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### Biextensions of 1-motives in Voevodsky's category of motives

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# BIEXTENSIONS OF 1-MOTIVES IN VOEVODSKY'S CATEGORY OF MOTIVES

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ABSTRACT. Let  $k$  be a perfect field. In this paper we prove that biextensions of 1-motives define multilinear morphisms between 1-motives in Voevodsky's triangulated category  $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$  of effective geometrical motives over  $k$  with rational coefficients.

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## INTRODUCTION

Let  $k$  be a perfect field. In [O] Orgogozo constructs a fully faithful functor

$$(0.1) \quad \mathcal{O} : \mathcal{D}^b(1 - \mathrm{Isomot}(k)) \longrightarrow \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$$

from the bounded derived category of the category  $1 - \mathrm{Isomot}(k)$  of 1-motives over  $k$  defined modulo isogenies to Voevodsky's triangulated category  $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$  of effective geometrical motives over  $k$  with rational coefficients. If  $M_i$  (for  $i = 1, 2, 3$ ) is a 1-motive defined over  $k$  modulo isogenies, in this paper we prove that the group of isomorphism classes of biextensions of  $(M_1, M_2)$  by  $M_3$  is isomorphic to the group of morphisms of the category  $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$  from the tensor product  $\mathcal{O}(M_1) \otimes_{tr} \mathcal{O}(M_2)$  to  $\mathcal{O}(M_3)$ :

**Theorem 0.1.** *Let  $M_i$  (for  $i = 1, 2, 3$ ) be a 1-motive defined over a perfect field  $k$ . Then*

$$\mathrm{Biext}^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})}(\mathcal{O}(M_1) \otimes_{tr} \mathcal{O}(M_2), \mathcal{O}(M_3)).$$

This isomorphism answers a question raised by Barbieri-Viale and Kahn in [BK1] Remark 7.1.3 2). In loc. cit. Proposition 7.1.2 e) they prove the above theorem in the case where  $M_3$  is a semi-abelian variety. Our proof is a generalization of theirs.

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If  $k$  is a field of characteristic 0 embeddable in  $\mathbb{C}$ , by [D] (10.1.3) we have a fully faithful functor

$$(0.2) \quad \mathrm{T} : 1 - \mathrm{Mot}(k) \longrightarrow \mathcal{MR}(k)$$

from the category  $1 - \mathrm{Mot}(k)$  of 1-motives over  $k$  to the Tannakian category  $\mathcal{MR}(k)$  of mixed realizations over  $k$  (see [J] I 2.1), which attaches to each 1-motive its Hodge realization for any embedding  $k \hookrightarrow \mathbb{C}$ , its de Rham realization, its  $\ell$ -adic realizations for any prime number  $\ell$ , and its comparison isomorphisms. According to [B1] Theorem 4.5.1, if  $M_i$  (for  $i = 1, 2, 3$ ) is a 1-motive defined over  $k$  modulo isogenies, the group of isomorphism classes of biextensions of  $(M_1, M_2)$  by  $M_3$  is isomorphic to the group of morphisms of the category  $\mathcal{MR}(k)$  from the tensor product  $\mathrm{T}(M_1) \otimes \mathrm{T}(M_2)$  of the realizations of  $M_1$  and  $M_2$  to the realization  $\mathrm{T}(M_3)$  of  $M_3$ . Putting together this result with Theorem 0.1, we get the following isomorphisms

$$(0.3) \quad \begin{aligned} \mathrm{Biext}^1(M_1, M_2; M_3) \otimes \mathbb{Q} &\cong \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; \mathbb{Q})}(\mathcal{O}(M_1) \otimes_{tr} \mathcal{O}(M_2), \mathcal{O}(M_3)) \\ &\cong \mathrm{Hom}_{\mathcal{MR}(k)}(\mathrm{T}(M_1) \otimes \mathrm{T}(M_2), \mathrm{T}(M_3)). \end{aligned}$$

