Published for SISSA by Springer

Received: *March 30, 2010*
Accepted: *May 11, 2010*
Published: *May 31, 2010*

Stringy instanton corrections to $\mathcal{N}=2$ gauge couplings

Marco Billò,[a] Marialuisa Frau,[a] Francesco Fucito,[b] Alberto Lerda,[c] Jose F. Morales[b] and Rubik Poghossian[b,d]

[a] *Dipartimento di Fisica Teorica, Università di Torino and I.N.F.N. - sezione di Torino Via P. Giuria 1, I-10125 Torino, Italy*
[b] *I.N.F.N. - sezione di Roma Tor Vergata Via della Ricerca Scientifica, I-00133 Roma, Italy*
[c] *Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale and I.N.F.N. - Gruppo Collegato di Alessandria - sezione di Torino Viale T. Michel 11, I-15121 Alessandria, Italy*
[d] *Yerevan Physics Institute, Alikhanian Br. 2, 0036 Yerevan, Armenia*

E-mail: billo@to.infn.it, frau@to.infn.it, fucito@roma2.infn.it, lerda@to.infn.it, morales@roma2.infn.it, poghosyan@roma2.infn.it

Abstract: We discuss a string model where a conformal four-dimensional $\mathcal{N}=2$ gauge theory receives corrections to its gauge kinetic functions from "stringy" instantons. These contributions are explicitly evaluated by exploiting the localization properties of the integral over the stringy instanton moduli space. The model we consider corresponds to a setup with D7/D3-branes in type I′ theory compactified on $\mathcal{T}_4/\mathbb{Z}_2\times\mathcal{T}_2$, and possesses a perturbatively computable heterotic dual. In the heteoric side the corrections to the quadratic gauge couplings are provided by a 1-loop threshold computation and, under the duality map, match precisely the first few stringy instanton effects in the type I′ setup. This agreement represents a very non-trivial test of our approach to the exotic instanton calculus.

Keywords: D-branes, Brane Dynamics in Gauge Theories, Superstrings and Heterotic Strings

ArXiv ePrint: 1002.4322

doi:10.1007/JHEP05(2010)107

JHEP05(2010)107

Contents

JHEP05(2010)107

1 Introduction and motivations

It has been recently found [1–3] that certain classes of D-brane instantons arising in intersecting brane models can generate effective interactions at energies that are not linked to the gauge theory scale, and for this reason they are usually called "stringy" or "exotic" instantons. This feature is very welcome in the search of semi-realistic string scenarios for the TeV physics, where a hierarchy between various Majorana masses and Yukawa couplings is expected. It is therefore of the greatest importance to devise efficient and reliable techniques to determine quantitatively such exotic non-perturbative corrections through their explicit realization at the string level. This consideration is one of the main motivations behind the present work.

In refs. [4, 5] explicit models with stringy instantons were constructed; since then much work has been done extending and exploiting these results [6–25] (for a recent exhaustive review on the subject see ref. [26]). Even if the effects of exotic and gauge instantons are quite different from each other, in both cases they can be obtained from Euclidean branes entirely wrapping some cycle of the internal space. Depending on whether this cycle coincides or not with the one wrapped by the space-filling D-branes on which the gauge theory is defined, such Euclidean branes correspond to gauge or exotic instantons, respectively.

In the simplest cases, four-dimensional gauge instantons can be realized with bound states of space-filling D3-branes and point-like D(–1)-branes (or D-instantons) [27, 28]. Indeed, in these systems there are four directions in which the string coordinates may have mixed Neumann-Dirichlet (ND) boundary conditions, and the massless sector of open strings having at least one endpoint on the D(–1)'s is in one-to-one correspondence with the moduli (positions, sizes and gauge orientations) of the four dimensional gauge instanton solution. Actually, also the effective action on the moduli space, the rules of the instanton calculus and the profile of the classical solution can be explicitly obtained using D(–1)/D3-brane systems [29–32].

In the exotic cases, the gauge and instantonic branes intersect non-trivially in the internal space or carry different magnetic fluxes, and the open strings stretching between them have extra "twisted" directions besides the four ND ones along the space-time. This twist lifts some of their massless excitations, and some instanton moduli (specifically those related to sizes and gauge orientations) disappear from the spectrum. Their supersymmetric fermionic partners remain massless though, and when integrated out they can, under certain conditions, lead to the effective interactions we alluded to above.

A very simple example of this phenomenon occurs in the D(–1)/D7 brane system, which exhibits the world-sheet features of exotic instantons since mixed open strings have eight ND directions. By adding O7-planes, this system can be embedded in type I′ string theory compactified on a 2-torus $\mathcal{T}_2$, a setup which possesses a computable perturbative heterotic dual [33–39]. If the D7-branes are distributed democratically over the four orientifold fixed points on $\mathcal{T}_2$, they support a maximally supersymmetric gauge theory in eight dimensions with gauge group SO(8). In this gauge theory a D(–1)-brane represents a non-perturbative point-like configuration that has been recently identified [22] with the zero-size limit of the eight-dimensional octonionic instanton solution found long ago in refs. [40, 41].

The non-perturbative contributions of D-instantons to the effective action on the D7-branes can be explicitly computed as integrals over the moduli space via localization techniques, in analogy with what is done for usual gauge instantons [42], though with an exotic moduli spectrum. All D-instanton numbers correct the quartic gauge couplings of the eight-dimensional gauge theory [23], and this whole series of terms can be compared to those obtained in the dual heterotic string theory, where they correspond to world-sheet instantons describing the wrapping of the heterotic string on $\mathcal{T}_2$ [43, 44]. The success of this comparison provides a very non-trivial check of both the type I$'$/heterotic duality and the correctness of this approach to the exotic instanton calculus [23]. Similar techniques can be used also in non-conformal settings and for exotic instantons with fewer number of super-symmetries [24], although the heterotic counterpart of the induced interactions in these cases is far from clear (see also ref. [25] for related recent work).

An interesting feature of these eight-dimensional gauge theories is their similarity with the four-dimensional $\mathcal{N} = 2$ super Yang-Mills theories: indeed, the eight-dimensional prepotentials and the correlators of the chiral ring satisfy Matone-type relations for arbitrary SO(N) gauge groups [24]; this observation points to the existence of some direct relation between the eight-dimensional effective action and some underlying Seiberg-Witten curve, connected presumably to an F-theory description (see, for example, refs. [36, 45, 46] for earlier results in this direction).

In this paper we investigate the exotic calculus in a four-dimensional setup. We consider a perturbatively conformal $\mathcal{N} = 2$ gauge theory that, on the one hand, admits a brane realization where exotic instantons generate a whole series of corrections to the quadratic gauge couplings, while on the other hand it possesses a calculable heterotic dual against which these corrections can be checked (see [47] for a recent test of four fermionic couplings in the six-dimensional version of this type I/heterotic dual pair). This allows to provide a test of the exotic instanton calculus as reliable as the eight-dimensional one described above, but in a four-dimensional context.

The gauge theory we consider is realized on the world-volume of D7-branes at an O7 fixed-point within a D7/D3-brane system of type I$'$ compactified on $\mathcal{T}_4/\mathbb{Z}_2 \times \mathcal{T}_2$. This is a T-dual variant of the first example of a consistent $\mathcal{N} = 2$ open string compactification in which all tadpoles cancel [48, 49]. In section 2 we describe in detail the four-dimensional model, which actually admits different realizations corresponding to different consistent distributions of branes, and show how the conformal $\mathcal{N} = 2$ theory we are interested in arises. Then, we determine the holomorphic quadratic gauge couplings of the low-energy effective theory [50–52] starting, in section 3, with the perturbative terms (limited to 1-loop by supersymmetry). The theory, however, admits also non-perturbative corrections produced by brane instantons. These can be Euclidean 3-branes wrapped on $\mathcal{T}_4/\mathbb{Z}_2$, namely the same cycle wrapped by the D7-branes supporting the gauge theory, or D-instantons. In the first case, they correspond to ordinary gauge instantons and might yield corrections weighted by powers of $\exp(-8\pi^2/g^2)$, where g is the Yang-Mills coupling. The D-instanton corrections, instead, are weighted by powers of $\exp(-\pi/g_{\mathrm{s}}) = \exp(-4\pi^2/g^2\,\mathcal{V}_4)$, where g_{s} is the string coupling and $\mathcal{V}_4$ the volume of $\mathcal{T}_4/\mathbb{Z}_2$; they represent non-perturbative exotic contributions which are the subject of the analysis in sections 4 and 5. In particular, in

JHEP05(2010)107

section 4 we show that the spectrum of moduli supported by D-instantons is such that they can affect the quadratic gauge couplings of the D7-branes, and in section 5 we compute these corrections by carrying out the integrations over the exotic moduli space by means of localization techniques analogous to those used for ordinary instanton calculus; the formulas are rather involved, but we have been able to get explicit results up to $k = 3$ D-instantons. Section 6 introduces the heterotic dual model and describes the computation of the 1-loop thresholds from which the holomorphic quadratic gauge couplings can be deduced. Upon using the duality map, we show that the type I′ and the heterotic results perfectly agree. We take this as a highly non-trivial test of the correctness of our D-instanton computation. A summary of our main findings and some considerations regarding possible developments can be found in the conclusive section 7. Finally, in the six appendices we have gathered many technical results needed to reproduce the computations in the main part of the paper.

2 A $\mathcal{N} = 2$ conformal model from an orbifold of type I′

We consider type IIB string theory compactified on a 6-torus $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)} \times \mathcal{T}_2^{(3)}$ and modded out by $\mathbb{Z}_2 \times \mathbb{Z}_2$ where the generators of the two $\mathbb{Z}_2$ groups are

$$\Omega' = \Omega\,(-1)^{F_L}\, I^{(3)} \quad \text{and} \quad \hat{g} = I^{(1)} I^{(2)}\,, \tag{2.1}$$

with Ω the word-sheet parity, F_L the space-time left-fermion number and $I^{(i)}$ the reflection along the coordinates of $\mathcal{T}_2^{(i)}$. This compactification preserves eight supercharges, i.e. $\mathcal{N} = 2$ supersymmetry in four dimensions.

Type IIB string theory compactified on $\mathcal{T}_2^{(3)}$ and modded out by Ω' is usually called type I′ and is dual to a torus compactification of the heterotic SO(32) string with Wilson lines breaking the gauge group to $\mathrm{SO}(8)^4$. For this set-up, the D-instanton corrections to the *quartic* gauge prepotential on D7-branes were computed in ref. [23] and checked against the dual heterotic results [36–39], finding perfect agreement. In this paper we consider instead a K3 compactification of the type I′ theory in the orbifold limit represented by $(\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by $\hat{g}$, and analyze the *quadratic* gauge couplings on stacks of D7-branes. The compactification of the unoriented string on a $\mathcal{T}_4/\mathbb{Z}_2$ orbifold was considered long ago in refs. [48, 49], and the global constraints imposed by the tadpole cancellation condition were solved in that case. Thus, upon compactification on $\mathcal{T}_2^{(3)}$, our present set-up can be seen as the T-dual version of that model, for which the quadratic gauge couplings on D9-branes were recently considered in ref. [53].

