(d/2-2)(d-3)(x_{12}^2)^{d-3}},
 \tag{C.4}$$

$$\Delta^{(1),g}(x_{12}) = \frac{g_B^2\beta_0\Gamma^2(d/2-1)}{2^5\pi^{d-2}(3-d/2)(d-3)(2-d/2)^2(x_{12}^2)^{d-4}}.
 \tag{C.5}$$

Using the notation we introduced in eq. (C.3), we can organize the correction (C.1) in terms of two distinct classes of diagrams, i.e.

$$M_1 \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3},
 \tag{C.6}$$

$$M_2 \equiv \text{Diagram 4} + \text{Diagram 5}.
 \tag{C.7}$$

In the following subsections, we will analyse these two classes of corrections in turn.

C.1 Computing M_1

To deduce the expression of the different diagrams contributing to M_1 , we begin with considering the interaction action associated with the internal (gauge-scalar/pure-gauge) triple vertices. Using the conventions of appendix A.1, we find that

$$\begin{aligned}
 S_{gs} &= g_B \int d^d\omega f^{abc} \left(\partial_\mu \bar{\phi}_b A_a^\mu \phi_c - \bar{\phi}_b A_a^\mu \partial_\mu \phi_c \right) (\omega) \\
 S_{gg} &= g_B \int d^d\omega f^{acb} \left(\partial_\mu A_{\nu,c} A_a^\mu A_\nu^b \right) (\omega),
 \end{aligned}
 \tag{C.8}$$

where f^{abc} , defined by $[T^a, T^b] = if^{abc}T_c$, are the (antisymmetric) structure constants of $\mathfrak{su}(N)$. Inserting these actions in the correlation functions of eq. (C.2) and decorating the proper Wick contractions with the one-loop correction to the adjoint scalar $\delta_{ab}\Delta^{(1)}(x)$, defined in eq. (B.16), and with the tensor $\delta_{\mu\nu}\Delta^{(1)}(x)$, we arrive at the following expression

$$M_1 = -\frac{g_B^4 C_F N}{2} \oint d^3\tau \varepsilon(\tau) (R^2 - \dot{x}_1 \cdot \dot{x}_3) (\dot{x}_2 \cdot \partial_{x_1}) \int d^d\omega \sum_{i=1}^3 \prod_{j \neq i} \Delta^{(1)}(x_{i\omega}) \Delta(x_{j\omega}), \quad (C.9)$$

where $x_{i\omega} \equiv x_i - \omega$, while the function $\Delta(x)$ and the path-ordering symbol¹⁶ $\varepsilon(\tau)$ are defined in eqs. (3.7) and (3.22), respectively. By integrating over ω with usual Feynman parameters and expressing the coordinates x_i via the parametrization (1.7), we find that

$$M_1 = A_d \int_0^1 d\mathcal{F} \oint d^3\tau \varepsilon(\tau) \frac{(1 - \cos(\tau_{13}))(\alpha(1 - \alpha) \sin(\tau_{12}) - \alpha\gamma \sin(\tau_{32}))}{Q^{3d/2-4}}. \quad (C.10)$$

In the previous expression, the denominator Q is defined in eq. (3.20), while the integration measure over the Feynman parameters and the multiplicative prefactor are given by

$$d\mathcal{F} = dF(\alpha\beta\gamma)^{d/2-2} \left(\alpha^{d/2-2} + \beta^{d/2-2} + \gamma^{d/2-2} \right), \quad (C.11)$$

$$A_d = \frac{\hat{g}_B^6 C_F N \beta_0 \Gamma(3d/2 - 4) \Gamma^2(d/2 - 1)}{(d/2 - 2) \Gamma(d - 2) (\pi)^{3d/2-6} 2^{3d/2+2} \pi^4}. \quad (C.12)$$

In the previous expression, dF is the standard measure over the unit cube (3.21). To perform the contour integration in eq. (C.10), we employ the following identities¹⁷

$$\int_0^1 d\mathcal{F} \oint d^3\tau \frac{\partial}{\partial \tau_1} \left(\frac{\varepsilon(\tau)(1 - \cos \tau_{13})}{Q^{3d/2-5}} \right) = 0, \quad (C.13)$$

$$\partial_{\tau_1} \varepsilon(\tau) - 2(\delta(\tau_{12}) - \delta(\tau_{13})) = 0. \quad (C.14)$$

To proceed with the calculation, it is sufficient to insert eq. (C.13) in (C.10) and observe that the measure $d\mathcal{F}$ is completely symmetric. This enables to relabel the variables τ_i and keep the denominator Q unchanged. As a result, we find that

$$\begin{aligned} M_1 &= \frac{2A_d}{3d/2 - 5} \int_0^1 d\mathcal{F} \oint d^2\tau \frac{(1 - \cos \tau_{23})^{6-3d/2}}{(\gamma(1 - \gamma))^{3d/2-5}} - A_d \left(\frac{3d - 12}{3d - 10} \right) \int_0^1 d\mathcal{F} \oint d^3\tau \frac{\varepsilon(\tau) \sin \tau_{13}}{Q^{3d/2-5}} \\ &= -2F_2^{(2)} - \frac{N}{i\mathcal{R}} F_1^{(2)} + \frac{\hat{g}_B^6 \beta_0 C_F N 9 \zeta(3)}{16\pi^2}. \end{aligned} \quad (C.15)$$

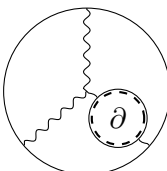
In the previous expression, the quantities $F_1^{(2)}$ and $F_2^{(2)}$ are the (UV divergent) bubble-like contributions we defined in eq. (3.37). They arise from the integration over the measure $d\mathcal{F}$ (C.11) in the first term of the previous expression, while the $\zeta(3)$ -like contribution is obtained by applying the master integral (3.23) to the second term.

¹⁶ $\varepsilon(\tau)$ arises since the internal diagrams are proportional to the antisymmetric structure constant f^{abc} .

¹⁷This procedure is analogous to that outlined in [23, 57] for the calculation of the two-loop *Mercedes-like diagrams* $\mathcal{W}_4^{\text{v.m.}}$ we defined in eq. (3.18), in the context of $\mathcal{N} = 4$ theories. In our case, the model is not superconformal and consequently, the analysis is more involved.

C.2 Computing M_2

In this section, we turn our attention to the corrections M_2 , depicted in eq. (C.7). Let us begin with discussing the diagrams involving three gauge fields. Inserting the pure-gauge vertex (C.8) in the first correlator of eq. (C.2) and decorating the Wick contractions with the tensor $\partial_{1,\mu}\partial_{2,\nu}\Delta^{(2),g}(x_{12})$, we arrive at the following representation

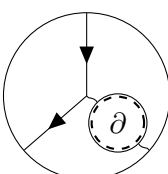


$$= \frac{g_B^4 C_F N}{2} \oint d^3\tau \varepsilon(\tau) \int d^d\omega \frac{d}{d\tau_1} \left(\mathcal{O}(x_j) \Delta(x_{3\omega}) \Delta(x_{2\omega}) \Delta^{(1),g}(x_{1\omega}) \right), \quad (\text{C.16})$$

where we recall that $\Delta(x)$ is the massless tree-level propagator defined in eq. (3.7), the function $\Delta^{(1),g}(x)$ is given by eq. (C.5), while $\mathcal{O}(x_j)$ denotes the following operator

$$\mathcal{O}(x_j) = \left[(\dot{x}_3 \cdot \partial_1) (\partial_1 - \partial_3) \cdot \dot{x}_2 + (\dot{x}_2 \cdot \dot{x}_3) (\partial_3 \cdot \partial_1) \right]. \quad (\text{C.17})$$

Let us now consider the diagrams in eq. (C.7) involving the propagation of two scalars and one gauge field. Inserting the gauge-scalar vertex (C.8) in the second correlator of eq. (C.2), and decorating the Wick contraction of the gauge field with the tensor $\partial_{1,\mu}\partial_{1,\nu}\Delta^{(1),g}(x_{1\omega})$, we find



$$= -\frac{g_B^4 C_F R^2 N}{2} \oint d^3\tau \varepsilon(\tau) \frac{d}{d\tau_1} \partial_3 \cdot \partial_1 \left(\Delta(x_{3\omega}) \Delta(x_{2\omega}) \Delta^{(1),g}(x_{1\omega}) \right). \quad (\text{C.18})$$

Finally, by combining together eqs. (C.16) and (C.18) and neglecting terms which provide total derivatives, we obtain the following result

$$\begin{aligned} M_2 &= \frac{g_B^4 C_F N}{2} \oint d^3\tau \varepsilon(\tau) \left(\dot{x}_2 \cdot \dot{x}_3 - R^2 \right) \frac{d}{d\tau_1} \partial_3 \cdot \partial_1 \int d^d\omega \Delta(x_{2\omega}) \Delta(x_{3\omega}) \Delta^{(1),g}(x_{1\omega}) \\ &= g_B^4 C_F N \oint d^3\tau (\delta(\tau_{12}) - \delta(\tau_{13})) \left(R^2 - \dot{x}_2 \cdot \dot{x}_3 \right) \partial_3 \cdot \partial_1 \int d^d\omega \Delta(x_{2\omega}) \Delta(x_{3\omega}) \Delta^{(1),g}(x_{1\omega}) \\ &= -2F_3^{(2)}, \end{aligned} \quad (\text{C.19})$$

where $F_3^{(2)}$ denotes the bubble-like contribution defined in eq. (3.37) and we obtained the second line via integration by parts and using eq. (C.14). Combining this result with eq. (C.15), we find that

$$M = M_1 + M_2 = -2F_2^{(2)} - 2F_3^{(2)} - \frac{N}{i\mathcal{R}} F_1^{(2)} + 9 \frac{\beta_0^R \hat{g}_B^6 C_F N}{16\pi^2} \zeta(3). \quad (\text{C.20})$$

D Lifesaver diagrams

In this section, we examine in detail the calculation of the *lifesaver-like* diagrams¹⁸

$$L = \left(\text{Diagram: A circle with a dashed inner circle. A wavy line enters from the top, and two arrows enter from the bottom. The dashed circle has a wavy line and two arrows on its boundary. } \right), \quad (\text{D.1})$$

where we used again the difference theory notation. In particular, the internal bubble encodes the one-loop irreducible corrections to the (gauge-scalar/pure-gauge) triple vertex in the difference theory approach.