These isomorphisms fit into the following context: in [H] Huber constructs a functor

$$\mathcal{H} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q}) \longrightarrow \mathcal{D}(\mathcal{MR}(k))$$

from Voevodsky's category  $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})$  to the triangulated category  $\mathcal{D}(\mathcal{MR}(k))$  of mixed realizations over  $k$ , which respects the tensor structures. Extending the functor  $\mathrm{T}$  (0.2) to the derived category  $\mathcal{D}^b(1 - \mathrm{Isomot}(k))$ , we obtain the following diagram

$$(0.4) \quad \begin{array}{ccc} \mathcal{D}^b(1 - \mathrm{Isomot}(k)) & \xrightarrow{\mathrm{T}} & \mathcal{D}(\mathcal{MR}(k)) \\ \mathcal{O} \downarrow & \nearrow \mathcal{H} & \\ \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q}) & & \end{array}$$

The isomorphisms (0.3) mean that biextensions of 1-motives define in a compatible way bilinear morphisms between 1-motives in each category involved in the above diagram. Barbieri-Viale and Kahn informed the authors that in [BK2] they have proved the commutativity of the diagram (0.4) in an axiomatic setting. If  $k = \mathbb{C}$ , they can prove its commutativity without assuming axioms.

We finish generalizing Theorem 0.1 to multilinear morphisms between 1-motives.

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#### NOTATION

If  $C$  is an additive category, we denote by  $C \otimes \mathbb{Q}$  the associated  $\mathbb{Q}$ -linear category which is universal for functors from  $C$  to a  $\mathbb{Q}$ -linear category. Explicitly, the category  $C \otimes \mathbb{Q}$  has the same objects as the category  $C$ , but the sets of arrows of  $C \otimes \mathbb{Q}$  are the sets of arrows of  $C$  tensored with  $\mathbb{Q}$ , i.e.  $\mathrm{Hom}_{C \otimes \mathbb{Q}}(-, -) = \mathrm{Hom}_C(-, -) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

We give a quick review of Voevodsky's category of motives (see [V]). Denote by  $\mathrm{Sm}(k)$  the category of smooth varieties over a field  $k$ . Let  $A = \mathbb{Z}$  or  $\mathbb{Q}$  be the coefficient ring. Let  $\mathrm{SmCor}(k, A)$  be the category whose objects are smooth

varieties over  $k$  and whose morphisms are finite correspondences with coefficients in  $A$ . It is an additive category.

The **triangulated category**  $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, A)$  **of effective geometrical motives over**  $k$  is the pseudo-abelian envelope of the localization of the homotopy category  $\mathcal{H}^b(\mathrm{SmCor}(k, A))$  of bounded complexes over  $\mathrm{SmCor}(k, A)$  with respect to the thick subcategory generated by the complexes  $X \times_k \mathbb{A}_k^1 \rightarrow X$  and  $U \cap V \rightarrow U \oplus V \rightarrow X$  for any smooth variety  $X$  and any Zariski-covering  $X = U \cup V$ .

The **category of Nisnevich sheaves on**  $\mathrm{Sm}(k)$ ,  $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k))$ , is the category of abelian sheaves on  $\mathrm{Sm}(k)$  for the Nisnevich topology.

A presheaf with transfers on  $\mathrm{Sm}(k)$  is an additive contravariant functor from  $\mathrm{SmCor}(k, A)$  to the category of abelian groups. It is called a Nisnevich sheaf with transfers if the corresponding presheaf of abelian groups on  $\mathrm{Sm}(k)$  is a sheaf for the Nisnevich topology. Denote by  $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))$  the **category of Nisnevich sheaves with transfers**. By [V] Theorem 3.1.4 it is an abelian category.

A presheaf with transfers  $F$  is called homotopy invariant if for any smooth variety  $X$  the natural map  $F(X) \rightarrow F(X \times_k \mathbb{A}_k^1)$  induced by the projection  $X \times_k \mathbb{A}_k^1 \rightarrow X$  is an isomorphism. A Nisnevich sheaf with transfers is called homotopy invariant if it is homotopy invariant as a presheaf with transfers.