The action of Ω' selects 4 O7-planes, located at the invariant points of the torus $\mathcal{T}_2^{(3)}$ with respect to the $I^{(3)}$ reflection. These points are labeled by a 2-vector $\vec{\alpha}$ as indicated in figure 1.

Similarly, $\Omega'\hat{g}$ preserves 64 O3-planes, located at the fixed points of the inversions in all three tori $\mathcal{T}_2^{(i)}$ which we will denote by a 6-vector $\vec{\xi}$ (see figure 2).

The (dimensionless) volume $\mathcal{V}$ of the internal compactification manifold is given by

$$\mathcal{V} = T_2^{(1)} T_2^{(2)} T_2^{(3)}\,, \tag{2.2}$$

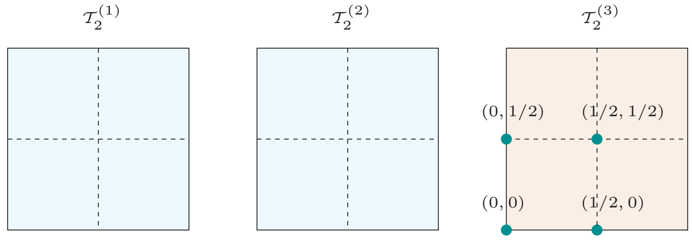

Figure 1. The location of the 4 O7-planes in $\mathcal{T}_2^{(3)}$ is identified by a 2-vector $\vec{\alpha}$ whose components can take the values 0 and 1/2, if the torus is parameterized with "flat" coordinates ranging from 0 to 1 (see appendix A for our notations and conventions).

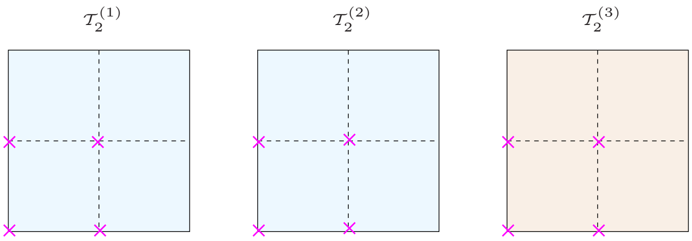

Figure 2. The location of the 64 O3-planes in $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)} \times \mathcal{T}_2^{(3)}$ is identified by a 6-vector $\vec{\xi}$ whose components again take values 0 or 1/2.

where $T_2^{(i)}$ is the Kähler modulus[1] of the torus $\mathcal{T}_2^{(i)}$, whose complex structure we denote by $U^{(i)}$. Since the third torus plays a distinguished rôle, in the following we will write simply T and U in place of $T^{(3)}$ and $U^{(3)}$. The low-energy effective super-gravity action for the above orientifold compactification is best expressed in terms of $U^{(i)}$ and of the complex fields $t^{(i)}$, whose imaginary parts $t_2^{(i)}$ are given by[2] [54, 55]

$$t_2^{(1)} = \mathrm{e}^{-\phi_{10}}\, T_2^{(2)} T_2\,, \quad t_2^{(2)} = \mathrm{e}^{-\phi_{10}}\, T_2^{(1)} T_2\,, \quad t_2 \equiv t_2^{(3)} = \mathrm{e}^{-\phi_{10}}\, T_2^{(1)} T_2^{(2)}\,, \tag{2.3}$$

where ϕ_{10} is the ten-dimensional dilaton.

The four-dimensional Planck mass $M_{\rm Pl}$, which represents the natural UV cut-off in the low-energy effective theory, is

$$M_{\rm Pl}^2 = \frac{1}{\alpha'}\,\mathrm{e}^{-2\phi_{10}}\,\mathcal{V} = \frac{1}{\alpha'}\,t_2\,\lambda_2\,T_2\,, \tag{2.4}$$

where

$$\lambda = C_0 + \mathrm{i}\mathrm{e}^{-\phi_{10}} \equiv \lambda_1 + \mathrm{i}\lambda_2 \tag{2.5}$$

is the usual axio-dilaton field. The tree-level bulk Kähler potential K of our theory can be written as

$$K = -\log\left(\lambda_2\right) - \sum_{i=1}^{3} \log\left(t_2^{(i)}\,U_2^{(i)}\right)\,. \tag{2.6}$$

---

[1] As usual, the Kähler moduli $T_2^{(i)}$ are complexified into $T^{(i)} = T_1^{(i)} + \mathrm{i}T_2^{(i)}$ by the B-field along the i-th torus; see appendix A for our conventions.

[2] The real parts $t_1^{(i)}$ are related, instead, to suitable RR potentials.

JHEP05(2010)107

As we will briefly recall in the next subsection, the cancellation of the RR tadpoles requires the presence of D7-branes transverse to $\mathcal{T}_2^{(3)}$ and of D3-branes transverse to the internal 6-torus, with a specific action of Ω', $\hat{g}$ and $\Omega'\hat{g}$ on their Chan-Paton (CP) factors. In this framework the modulus t_2 defined in (2.3) basically corresponds to the tree-level coupling of the gauge theory on the D7-branes, while $\lambda_2 = \mathrm{e}^{-\phi_{10}}$ describes the gauge coupling on the D3-branes. Notice that the orientifold projections (2.1) are compatible also with D-instantons and Euclidean E3-branes wrapped on $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}$, which must therefore be added to our model giving rise to non-perturbative corrections.

2.1 Tadpole cancellation constraints

Let us denote the number of D7-branes in each fixed point $\vec{\alpha}$ by $N_{\vec{\alpha}}$, and the number of D3-branes in each fixed point $\vec{\xi}$ by $M_{\vec{\xi}}$. Open string states connecting the various branes will be described by CP matrices with index range $N_{\vec{\alpha}}$ or $M_{\vec{\xi}}$ depending on whether the string ends on a D7- or on a D3-brane respectively. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ generators (2.1) act on these CP indices by means of unitary matrices γ. More precisely, we denote the $(N_{\vec{\alpha}} \times N_{\vec{\alpha}})$ matrix representing the generator $\hat{g}$ on the D7-branes at the fixed point $\vec{\alpha}$ by $\gamma_{\vec{\alpha}}(\hat{g})$, and use the same notation, *mutatis mutandis*, for the other orbifold generators and for the CP indices of the D3-branes. All these unitary matrices square to the identity since they represent $\mathbb{Z}_2$ generators, and thus for all of them we have

$$\gamma^{-1} = \gamma\,, \quad \gamma^* = \gamma^T\ . \tag{2.7}$$

Moreover, in any representation (both on the D7's and on the D3's), the group relations require that

$$\gamma(\Omega')\,\gamma(\hat{g}) = \gamma(\Omega'\hat{g})\ . \tag{2.8}$$

The tadpole constraints arise from the analysis of the IR divergences in the exchange channel of the Klein bottle amplitude and of the annuli and Möbius diagrams with boundaries on D7- and/or on D3-branes. In the RR sector such divergences signal the propagation of massless RR forms, and hence the presence of unphysical tadpoles that should be canceled globally for consistency. In our model (see appendix B.1 for details) this cancellation is achieved if, following ref. [49], we take the $\left(N_{\vec{\alpha}} \times N_{\vec{\alpha}}\right)$ matrices $\gamma_{\vec{\alpha}}$ to be of the form

$$\gamma_{\vec{\alpha}}(\Omega') = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\,, \quad \gamma_{\vec{\alpha}}(\hat{g}) = \gamma_{\vec{\alpha}}(\Omega'\hat{g}) = \begin{pmatrix} 0 & \mathrm{i}\,\mathbb{1} \\ -\mathrm{i}\,\mathbb{1} & 0 \end{pmatrix}\,, \tag{2.9}$$

the $\left(M_{\vec{\xi}} \times M_{\vec{\xi}}\right)$ matrices $\gamma_{\vec{\xi}}$ to be of the form

$$\gamma_{\vec{\xi}}(\Omega') = \gamma_{\vec{\xi}}(\hat{g}) = \begin{pmatrix} 0 & \mathrm{i}\,\mathbb{1} \\ -\mathrm{i}\,\mathbb{1} & 0 \end{pmatrix}\,, \quad \gamma_{\vec{\xi}}(\Omega'\hat{g}) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\,, \tag{2.10}$$

and then if we require that

$$\sum_{\vec{\alpha}} N_{\vec{\alpha}} = 32 \quad \text{and} \quad \sum_{\vec{\xi}} M_{\vec{\xi}} = 32\ . \tag{2.11}$$

When these conditions are satisfied, all RR tadpoles are canceled globally. However, it is possible also enforce a more stringent constraint and *locally* cancel the RR charge carried by each O7-plane if we require that

$$N_{\vec{\alpha}} = 8\,, \tag{2.12}$$

i.e. if we place exactly 4 dynamical[3] D7-branes on top of each O7-plane. Since there are 64 O3-planes but only 16 dynamical "half" D3-branes as indicated by (2.11), it is impossible to cancel the RR charge locally at each O3 location; however, we can at least cancel the O3-charge in the last torus by choosing

$$\sum_{\vec{\xi}_4} M_{\vec{\xi}} = 8 \tag{2.13}$$

with the sum running over all $\vec{\xi} = (\vec{\xi}_4, \vec{\xi}_2)$ for any fixed $\vec{\xi}_2$, i.e. over all O3-planes on top of the O7 specified by $\vec{\xi}_2$. This is the choice we make from now on. Thus, on each O7-plane we put 4 dynamical D7-branes and 4 "half" D3-branes. The latter can then be distributed over the 16 orbifold fixed points that are common to a given O7-plane, leading to different possibilities which will be briefly mentioned in the next subsection.