D.1 Construction of the building blocks

Expanding the Wilson loop operator (1.6) at order g_B^3 , we obtain the following representation for the diagrams depicted in eq. (D.1)

$$L = L_g + L_{gs}, \quad (\text{D.2})$$

where the quantities L_{gs} and L_g encode two correlators in the difference theory approach

$$L_g = \left(\frac{(ig_B)^3}{3!N} \right) \left(\oint d^3\tau \langle \text{tr } \mathcal{P} \mathcal{A}(\tau_1) \mathcal{A}(\tau_2) \mathcal{A}(\tau_3) \rangle_L \right), \quad (\text{D.3})$$

$$L_{gs} = \left(\frac{ig_B^3 R^2}{2N} \right) \left(\oint d^3\tau \langle \text{tr } \mathcal{P} \mathcal{A}(\tau_1) \Phi(\tau_2) \bar{\Phi}(\tau_3) \rangle_L \right). \quad (\text{D.4})$$

We begin with discussing eq. (D.4), which involves the irreducible correction to the gauge-scalar vertex in the difference method. To construct these corrections, it is sufficient to determine the relevant diagrams characterized by internal matter line in the representation \mathcal{R} and subsequently, to subtract an identical contribution in which $\mathcal{R} = \text{Adj}$. Introducing $\mathcal{S}_{abc}^\mu(x_i) = \langle A_a^\mu(x_1) \phi_b(x_2) \bar{\phi}_c(x_3) \rangle_L$, we have

$$\begin{aligned} \mathcal{S}_{abc}^\mu(x_i) &= \left(\text{Diagram: Triangle with wavy line } A_a^\mu(x_1) \text{ at top, } \bar{\phi}^c(x_3) \text{ at bottom-left, } \phi^b(x_2) \text{ at bottom-right.} \right) + \left(\text{Diagram: Triangle with wavy line } A_a^\mu(x_1) \text{ at top, } \bar{\phi}^c(x_3) \text{ at bottom-left, } \phi^b(x_2) \text{ at bottom-right.} \right) - (\mathcal{R} = \text{Adj}) \quad (\text{D.5}) \\ &= f_{abc} (\mathcal{S}_1^\mu(x_i) + \mathcal{S}_2^\mu(x_i)). \end{aligned}$$

where we defined the functions

$$\mathcal{S}_1^\mu(x_i) = (2ig_B^3)(i_{\mathcal{R}} - N) \int dP \int \frac{d^d k}{(2\pi)^d} \frac{p_2^\mu k^2 - p_3^\mu (k - p_1)^2}{k^2 (k - p_1)^2 (k + p_3)^2}, \quad (\text{D.6})$$

$$\mathcal{S}_2^\mu(x_i) = (2ig_B^3)(i_{\mathcal{R}} - N) \int dP \int \frac{d^d k}{(2\pi)^d} \frac{p_1^\mu (k + p_3)^\mu - p_3^2 (k - p_1)^\mu - p_2^2 k^\mu}{k^2 (k - p_1)^2 (k + p_3)^2}. \quad (\text{D.7})$$

¹⁸We provide an extended analysis since we did not find evidence of analogous calculations in the existing literature.

In the previous expression, dP denotes the usual integration measure over the external momenta p_i , i.e.

$$dP = \prod_{i=1}^3 \frac{d^d p_i}{(2\pi)^d} \frac{e^{-ip_i \cdot x_i}}{p_i^2} (2\pi)^d \delta^d(\Sigma_j p_j). \quad (\text{D.8})$$

Let us observe that the functions in eqs. (D.6) and (D.7) have a different behaviour in the limit $d \rightarrow 4$. Indeed, integrating over the large loop momentum yields a pole $1/(d-4)$ in eq. (D.6), while the function $\mathcal{S}_2^\mu(x_i)$ is regular in four dimensions. Substituting eq. (D.5) into eq. (D.4), we can naturally arrange the results in terms of two distinct contributions:

$$\mathbb{L}_{gs} = \mathbb{L}_{gs,1} + \mathbb{L}_{gs,2}. \quad (\text{D.9})$$

Specifically, the quantity $\mathbb{L}_{gs,1}$ is given by

$$\begin{aligned} \mathbb{L}_{gs,1} &= -\frac{g_B^3 R^2}{4} N C_F \oint d^3 \tau \varepsilon(\tau) (\dot{x}_1 \cdot \mathcal{S}_1) \\ &= \mathcal{A}_1 R^2 \oint d^3 \tau \varepsilon(\tau) (\dot{x}_2 \cdot \partial_1) \int d^d \omega \Delta^{(1)}(x_{1\omega}) \Delta(x_{2\omega}) \Delta(x_{3\omega}), \end{aligned} \quad (\text{D.10})$$

with $\Delta(x)$ being the tree-level propagator (3.7) and $\Delta^{(1)}(x)$ the (UV) divergent one-loop correction to the adjoint scalar propagator (B.16), while

$$\begin{aligned} \mathbb{L}_{gs,2} &= -\frac{g_B^3 R^2}{4} N C_F \oint d^3 \tau \varepsilon(\tau) (\dot{x}_1 \cdot \mathcal{S}_2) \\ &= \mathcal{A}_2 \oint d^3 \tau \varepsilon(\tau) \int dP (-ip_2^2) \int \frac{d^d k}{(2\pi)^2} \frac{R^2 (2k \cdot \dot{x}_1 - k \cdot \dot{x}_2)}{k^2 (k+p_1)^2 (k-p_3)^2}. \end{aligned} \quad (\text{D.11})$$

In the previous expressions, we introduced, for the sake of conciseness, the quantities

$$\mathcal{A}_1 = \frac{C_F N g_B^4}{2}, \quad (\text{D.12})$$

$$\mathcal{A}_2 = 4\pi^2 C_F N \beta_0 g_B^6. \quad (\text{D.13})$$

The analysis of the internal diagrams which enter eq. (D.3) goes along the same lines. In particular, the irreducible one-loop correction to the pure-gauge vertex in the difference method receives corrections from both scalar and fermionic loops and for convenience, we will consider them in turn. Matter scalars contribute via the following diagrams

$$\begin{aligned} \mathcal{S}_{\mu\nu\rho}^{abc}(x_i) &= \begin{array}{c} A_\mu^a(x_1) \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ A_\nu^b(x_2) \quad A_\rho^c(x_3) \end{array} + \begin{array}{c} A_\mu^a(x_1) \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ A_\rho^c(x_3) \quad A_\nu^b(x_2) \end{array} - (\mathcal{R} = \text{Adj}) \\ &= 2ig_B^3 f^{abc} (N - i\mathcal{R}) \int dP \int \frac{d^d k}{(2\pi)^d} \frac{(2k+p_1)_\mu (2k+p_1-p_3)_\mu (2k-p_3)_\rho}{k^2 (k+p_1)^2 (k-p_3)^2}. \end{aligned} \quad (\text{D.14})$$

In the previous expression, we used a double dashed/dotted line to denote, respectively, the contributions associated with the scalars q and \tilde{q} , which transform in the representation \mathcal{R} and \mathcal{R}^* , and we recall that dP is defined in eq. (D.8).

The matter fermions contribute with the following diagrams:

$$\begin{aligned}
 \mathcal{F}_{\mu\nu\rho}^{abc}(x_i) = & \quad \begin{array}{c} A_\mu^a(x_1) \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ A_\nu^b(x_2) \quad A_\rho^c(x_3) \end{array} + \begin{array}{c} A_\mu^a(x_1) \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ A_\rho^c(x_3) \quad A_\nu^b(x_2) \end{array} - (\mathcal{R} = \text{Adj}) \\
 & = ig_B^3 f^{abc} (N - i_{\mathcal{R}}) \int dP \int \frac{d^d k}{(2\pi)^d} \frac{(\text{Tr } \bar{\sigma}_\mu k \bar{\sigma}_\rho (k - p_3) \bar{\sigma}_\nu (k + p_1))}{k^2 (k + p_1)^2 (k - p_3)^2} \\
 & + ig_B^3 f^{abc} (N - i_{\mathcal{R}}) \int dP \int \frac{d^d k}{(2\pi)^d} \frac{(\text{Tr } \bar{\sigma}_\rho k \bar{\sigma}_\mu (k + p_1) \bar{\sigma}_\nu (k - p_3))}{k^2 (k + p_1)^2 (k - p_3)^2},
 \end{aligned} \tag{D.15}$$

where we used again a double dashed/dotted line to denote, respectively, the contributions of the fermions η and $\tilde{\eta}$, which transforms in the representation \mathcal{R} and \mathcal{R}^* . Employing the identities in eq. (A.9) and neglecting terms which provide total derivatives integrated over closed paths when inserted in eq. (D.3), we eventually find that

$$\mathcal{F}_{\mu\nu\rho}^{abc}(x_i) = -\mathcal{S}_{\mu\nu\rho}^{abc}(x_i) + f^{abc} (\mathcal{G}_{1,\mu\nu\rho}(x_i) + \mathcal{G}_{2,\mu\nu\rho}(x_i)). \tag{D.16}$$