The **category**  $\mathrm{DM}_{-}^{\mathrm{eff}}(k, A)$  **of effective motivic complexes** is the full subcategory of the derived category  $\mathcal{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A)))$  of complexes of Nisnevich sheaves with transfers bounded from the above, which consists of complexes with homotopy invariant cohomology sheaves.

There exists a functor  $L : \mathrm{SmCor}(k, A) \rightarrow \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))$  which associates to each smooth variety  $X$  a Nisnevich sheaf with transfers given by  $L(X)(U) = c(U, X)_A$ , where  $c(U, X)_A$  is the free  $A$ -module generated by prime correspondences from  $U$  to  $X$ . This functor extends to complexes furnishing a functor

$$L : \mathcal{H}^b(\mathrm{SmCor}(k, A)) \longrightarrow \mathcal{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))).$$

There exists also a functor  $C_* : \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A)) \rightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k, A)$  which associates to each Nisnevich sheaf with transfers  $F$  the effective motivic complex  $C_*(F)$  given by  $C_n(F)(U) = F(U \times \Delta^n)$  where  $\Delta^*$  is the standard cosimplicial object. This functor extends to a functor

$$(0.5) \quad \mathbf{RC}_* : \mathcal{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))) \longrightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k, A)$$

which is left adjoint to the natural embedding. Moreover, this functor identifies the category  $\mathrm{DM}_{-}^{\mathrm{eff}}(k, A)$  with the localization of  $\mathcal{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A)))$  with respect to the localizing subcategory generated by complexes of the form  $L(X \times_k \mathbb{A}_k^1) \rightarrow L(X)$  for any smooth variety  $X$  (see [V] Proposition 3.2.3).

If  $X$  and  $Y$  are two smooth varieties over  $k$ , the equality

$$(0.6) \quad L(X) \otimes L(Y) = L(X \times_k Y)$$

defines a tensor structure on the category  $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))$ , which extends to the derived category  $\mathcal{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A)))$ . The tensor structure on  $\mathrm{DM}_{-}^{\mathrm{eff}}(k, A)$ , that we denote by  $\otimes_{tr}$ , is the descent with respect to the projection  $\mathbf{RC}_*$  (0.5) of the tensor structure on  $\mathcal{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A)))$ .

If we assume  $k$  to be a perfect field, by [V] Proposition 3.2.6 there exists a functor

$$(0.7) \quad i : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, A) \rightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k, A)$$

which is a full embedding with dense image and which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{H}^b(\mathrm{SmCor}(k, A)) & \xrightarrow{L} & \mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))) \\ \downarrow & & \downarrow \mathbf{RC}_* \\ \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, A) & \xrightarrow{-i} & \mathrm{DM}_{-}^{\mathrm{eff}}(k, A). \end{array}$$

*Remark 0.2.* For Voevodsky's theory of motives with rational coefficients, the étale topology gives the same motivic answer as the Nisnevich topology: if we construct the category of effective motivic complexes using the étale topology instead of the Nisnevich topology, we get a triangulated category  $\mathrm{DM}_{-, \mathrm{ét}}^{\mathrm{eff}}(k, A)$  which is equivalent as triangulated category to the category  $\mathrm{DM}_{-}^{\mathrm{eff}}(k, A)$  if we assume  $A = \mathbb{Q}$  (see [V] Proposition 3.3.2).

### 1. 1-MOTIVES IN VOEVOFSKY'S CATEGORY

A **1-motive**  $M = (X, A, T, G, u)$  over a field  $k$  (see [D] §10) consists of

- a group scheme  $X$  over  $k$ , which is locally for the étale topology, a constant group scheme defined by a finitely generated free  $\mathbb{Z}$ -module,
- an extension  $G$  of an abelian  $k$ -variety  $A$  by a  $k$ -torus  $T$ ,
- a morphism  $u : X \rightarrow G$  of commutative  $k$ -group schemes.