2.2 A conformal set-up

Let us focus on one of the O7-planes, say for example on the one at $\vec{\alpha} = (0,0)$, and on the 4 dynamical D7-branes located there. The latter support open string excitations whose CP factors are (8×8) matrices Λ subject to the following conditions

$$\gamma^*_{\vec{\alpha}}(\Omega')\,\Lambda^T\,\gamma^T_{\vec{\alpha}}(\Omega') = \varepsilon_{\Omega'}\,\Lambda\,, \qquad \gamma^*_{\vec{\alpha}}(\hat{g})\,\Lambda\,\gamma^T_{\vec{\alpha}}(\hat{g}) = \varepsilon_{\hat{g}}\,\Lambda\,, \tag{2.14}$$

where $\varepsilon_{\Omega'}$ and $\varepsilon_{\hat{g}}$ are the eigenvalues of Ω' and $\hat{g}$ on the oscillator part of the corresponding states, in such a way that these are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold. For instance, for the massless vector V_μ (represented by the state $\psi^\mu_{-\frac{1}{2}}|0\rangle$ with $\mu = 0, \ldots, 3$) and the massless complex scalar φ (represented by the state $\psi^{(3)}_{-\frac{1}{2}}|0\rangle$ along the torus $\mathcal{T}_2^{(3)}$), we have $\varepsilon_{\hat{g}} = -\varepsilon_{\Omega'} = 1$. On the other hand, for two massless complex scalars $h^{(1)}$ and $h^{(2)}$ along the directions of $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}$ (represented by the states $\psi^{(1)}_{-\frac{1}{2}}|0\rangle$ and $\psi^{(2)}_{-\frac{1}{2}}|0\rangle$) we have $\varepsilon_{\hat{g}} = \varepsilon_{\Omega'} = -1$. Then, using (2.9) the CP structure of the various massless fields selected by (2.14) turns out to be

$$V_\mu = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}, \quad \varphi = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}, \quad h^{(1)} = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}, \quad h^{(2)} = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix} \tag{2.15}$$

where A, A_1 and A_2 are (4×4) antisymmetric matrices, and S is a (4×4) symmetric matrix. We therefore see that the vector V_μ and the scalar φ are in the adjoint representation of

[3] Here we follow the same terminology introduced in ref. [33]. Therefore, when the D7-brane CP indices take $N_{\vec{\alpha}}$ values, we say that there are $N_{\vec{\alpha}}/2$ dynamical D7-branes since half of the CP indices can be regarded as images of the others under the orientifold parity Ω'. Likewise, when the D3-brane CP indices take $M_{\vec{\xi}}$ values, we say that there are $M_{\vec{\xi}}/2$ dynamical "half" D3-branes since a further half of the CP indices can be regarded as images under the orbifold parity $\hat{g}$.

U(4), embedded in SO(8), while the two scalars $h^{(1)}$ and $h^{(2)}$ are in the antisymmetric representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of U(4) plus its conjugate $\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$, again embedded in SO(8). Adding the corresponding fermions from the R sector, the massless spectrum of the 7/7 strings consists of one $\mathcal{N} = 2$ vector multiplet in the adjoint representation of U(4) schematically given by

$$\Phi(x, \theta) \sim \varphi(x) + \theta^2 F(x) + \text{ fermions} \tag{2.16}$$

where F is the gauge field-strength, one hyper-multiplet in the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation and one hyper-multiplet in the conjugate $\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ representation.

Now let us consider the 7/3 open strings stretching between D7- and D3-branes. In this case the massless excitations correspond to twisted states with mixed Neumann-Dirichlet boundary conditions along the directions of $\mathcal{T}_2^{(1)}$ and $\mathcal{T}_2^{(2)}$, and organize in 4 hyper-multiplets (one for each "half" D3-brane) transforming in the fundamental representation $\square$ of U(4). To see this, let us consider m "half" D3-branes located at a given orbifold fixed point $\vec{\xi}$. In order to survive the orbifold[4] projection, the CP factor Λ of the massless states of the 7/3 sector must satisfy the following constraint

$$\gamma^*_{\vec{\alpha}}(\hat{g})\,\Lambda\,\gamma^T_{\vec{\xi}}(\hat{g}) = \varepsilon_{\hat{g}}\,\Lambda \quad \text{with} \quad \varepsilon_{\hat{g}} = 1\,, \tag{2.17}$$

which, upon using (2.9) and (2.10), is solved by

$$\Lambda = \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \tag{2.18}$$

with X_1 and X_2 being generic $(4 \times m)$ matrices. Thus, these mixed states transform as m hyper-multiplets in the fundamental representation of U(4). In our model, of course, we have $m = 0$ for 12 fixed points and $m = 1$ for 4 fixed points contributing in total 4 hyper-multiplets. Nothing changes in this respect, if the "half" D3-branes are distributed differently among the various orbifold fixed points.

On the contrary, what changes according to the configuration of D3-branes is the theory on the world-volume of the latter. If the 4 D3-branes are all located at the same fixed point, we have a gauge theory with group U(4) and a matter content similar to the one discussed above for the D7-branes. If, instead, 3 D3-branes are located at one fixed point and and the fourth D3 is at a different one, we have a gauge theory with group U(3) × U(1), and so and so forth. The case in which the 4 D3-branes are all in 4 different fixed points, thus giving rise to a theory with a U(1)^4 symmetry, is of particular interest since it is this configuration which admits a simple perturbative heterotic dual. Thus, from now on we will restrict our analysis to this case only. The theory we consider is therefore the one living on the 4 D7-branes on top of one of the orientifold O7-planes, with the 4 D3-branes placed at four different orbifold fixed points, as shown for example in figure 3.

The gauge group is U(4) × U(1)^4, with the latter factors representing flavor symmetries from the point of view of the theory on the D7-branes. The massless content of this $\mathcal{N} = 2$

[4]The orientifold projection Ω' does not impose any restriction but only identifies states of the 7/3 sector with states of opposite orientation belonging to the 3/7 sector.

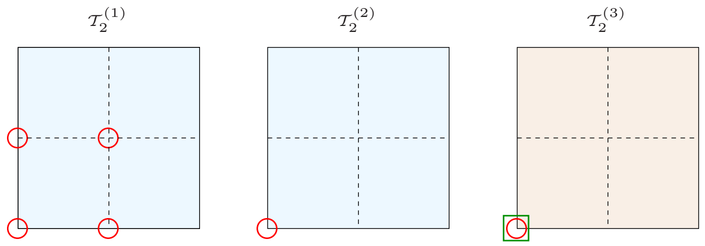

Figure 3. Brane locations in our model. The square denotes the orientifold fixed point $\vec{\alpha} = (0, 0)$ where the 4 D7-branes are located, while the circles denote the positions of the 4 "half" D3-branes.

$\mathcal{N} = 2$ rep.	sector	$\varepsilon_{\Omega'}$	$\varepsilon_{\hat{g}}$	CP factor	#	U(4)	$q_{\mathrm{U}(1)}$
vector	7/7	$-$	$+$	$\begin{pmatrix} A & S \\ -S & A \end{pmatrix}$	1	adj.	0
hyper	7/7	$-$	$-$	$\begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}$	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	-2
hyper	7/7	$-$	$-$	$\begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}$	1	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$+2$
hyper	7/3	undef.	$+$	$\begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix}$	4	$\square$	-1

Table 1. Massless spectrum on the world-volume of the 4 D7-branes at one of the orientifold fixed points of our model.

model is summarized in table 1, where in the last column we have indicated also the $\mathrm{U}(1)_{D7}$-charge of the various multiplets. Notice that while the adjoint fields are clearly neutral, the charge of 7/7 hyper-multiplets is, in absolute value, twice the charge of the 7/3 hyper-multiplets. This fact can be easily understood, since the 7/7 fields correspond to open strings with two charged endpoints on the D7-branes, as opposed to the 7/3 fields which have only one charged endpoint on the D7-branes.

It is not difficult to check that this model is conformal. Indeed, the 1-loop β-function coefficient for a $\mathcal{N} = 2$ theory with gauge group G is given by

$$b = 2\Big[T(G) - \sum_r n_r T(r)\Big] , \tag{2.19}$$

where the index $T(r)$ of a representation r of G is defined by

$$T(r)\,\delta_{AB} = \mathrm{tr}\,(T_A(r)T_B(r)) , \tag{2.20}$$

$T(G)$ stands for $T(\mathrm{adj})$ and n_r is the number of hyper-multiplets transforming in the

JHEP05(2010)107

representation r. In our case (see table 1) we have $G = \mathrm{U}(N)$ with $N = 4$, and

$$n_{\ydiagram{1,1}} = n_{\overline{\ydiagram{1,1}}} = 1\,, \quad n_{\Box} = 4\,. \tag{2.21}$$

Since the relevant indices are[5]

$$T(G) = N = 4\,, \quad T(\ydiagram{1,1}) = T(\overline{\ydiagram{1,1}}) = \frac{N-2}{2} = 1\,, \quad T(\Box) = \frac{1}{2}\,, \tag{2.22}$$

the β-function coefficient vanishes:

$$b = 4 - n_{\Box} = 0\,. \tag{2.23}$$

As we will see in the next section, the quadratic effective action for the D7 gauge fields contains also a double-trace structure $(\mathrm{tr}\, F)^2$, which clearly arises only in the U(1) sector and renormalizes separately from the Yang-Mills term $\mathrm{tr}\, F^2$. The 1-loop β-function coefficient for the double-trace coupling, which we denote as b', can be deduced from the coefficient of the β-function for the U(1) factor of the gauge group, which in turn is computed from the charges of the various multiplets. Such abelian β-function coefficient is given by

$$\beta_{\mathrm{U}(1)} = -\sum_r n_r\, q_r^2\, d(r) \tag{2.24}$$

where q_r is the U(1) charge of the hyper-multiplet in the representation r whose dimension is $d(r)$. Inserting in this expression the U(1) charges and multiplicities given in table 1, one finds

$$\beta_{\mathrm{U}(1)} = \left(4 - n_{\Box}\right) N - 4N^2 = \left(4 - n_{\Box}\right) \mathrm{tr}\, \mathbb{1} - 4\left(\mathrm{tr}\, \mathbb{1}\right)^2\,. \tag{2.25}$$

Our specific model is precisely of this type, with $N = 4$ and $n_{\Box} = 4$. Thus, from (2.25) we easily deduce that the U(1) contribution to the β-function of the single-trace term is the same as the non-abelian one (2.23) and vanishes in our model, while the contribution b' to the double-trace term is

$$b' = -4\,. \tag{2.26}$$

This concludes our analysis of the properties dictated by the (massless) spectrum of string excitations on the D7 branes. In the next section we turn to the structure of the interaction terms in the low-energy effective action, starting from the perturbative contributions.