In the previous expression, $\mathcal{S}_{\mu\nu\rho}^{abc}(x_i)$ is the contribution to the pure gauge-vertex resulting from the scalar loops (D.14), while the quantities $\mathcal{G}_{1,\mu\nu\rho}$ and $\mathcal{G}_{2,\mu\nu\rho}$ are, respectively, the counterparts of the functions \mathcal{S}_1^μ and \mathcal{S}_2^μ defined in eqs. (D.6) and (D.7) and have a similar behaviour for $d \rightarrow 4$. Their explicit expressions are:

$$\begin{aligned}
 \mathcal{G}_{1,\mu\nu\rho}(x_i) = & -2ig_B^3 (i_{\mathcal{R}} - N) \int dP \int \frac{d^d k}{(2\pi)^d} \left[\frac{\delta_{\mu\nu} (k^2 p_{2,\rho} - p_{1,\rho} (k - p_3)^2)}{k^2 (k + p_1)^2 (k - p_3)^2} \right. \\
 & \left. + \frac{\delta_{\mu\rho} (p_{1,\nu} (k - p_3)^2 - p_{3,\nu} (k + p_1)^2)}{k^2 (k + p_1)^2 (k - p_3)^2} + \frac{\delta_{\nu\rho} (p_{3,\mu} (k + p_1)^2 - p_{2,\mu} k^2)}{k^2 (k + p_1)^2 (k - p_3)^2} \right]
 \end{aligned} \tag{D.17}$$

and

$$\begin{aligned}
 \mathcal{G}_{\mu\nu\rho}(x_i) = & 2ig_B^3 (i_{\mathcal{R}} - N) \int dP \int \frac{d^d k}{(2\pi)^d} \left[\frac{\delta_{\mu\nu} (p_1^2 (k - p_3)_\rho + p_2^2 k_\rho - p_3^2 (k + p_1)_\rho)}{k^2 (k + p_1)^2 (k - p_3)^2} \right. \\
 & \left. + \frac{\delta_{\mu\rho} (p_3^2 (k + p_1)_\nu - p_2^2 k_\nu + p_1^2 (k - p_3)_\nu)}{k^2 (k + p_1)^2 (k - p_3)^2} - \frac{\delta_{\nu\rho} (p_1^2 (k - p_3)_\mu - p_2^2 k_\mu - p_3^2 (k + p_1)_\mu)}{k^2 (k + p_1)^2 (k - p_3)^2} \right].
 \end{aligned} \tag{D.18}$$

Combining together the contribution to the pure-gauge vertex resulting from the scalars (D.16) with that of the fermions (D.14) and inserting the result in eq. (D.3), we again

can organize the final result in terms of two distinct contributions:

$$\mathbb{L}_g = \mathbb{L}_{g,1} + \mathbb{L}_{g,2}. \quad (\text{D.19})$$

In analogy to eq. (D.9), $\mathbb{L}_{g,1}$ takes the following form

$$\begin{aligned} \mathbb{L}_{g,1} &= \frac{g_B^3}{12} N C_F \oint d^3\tau \varepsilon(\tau) (\dot{x}_1^\mu \dot{x}_2^\nu \dot{x}_3^\rho) \mathcal{G}_{1,\mu\nu\rho} \\ &= -\mathcal{A}_1 \oint d^3\tau \varepsilon(\tau) (\dot{x}_1 \cdot \dot{x}_3) (\dot{x}_2 \cdot \partial_1) \int d^d\omega \Delta^{(1)}(x_{1\omega}) \Delta(x_{2\omega}) \Delta(x_{3\omega}). \end{aligned} \quad (\text{D.20})$$

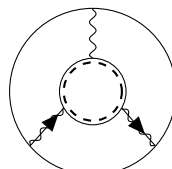
where the coefficient \mathcal{A}_1 was introduced in eq. (D.12) while $\Delta^{(1)}(x)$ and $\Delta(x)$ are defined in eqs. (B.16) and (3.7), respectively. Let us note that the previous expression has the same structure of eq. (D.10) with the replacement $R^2 \rightarrow -\dot{x}_1 \cdot \dot{x}_2$, as expected by supersymmetry. On the other hand, $\mathbb{L}_{g,2}$ is given by

$$\mathbb{L}_{g,2} = \frac{g_B^3}{12} N C_F \oint d^3\tau \varepsilon(\tau) (\dot{x}_1^\mu \dot{x}_2^\nu \dot{x}_3^\rho) \mathcal{G}_{2,\mu\nu\rho} \quad (\text{D.21})$$

$$= \mathcal{A}_2 \oint d^3\tau \varepsilon(\tau) \int dP \left(i p_2^2 \right) \int \frac{d^d k}{(2\pi)^d} \frac{(2k \cdot \dot{x}_1) (\dot{x}_2 \cdot \dot{x}_3) - (k \cdot \dot{x}_2) (\dot{x}_1 \cdot \dot{x}_3)}{k^2 (k + p_1)^2 (k - p_3)^2}, \quad (\text{D.22})$$

where the coefficient \mathcal{A}_2 is defined in eq. (D.13).

Summary of the results. Using the results we derived in this section, we can construct the final expression for the lifesaver diagram depicted in eq. (D.1). Starting from eq. (D.1) and expressing the one-loop correction to the gauge-scalar and pure-gauge vertices by eqs. (D.9) and (D.19), we can arrange the four contributions in such a way to reconstruct the usual $R^2 - \dot{x}_i \cdot \dot{x}_j$ factor. Thus, we can write



$$= \mathbb{L}_g + \mathbb{L}_{gs} \quad (\text{D.23})$$

$$= (\mathbb{L}_{g,1} + \mathbb{L}_{gs,1}) + (\mathbb{L}_{g,2} + \mathbb{L}_{gs,2}) \equiv \mathbb{L}_1 + \mathbb{L}_2.$$

Explicitly, we have

$$\mathbb{L}_1 = \mathcal{A}_1 \oint d^3\tau \varepsilon(\tau) \left(R^2 - \dot{x}_1 \cdot \dot{x}_3 \right) \dot{x}_2 \cdot \partial_1 \int d^d\omega \Delta^{(1)}(x_{1\omega}) \Delta(x_{2\omega}) \Delta(x_{3\omega}) \quad (\text{D.24})$$

and

$$\mathbb{L}_2 = \mathcal{A}_2 \oint d^3\tau \varepsilon(\tau) \int dP \left(i p_2^2 \right) \int \frac{d^d k}{(2\pi)^d} \frac{(2k \cdot \dot{x}_1) (\dot{x}_2 \cdot \dot{x}_3 - R^2) - (k \cdot \dot{x}_2) (\dot{x}_1 \cdot \dot{x}_3 - R^2)}{k^2 (k + p_1)^2 (k - p_3)^2}. \quad (\text{D.25})$$

D.2 Integration over the Wilson loop contour: calculating L_1

In this subsection, we examine in detail the integration over the Wilson loop contour of the contribution defined in eq. (D.24): the calculation is analogous to that we described in section C.1 for the correction M_1 , defined in eq. (C.9). To begin with, we integrate over the bulk point ω by introducing the usual Feynman parametrizations for the propagators $\Delta(x)$ and $\Delta^{(1)}(x)$ defined, respectively, in eq. (3.7) and (B.13). Using the parametrization (1.7) for the points x_i on the Wilson loop contour, we obtain

$$L_1 = -A_d \int_0^1 dF (\alpha^2 \beta \gamma)^{d/2-2} \oint d^3 \tau \frac{\varepsilon(\tau) (1 - \cos \tau_{13}) (\alpha(1 - \alpha) \sin \tau_{12} + \alpha \gamma \sin \tau_{23})}{Q^{3d/2-4}}, \quad (D.26)$$

where the measure dF is given by eq. (3.21), while A_d and Q are defined in eqs. (C.12) and (3.20), respectively. Integrating by parts via the identity (C.13), we find that

$$\begin{aligned} L_1 &= \frac{2A_d}{5 - 3d/2} \int_0^1 dF (\alpha^2 \beta \gamma)^{d/2-2} \oint d^2 \tau \frac{(1 - \cos \tau_{23})^{6-3d/2}}{[\gamma(1 - \gamma)]^{3d/2-5}} \\ &+ A_d \frac{6 - 3d/2}{5 - 3d/2} \int_0^1 dF (\alpha^2 \beta \gamma)^{d/2-2} \oint d^3 \tau \frac{\varepsilon(\tau) \sin \tau_{13}}{Q^{3d/2-5}} - A_d I_1(d), \end{aligned} \quad (D.27)$$

where

$$\begin{aligned} I_1(d) &= \int_0^1 dF (\alpha^2 \beta \gamma)^{d/2-2} \oint d^3 \tau \varepsilon(\tau) \frac{\beta \gamma \sin \tau_{13} (1 - \cos \tau_{23}) + \alpha \gamma \sin \tau_{23} (1 - \cos \tau_{13})}{Q^{3d/2-4}} \\ &= \int_0^1 dF (\alpha^{d-4} (\beta \gamma)^{d/2-1} - \beta^{d-3} \alpha^{d/2-2} \gamma^{d/2-1}) \oint d^3 \tau \varepsilon(\tau) \frac{(1 - \cos \tau_{23}) \sin \tau_{13}}{Q^{3d/2-4}}. \end{aligned} \quad (D.28)$$