A 1-motive  $M = (X, A, T, G, u)$  can be viewed also as a length 1 complex  $[X \xrightarrow{u} G]$  of commutative  $k$ -group schemes. In this paper, as a complex we shall put  $X$  in degree 0 and  $G$  in degree 1. A morphism of 1-motives is a morphism of complexes of commutative  $k$ -group schemes. Denote by  $1 - \mathrm{Mot}(k)$  the category of 1-motives over  $k$ . It is an additive category but it isn't an abelian category.

Denote by  $1 - \mathrm{Isomot}(k)$  the  $\mathbb{Q}$ -linear category  $1 - \mathrm{Mot}(k) \otimes \mathbb{Q}$  associated to the category of 1-motives over  $k$ . The objects of  $1 - \mathrm{Isomot}(k)$  are called 1-isomotifs and the morphisms of  $1 - \mathrm{Mot}(k)$  which become isomorphisms in  $1 - \mathrm{Isomot}(k)$  are the isogenies between 1-motives, i.e. the morphisms of complexes  $[X \rightarrow G] \rightarrow [X' \rightarrow G']$  such that  $X \rightarrow X'$  is injective with finite cokernel, and  $G \rightarrow G'$  is surjective with finite kernel. The category  $1 - \mathrm{Isomot}(k)$  is an abelian category (see [O] Lemma 3.2.2).

Assume now  $k$  to be a perfect field. The two main ingredients which furnish the link between 1-motives and Voevodsky's motives are:

- (1) any commutative  $k$ -group scheme represents a Nisnevich sheaf with transfers, i.e. an object of  $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))$  ([O] Lemma 3.1.2),
- (2) if  $A$  (resp.  $T$ , resp.  $X$ ) is an abelian  $k$ -variety (resp. a  $k$ -torus, resp. a group scheme over  $k$ , which is locally for the étale topology, a constant group scheme defined by a finitely generated free  $\mathbb{Z}$ -module), then the Nisnevich sheaf with transfers that it represents is homotopy invariant ([O] Lemma 3.3.1).

Since we can view 1-motives as complexes of smooth varieties over  $k$ , we have a functor from the category of 1-motives to the category  $\mathcal{C}(\mathrm{Sm}(k))$  of complexes over  $\mathrm{Sm}(k)$ . According to (1), this functor factorizes through the category of complexes over  $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A))$ :

$$1 - \mathrm{Mot}(k) \longrightarrow \mathcal{C}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{SmCor}(k, A)))$$

If we tensor with  $\mathbb{Q}$ , we get an additive exact functor between abelian categories

$$1 - \text{Isomot}(k) \longrightarrow \mathcal{C}(\text{Sh}_{\text{Nis}}(\text{SmCor}(k, A)) \otimes \mathbb{Q}).$$

Taking the associated bounded derived categories, we obtain a triangulated functor

$$\mathcal{D}^b(1 - \text{Isomot}(k)) \longrightarrow \mathcal{D}^b(\text{Sh}_{\text{Nis}}(\text{SmCor}(k, A)) \otimes \mathbb{Q}).$$

Finally, according to (2) this last functor factorizes through the triangulated functor

$$\mathcal{O} : \mathcal{D}^b(1 - \text{Isomot}(k)) \longrightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k, A) \otimes \mathbb{Q}.$$

By [O] Proposition 3.3.3 this triangulated functor is fully faithful, and by loc. cit. Theorem 3.4.1 it factorizes through the thick subcategory  $d_1\text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q})$  of  $\text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q})$  generated by smooth varieties of dimension  $\leq 1$  over  $k$  and it induces an equivalence of triangulated categories, that we denote again by  $\mathcal{O}$ ,

$$\mathcal{O} : \mathcal{D}^b(1 - \text{Isomot}(k)) \longrightarrow d_1\text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q}).$$

In order to simplify notation, if  $M$  is a 1-motive, we denote again by  $M$  its image in  $d_1\text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q})$  through the above equivalence of categories and also its image in  $\text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q})$  through the full embedding (0.7).