3 Type I′ gauge effective action: perturbative part

The tree-level action for the $\mathcal{N} = 2$ Super Yang-Mills theory discussed in the previous section can be obtained by computing disk scattering amplitudes among the various massless excitations of the open strings with at least one end-point on the D7-branes and then

[5] The following formulas actually refer to the SU(N) part of the gauge group. Later we will consider also the U(1) factor.

taking the field theory limit $\alpha' \to 0$. Alternatively, at least for the pure Yang-Mills part we can consider the Dirac-Born-Infeld (DBI) action for a D7-brane, namely

$$S_{\rm DBI} = \frac{2\pi}{(2\pi\sqrt{\alpha'})^8} \int d^8x \; {\rm e}^{-\phi_{10}} \sqrt{\left(G_{MN} + 2\pi\alpha' F_{MN}\right)} \tag{3.1}$$

with G_{MN} being the world-volume metric and F_{MN} the gauge field-strength ($M, N = 0, \ldots, 7$), and then compactify it to four dimensions on $\left(\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}\right)/\mathbb{Z}_2$. In this way, after promoting the field-strength to be non-abelian and rescaling the four-dimensional metric to the flat one, we obtain, at the quadratic level,

$$S_{\rm tree} = \frac{1}{2g^2} \int d^4x \; {\rm tr}\,(F_{\mu\nu}^2)\,, \tag{3.2}$$

where $\mu, \nu = 0, \ldots, 3$, and the Yang-Mills coupling constant g is given by

$$\frac{4\pi}{g^2} = t_2 \tag{3.3}$$

with t_2 defined in eq. (2.3). Now we turn to the 1-loop terms.

3.1 1-loop contributions

To derive the 1-loop threshold corrections to the quadratic gauge couplings on the D7-branes we use the background field method. The 1-loop amplitudes are extracted from the second derivatives of the weigthed partition function of open strings with at least one endpoint on the D7-branes in presence of a constant magnetic field on the D7-brane world-volume (see ref. [56] for previous studies of F^2-amplitudes in $\mathcal{N} = 2$ brane set-ups).

For concreteness we switch on a magnetic field $\mathcal{H}$ along, say, the directions 2 and 3, i.e.

$$F_{23} = -F_{32} = \mathcal{H}\,, \quad F_{\mu\nu} = 0 \;\; \text{for} \;\; \mu, \nu \neq 2, 3\,, \tag{3.4}$$

and taking value only in the Cartan directions of the gauge group. Furthermore, we suppose again to have N dynamical D7-branes and set $N = 4$ in the end. As discussed in section 2.2, in the real basis for the CP indices the adjoint of $\mathrm{U}(N)$ is embedded into $\mathrm{SO}(2N)$, so that Cartan subalgebra of $\mathrm{U}(N)$ is represented by skew-diagonal matrices. However, with a complex change of basis we diagonalize them and hence bring our Cartan magnetization in the form

$$\frac{\rm i}{2\pi\alpha'}\,{\rm diag}\big(h_1, h_2, \ldots, h_N, -h_1, -h_2, \ldots, -h_N\big) \equiv \frac{\rm i}{2\pi\alpha'}\,{\rm diag}\big(h_i\big)\;. \tag{3.5}$$

Here we have introduced the index $i = 1, \ldots, 2N$ running over all D7-brane labels; the fundamental and antifundamental indices of $\mathrm{U}(N)$, taking values $1, \ldots, N$, will be denoted instead by I and $\bar{I}$ respectively. Thus, we have $i = I$ for $i = 1, \ldots, N$ and $i = \bar{I} + N$ for $i = N + 1, \ldots, 2N$, so that $h_{\bar{I}} = -h_I$. The $\mathrm{U}(N)$ field strength $\mathcal{H}$ corresponds to the $(N \times N)$ upper block in (3.5), namely

$$\mathcal{H} = \frac{\rm i}{2\pi\alpha'}\,{\rm diag}\big(h_1, h_2, \ldots, h_N\big) \equiv \frac{\rm i}{2\pi\alpha'}\,{\rm diag}\big(h_I\big)\;. \tag{3.6}$$

In presence of the magnetization (3.5), a 7/7 open string stretching between the i-th and j-th D7-brane is twisted by an angle

$$\nu_{ij} = -\frac{1}{\pi}\left(\arctan h_i - \arctan h_j\right) \sim -\frac{h_i - h_j}{\pi} + O(h^3)\,, \tag{3.7}$$

and the spectrum of physical excitations changes correspondingly.

The 1-loop effective action of the D7-branes can be deduced from the 1-loop vacuum energy in the background (3.5). For the 7/7 open strings this vacuum energy has the following schematic form

$$\int_0^\infty \frac{d\tau_2}{2\tau_2} \sum_{i,j} \mathrm{Tr}_{\,(h_i,h_j)}\left(\frac{1+\Omega'}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,q^{L_0-\frac{c}{24}}\right) = \mathcal{A}_{7/7}(h) + \mathcal{M}_{7/7}(h) \tag{3.8}$$

where $\frac{1+(-1)^F}{2}$ is the GSO projector, $q = \mathrm{e}^{-\pi\tau_2}$, and the trace $\mathrm{Tr}_{\,(h_i,h_j)}$ is computed over the spectrum of 7/7 open strings with boundary conditions determined by the values (h_i, h_j). In the right hand side of (3.8) we have distinguished, as usual, an annulus contribution $\mathcal{A}_{7/7}(h)$ and a Möbius strip contribution $\mathcal{M}_{7/7}(h)$, which is non-vanishing only if $h_j = -h_i$ due to the presence of Ω' inside the trace [22].

Our model contains also $m = 4$ "half" D3-branes at the same fixed point of $\mathcal{T}_2^{(3)}$ of the D7-branes, so that also the D7/D3 strings can have massless modes contributing to the low-energy effective action. At 1-loop we should therefore take into account also annuli with one boundary on a magnetized D7-brane and the other on one of the D3-branes, corresponding to the amplitude

$$\mathcal{A}_{7/3}(h) + \mathcal{A}_{3/7}(h) = \int_0^\infty \frac{d\tau_2}{2\tau_2} \sum_{i,a} \mathrm{Tr}_{\,(h_i,a)}\left(\frac{1+g}{2}\,\frac{1+(-1)^F}{2}\,q^{L_0-\frac{c}{24}}\right) \tag{3.9}$$

where a labels the CP indices of the D3-branes, taking $2m$ values.

The amplitudes (3.8) and (3.9) are computed in appendix B.2 and the result is given in eqs. (B.37), (B.39) and (B.41). All in all, the total 1-loop effective action turns out to be

$$\begin{aligned} S_{1-\mathrm{loop}} =&\; \mathcal{A}_{7/7}(h) + \mathcal{M}_{7/7}(h) + \mathcal{A}_{7/3}(h) + \mathcal{A}_{3/7}(h) \\ =&\; -\frac{V_4}{8\pi^2}\Big[(4-m)\,\mathrm{tr}\,\mathcal{H}^2 - 4\,(\mathrm{tr}\,\mathcal{H})^2\Big] \int_0^\infty \frac{d\tau_2}{2\tau_2}\, W(\tau_2) + O(h^3)\,. \end{aligned} \tag{3.10}$$

with

$$W(\tau_2) = \sum_{\vec{w}\in\mathbb{Z}^2} \mathrm{e}^{-\pi\tau_2\,\frac{|w^1+w^2U|^2 T_2}{U_2}} \tag{3.11}$$

representing the sum over winding states on $\mathcal{T}_2^{(3)}$ (see eq. (A.25)). Notice that the expression in square brackets has the same structure appearing in eq. (2.25), and that the coefficient of the single-trace term is the correct β-function coefficient for this model (see eq. (2.23)), since the number m of D3-branes equals the number $n_{\square}$ of fundamental hypermultiplets. In our conformal case, i.e. $m = n_{\square} = 4$, there is no running for the single-trace coupling, but there is a non-vanishing 1-loop contribution proportional to $(\mathrm{tr}\,\mathcal{H})^2$. Promoting $\mathcal{H}$ to a full dynamical field $F_{\mu\nu}$, this contribution in the end reads

$$S_{1-\mathrm{loop}} = \frac{1}{8\pi^2}\int d^4x\,(\mathrm{tr}\,F)^2\int_0^\infty \frac{d\tau_2}{\tau_2}\,W(\tau_2) + O(F^3)\ . \tag{3.12}$$

It is important to stress that this 1-loop action is entirely due to zero-mode states wrapping around $\mathcal{T}_2^{(3)}$ and giving rise to the winding sum $W(\tau_2$. The contributions of the massive string states, instead, exactly cancel as a consequence of the fact that in $\mathcal{N}=2$ theories the F^2-terms are "BPS saturated" quantities (see e.g. ref. [56] for an extension of this result to more general brane setups). This property makes the quadratic gauge couplings reliable variables to follow under the non-perturbative type I′/heterotic duality.

The integral over the modular parameter τ_2 in (3.12) can be evaluated following the methods of ref. [50], as reviewed for example in the appendix of ref. [57], and, after regularizing the divergences, the result (up to moduli independent constants) is

$$\begin{aligned}\int_0^\infty \frac{d\tau_2}{\tau_2}\,W(\tau_2) &= -\log(\alpha'\mu^2) - \log\left(\frac{U_2\,|\eta(U)|^4}{T_2}\right)\\ &= -\log\left(\frac{\mu^2 t_2}{M_{\mathrm{Pl}}^2}\right) - \log\left(\lambda_2\,U_2\,|\eta(U)|^4\right)\end{aligned} \tag{3.13}$$

where in the second step we introduced as UV cut-off the four-dimensional Planck mass M_{Pl} (2.4). Thus, the 1-loop action (3.12) becomes

$$S_{1-\mathrm{loop}} = \frac{1}{32\pi^2}\int d^4x\,(\mathrm{tr}\,F)^2\left[-4\log\left(\frac{\mu^2 t_2}{M_{\mathrm{Pl}}^2}\right) - 4\log\left(\lambda_2\,U_2\,|\eta(U)|^4\right)\right] + O(F^3)\ . \tag{3.14}$$

From this explicit result, we can read the β-function coefficients and the 1-loop threshold corrections to the gauge couplings, as explained in appendix E (see in particular eq. (E.1)). The absence of single-trace quadratic terms implies that $b=0$ in agreement with eq. (2.23), and

$$\Delta = 0\ . \tag{3.15}$$

On the other hand, for the double-trace structure we see that $b'=-4$ in agreement with eq. (2.26). The 1-loop threshold follows then from eq. (E.1) and reads

$$\Delta' = -4\log\left(\lambda_2\,U_2\,|\eta(U)|^4\right)\ . \tag{3.16}$$

Notice that this threshold is invariant under the target-space modular trasformations acting on U, but it is not invariant under the $\mathrm{Sl}(2,\mathbb{Z})$ transformations of the axio-dilaton λ. This lack of invariance signals the necessity of non-perturbative corrections which, as we will show in sections 4 and 5, are induced by D-instantons.