Comparing the previous expression with eq. (C.15), we note that the last term is a novelty. It arises because the integrand in eq. (D.24) is not completely symmetric in the exchange of the coordinates x_i . On the other hand, the extra term $I_1(d)$ does not contribute to the final result; to see this, we replace the denominator Q with a two-fold Mellin-Barnes integral via (F.7) obtaining

$$\frac{1}{Q^{3d/2-4}} = \frac{2^{4-3d/2}}{\Gamma(3d/2-4)} \int \frac{dudv}{(2\pi i)^2} \frac{\Gamma(3d/2-4+u+v) \Gamma(-u) \Gamma(-v) (\beta \gamma \sin^2 \frac{\tau_{23}}{2})^u}{(\alpha \beta \sin^2 \frac{\tau_{12}}{2})^{3d/2-4+u+v} (\alpha \gamma \sin^2 \frac{\tau_{13}}{2})^{-v}}. \quad (D.29)$$

Inserting the previous expression into eq. (D.28), we integrate over the Wilson loop contour via eq. (F.6) and we obtain

$$\begin{aligned} I_1(d) &= \int \frac{dudv}{(2\pi i)^2} \frac{\Gamma(3d/2-4+u+v) \Gamma(-u) \Gamma(-v) \Gamma(d/2+u+v)}{2^{3d/2-5} \Gamma(3d/2-4) \Gamma(5-d)} \mathcal{J}(4-3d/2-u-v, u+1, v) \\ &\quad \times \left(\Gamma(1-d/2-u) \Gamma(4-d-v) - \Gamma(2-d/2-v) \Gamma(3-d-u) \right), \end{aligned} \quad (D.30)$$

where the function $\mathcal{J}(x, y, z)$ is defined in eq. (F.6). Expanding the previous expression about $d = 4$, we arrive at

$$I_1(d) = (d - 4) \int_{\delta - i\infty}^{\delta + i\infty} \frac{dv du}{(2\pi i)^2} \frac{\csc(\pi u) \csc(\pi v) \csc(\pi(u + v))(\psi^{(0)}(-u) - \psi^{(0)}(-v))}{u + v + 1} + \dots, \quad (\text{D.31})$$

where the dots stand for terms of order $\mathcal{O}(d - 4)^2$, while $\delta \in (-1, 0)$ denotes the real part of the integration variables u and v . The previous expression vanishes identically because of the antisymmetry of the integrand, meaning that $I_1(d) = \mathcal{O}(d - 4)^2$ and $A_d I_1(d) = \mathcal{O}(d - 4)$, as it can be seen by employing eq. (C.12).

Concerning the first two terms in eq. (D.27), one can explicit perform the integration over the Feynman parameters and apply the master integral (3.23) to obtain

$$\mathbf{L}_1 = F_2^{(2)} - \hat{g}_B^6 \frac{3C_F N \beta_0 \zeta(3)}{16\pi^2} + \mathcal{O}(d - 4), \quad (\text{D.32})$$

where we recall that $F_2^{(2)}$ is the bubble-like contribution defined in eq. (3.37).

D.3 Integration over the Wilson loop contour: calculating \mathbf{L}_2

The calculation of \mathbf{L}_2 is more complicated than that we performed in the previous subsection. To begin with, we consider eq. (D.25) and we integrate over the internal momentum k by introducing the usual Feynman parameters for the three propagators. We find that

$$\mathbf{L}_2 = 2\mathcal{A}_2 \frac{i\Gamma(3 - d/2)}{(4\pi)^{d/2}} \oint d^3\tau \varepsilon(\tau) (R^2 - \dot{x}_1 \cdot \dot{x}_3) \int dP dX \frac{p_3^2 (z\dot{x}_2 \cdot p_2 - x\dot{x}_2 \cdot p_1) + zp_2^2 \dot{x}_2 \cdot p_3}{(xyp_1^2 + zyp_2^2 + xzp_3^2)^{3-d/2}}, \quad (\text{D.33})$$

with $dX = dx dy dz \delta(1 - x - y - z)$. The previous expression involves the quantity $p_2 \cdot \dot{x}_2$ which, upon integration over the external momenta dP , yields a total derivative with respect to the variable τ_2 . As a result, the contour integration of this contribution is technically simpler to treat. Therefore, we find convenient to express eq. (D.33) as the sum of two terms, i.e. $\mathbf{L}_2 = \mathbf{L}'_2 + \mathbf{L}''_2$, with

$$\mathbf{L}'_2 = 2\mathcal{A}_2 \frac{i\Gamma(3 - d/2)}{(4\pi)^{d/2}} \oint d^3\tau \varepsilon(\tau) (R^2 - \dot{x}_1 \cdot \dot{x}_3) \int dP \int_0^1 dX \frac{p_3^2 (z\dot{x}_2 \cdot p_2)}{M^{3-d/2}}, \quad (\text{D.34})$$

$$\mathbf{L}''_2 = 2\mathcal{A}_2 \frac{i\Gamma(3 - d/2)}{(4\pi)^{d/2}} \oint d^3\tau \varepsilon(\tau) (R^2 - \dot{x}_1 \cdot \dot{x}_3) \int dP \int_0^1 dX \frac{zp_2^2 (\dot{x}_2 \cdot p_3) - p_3^2 x (\dot{x}_2 \cdot p_1)}{M^{3-d/2}}, \quad (\text{D.35})$$

which we will analyse in turn. In the previous expression, the denominator M is

$$M = xyp_1^2 + zyp_2^2 + xzp_3^2. \quad (\text{D.36})$$

D.3.1 Computing L'_2

As we already stressed, the computation of the function L'_2 goes through the observation that the product $p_2 \cdot \dot{x}_2$ becomes a total derivative upon integration over the momenta p_i (see eq. (D.8)). By relabelling $\tau_1 \leftrightarrow \tau_2$ and recalling that $\varepsilon(\tau)$ is antisymmetric, we find that eq. (D.34) can be rewritten as follows

$$\begin{aligned} L'_2 &= 2\mathcal{A}_2 \frac{\Gamma(3-d/2)}{(4\pi)^{d/2}} \oint d^3\tau \varepsilon(\tau) (R^2 - \dot{x}_2 \cdot \dot{x}_3) \frac{d}{d\tau_1} \int dP \int_0^1 dX \frac{x p_3^2}{M^{3-d/2}} \\ &= 4\mathcal{A}_2 \frac{\Gamma(3-d/2)}{(4\pi)^{d/2}} \oint d^3\tau (\delta(\tau_{13}) - \delta(\tau_{12})) (R^2 - \dot{x}_2 \cdot \dot{x}_3) \int dP \int_0^1 dX \frac{x p_3^2}{M^{3-d/2}}, \end{aligned} \quad (\text{D.37})$$

where we obtained the second line via integration by parts and eq. (C.14). To proceed with the computation, we employ the following identity (see eq. (F.13))

$$\frac{\Gamma(3-d/2)}{M^{3-d/2}} = \int \frac{dudv}{(2\pi i)^2} \frac{\Gamma(3-d/2+u+v)\Gamma(-u)\Gamma(-v)}{x^{3-d/2+u}y^{3-d/2+v}z^{-u-v}} \frac{(p_2^2)^u (p_3^2)^v}{(p_1^2)^{3-d/2+u+v}}, \quad (\text{D.38})$$

where the integration contour separates the increasing and decreasing poles of the Γ -functions. Substituting the previous expression in eq. (D.37) and sequentially performing the integration over the Feynman parameters and the momenta p_i (see eq. (D.8)), we find

$$L'_2 = \mathcal{A}_2 \frac{\Gamma(3d/2-5) \left(\mathcal{M}^{(ii)}(d) - \mathcal{M}^{(i)}(d) \right)}{4^4 \pi^{3d/2} \Gamma(d-2) \Gamma(5-d)} \oint \frac{R^2 - \dot{x}_2 \cdot \dot{x}_3}{[x_{23}^2]^{3d/2-5}}, \quad (\text{D.39})$$

with the two Mellin-like amplitudes defined as

$$\mathcal{M}^{(i)}(d) = \int \frac{dudv}{(2\pi i)^2} \frac{\Gamma(d-4-u-v)\Gamma(d/2-1+u)\Gamma(5-d+v)\Gamma(-v)\Gamma(1+u+v)}{u(d/2-3-u-v)\Gamma(3d/2-5-v)[\Gamma(d/2-2-v)\Gamma(d/2-1-u)]^{-1}}, \quad (\text{D.40})$$

$$\mathcal{M}^{(ii)}(d) = \int \frac{dudv}{(2\pi i)^2} \frac{\Gamma(-u)\Gamma(4-d+u)\Gamma(1+u+v)\Gamma(d/2-1-u)\Gamma(d/2-2-v)}{\Gamma(3d/2-4-u)(3-d/2+u+v)[\Gamma(d-4-u-v)\Gamma(d/2+v)]^{-1}}. \quad (\text{D.41})$$

These two functions exhibit different behaviours when $d \rightarrow 4$. On the one hand, eq. (D.41) becomes singular in this limit due to the product $\Gamma(-u)\Gamma(4-d+u)$, which does not enable to separate the first increasing and the first decreasing pole. On the other hand, eq. (D.40) is perfectly finite in four dimensions. For future reference, we provide its expansion about $d = 4$, i.e.

$$\mathcal{M}^{(i)}(d) = \int_{\delta-i\infty}^{\delta+i\infty} \frac{dudv}{(2\pi i)^2} \frac{\pi^3 \csc(\pi u) \csc(\pi v) \csc(\pi(u+v))}{v(u+v+1)} + \dots = 2\zeta(3) + \mathcal{O}(d-4), \quad (\text{D.42})$$

where $\delta \in (-1, 0)$ represents the real part of the variables u and v . As we will shortly see, similar quantities will arise from the integral L''_2 , defined in eq. (D.35).