For the proof of Theorem 0.1, we will need the following

**Proposition 1.1.** *Let  $M_i$  (for  $i = 1, 2, 3$ ) be a 1-motive defined over  $k$ . The forgetful triangulated functors*

$$\text{DM}_{\text{gm}}^{\text{eff}}(k, A) \xrightarrow{a} \mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k, A))) \xrightarrow{b} \mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{Sm}(k)))$$

*induce an isomorphism*

$$\text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k, A)}(M_1 \otimes_{tr} M_2, M_3) \cong \text{Hom}_{\mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3).$$

*Proof.* The forgetful functor  $a$  admits as left adjoint the functor  $\mathbf{RC}_*$  (0.5). The forgetful functor  $b$  from the category of Nisnevich sheaves with transfers to the category of Nisnevich sheaves admits as left adjoint the free sheaf with transfers functor

$$(1.1) \quad \Phi : \mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{Sm}(k))) \longrightarrow \mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k, A)))$$

([V] Remark 1 page 202). If  $X$  is a smooth variety over  $k$ , let  $\mathbb{Z}(X)$  be the sheafification with respect to the Nisnevich topology of the presheaf  $U \mapsto \mathbb{Z}[\text{Hom}_{\text{Sm}(k)}(U, X)]$ . Clearly  $\Phi(\mathbb{Z}(X))$  is the Nisnevich sheaf with transfers  $L(X)$ . If  $Y$  is another smooth variety over  $k$ , we have that  $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times_k Y)$  (see [MVW] Lemma 12.14) and so by formula (0.6) we get

$$\Phi(\mathbb{Z}(X) \otimes \mathbb{Z}(Y)) = \Phi(\mathbb{Z}(X)) \otimes_{tr} \Phi(\mathbb{Z}(Y)).$$

The tensor structure on  $\text{DM}_{\text{gm}}^{\text{eff}}(k, A)$  is the descent of the tensor structure on  $\mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k, A)))$  with respect to  $\mathbf{RC}_*$  and therefore

$$\mathbf{RC}_* \circ \Phi(\mathbb{Z}(X) \otimes \mathbb{Z}(Y)) = \mathbf{RC}_* \circ \Phi(\mathbb{Z}(X)) \otimes_{tr} \mathbf{RC}_* \circ \Phi(\mathbb{Z}(Y)).$$

Using this equality and the fact that the composite  $\mathbf{RC}_* \circ \Phi$  is the left adjoint of  $b \circ a$ , we have

$$\begin{aligned} \text{Hom}_{\mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3) &\cong \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k, A)}(\mathbf{RC}_* \circ \Phi(M_1 \overset{\mathbb{L}}{\otimes} M_2), M_3) \\ &\cong \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k, A)}(\mathbf{RC}_* \circ \Phi(M_1) \otimes_{tr} \mathbf{RC}_* \circ \Phi(M_2), M_3). \end{aligned}$$

Since 1-motives are complexes of homotopy invariant Nisnevich sheaves with transfers, the counit arrows  $\mathbf{R}C_* \circ \Phi(M_i) \rightarrow M_i$  (for  $i = 1, 2$ ) are isomorphisms and so we can conclude.  $\square$

## 2. BILINEAR MORPHISMS BETWEEN 1-MOTIVES

Let  $K_i = [A_i \xrightarrow{u_i} B_i]$  (for  $i = 1, 2, 3$ ) be a length 1 complex of abelian sheaves (over any topos  $\mathbf{T}$ ) with  $A_i$  in degree 1 and  $B_i$  in degree 0. A **biextension**  $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$  **of**  $(K_1, K_2)$  **by**  $K_3$  consists of

- (1) a biextension of  $\mathcal{B}$  of  $(B_1, B_2)$  by  $B_3$ ;
- (2) a trivialization  $\Psi_1$  (resp.  $\Psi_2$ ) of the biextension  $(u_1, id_{B_2})^*\mathcal{B}$  of  $(A_1, B_2)$  by  $B_3$  (resp. of the biextension  $(id_{B_1}, u_2)^*\mathcal{B}$  of  $(B_1, A_2)$  by  $B_3$ ) obtained as pull-back of  $\mathcal{B}$  via  $(u_1, id_{B_2}) : A_1 \times B_2 \rightarrow B_1 \times B_2$  (resp. via  $(id_{B_1}, u_2) : B_1 \times A_2 \rightarrow B_1 \times B_2$ ). These two trivializations have to coincide over  $A_1 \times A_2$ ;
- (3) a morphism  $\lambda : A_1 \otimes A_2 \rightarrow A_3$  such that the composite  $A_1 \otimes A_2 \xrightarrow{\lambda} A_3 \xrightarrow{u_3} B_3$  is compatible with the restriction over  $A_1 \times A_2$  of the trivializations  $\Psi_1$  and  $\Psi_2$ .