3.2 Holomorphic gauge couplings

The moduli dependence of string loop corrections to the gauge kinetic terms of a supersymmetric effective quantum field theory like ours is best described in terms of *holomorphic* couplings, as explained in refs. [50–52] and briefly reviewed in appendix E. These Wilsonian functions, in general, have the following structure

$$f = f_{(0)} + \frac{1}{4\pi} f_{(1)} + f_{\text{n.p.}} \tag{3.17}$$

where the subscripts $_{(0)}$ and $_{(1)}$ refer, respectively, to the tree-level and 1-loop contributions, while the last term accounts for possibile non perturbative corrections.

In our specific theory, there two such functions: one for the usual single-trace Yang-Mills term $\operatorname{tr} F^2$, which we will denote by f, and one for the double-trace term $(\operatorname{tr} F)^2$, called f' in the following. At tree level we have

$$f_{(0)} = -\mathrm{i}\, t\,, \quad f'_{(0)} = 0\,, \tag{3.18}$$

as one can see from eq. (3.3) and the fact that no double-trace term is present in the tree-level action (3.2). The 1-loop contributions, instead, can be obtained from the formulas (see also eq. (E.8))

$$\operatorname{Re} f_{(1)} = \Delta + \Delta_{\text{univ}} + b\,\widehat{K}\,, \quad \operatorname{Re} f'_{(1)} = \Delta' + \Delta_{\text{univ}} + b'\,\widehat{K}\,, \tag{3.19}$$

where

$$\widehat{K} = -\log\left(\lambda_2\, U_2\right)\,. \tag{3.20}$$

is related to the Kähler metric of the adjoint scalar fields, while Δ_{univ} is a universal effect due to the mixing of the dilaton with the compactification moduli [50, 51]. Since both Δ, Δ', Δ_{univ} and $\widehat{K}$ contain non-holomorphic terms, the relations (3.19) imply that all such terms should compensate each other for consistency to yield holomorphic expressions for $f_{(1)}$ and $f'_{(1)}$. In type II or type I theories (as opposed to heterotic models) the universal correction Δ_{univ} is actually of $O(g_s)$, g_s being the string coupling, and thus it does not contribute to the coupling functions at 1-loop. This can be seen for example from the explicit calculation of the corrections to the Kähler potential performed in ref. [58]. This same observation was used in ref. [59] to obtain the Kähler metrics of twisted matter fields from instantonic annulus diagrams in agreement with the explicit perturbative string derivation presented in ref. [60]. Thus, for our purposes here, we can drop the Δ_{univ} term from the various formulas. Recalling the expression (3.20) for $\widehat{K}$ and using the results for b, b', Δ and Δ' obtained in the previous subsection, from eq. (3.19) we finally get

$$\operatorname{Re} f_{(1)} = 0\,, \quad \operatorname{Re} f'_{(1)} = -4\log\left(|\eta(U)|^4\right)\,, \tag{3.21}$$

in agreement with the holomorphy requirements. In the next sections we will study the non-perturbative corrections $f_{\text{n.p.}}$ and $f'_{\text{n.p.}}$ to the coupling functions induced by D-instantons.

4 D-instantons and their moduli spectrum

We now discuss the effects of instantonic branes on the system described so far. There are two types of branes that are point-like with respect to the four-dimensional uncompact space and can be put on the O7-planes in a supersymmetric fashion, namely

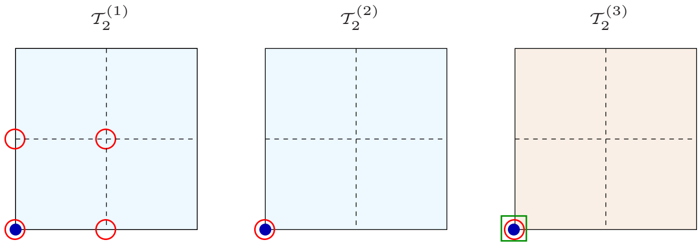

Figure 4. A possible arrangement of the D(–1)/D3/D7 system in case *a)*. The empty square denotes the fixed point $\vec{\alpha}$ of $\mathcal{T}_2^{(3)}$ occupied by the D7-branes, the empty circles the four fixed points $\vec{\xi}$ in $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)} \times \mathcal{T}_2^{(3)}$ occupied by the D3's and the filled circle the one among these latter where k D(–1)'s are positioned. There are four inequivalent possibilities for the D(–1) location.

- extended Euclidean 3-branes (or E3-branes) wrapping $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}$;
- point-like D(–1)-branes that are completely localized in all directions.

The E3-branes represent ordinary gauge instantons for the field theory living on the D7-branes; indeed in the E3/D7 system there are precisely four directions with mixed Dirichlet-Neumann boundary conditions and the spectrum of the physical excitations of the open strings with at least one endpoint on the Euclidean branes is in full agreement with that of the ADHM construction for gauge instantons [27, 28, 32]. On the other hand the D(–1)-branes describe truly stringy (or exotic) instanton configurations for the D7-brane gauge theory [22]. In fact, in this case between the instantonic and the space-filling branes there are eight directions with mixed boundary conditions, and the corresponding spectrum of moduli is not the conventional one.

In this paper we only discuss the contributions produced by the D(–1)-branes, leaving the study of the E3-branes to a future work. In particular we will show that fractional D-instantons located at orbifold fixed points have the right content of zero-modes to correct non-perturbatively the gauge kinetic function of the $\mathcal{N} = 2$ U(4) theory discussed in section 3, and later will check the result against the dual heterotic string calculation. Again we focus on the four D7-branes located at one of the orientifold fixed points, and place on them a number of fractional D-instantons. However, since there are also four D3-branes distributed in four different orbifold fixed points, we have to distinguish between two possibilities, depending on whether the D-instantons are at the same position of one of the D3-branes or are at an empty fixed point.

In the first case (case *a)* in the following), schematically represented in figure 4, there is one orbifold fixed point, say $\vec{\xi}$, occupied both by the D(–1)'s and by one D3; therefore we can find massless excitations not only in the spectrum of the $(-1)/(-1)$ and $(-1)/7$ open strings, but also in that of the $(-1)/3$ strings stretching between the D-instantons and the D3-brane located in that point. Since there are four different D3-branes, this situation can be realized in four different but completely equivalent ways.

In the second case (case *b)* in the following), represented in figure 5, only the $(-1)/(-1)$ and $(-1)/7$ open strings can support massless moduli because the $(-1)/3$ strings have always a non-vanishing stretching energy due to the non-zero space separation between their

JHEP05(2010)107

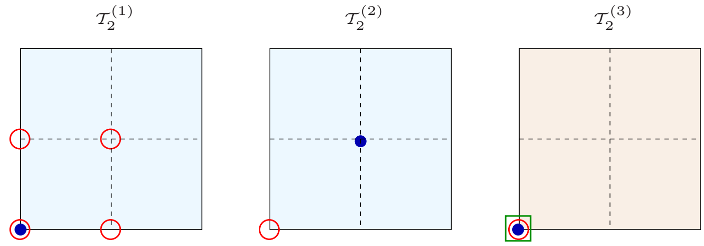

Figure 5. A possible arrangement of the D(−1)/D3/D7 system, case *b)*. This time the D(−1)'s occupy a fixed point where no D3's are sitting. There are twelve inequivalent possibilities.

endpoints. Since in our model there are twelve orbifold fixed points that are not occupied by D3-branes, this case can be realized in twelve different but completely equivalent ways.

In order to select which moduli survive the orientifold and orbifold projections, it is necessary to specify how the discrete parities Ω' and $\hat{g}$ act on the CP indices of the D-instantons. Extending the consistency arguments of ref. [49] to our case, we can show that this action can be represented by matrices $\Gamma_{\vec{\xi}}$ having the same form as the matrices $\gamma_{\vec{\alpha}}$ acting on the CP indices of the D7-branes and introduced in eq. (2.9), namely

$$\Gamma_{\vec{\xi}}(\Omega') = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} \,, \quad \Gamma_{\vec{\xi}}(\hat{g}) = \Gamma_{\vec{\xi}}(\Omega'\hat{g}) = \begin{pmatrix} 0 & \mathrm{i}\,1\!\!1 \\ -\mathrm{i}\,1\!\!1 & 0 \end{pmatrix} \,. \tag{4.1}$$

Clearly this implies that the number of CP indices for the D(−1)'s must be an even integer, say $2k$, so that the various blocks in (4.1) are $(k \times k)$ matrices. Adopting the same terminoloy used for the D7- and the D3-branes, we say that this case corresponds to having k "half" D-instantons. Let us also note that the physical moduli organize in representations of the Lorentz symmetry group, which in our local system is broken to

$$\mathrm{SO}(4) \times \widehat{\mathrm{SO}}(4) \times \mathrm{SO}(2) = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \times \widehat{\mathrm{SU}}(2)_+ \times \widehat{\mathrm{SU}}(2)_- \times \mathrm{SO}(2) \tag{4.2}$$

by the presence of O7/O3-planes and D7/D3-branes. In the following, we will refer collectively to the SU(2) factors above as $\mathrm{SU}(2)^4$. The subscript $\pm$ refers to the fact that the irreducible factors $\mathrm{SU}(2)_\pm$ inside SO(4) rotate (anti)-self-dual tensors.

Let us now give some details.

(−1)/(−1) strings: this is the neutral sector since it comprises states that do not transform under the U(4) gauge group. A generic modulus in this sector has a Chan-Paton matrix structure Λ which must fulfill the invariance conditions

$$\Gamma^*_{\vec{\xi}}(\Omega')\,\Lambda^T\,\Gamma^T_{\vec{\xi}}(\Omega') = \varepsilon_{\Omega'}\,\Lambda \quad \text{and} \quad \Gamma^*_{\vec{\xi}}(\hat{g})\,\Lambda\,\Gamma^T_{\vec{\xi}}(\hat{g}) = \varepsilon_{\hat{g}}\Lambda\,, \tag{4.3}$$

where $\varepsilon_{\Omega'}$ and $\varepsilon_{\hat{g}}$ are the eigenvalues of Ω' and $\hat{g}$ on the oscillator part of the corresponding state.