D.3.2 Computing \mathbf{L}_2''

We begin with writing eq. (D.35) as

$$\mathbf{L}_2'' = \mathcal{A}_2 \oint d^3\tau \varepsilon(\tau) (R^2 - \dot{x}_1 \cdot \dot{x}_3) (\dot{x}_2 \cdot \partial_1) \tilde{\mathbf{L}}_2'', \quad (\text{D.43})$$

where

$$\tilde{\mathbf{L}}_2'' = 2 \frac{\Gamma(3-d/2)}{(4\pi)^{d/2}} (\dot{x}_2 \cdot \partial_1) \int dP \int_0^1 dX \frac{xp_3^2 + yp_2^2}{M^{3-d/2}}. \quad (\text{D.44})$$

Let us concentrate on $\tilde{\mathbf{L}}_2''$, which contains the integration over the Feynman parameters and over the measure dP . By expressing the denominator M , defined in eq. (D.36), as a two-fold Mellin-Barnes integral via eq. (D.38), the integration over the measures dX and dP becomes elementary. The net result can be expressed as a combination of *generalized* propagators $\mathcal{D}(x, s)$ defined in eq. (B.7):

$$\tilde{\mathbf{L}}_2'' = \int d\Omega \mathcal{D}(x_{1\omega}, \sigma) f_d(u, v) \left(\mathcal{D}(x_{2\omega}, 1-u) \mathcal{D}(x_{3\omega}, -v) + \mathcal{D}(x_{2\omega}, -v) \mathcal{D}(x_{3\omega}, 1-u) \right) \quad (\text{D.45})$$

where $d\Omega = d^d\omega du dv / (2\pi i)^2 (2\pi)^d$ and $\sigma = 4 - d/2 + u + v$. The integration over the variable ω arises from the conservation of the momenta p_i , while

$$f_d(u, v) = \frac{\Gamma(d/2 - 1 - u) \Gamma(d/2 - 2 - v) \Gamma(1 + u + v) \Gamma(3 - d/2 + v + u) \Gamma(-u) \Gamma(-v)}{\Gamma(d - 2) \pi^{-d/2}}. \quad (\text{D.46})$$

To proceed with the calculation, we integrate over $d^d\omega$ by introducing three Feynman parameters for the different propagators $\mathcal{D}(x, s)$. By employing eq. (B.7) and the parametrization of the coordinates τ_i , given by eq. (1.7), we obtain

$$\mathbf{L}_2'' = \frac{\mathcal{A}_2 \Gamma(3d/2 - 4) R^{12-3d}}{\Gamma(d - 2) 2^9 \pi^{3d/2} 2^{3d/2-5}} \int d\mathcal{M} \oint d^3\tau \frac{\varepsilon(\tau) (\cos \tau_{13} - 1) (\alpha(1 - \alpha) \sin \tau_{12} + \alpha\gamma \sin \tau_{23})}{Q^{3d/2-4}}. \quad (\text{D.47})$$

The denominator Q is defined in eq. (3.20), while the measure $d\mathcal{M}$ is given by

$$d\mathcal{M} = \frac{du dv}{(2\pi i)^2} dF \alpha^{d-5-u-v} \beta^{d/2-2+u} \gamma^{d/2-2+v} \left(\tilde{f}(u, v) \gamma + \tilde{f}(v, u) \beta \right), \quad (\text{D.48})$$

with

$$\tilde{f}(u, v) = - \frac{\Gamma(d/2 - 1 - v) \Gamma(d/2 - 2 - u) \Gamma(1 + u + v)}{u(3 - d/2 + u + v)}, \quad (\text{D.49})$$

while dF was defined in eq. (3.21). Note that the integration measure is symmetric under the simultaneous exchange of $\beta \leftrightarrow \gamma$ and $u \leftrightarrow v$. Finally, we integrate over the coordinates τ_i by employing the identity (C.13) and eventually obtain

$$\begin{aligned} \mathbf{L}_2'' = & - \mathcal{A}_2 \frac{\Gamma(3d/2 - 4) R^{12-3d}}{\Gamma(d - 2) 2^9 \pi^{3d/2} 2^{3d/2-5}} \int d\mathcal{M} \left(\frac{2}{3d/2 - 5} \oint d^2\tau \frac{(1 - \cos \tau_{12})^{6-3d/2}}{[\gamma(1 - \gamma)]^{3d/2-5}} \right) \\ & - \mathcal{A}_2 \frac{\Gamma(3d/2 - 4) R^{12-3d}}{\Gamma(d - 2) 2^9 \pi^{3d/2} 2^{3d/2-5}} \int d\mathcal{M} \left(\frac{3(2 - d/2)}{3d/2 - 5} \oint d^3\tau \varepsilon(\tau) \frac{\sin \tau_{13}}{Q^{3d/2-5}} + T_d(\alpha, \beta, \gamma) \right), \end{aligned} \quad (\text{D.50})$$

where

$$T_d(\alpha, \beta, \gamma) = \oint d^3\tau \varepsilon(\tau) \frac{\beta\gamma \sin \tau_{13}(1 - \cos \tau_{23}) + \alpha\gamma \sin \tau_{23}(1 - \cos \tau_{13})}{Q^{3d/2-4}}. \quad (\text{D.51})$$

The first term in the second line is proportional to $(d-4)$: it vanishes in the limit $d \rightarrow 4$ since the path-ordered integral is regular and consequently, it can be neglected for the three-loop analysis. Actually, as we will show in the following subsection, also the term involving the function $T_d(\alpha, \beta, \gamma)$ is of order $(d-4)$. Thus, we can write

$$\begin{aligned} \mathbb{L}_2'' &= -\mathcal{A}_2 \frac{\Gamma(3d/2-4)R^{12-3d}}{\Gamma(d-2)2^9\pi^{3d/2}2^{3d/2-5}} \int d\mathcal{M} \left(\frac{2}{3d/2-5} \oint d^2\tau \frac{(1-\cos \tau_{12})^{6-3d/2}}{[\gamma(1-\gamma)]^{3d/2-5}} \right) + \mathcal{O}(d-4) \\ &= -\mathcal{A}_2 \frac{\Gamma(3d/2-5) \left(\mathcal{M}^{(ii)}(d) + \mathcal{M}^{(i)}(d) \right)}{4^4\pi^{3d/2}\Gamma(d-2)\Gamma(5-d)} \oint \frac{R^2 - \dot{x}_2 \cdot \dot{x}_3}{(x_{23}^2)^{3d/2-5}} + \mathcal{O}(d-4), \end{aligned} \quad (\text{D.52})$$

where we employed the explicit form of the measure $d\mathcal{M}$ (D.48) to integrate over the Feynman parameters and used the definitions of the amplitudes $\mathcal{M}^{(i)}(d)$ and $\mathcal{M}^{(ii)}(d)$ given in eqs. (D.40) and (D.41). Finally, combining this result with eq. (D.39), we find that the function \mathbb{L}_2 (D.33) can be expanded as

$$\begin{aligned} \mathbb{L}_2 &= \mathbb{L}_2' + \mathbb{L}_2'' \\ &= -\frac{\mathcal{A}_2\Gamma(3d/2-5)\mathcal{M}^{(i)}(d)}{2^7\pi^{3d/2}\Gamma(d-2)\Gamma(5-d)} \oint \frac{R^2 - \dot{x}_2 \cdot \dot{x}_3}{(x_{23}^2)^{3d/2-5}} + \mathcal{O}(d-4) \\ &= -\frac{C_F N \hat{g}_B^6 \zeta(3)}{8\pi^2} + \mathcal{O}(d-4). \end{aligned} \quad (\text{D.53})$$

The last equality follows from the definition of the coefficient \mathcal{A}_2 , given by eq. (D.13), from the expansion of the amplitude $\mathcal{M}^{(i)}(d)$ about $d=4$ (D.42) and from the integration over the contour.

D.3.3 Evanescent integrals

Let us conclude this section by explicitly showing that the last contribution in the second line of eq. (D.50) is of order $(d-4)$. We consider

$$\begin{aligned} \mathbb{E}(d) &= \int d\mathcal{M} T_d(\alpha, \beta, \gamma) \\ &= \int_0^1 dF \int \frac{dudv}{(2\pi i)^2} \alpha^{d-5-u-v} \beta^{d/2-2+u} \gamma^{d/2-2+v} \left(\gamma \tilde{f}(u, v) + \beta \tilde{f}(v, u) \right) T_d(\alpha, \beta, \gamma), \end{aligned} \quad (\text{D.54})$$

which the second line follows from the definition of $d\mathcal{M}$, given by eq. (D.48). The first term can be written as

$$\mathbb{E}_1(d) = 2 \int \frac{dF dudv}{(2\pi i)^2} \left(\frac{\gamma^{d/2+v} \tilde{f}(u, v)}{\alpha^{5-d+u+v} \beta^{1-d/2-u}} - \frac{\gamma^{d/2+v} \tilde{f}(u, v)}{\beta^{4-d+u+v} \alpha^{2-d/2-u}} \right) \oint d^3\tau \varepsilon(\tau) \frac{\sin \tau_{13} \sin^2 \frac{\tau_{23}}{2}}{Q^{3d/2-4}}, \quad (\text{D.55})$$

where we used the integral representation for the function $T_d(\alpha, \beta, \gamma)$ (D.28) and the anti-symmetry of the ε -symbols (3.22). Changing variable according to $u' = d/2 - 3 - u - v$ in the second term, we find that $E_1(d) = 0$ for any d .