We denote by  $\mathbf{Biext}(K_1, K_2; K_3)$  the category of biextensions of  $(K_1, K_2)$  by  $K_3$ . The Baer sum of extensions defines a group law for the objects of the category  $\mathbf{Biext}(K_1, K_2; K_3)$ , which is therefore a Picard category (see [SGA7] Exposé VII 2.4, 2.5 and 2.6). Let  $\mathbf{Biext}^0(K_1, K_2; K_3)$  be the group of automorphisms of any biextension of  $(K_1, K_2)$  by  $K_3$ , and let  $\mathbf{Biext}^1(K_1, K_2; K_3)$  be the group of isomorphism classes of biextensions of  $(K_1, K_2)$  by  $K_3$ .

According to the main result of [B2], we have the following homological interpretation of the groups  $\mathbf{Biext}^i(K_1, K_2; K_3)$ :

$$(2.1) \quad \mathbf{Biext}^i(K_1, K_2; K_3) \cong \mathrm{Ext}^i(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) \quad (i = 0, 1)$$

Since we can view 1-motives as complexes of commutative  $S$ -group schemes of length 1, all the above definitions apply to 1-motives.

*Remark 2.1.* The homological interpretation (2.1) of biextensions computed in [B2] is done for chain complexes  $K_i = [A_i \xrightarrow{u_i} B_i]$  with  $A_i$  in degree 1 and  $B_i$  in degree 0. In this paper 1-motives are considered as cochain complexes  $M_i = [X_i \xrightarrow{u_i} G_i]$  with  $X$  in degree 0 and  $G$  in degree 1. Therefore after switching from homological notation to cohomological notation, the homological interpretation of the group  $\mathbf{Biext}^1(M_1, M_2; M_3)$  can be stated as follow:

$$\mathbf{Biext}^1(M_1, M_2; M_3) \cong \mathrm{Ext}^1(M_1[1] \overset{\mathbb{L}}{\otimes} M_2[1], M_3[1])$$

where the shift functor  $[i]$  on a cochain complex  $C^*$  acts as  $(C^*[i])^j = C^{i+j}$ .

*Proof of Theorem 0.1* By proposition 1.1, we have that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, \mathbb{Q})}(M_1 \otimes_{tr} M_2, M_3) &\cong \mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k, A) \otimes \mathbb{Q}}(M_1 \otimes_{tr} M_2, M_3) \\ &\cong \mathrm{Hom}_{\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3) \otimes \mathbb{Q}. \end{aligned}$$

On the other hand, according to the remark 2.1 we have the following homological interpretation of the group  $\mathbf{Biext}^1(M_1, M_2; M_3)$ :

$$\mathbf{Biext}^1(M_1, M_2; M_3) \cong \mathrm{Ext}^1(M_1[1] \overset{\mathbb{L}}{\otimes} M_2[1], M_3[1]) \cong \mathrm{Hom}_{\mathcal{D}^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}(k)))}(M_1 \overset{\mathbb{L}}{\otimes} M_2, M_3)$$

and so we can conclude.