The physical zero-modes are easily obtained by dimensionally reducing the $\mathcal{N} = 1$ supersymmetric gauge theory from ten to zero dimensions. In the NS sector, the ten real bosonic moduli split in different sets according to their transformation properties under the Lorentz group (4.2). Adopting an ADHM inspired notation, we label them as follows. There are two complex scalars B_ℓ ($\ell = 1, 2$) associated to the four real string states $\psi^\mu_{-1/2}\,|0\rangle$ along the space-time directions; they transform as a vector of SO(4), i.e., in the $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$ of $\mathrm{SU}(2)^4$ and have $\varepsilon_{\Omega'} = \varepsilon_g = 1$. Two complex scalars $B_{\dot\ell}$ ($\dot\ell = 3, 4$) are associated to $\psi^{(1)}_{-1/2}\,|0\rangle$ and $\psi^{(2)}_{-1/2}\,|0\rangle$ along the directions of $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}$; they transform in the $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$ and have $\varepsilon_{\Omega'} = 1$, $\varepsilon_{\hat g} = -1$. Finally, there is one complex scalar χ associated to the string excitation $\psi^{(3)}_{-1/2}\,|0\rangle$ along $\mathcal{T}_2^{(3)}$, which is a vector of SO(2) and a singlet of $\mathrm{SU}(2)^4$, and thus it has $\varepsilon_{\Omega'} = -1$, $\varepsilon_{\hat g} = 1$ since Ω' contains the reflection in the $\mathcal{T}_2^{(3)}$ directions.

All these bosonic fields are matrices, and must satisfy the invariance constraint (4.3) with Λ replaced in turn by B_ℓ, $B_{\dot\ell}$ and χ, with the values of $\varepsilon_{\Omega'}$ and $\varepsilon_{\hat g}$ specified above and listed in table 2. Using the explicit form (4.1) for the matrices $\Gamma_{\vec\xi}$, we see that this requires that

$$B_\ell = \begin{pmatrix} S & A \\ -A & S \end{pmatrix}\,, \quad B_{\dot\ell} = \begin{pmatrix} S_1 & S_2 \\ S_2 & -S_1 \end{pmatrix}\,, \quad \chi = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}\,, \tag{4.4}$$

where S, S_1 and S_2 are symmetric $(k \times k)$ matrices, and A is an anti-symmetric $(k \times k)$ matrix. Thus the scalars B_ℓ and χ transform in the adjoint representation of $\mathrm{U}(k)$ (the first embedded in the symmetric representation of $\mathrm{SO}(2k)$, the second in the anti-symmetric one), while the scalars $B_{\dot\ell}$ transform in the symmetric representation $\Box\!\Box$ of $\mathrm{U}(k)$ plus its conjugate $\overline{\Box\!\Box}$.

A similar analysis can be performed in the R sector of the $(-1)/(-1)$ strings. Here we have sixteen fermionic moduli which we can group into four sets $M_{\dot\alpha a}$, $M_{\alpha\dot a}$, $N_{\alpha a}$ and $N_{\dot\alpha\dot a}$ with $\alpha, \dot\alpha, a, \dot a$ labelling the spinor representations of the four SU(2)'s. We have denoted by M's and N's the components with positive and negative SO(2) chiralities, respectively, that correspond to eigenvalues plus and minus under Ω'. On the other hand, under $\hat g$ all modes carrying an index $\dot a$ pick up a minus sign. The resulting $\varepsilon_{\Omega'}$ and $\varepsilon_{\hat g}$ eigenvalues have to be inserted into the constraint (4.3) and determine the form of the CP matrices. We notice that $M_{\dot\alpha a}$, $M_{\alpha\dot a}$ have the same eigenvalues as B_ℓ and $B_{\dot\ell}$, and therefore they share the form of their CP factors. The other two sets of fermions satisfy again (4.3) with polarizations of the form

$$N_{\dot\alpha\dot a} = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}\,, \quad N_{\alpha a} = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}\,, \tag{4.5}$$

where A, A_1 and A_2 are anti-symmetric $(k \times k)$ matrices, and S is a symmetric $(k \times k)$ matrix. Thus, $N_{\dot\alpha\dot a}$ transform in the anti-symmetric representation $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ of $\mathrm{U}(k)$ plus its conjugate $\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}$, while $N_{\alpha a}$ transform in the adjoint of $\mathrm{U}(k)$.

For later convenience we have summarized the above properties of the neutral moduli in table 2, where in the last column we have indicated also their scaling length dimensions.

It is useful to remark that the components of the moduli B_ℓ and $M_{\dot\alpha a}$ along the identity play a distinguished rôle in the computation of D-instanton induced interactions; in fact they do not interact with other moduli and correspond to the 4-dimensional super-coordinates x and θ [31, 32]. Due to their Chan-Paton structure, the moduli $B_{\dot\ell}$ and $M_{\dot\alpha a}$,

moduli	$\mathrm{SU}(2)^4$	$\varepsilon_{\Omega'}$	$\varepsilon_{\hat{g}}$	CP factor	$\mathrm{U}(k)$	dimensions
B_ℓ $M_{\dot\alpha a}$	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ $(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})$	$+$	$+$	$\begin{pmatrix} S & A \\ -A & S \end{pmatrix}$	adjoint	L^1 $L^{1/2}$
$B_{\dot\ell}$ $M_{\alpha\dot a}$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$ $(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})$	$+$	$-$	$\begin{pmatrix} S_1 & S_2 \\ S_2 & -S_1 \end{pmatrix}$	$\square\!\square + \overline{\square\!\square}$	L^1 $L^{1/2}$
$N_{\dot\alpha\dot a}$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$	$-$	$-$	$\begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix} + \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$L^{-3/2}$
$N_{\alpha a}$ $\bar\chi$	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$ $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-$	$+$	$\begin{pmatrix} A & S \\ -S & A \end{pmatrix}$	adjoint	$L^{-3/2}$ L^{-1}

Table 2. The spectrum of neutral moduli aring from $(-1)/(-1)$ open strings stretching between two D-instantons.

instead, have no trace part; their components $\hat{x}$ and $\hat{\theta}$ along σ_3 (in block-diagonal terms) do appear in the moduli action, and thus cannot generically play the rôle of supercoordinates in the internal directions along the orbifold. We will however see later that effectively they behave as such when we compute certain types of instanton-induced interactions.

$(-1)/7$ strings: this is the charged sector that accounts for open strings stretching between the k D-instantons and the four D7-branes, whose CP factors are $(2k\times 8)$ matrices. Since there are eight directions with mixed Dirichlet-Neumann boundary conditions, in the NS sector it is not possible to construct vertex operators of conformal weight one, and thus there are no physical bosonic moduli in the spectrum. On the other hand, in the R sector we do find physical moduli. These are fermionic scalars μ' that, in order to survive the orbifold projection, must satisfy the following relation

$$\Gamma^*_{\vec{\xi}}(\hat{g})\,\mu'\,\gamma^T_{\vec{\alpha}}(\hat{g}) = \varepsilon_{\hat{g}}\mu' \quad \text{with} \quad \varepsilon_{\hat{g}} = +1\ . \tag{4.6}$$

We do not get any further condition by applying the orientifold parity Ω', since it exchanges the string orientation and thus leads to suitable identifications between states of the $(-1)/7$ sector with those of the $7/(-1)$ one. Recalling the explicit form (4.1) and (2.9) of the matrices $\Gamma_{\vec{\xi}}(\hat{g})$ and $\gamma_{\vec{\alpha}}(\hat{g})$, we can easily see that the above constraint implies that

$$\mu' = \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \tag{4.7}$$

where X_1 and X_2 are generic $(k\times 4)$ matrices. Thus, these fermionic moduli organize into a complex scalar transforming in the fundamental/anti-fundamental representation $(\square,\overline{\square})$ of the symmetry group $\mathrm{U}(k)\times\mathrm{U}(4)$. Their properties are summarized in table 3.

JHEP05(2010)107

moduli	$\mathrm{SU}(2)^4$	$\varepsilon_{\hat{g}}$	CP factor	$\mathrm{U}(k)\times\mathrm{U}(4)$	dimensions
μ'	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$+$	$\begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix}$	$(\square,\overline{\square})$	$L^{1/2}$

Table 3. The spectrum of charged moduli arising from $(-1)/7$ open strings stretching between k D-instantons and four D7-branes.

(−1)/3 strings: let us finally consider the flavored sector of the instanton moduli space which arises from the open strings connecting the D-instantons with the half D3-branes. As we have explained at the beginning of this section, this sector exists only when the D(−1)'s and the D3's occupy the same fixed point. In our model, this happens in case *a)* considered in figure 4, with just one half D3-brane at the fixed point of the D(–1)'s. The CP factors of the $(-1)/3$ moduli are then $(2k\times 2)$ matrices transforming in some representation of $\mathrm{U}(k)\times\mathrm{U}(1)$. It will be useful in the following to consider the generalized case with m half D3-branes suppporting a $\mathrm{U}(m)$ symmetry, with $2k\times 2m$ CP factors; the configuration *a)* corresponds to $m=1$. In the case *b)* represented in figure 5, the D3/D(–1) moduli are absent and we may say that this case corresponds to $m=0$. As usual in D(−1)/D3 systems, in the NS sector one finds two complex variables w_α which transform as a chiral spinor with respect to the SO(4) acting on the ND directions, namely belong to the $(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$ of $\mathrm{SU}(2)^4$ in our language. In order to survive the orbifold projection, they must satisfy the following constraint:

$$\Gamma^*_{\vec{\xi}}(\hat{g})\, w_\alpha\, \gamma^T_{\vec{\xi}}(\hat{g}) = \varepsilon_{\hat{g}}\, w_\alpha \quad \text{with} \quad \varepsilon_{\hat{g}} = +1\ . \tag{4.8}$$

Recalling eqs. (2.10) and (4.1), one can easily conclude that

$$w_\alpha = \begin{pmatrix} Y_1 & Y_2 \\ -Y_2 & Y_1 \end{pmatrix}\ . \tag{4.9}$$

From this we deduce that the moduli w_α transform in the fundamental representation under $\mathrm{U}(k)$ and the anti-fundamental under $\mathrm{U}(m)$.

In the R sector we find eight fermionic moduli which can be organized in two spinors, μ_a and $\mu_{\dot{a}}$, of of opposite chiralities with respect to the internal $\widehat{\mathrm{SO}}(4)$. In particular, μ_a transforms in the $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$ of $\mathrm{SU}(2)^4$ and is invariant under $\hat{g}$ like w_α, with which it shares the same CP structure (4.9). Instead, $\mu_{\dot{a}}$ belongs to $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$ and is odd under $\hat{g}$, leading to

$$\Gamma^*_{\vec{\xi}}(\hat{g})\, \mu_{\dot{a}}\, \gamma^T_{\vec{\xi}}(\hat{g}) = \varepsilon_{\hat{g}}\, \mu_{\dot{a}} \quad \text{with} \quad \varepsilon_{\hat{g}} = -1\ . \tag{4.10}$$

This is solved by taking

$$\mu_{\dot{a}} = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & -Y_1 \end{pmatrix}\ , \tag{4.11}$$

moduli	$\mathrm{SU}(2)^4$	$\varepsilon_{\hat{g}}$	CP factor	$\mathrm{U}(k)\times\mathrm{U}(m)$	dimensions
w_α μ_a	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$ $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})$	$+$	$\begin{pmatrix} Y_1 & Y_2 \\ -Y_2 & Y_1 \end{pmatrix}$	$(\square,\overline{\square})$	L^1 $L^{1/2}$
$\mu_{\dot{a}}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	$-$	$\begin{pmatrix} Y_1 & Y_2 \\ Y_2 & -Y_1 \end{pmatrix}$	$(\square,\square)$	$L^{1/2}$

Table 4. The spectrum of flavored moduli aring from $(-1)/3$ open strings stretching between k D-instantons and one "half" D3-brane at the same fixed point.

implying that the $\mu_{\dot{a}}$'s transform in the fundamental representation both of $\mathrm{U}(k)$ and of $\mathrm{U}(m)$. All this is summarized in table 4, where again the last column contains the length dimensions of the various moduli.