The calculation of the second contribution in eq. (D.54) is more subtle. We find that

$$E_2(d) = 2 \int \frac{dF du dv}{(2\pi i)^2} \left(\frac{\gamma^{d/2-1+v} \tilde{f}(v, u)}{\alpha^{5-d+u+v} \beta^{-d/2-u}} - \frac{\gamma^{d/2-1+v} \tilde{f}(v, u)}{\beta^{4-d+u+v} \alpha^{1-d/2-u}} \right) \oint d^3 \tau \varepsilon(\tau) \frac{\sin \tau_{13} \sin^2 \frac{\tau_{23}}{2}}{Q^{3d/2-4}}, \tag{D.56}$$

where we employed again the integral representation of the function $T_d(\alpha, \beta, \gamma)$. To continue the calculation, we consider separately the quantities

$$\begin{aligned} E'_2(d) &= \int \frac{dF du dv}{(2\pi i)^2} \frac{\gamma^{d/2-1+v} \tilde{f}(v, u)}{\alpha^{5-d+u+v} \beta^{-d/2-u}} \oint d^3 \tau \varepsilon(\tau) \frac{\sin \tau_{13} (1 - \cos \tau_{23})}{Q^{3d/2-4}}, \\ E''_2(d) &= \int \frac{dF du dv}{(2\pi i)^2} \frac{\gamma^{d/2-1+v} \tilde{f}(v, u)}{\beta^{4-d+u+v} \alpha^{1-d/2-u}} \oint d^3 \tau \varepsilon(\tau) \frac{\sin \tau_{13} (1 - \cos \tau_{23})}{Q^{3d/2-4}}. \end{aligned} \tag{D.57}$$

Firstly focussing on $E'_2(d)$, we replace the denominator Q with its Mellin-Barnes image (D.29). This enables to integrate over the contour by employing eq. (F.6) and the result can be written as a four-fold Mellin-Barnes integral

$$\begin{aligned} E'_2(d) &= \int \frac{du dv ds dt}{(2\pi i)^4} \frac{\Gamma(3d/2 - 4 - s - t) \Gamma(-s) \Gamma(-t) \Gamma(-d/2 - u - v - s) \Gamma(5 - d - t + u)}{2^{3d/2-5} \Gamma(5 - d) \Gamma(3d/2 - 4)} \\ &\quad \times \Gamma(d/2 + v + t + s) \mathcal{J}(3d/2 - 4 - s - t, s + 1, t), \end{aligned} \tag{D.58}$$

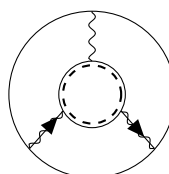
where the function $\mathcal{J}(x, y, z)$ is defined in eq. (F.6). Expanding the previous expression about $d \rightarrow 4$ enables to integrate over s and t by a repeated application of Barnes's first lemma. We eventually find that

$$E'_2(d) = 8\pi^2 \int_{\delta-i\infty}^{\delta+i\infty} \frac{du dv}{(2\pi i)^2} \frac{\Gamma(1-u) \Gamma(u+2) \Gamma(-v) \Gamma(v) \Gamma(-u-v) \Gamma(u+v+1)}{v(1+u+v)} + \mathcal{O}(d-4), \tag{D.59}$$

where $\delta \in (-1, 0)$ is the real part of the variables u and v . However, it is not necessary to perform the integration over the last two variables since, repeating the same analysis for the quantity $E''_2(d)$, it is possible to show its double Mellin-Barnes representation coincides with the previous expression. Exploiting this fact in eq. (D.56), we have, expanding about $d \rightarrow 4$, an identical cancellation. This explicitly showed that the function $E(d)$ (D.54) is of order $\mathcal{O}(d-4)$.

D.4 Summary of the results

Let us briefly summarize the results for the calculation of the lifesaver diagrams (D.1). Starting from eq. (D.23), we finally find that

$$\left(\text{Diagram} \right) = L_1 + L_2 = F_2^{(2)} - \hat{g}_B^6 \frac{5C_F N \beta_0^R \zeta(3)}{16\pi^2} + \mathcal{O}(d-4), \tag{D.60}$$


where the second equality follows from eqs. (D.32) and (D.53) and we recall that $F_2^{(2)}$ is the bubble-like contribution defined in eq. (3.37).

E Diagrams with four emissions

In this section, we provide calculation details of the following class of diagrams

$$\mathcal{W}'_{6(4)} = \text{Diagram}, \quad (\text{E.1})$$

where we recall that the double dashed/continuous internal bubble denotes the one-loop correction to the adjoint scalar and gauge field in the difference approach, see eqs. (B.16) and (B.17).

Following the approach outlined in section C and employing eq. (C.3), we can organize the diagrams depicted in eq. (E.1) as follows

$$\Sigma'_4 = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4}, \quad (\text{E.2})$$

$$\Sigma''_4 = \text{Diagram 5} + \text{Diagram 6}. \quad (\text{E.3})$$

E.1 Computing Σ'_4

Expanding the Wilson loop operator at order g_B^4 and decorating the Wick contractions with the one-loop correction to the adjoint scalar propagator $\Delta^{(1)}(x)$, defined in eq. (B.16), and with the tensor $\delta_{\mu\nu}\Delta^{(1)}(x)$, we find that

$$\begin{aligned} \Sigma'_4 = \frac{g_B^4}{N} \int_{\tau_1 > \tau_2 > \tau_3 > \tau_4} d^4\tau C^{aabb} & \left(\widehat{\Delta}(x_{12})\widehat{\Delta}^{(1)}(x_{34}) + \widehat{\Delta}(x_{34})\widehat{\Delta}^{(1)}(x_{12}) \right) + \\ & C^{aabb} \left(\widehat{\Delta}(x_{14})\widehat{\Delta}^{(1)}(x_{23}) + \widehat{\Delta}(x_{23})\widehat{\Delta}^{(1)}(x_{14}) \right) + \\ & C^{abab} \left(\widehat{\Delta}(x_{13})\widehat{\Delta}^{(1)}(x_{14}) + \widehat{\Delta}(x_{34})\widehat{\Delta}^{(1)}(x_{12}) \right), \end{aligned} \quad (\text{E.4})$$

where we recall that the tensor C^{abcd} is defined in eq. (3.14), while $\widehat{\Delta}(x)$ and $\widehat{\Delta}^{(1)}(x)$ are, respectively, given by eqs. (3.9) and (3.28). Using the non-Abelian exponentiation rules for the Wilson loop, we rewrite the previous expression as follows

$$\Sigma'_4 = \mathcal{W}_2 \mathcal{W}'_4 + \frac{g_B^4}{2N} \text{tr} \left([T^b, T^a] \right)^2 \int_{\mathcal{D}} d^4\tau \left(\widehat{\Delta}(x_{13})\widehat{\Delta}^{(1)}(x_{24}) + \widehat{\Delta}(x_{24})\widehat{\Delta}^{(1)}(x_{13}) \right), \quad (\text{E.5})$$

where \mathcal{D} denotes the ordered region $\tau_1 > \tau_2 > \tau_3 > \tau_4$, the functions \mathcal{W}_2 and \mathcal{W}'_4 are defined in eqs. (3.11) and (3.29), while the second term in the previous expression denotes the maximally non-Abelian part of the diagram. Going through the calculation of eq. (E.5) we encounter, by employing the parametrization eq. (1.7), the following integral

$$\int_{\tau_1 > \tau_2 > \tau_3 > \tau_4} d^4\tau \frac{1}{(4 \sin^2 \frac{\tau_{13}}{2})^{d/2-2} (4 \sin^2 \frac{\tau_{24}}{2})^{d-4}} + \frac{1}{(4 \sin^2 \frac{\tau_{24}}{2})^{d/2-2} (4 \sin^2 \frac{\tau_{12}}{2})^{d-4}}. \quad (\text{E.6})$$

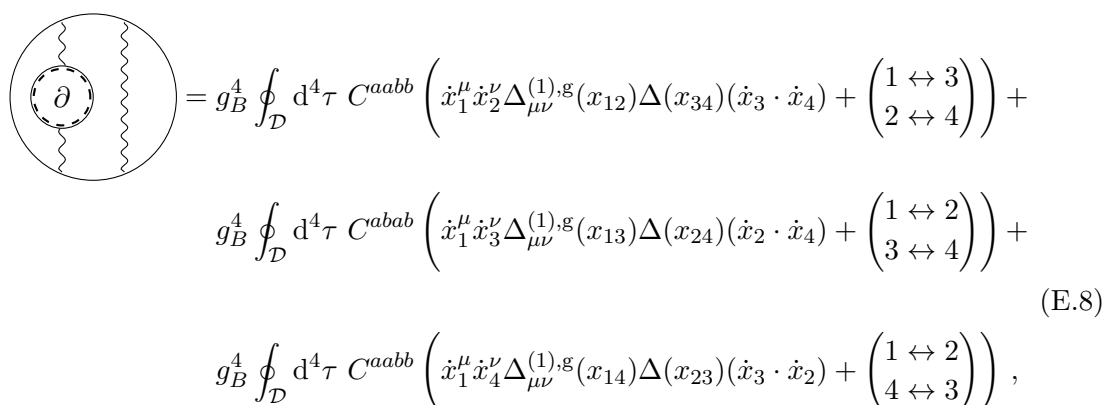
Using Fourier expansion methods outlined in appendix F, the previous expressions can be evaluated in terms of generalized hypergeometric functions (see eq. (F.20)). We find that

$$\Sigma'_4 = \hat{g}_B^6 C_F \frac{2N^2 - 3}{6N} P_2(d) B_1(d) B_2(d) + \frac{\hat{g}_B^6 \beta_0 C_F N 3 \zeta(3)}{2^4 \pi^2} + \dots, \quad (\text{E.7})$$

where the dots stand for terms proportional to $(d - 4)^2$, while the function $B_n(x)$ and $P_2(x)$ are defined in eqs. (3.11) and (3.29), respectively. Note that the $\zeta(3)$ -like term in the previous expression is analogous to that we generated from the maximally non-Abelian part of the two-loop ladder-like diagram (3.16). In particular, the result of eq. (3.16) is proportional to the evanescent factor $\epsilon = 2 - d/2$ resulting from the integration over the contour. This factor also arises in eq. (E.7) but it cancels against the UV pole of the one-loop correction $\Delta^{(1)}(x)$ (B.16) and leaves a finite result.