### 3. MULTILINEAR MORPHISMS BETWEEN 1-MOTIVES

1-motives are endowed with an increasing filtration, called the weight filtration. Explicitly, the weight filtration  $W_*$  on a 1-motive  $M = [X \xrightarrow{u} G]$  is

$$\begin{aligned} W_i(M) &= M \text{ for each } i \geq 0, \\ W_{-1}(M) &= [0 \longrightarrow G], \\ W_{-2}(M) &= [0 \longrightarrow Y(1)], \\ W_j(M) &= 0 \text{ for each } j \leq -3. \end{aligned}$$

Defining  $\text{Gr}_i^W = W_i/W_{i+1}$ , we have  $\text{Gr}_0^W(M) = [X \rightarrow 0]$ ,  $\text{Gr}_{-1}^W(M) = [0 \rightarrow A]$  and  $\text{Gr}_{-2}^W(M) = [0 \rightarrow Y(1)]$ . Hence locally constant group schemes, abelian varieties and tori are the pure 1-motives underlying  $M$  of weights 0,-1,-2 respectively.

The main property of morphisms of motives is that they have to respect the weight filtration, i.e. any morphism  $f : A \rightarrow B$  of motives satisfies the following equality

$$f(A) \cap W_i(B) = f(W_i(A)) \quad \forall i \in \mathbb{Z}.$$

Assume  $M$  and  $M_1, \dots, M_l$  to be 1-motives over a perfect field  $k$  and consider a morphism

$$F : \otimes_{j=1}^l M_j \rightarrow M.$$

Because morphisms of motives have to respect the weight filtration, the only non trivial components of the morphism  $F$  are the components of the morphism

$$\otimes_{j=1}^l M_j / W_{-3}(\otimes_{j=1}^l M_j) \longrightarrow M.$$

Using [B1] Lemma 3.1.3 with  $i = -3$ , we can write explicitly this last morphism in the following way

$$\sum_{\substack{\nu_1 < \nu_2 \text{ and } \nu_1 < \dots < \nu_{l-2} \\ \nu_1, \nu_2 \notin \{\nu_1, \dots, \nu_{l-2}\}}} X_{\nu_1} \otimes \dots \otimes X_{\nu_{l-2}} \otimes (M_{\nu_1} \otimes M_{\nu_2} / W_{-3}(M_{\nu_1} \otimes M_{\nu_2})) \longrightarrow M.$$

To have the morphism

$$X_{\nu_1} \otimes \dots \otimes X_{\nu_{l-2}} \otimes (M_{\nu_1} \otimes M_{\nu_2} / W_{-3}(M_{\nu_1} \otimes M_{\nu_2})) \longrightarrow M$$

is equivalent to have the morphism

$$M_{\nu_1} \otimes M_{\nu_2} / W_{-3}(M_{\nu_1} \otimes M_{\nu_2}) \longrightarrow X_{\nu_1}^\vee \otimes \dots \otimes X_{\nu_{l-2}}^\vee \otimes M$$

where  $X_{\nu_n}^\vee$  is the  $k$ -group scheme  $\underline{\text{Hom}}(X_{\nu_n}, \mathbb{Z})$  for  $n = 1, \dots, l-2$ . But as observed in [B1] §1.1 “to tensor a motive by a motive of weight zero” means to take a certain number of copies of this motive, and so applying Theorem 0.1 we get

**Theorem 3.1.** *Let  $M$  and  $M_1, \dots, M_l$  be 1-motives over a perfect field  $k$ . Then,*

$$\begin{aligned} \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k, \mathbb{Q})}(M_1 \otimes_{tr} M_2 \otimes_{tr} \dots \otimes_{tr} M_l, M) &\cong \\ \sum \text{Biext}^1(M_{\nu_1}, M_{\nu_2}; X_{\nu_1}^\vee \otimes \dots \otimes X_{\nu_{l-2}}^\vee \otimes M) &\otimes \mathbb{Q} \end{aligned}$$

where the sum is taken over all the  $(l-2)$ -uplets  $\{\nu_1, \dots, \nu_{l-i+1}\}$  and all the 2-uplets  $\{\nu_1, \nu_2\}$  of  $\{1, \dots, l\}$  such that  $\{\nu_1, \dots, \nu_{l-2}\} \cap \{\nu_1, \nu_2\} = \emptyset$  and  $\nu_1 < \dots < \nu_{l-2}$ ,  $\nu_1 < \nu_2$ .

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