5 D-instanton corrections from localization formulæ

The effective action on the D7-branes gets corrected by D-instantons. In our $\mathcal{N}=2$ setup these corrections are encoded in a prepotential function of the chiral superfield $\Phi(x,\theta)$ containing the adjoint scalar φ and the gauge field strength F of the $\mathrm{U}(4)$ gauge group (see (2.16)), and can be expressed as follows

$$S_{\text{n.p.}} \sim \int d^4x\, d^4\theta\, \mathcal{F}_{\text{n.p.}}(\Phi) \;+\; \text{c.c.} \tag{5.1}$$

The prepotential $\mathcal{F}_{\text{n.p.}}(\Phi)$ receives contributions from D-instanton configurations of type *a)* and *b)*, described in figures 4 and 5, corresponding , respectively, to instantons sitting on a fixed point occupied by a D3-brane or empty. Taking into account the multiplicity of these configurations, we can write

$$\mathcal{F}_{\text{n.p.}}(\Phi) = 12\,\mathcal{F}^{(m=0)}(\Phi) + 4\,\mathcal{F}^{(m=1)}(\Phi)\;. \tag{5.2}$$

The prepotentials $\mathcal{F}^{(m)}$ can be expressed as an integral over the "centered" moduli space $\widehat{\mathcal{M}}_{k,m}$ of the instantonic branes as follows

$$\mathcal{F}^{(m)}(\Phi) = \sum_k q^k \int d\widehat{\mathcal{M}}_{k,m}\; \mathrm{e}^{-S_{\text{mod}}(\widehat{\mathcal{M}}_{k,m},\Phi)}\;. \tag{5.3}$$

with $q=\mathrm{e}^{\pi\mathrm{i}\lambda}$. Here $(-\pi\mathrm{i}k\lambda)$ is the classical action of k fractional half D-instantons, while $S_{\text{mod}}(\widehat{\mathcal{M}}_{k,m},\Phi)$ is the action describing the interactions of the centered moduli (i.e. all moduli except x and θ) among themselves and with the superfield Φ; all these interactions occur via disk diagrams[6] with at least part of their boundary attached to D(–1) branes.

[6]In principle one should include also annuli and Möbius diagrams with a boundary on the D-instantons [1]. In the present model, such diagrams do not contribute: the D(-1)/D7 and D(-1)/O7 amplitudes are related to the running of the quartic coupling on the D7's, which vanish in our model [22]; the D(-1)/D3 and D(-1)/O3 amplitudes also vanish.

The moduli x and θ play the rôle of the $\mathcal{N} = 2$ super-coordinates and appear in $S_{\rm mod}(\widehat{\mathcal{M}}_{k,m}, \Phi)$ only through the superfield $\Phi(x, \theta)$. This implies that during the calculation we can take Φ to be constant, and promote it to a full fledged dynamical field only in the end. However, even with this position, the integrals in eq. (5.3) remain rather cumbersome, and can be explicitly performed only for very low instanton numbers, typically $k = 1$. Substantial progress can be achieved following the seminal observation [42] that, after suitable deformations of the moduli action, the integrals localize around isolated points in the instanton moduli space, and that an explicit result for the prepotential can be obtained after turning off in a controlled way the deformations. This idea has already been made systematic and applied with success in several interesting contexts [23, 24, 61–63]. Here we put it at work for a system of k D(−1)-instantons, m D3-branes and N D7-branes in presence of O7- and O3-planes, and present explicit computations for the relevant cases with $N = 4$ and $m = 0, 1$, up to $k = 3$.

We first take $\Phi = \mathrm{diag}\,(a_1, \ldots, a_N, -a_1, \ldots, -a_N)$ where a_u are constant expectation values along the Cartan directions of $\mathrm{U}(N)$, and then consider the ϵ-deformed instanton partition function

$$Z^{(m)}(a, \epsilon) = \sum_k q^k\, Z_k^{(m)}(a, \epsilon) = \sum_k q^k \int d\mathcal{M}_{k,m}\ \mathrm{e}^{-S^{\epsilon}_{\rm mod}(\mathcal{M}_{k,m}, a)}\ . \tag{5.4}$$

Here we have conventionally set $Z_0^{(m)}(a, \epsilon) = 1$, and introduced $S^{\epsilon}_{\rm mod}$ which is obtained by deforming $S_{\rm mod}$ with Lorentz breaking terms parameterized by four parameters ϵ_I describing rotations along the four Cartan directions of $\mathrm{SO}(4) \times \widehat{\mathrm{SO}}(4)$. From the string perspective, these deformations can be obtained by switching on suitable RR background fluxes on the D7-branes, as shown in refs. [23, 64]. Notice that integrals in eq. (5.4) run over all moduli, including the "center of mass" super-coordinates x and θ. In presence of the ϵ-deformations it is rather easy to see that the integration over the super-space yields a volume factor growing as $1/(\epsilon_1\epsilon_2)$ in the limit of small $\epsilon_{1,2}$. Therefore, to obtain the integral over the centered moduli this factor has to be removed. In addition, we have to take into account the fact that the k-th order in the q-expansion receives contributions not only from genuine k-instanton configurations but also from disconnected ones, corresponding to copies of instantons of lower numbers k_i such that $\sum k_i = k$. To isolate the connected components we have to take the logarithm of $Z^{(m)}(a, \epsilon)$. Thus, we are led to consider

$$\mathcal{F}^{(m)}(a, \epsilon) = \epsilon_1\epsilon_2 \log Z^{(m)}(a, \epsilon)\ . \tag{5.5}$$

The prepotential will be extracted from $\mathcal{F}^{(m)}(a, \epsilon)$ after taking the appropriate $\epsilon_I \to 0$ limit and replacing a with the complete superfield Φ. It is worth remarking that the function $\mathcal{F}^{(m)}(a, \epsilon)$ contains also information about more general interactions in the four-dimensional theory such as non-perturbative gravitational couplings, flux induced mass terms, etc. For instance, the gravitational terms can be extracted from $\mathcal{F}^{(m)}(a, \epsilon)$ after promoting $\epsilon_{1,2}$ to dynamical superfields describing a graviphoton multiplet, as done in ref. [64] or, in the eight-dimensional context, in refs. [23, 24]. Similarly, the terms involving $\epsilon_{3,4}$ can be interpreted as mass deformations for the antisymmetric matter that are induced by RR fluxes. In this paper we focus only on corrections to gauge kinetic functions, and therefore higher order terms in ϵ's will be systematically discarded.

5.1 Localization formulæ

The localization procedure is based on the co-homological structure of the instanton moduli action which is exact with respect to a suitable BRST charge Q:

$$S_{\rm mod} = Q\Xi \ . \tag{5.6}$$

Q can be obtained by choosing any component of the supersymmetry charges preserved on the brane system. Supersymmetry charges are invariant under $\mathrm{U}(k)\times\mathrm{U}(m)\times\mathrm{U}(N)$ but transform as a spinor of $\mathrm{SO}(4)^2$, so that the choice of Q breaks this symmetry to the $\mathrm{SU}(2)^3$ subgroup which preserves this spinor. In our case we take[7]

$$\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2\times\mathrm{SU}(2)_3 = \mathrm{SU}(2)_-\times\widehat{\mathrm{SU}}(2)_-\times\mathrm{diag}\left[\mathrm{SU}(2)_+\times\widehat{\mathrm{SU}}(2)_+\right] \ . \tag{5.7}$$

This reduction is achieved by identifying the spinor indices "α" and "a" of the first and third SU(2)'s in tables 2, 3 and 4 of section 4. After this identification is made, the fermionic moduli $M_{\dot\alpha a}$ and $M_{\alpha\dot a}$ can be renamed as $M_{\ell=\dot\alpha\alpha}$ and $M_{\dot\ell=a\dot a}$, and paired with B_ℓ and $B_{\dot\ell}$ into BRST multiplets. Similarly, the singlet component $\eta\equiv N_{\alpha a}\epsilon^{\alpha a}$ and the $(-1)/3$ fermionic moduli $\mu_{\alpha=a}$ have the right transformation properties to qualify for the BRST partners of $\bar\chi$ and w_α respectively.

The remaining fields $N_m\equiv\sigma_m^{\alpha a}N_{\alpha a}$, $N_{\dot\alpha\dot a}$, $\mu_{\dot a}$ and μ' are unpaired, and should be supplemented with auxiliary fields having identical transformation properties. We denote such fields as d_m, $D_{\dot\alpha\dot a}$, $h_{\dot a}$ and h', respectively. In ordinary cases the auxiliary fields collect the D- and F-terms of the gauge theory on the D(−1)'s, and the corresponding D- and F-flatness conditions are the ADHM constraints on the instanton moduli space (see for example refs. [61, 65, 66] for details). In our case we have an extension of this construction, defining a sort of generalized "exotic" instanton moduli space. More precisely, the seven auxiliary moduli d_m, $D_{\dot\alpha\dot a}$, of dimension L^2, linearize the quartic interactions among the scalars B_ℓ and $B_{\dot\ell}$ and correspond to vertex operators[8] that are bilinear in the fermionic string coordinates [23, 32]. In particular, the triplet d_m disentangles the quartic interactions of B_ℓ and $B_{\dot\ell}$ among themselves, while the quartet $D_{\dot\alpha\dot a}$ decouples the quartic interactions between B_ℓ and $B_{\dot\ell}$. Likewise, the dimensionless auxiliary $(-1)/3$ moduli h_a disentangle the quartic interactions between $B_{\dot\ell}$ and w_α. Finally, h' completes the $(-1)/7$ BRST multiplet. In the end, χ remains unpaired and therefore $Q\chi=0$.

In this way, all moduli but χ form BRST doublets, which we will schematically denote as $\big(\phi,\psi\equiv Q\phi\big)$ in the following and which are explicitly listed in the first column of table 5.