E.2 Computing Σ''_4

In this section, we turn our attention to the correction Σ''_4 , represented in eq. (E.3). Let us begin with considering in detail the first diagram which only involves gauge fields. We expand the Wilson loop operator at order g_B^4 , and we decorate the relevant Wick contractions by the tensor $\Delta_{\mu\nu}^{(1),g}(x) \equiv \partial_{1,\mu} \partial_{1,\nu} \Delta^{(1),g}(x)$. We have



$$\begin{aligned} &= g_B^4 \oint_{\mathcal{D}} d^4\tau C^{aabb} \left(\dot{x}_1^\mu \dot{x}_2^\nu \Delta_{\mu\nu}^{(1),g}(x_{12}) \Delta(x_{34}) (\dot{x}_3 \cdot \dot{x}_4) + \begin{pmatrix} 1 \leftrightarrow 3 \\ 2 \leftrightarrow 4 \end{pmatrix} \right) + \\ &g_B^4 \oint_{\mathcal{D}} d^4\tau C^{abab} \left(\dot{x}_1^\mu \dot{x}_3^\nu \Delta_{\mu\nu}^{(1),g}(x_{13}) \Delta(x_{24}) (\dot{x}_2 \cdot \dot{x}_4) + \begin{pmatrix} 1 \leftrightarrow 2 \\ 3 \leftrightarrow 4 \end{pmatrix} \right) + \\ &g_B^4 \oint_{\mathcal{D}} d^4\tau C^{aabb} \left(\dot{x}_1^\mu \dot{x}_4^\nu \Delta_{\mu\nu}^{(1),g}(x_{14}) \Delta(x_{23}) (\dot{x}_3 \cdot \dot{x}_2) + \begin{pmatrix} 1 \leftrightarrow 2 \\ 4 \leftrightarrow 3 \end{pmatrix} \right), \end{aligned} \quad (\text{E.8})$$

where we denoted with \mathcal{D} the ordered region $\tau_1 > \tau_2 > \tau_3 > \tau_4$ and we recall that $\Delta(x_{12})$ is massless tree level propagator defined in eq. (3.7), while the tensor C^{abcd} is given by eq. (3.14). The calculation of these diagrams can be further simplified by employing again the non-Abelian

exponentiation properties of the Wilson loop. Going through the calculation, we arrive at¹⁹

$$\begin{aligned}
 \text{Diagram} &= \frac{g_B^4}{2N} \text{tr} \left([T^a, T^b] \right)^2 \int_{\mathcal{D}} d^4\tau \left(\dot{x}_1^\mu \dot{x}_3^\nu \Delta_{\mu\nu}^{(1),g}(x_{13}) \Delta(x_{24}) (\dot{x}_2 \cdot \dot{x}_4) + \left(\begin{array}{c} 1 \leftrightarrow 2 \\ 3 \leftrightarrow 4 \end{array} \right) \right) \\
 &= g_B^4 \frac{C_F N}{2} \oint d^2\tau (\dot{x}_1 \cdot \dot{x}_2) \Delta^{(1),g}(x_{12}) \Delta(x_{12}), \tag{E.9}
 \end{aligned}$$

where we recall that $C_F = (N^2 - 1)/2N$. To obtain the last equality, we integrated by parts twice. Repeating the same analysis for the second diagram in eq. (E.3), we find that

$$\text{Diagram} = g_B^4 \frac{C_F N}{2} \oint d^2\tau \left(-R^2 \right) \Delta^{(1),g}(x_{12}) \Delta(x_{12}). \tag{E.10}$$

Combining together the relations we derived in this subsection, we finally arrive at the following representation for the correction Σ_4'' , defined in (E.3), i.e.

$$\Sigma_4'' = -g_B^4 \frac{C_F N}{2} \oint d^2\tau \left(R^2 - \dot{x}_1 \cdot \dot{x}_2 \right) \Delta^{(1),g}(x_{12}) \Delta(x_{12}) = F_3^{(2)}. \tag{E.11}$$

The last equality can be explicitly proved by recalling that the functions $\Delta(x)$ and $\Delta^{(1),g}(x)$ are, respectively, given by eq. (3.7) and (C.5), and using the explicit expression bubble-like correction $F_3^{(2)}$, given by eq. (3.37). Combining together the previous expression and eq. (E.7), we reproduce eq. (3.39).

F Trigonometric integrals

In this section, we evaluate the trigonometric integrals appearing in the calculation of the circular Wilson loop. It is convenient to firstly outline some useful relations. We will make extensively use of the following identity [58]

$$\begin{aligned}
 \mathcal{M}(a, b, c) &= \int_0^{2\pi} d^3\tau \left(\sin^2 \frac{\tau_{12}}{2} \right)^a \left(\sin^2 \frac{\tau_{13}}{2} \right)^b \left(\sin^2 \frac{\tau_{23}}{2} \right)^c \\
 &= 8\pi^{3/2} \frac{\Gamma(a + 1/2) \Gamma(b + 1/2) \Gamma(c + 1/2) \Gamma(1 + a + b + c)}{\Gamma(1 + a + c) \Gamma(1 + b + c) \Gamma(1 + a + b)}. \tag{F.1}
 \end{aligned}$$

We can use this identity to derive other useful results. For instance, as explained in appendix G of [59], the nested integral

$$\begin{aligned}
 \mathcal{I}[\alpha, \beta, \gamma] &= \int_{\tau_1 > \tau_2 > \tau_3} d^3\tau \left[\left(\sin^2 \frac{\tau_{12}}{2} \right)^\alpha \left(\sin^2 \frac{\tau_{13}}{2} \right)^\beta \left(\sin^2 \frac{\tau_{23}}{2} \right)^\gamma \cos \frac{\tau_{23}}{2} \right. \\
 &\quad \left. - \left(\sin^2 \frac{\tau_{23}}{2} \right)^\alpha \left(\sin^2 \frac{\tau_{12}}{2} \right)^\beta \left(\sin^2 \frac{\tau_{13}}{2} \right)^\gamma \cos \frac{\tau_{13}}{2} \right. \\
 &\quad \left. \left(\sin^2 \frac{\tau_{13}}{2} \right)^\alpha \left(\sin^2 \frac{\tau_{23}}{2} \right)^\beta \left(\sin^2 \frac{\tau_{12}}{2} \right)^\gamma \cos \frac{\tau_{12}}{2} \right], \tag{F.2}
 \end{aligned}$$

¹⁹To obtain eq. (E.9), we neglected terms which yield total derivatives integrated over a closed path.

can be reduced to a linear combination of functions we introduced in eq. (F.1). The net result can be written as follows

$$\mathcal{I}[\alpha, \beta, \gamma] = 4\pi^{3/2} \frac{\Gamma(1 + \alpha + \beta + \gamma)\Gamma(1 + \alpha)\Gamma(1 + \beta)\Gamma(1/2 + \gamma)}{\Gamma(3/2 + \alpha + \gamma)\Gamma(3/2 + \beta + \gamma)\Gamma(1 + \alpha + \beta)}. \quad (\text{F.3})$$

Finally, by employing this useful relation, we can derive a general expression for the following path-ordered integral

$$\mathcal{J}(\alpha, \beta, \gamma) = \oint d^3\tau \varepsilon(\tau) \sin \tau_{13} \left(\sin^2 \frac{\tau_{12}}{2}\right)^\alpha \left(\sin^2 \frac{\tau_{13}}{2}\right)^\gamma \left(\sin^2 \frac{\tau_{23}}{2}\right)^\beta, \quad (\text{F.4})$$

where we recall that $\varepsilon(\tau) \equiv \varepsilon(\tau_1, \tau_2, \tau_3)$ is defined in terms of the Heaviside θ -function in eq. (3.22). Employing this definition for the ε -symbol and relabelling the integration variables, we find that

$$\mathcal{J}(\alpha, \beta, \gamma) = -2\mathcal{I}(\beta, \alpha, \gamma + 1/2) - 2\mathcal{I}(\alpha, \beta, \gamma + 1/2) = -4\mathcal{I}(\alpha, \beta, \gamma + 1/2). \quad (\text{F.5})$$

To obtain the last line we noted that $\mathcal{I}(\alpha, \beta, \gamma)$ is symmetric in the exchange of the first two arguments. Therefore, the final result reads

$$\mathcal{J}(\alpha, \beta, \gamma) = -16\pi^{3/2} \frac{\Gamma(3/2 + \alpha + \beta + \gamma)\Gamma(1 + \alpha)\Gamma(1 + \beta)\Gamma(1 + \gamma)}{\Gamma(2 + \alpha + \gamma)\Gamma(2 + \beta + \gamma)\Gamma(1 + \alpha + \beta)}. \quad (\text{F.6})$$

Finally, in the calculation of the Wilson loop, we will extensively use the following identity

$$\frac{1}{(A + B + C)^\sigma} = \frac{1}{\Gamma(\sigma)} \int_{-i\infty}^{+i\infty} \frac{du dv}{(2\pi i)^2} \frac{B^u C^v}{A^{\sigma+u+v}} \Gamma(\sigma + u + v)\Gamma(-u)\Gamma(-v), \quad (\text{F.7})$$

where the integration contour runs parallelly to imaginary axis in such a way that the increasing and decreasing poles of the Γ -functions are separated.

F.1 Path-ordered integrals

In this subsection, we employ some of the identities we presented in the previous section to evaluate the path-ordered integral we introduced in eq. (3.23), i.e.

$$\begin{aligned} E(d) &= \int_0^1 dF (\alpha\beta\gamma)^{d/2-2} \oint d^3\tau \varepsilon(\tau) \frac{\sin \tau_{13}}{Q^{d-3}} \\ &= -8 \int_0^1 dF (\alpha\beta\gamma)^{d/2-2} \int_{\tau_1 > \tau_2 > \tau_3} d^3\tau \frac{\sin \frac{\tau_{13}}{2} \sin \frac{\tau_{12}}{2} \sin \frac{\tau_{23}}{2}}{Q^{d-3}}, \end{aligned} \quad (\text{F.8})$$

where α, β and γ are Feynman parameters integrated over the unit cube via the measure dF (3.21), while the denominator Q is defined in eq. (3.20). To obtain the second line, we employed the explicit definition of the ε -symbol in terms of the Heaviside θ -function (3.22). To integrate over the Feynman parameters, we replace the denominator Q with a two-fold Mellin-Barnes representation, i.e. (see eqs. (3.20) and (F.7))

$$\frac{1}{Q^\sigma} = \frac{2^{-\sigma}}{\Gamma(\sigma)} \int_{-i\infty}^{+i\infty} \frac{du dv}{(2\pi i)^2} \frac{\Gamma(\sigma + u + v)\Gamma(-u)\Gamma(-v)}{(\beta\alpha \sin^2 \frac{\tau_{12}}{2})^{\sigma+u+v} (\beta\gamma \sin^2 \frac{\tau_{23}}{2})^{-u} (\gamma\alpha \sin^2 \frac{\tau_{31}}{2})^{-v}}, \quad (\text{F.9})$$

where the integration path runs parallelly to the imaginary axes and separates the increasing and the decreasing poles of the Γ -function. Substituting this identity in eq. (F.8) and performing the integration over the Feynman parameters, we arrive at the following result

$$E(d) = \int \frac{du dv}{(2\pi i)^2} \frac{\Gamma(d-3+u+v)\Gamma(-u)\Gamma(-v)\Gamma(2-d/2-u)}{(\Gamma(2-d/2-v)\Gamma(d/2-1+u+v))^{-1}} \mathcal{E}(u, v, d), \quad (\text{F.10})$$

where in the previous expression we denoted the integral over the coordinates τ_i as follows

$$\begin{aligned} \mathcal{E}(u, v, d) &= -\frac{2^{6-d}}{\Gamma(3-d/2)\Gamma(d-3)} \int_{\tau_1 > \tau_2 > \tau_3} d^3\tau \frac{(\sin^2 \frac{\tau_{23}}{2})^{u+1/2} (\sin^2 \frac{\tau_{31}}{2})^{v+1/2}}{(\sin^2 \frac{\tau_{12}}{2})^{d-3+u+v-1/2}} \\ &= -\frac{2^{6-d}}{\Gamma(3-d/2)\Gamma(d-3)3!} \oint d^3\tau \frac{(\sin^2 \frac{\tau_{23}}{2})^{u+1/2} (\sin^2 \frac{\tau_{31}}{2})^{v+1/2}}{(\sin^2 \frac{\tau_{12}}{2})^{d-3+u+v-1/2}} \\ &= -\frac{2^{9-d}\pi^{3/2}\Gamma(11/2-d)}{\Gamma(3-d/2)\Gamma(d-3)3!} \frac{\Gamma(u+1)\Gamma(v+1)\Gamma(4-d-u-v)}{\Gamma(2+u+v)\Gamma(5-d-u)\Gamma(5-d-v)}. \end{aligned} \quad (\text{F.11})$$

In the previous expression, we obtained the second line by observing that the integrand is completely symmetric. This can be proved by properly shifting the Mellin-Barnes variables and enables us to replace the nested integration with an integral over the complete circle. Employing eq. (F.1), we finally find

$$E(d) = -\frac{2^{9-d}\pi^{3/2}\Gamma(11/2-d)}{\Gamma(3-d/2)\Gamma(d-3)3!} M(d). \quad (\text{F.12})$$

In the previous expression, the amplitude $M(d)$ is a meromorphic function of the dimension d which is defined in terms of the following two-fold Mellin-Barnes integral

$$\begin{aligned} M(d) &= \int \frac{du dv}{(2\pi i)^2} \frac{\Gamma(v+1)\Gamma(-v)\Gamma(2-d/2-v)\Gamma(-u)\Gamma(u+1)\Gamma(2-d/2-u)}{\Gamma(5-d-v)} \times \\ &\quad \times \frac{\Gamma(d-3+u+v)\Gamma(4-d-u-v)\Gamma(d/2-1+u+v)}{\Gamma(2+u+v)\Gamma(5-d-u)}. \end{aligned} \quad (\text{F.13})$$

Since the function $E(d)$ appears in the calculation of the Wilson loop with an evanescent coefficient (see eq. (3.18)), we only have to determine its behaviour for $d \rightarrow 4$. We find

$$\begin{aligned} M(d) \Big|_{d=4} &= \int_{-\delta'-i\infty}^{+\delta'+i\infty} \frac{dv du}{(2\pi i)^2} \frac{-\pi^3 \csc(\pi u) \csc(\pi v) \csc(\pi(u+v))}{(1+u+v)uv} \\ &= 6\zeta(3), \end{aligned} \quad (\text{F.14})$$

where $\delta' = \Re(u) = \Re(v) \in (-1, 0)$, in such a way that the increasing poles are to the right of the integration contour, while the decreasing ones are to the left. Substituting the previous expression in eq. (F.12), we finally arrive at

$$E(d) = -16\pi^2\zeta(3) + \mathcal{O}(d-4). \quad (\text{F.15})$$

F.2 Fourier expansions methods and the ladder-like diagrams

In this section, we will go through the calculation of the trigonometric integrals which enter the maximally non-Abelian part of the multiple-exchange diagrams (3.15) and (E.5).

The starting point is the Fourier expansion of the real even function $1/\sin^{2\alpha}(\frac{x}{2})$

$$\frac{1}{(4\sin^2\frac{x}{2})^\alpha} = \frac{1}{2}a_0(\alpha) + \sum_{n=1}^{\infty} a_n(\alpha) \cos nx, \quad (\text{F.16})$$

where the Fourier coefficients are given by [58]

$$a_n(\alpha) = \frac{1}{\pi} \int_0^{2\pi} dx \frac{\cos nx}{(4\sin^2\frac{x}{2})^\alpha} = \frac{\sec(\pi\alpha)\Gamma(n+\alpha)}{\Gamma(2\alpha)\Gamma(1-\alpha+n)}. \quad (\text{F.17})$$

Expressing the coordinates x_i in terms of trigonometric functions via eq. (1.7), we find that the integrals appearing in eqs. (E.5) and (3.15) take the following form

$$L(\alpha, \beta) = \int_{\mathcal{D}} \frac{d^4\tau}{(4\sin^2\frac{\tau_{13}}{2})^\alpha (4\sin^2\frac{\tau_{24}}{2})^\beta}, \quad (\text{F.18})$$

where the integration domain \mathcal{D} is defined by the ordered region $\tau_1 > \tau_2 > \tau_3 > \tau_4$. Replacing the trigonometric functions via their Fourier expansions (F.16) and performing the integration over the coordinates τ_i , we finally arrive at the following representation

$$L(\alpha, \beta) = \frac{\pi^4}{6} a_0(\alpha) a_0(\beta) - \sum_{n=1}^{\infty} \frac{\pi^2}{n^2} \left(a_0(\alpha) a_n(\beta) + a_0(\beta) a_n(\alpha) - a_n(\beta) a_n(\alpha) \right). \quad (\text{F.19})$$

The infinite sums in the previous expression can be easily performed in terms of usual generalized hypergeometric functions. After a straightforward calculation, we find that

$$\frac{L(\alpha, \beta)}{\pi^2 a_0(\alpha) a_0(\beta)} = \zeta(2) - \frac{\alpha {}_4F_3(x_\alpha, y_\alpha, 1)}{(1-\alpha)} - \frac{\beta {}_4F_3(x_\beta, y_\beta, 1)}{(1-\beta)} + \frac{\alpha\beta {}_5F_4(w_{\alpha,\beta}, z_{\alpha,\beta}, 1)}{(1-\alpha)(1-\beta)}, \quad (\text{F.20})$$

where the parameters of the two generalized hypergeometric functions are encoded in the following quantities $x_\alpha = (1, 1, 1, 1 + \alpha)$, $y_\alpha = (2, 2, 2 - \alpha)$, $w_{\alpha,\beta} = (1, 1, 1, 1 + \alpha, 1 + \beta)$ and $z_{\alpha,\beta} = (2, 2, 2 - \alpha, 2 - \beta)$.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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