[7] In the D(−1)/D7 system considered in ref. [23], the subgroup of the SO(8) Lorentz symmetry preserving a fixed spinor is SO(7), embedded in such a way that a vector of SO(8) becomes a spinor. In our case, eq. (5.7) represents the subgroup of this SO(7) which is compatible with the 4 + 4 split of the eight-dimensional space induced by the orbifold compactification on $\mathcal{T}_4/\mathbb{Z}_2$.

[8] One can see that vertices associated to the moduli $D_{\dot\alpha\dot a}$ transform under the discrete parities Ω' and g in the same way as the fermionic moduli $N_{\dot\alpha\dot a}$, while the triplet d_m transforms like the fermions N_m. Thus, the structure of their CP factors match that of their BRST partners as expected.

JHEP05(2010)107

Note that ϕ is a boson if the multiplet is built out of physical moduli, and is a fermion if instead it contains auxiliary fields. Indeed, the auxiliary fields, being related to D- and F-terms, can only appear as highest components in the BRST multiplets while the physical bosonic moduli enter as the lowest components of the pair. These statistical properties are listed in the second column of table 5. With all these ingredients at hand, one can show that the moduli action $S_{\rm mod}$ can be written in the form (5.6). The details of the fermion Ξ are irrelevant to the computation, since integrals are insensitive to Q-exact terms.

Since the length dimension of the BRST charge is $L^{-1/2}$, the length dimensions of the components (ϕ, ψ) of Q-multiplet are $\left(\Delta, \Delta - \frac{1}{2}\right)$. Thus, recalling that a fermionic variable and its differential have opposite dimensions, we find that the measure on the instanton moduli space

$$d\mathcal{M}_{k,m} \equiv d\chi \prod_{(\phi,\psi)} d\phi \; d\psi \tag{5.8}$$

has the following scaling dimensions

$$L^{-k^2 + \frac{1}{2}(n_+ - n_-)} \; . \tag{5.9}$$

Here, the first term in the exponent accounts for the unpaired k^2 bosonic moduli χ, of dimension L^{-1}, and $n_\pm$ denotes the number of Q-multiplets where the statistics of the lowest component is $(-)^{F_\phi} = \pm$. Using table 5, we can explicitly rewrite eq. (5.9) as

$$L^{-k^2 + \frac{1}{2}(n_B + n_{\bar{\chi}} + n_w - n_N - n_\mu - n_{\mu'})} = L^{\frac{4-N}{2}\, k} \tag{5.10}$$

where n_ϕ is the number of real component of a modulus of type ϕ. The measure is therefore dimensionless for $N = 4$, i.e. for the U(4) D7-brane gauge theory in our model. Note that this result is independent from the number m of D3-branes at the fixed point where the instantons sit, and therefore holds for both the two relevant cases $m = 0, 1$ in our setup.

To localize the integral over moduli space, it is necessary to make the charge Q equivariant with respect to all symmetries, which in our case are the gauge symmetry $\mathrm{U}(k) \times \mathrm{U}(N) \times \mathrm{U}(m)$, and the residual Lorentz symmetry $\mathrm{SU}(2)^3$. For our purposes it is enough to consider the Cartan directions of the various groups. We label the Cartan components of the $\mathrm{U}(k)$ parameters of Q by $\vec{\chi}$, those of $\mathrm{U}(m)$, those of $\mathrm{U}(m)$ by $\vec{b}$ and those of $\mathrm{U}(N)$ by $\vec{a}$. From the string perspective $\vec{\chi}$, $\vec{b}$ and $\vec{a}$ parametrize, respectively, the positions of the D(–1), D3 and D7-branes along the overall transverse two-dimensional plane, and their appearance in the moduli action can be deduced from disk amplitudes with (part of) their boundary on the D-instantons and with insertion of (–1)/(–1), 3/3 or 7/7 fields. Thus, $\vec{a}$ can be interpreted as the vacuum expectation value of the chiral superfield Φ of gauge theory on the D7-branes, and $\vec{b}$ as the analogue for the gauge theory on the D3-branes. Finally, the Cartan directions of the residual Lorentz group $\mathrm{SU}(2)^3$ are parametrized by ϵ_I $(I = 1, \ldots, 4)$ subject to the constraint

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0 \; . \tag{5.11}$$

Although only three out of the four ϵ's are independent variables, it is convenient during the computation to keep all of them as independent variables and impose the relation (5.11) only at the very end.

After the equivariant deformation, the charge Q becomes nilpotent up to an element of the symmetry group. It is convenient to use the basis provided by the weights of this group, and thus we denote by ϕ_q and ψ_q the components of ϕ and ψ along a weight

$$\vec{q} \equiv \left(\vec{q}_{\mathrm{U}(k)}, \vec{q}_{\mathrm{U}(N)}, \vec{q}_{\mathrm{U}(m)}, \vec{q}_{\mathrm{SU}(2)^3}\right) \in \mathcal{W}(\phi)\,, \tag{5.12}$$

where $\mathcal{W}(\phi)$ is the set of weights of the representation under which ϕ transforms, which can be read from the third and fourth columns of table 5. Then, in this basis the charge Q acts diagonally as follows

$$Q\phi_q = \psi_q\,, \quad Q\psi_q = \Omega_q \phi_q\,, \tag{5.13}$$

where Ω_q parametrizes the equivariant deformation, i.e. the eigenvalues of Q^2. From the brane perspective, Ω_q specifies the distance in the overall two-dimensional transverse plane between the branes at the two endpoints of the open string. Explicitly, we have

$$\Omega_q = \vec{\chi} \cdot \vec{q}_{\mathrm{U}(k)} + \vec{a} \cdot \vec{q}_{\mathrm{U}(N)} + \vec{b} \cdot \vec{q}_{\mathrm{U}(m)} + \vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^3}\,\,. \tag{5.14}$$

The $\vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^3}$ eigenvalues appearing above can be deduced from $\vec{\epsilon} \cdot \vec{q}_{\mathrm{SO}(4)^2}$, where $\vec{q}_{\mathrm{SO}(4)^2}$ is the $\mathrm{SO}(4) \times \widehat{\mathrm{SO}}(4)$ weight vectors of the physical moduli inside each multiplet, using the relation (5.11) among the ϵ_I parameters. For example, the complex moduli B_ℓ, transforming as a vector of the first SO(4), have $\vec{q}_{\mathrm{SO}(4)^2}$ weights $(\pm 1, 0, 0, 0)$ or $(0, \pm 1, 0, 0)$, and thus their contribution to Ω_q is $\pm\epsilon_1$ or $\pm\epsilon_2$. Similarly, for $B_{\hat{\ell}}$ which is a vector of $\widehat{\mathrm{SO}}(4)$, we find $\pm\epsilon_3$ or $\pm\epsilon_4$. The same results are found for the M-fermions which transform as a right spinor of $\mathrm{SO}(4) \times \widehat{\mathrm{SO}}(4)$ and have weights $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with an odd number of plus signs. For the N-fermions, transforming instead as a left spinor with an even number of plus signs, after using eq. (5.11) we find 0, $\pm\frac{1}{2}(\epsilon_1 + \epsilon_2)$, $\pm\frac{1}{2}(\epsilon_1 + \epsilon_3)$ and $\pm\frac{1}{2}(\epsilon_2 + \epsilon_3)$. Finally, w_α transforming as a right spinor of SO(4) has eigenvalues $\pm\frac{1}{2}(\epsilon_1 + \epsilon_2)$, while $\mu_{\dot{a}}$ transforming as a left spinor of $\widehat{\mathrm{SO}}(4)$ corresponds to $\pm\frac{1}{2}(\epsilon_3 - \epsilon_4)$. Alternatively, these eigenvalues can be read from the formula

$$\vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^3} = q_1(\epsilon_1 - \epsilon_2) + q_2(\epsilon_3 - \epsilon_4) + q_3(\epsilon_1 + \epsilon_2) \tag{5.15}$$

with $q_i = 0$ for states in the $\mathbf{1}$, $q_i = \pm\frac{1}{2}$ for states in the $\mathbf{2}$ and so on.[9] All this is summarized in the last column of table 5, where we have displayed the positive eigenvalues of $\vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^3}$ (assuming $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$) corresponding to the holomorphic components of the various fields.

The complete localization of the integral around isolated fixed points implies that the integral is given by the (super)-determinant of Q^2 evaluated at the fix points of Q [42, 61, 66]. As we already mentioned, the moduli χ and $\bar{\chi}$ appear very asymmetrically in the BRST formalism: χ parametrizes the $\mathrm{U}(k)$ gauge rotations, while $\bar{\chi}$ falls

[9] To see this, associate to each modulus the $\mathrm{SU}(2)^4$ charges $q_\pm, \hat{q}_\pm$ and the eigenvalue $\vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^4} = \epsilon_1(q_+ + q_-) + \epsilon_2(q_+ - q_-) + \epsilon_3(\hat{q}_+ + \hat{q}_-) + \epsilon_4(\hat{q}_+ - \hat{q}_-)$. Then, eq. (5.15) follows after the identification $q_1 = q_-$, $q_2 = \hat{q}_-$, $q_3 = q_+ - \hat{q}_+$ and the use of (5.11). For example, $B_{1,2} \in (\mathbf{2}, \mathbf{1}, \mathbf{2})$ have $\mathrm{SU}(2)^3$ weights $(\pm\frac{1}{2}, 0, \pm\frac{1}{2})$ that once plugged into eq. (5.15) lead to $\pm\epsilon_1$ and $\pm\epsilon_2$.

(ϕ,ψ)	$(-)^{F_\phi}$	$\mathrm{U}(k)\times\mathrm{U}(N)\times\mathrm{U}(m)$	$\mathrm{SU}(2)^3$	$\vec{\epsilon}\cdot\vec{q}_{\mathrm{SU}(2)^3}$
(B_ℓ, M_ℓ)	$+$	$(\mathrm{adj},\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{1},\mathbf{2})$	ϵ_1,ϵ_2
$(B_{\dot\ell}, M_{\dot\ell})$	$+$	$(\square\!\square,\mathbf{1},\mathbf{1})+\mathrm{h.c.}$	$(\mathbf{1},\mathbf{2},\mathbf{2})$	ϵ_3,ϵ_4
$(N_{\dot\alpha\dot a}, D_{\dot\alpha\dot a})$	$-$	$(\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{1},\mathbf{1})+\mathrm{h.c.}$	$(\mathbf{2},\mathbf{2},\mathbf{1})$	$\epsilon_2+\epsilon_3,\epsilon_1+\epsilon_3$
(N_m, d_m)	$-$	$(\mathrm{adj},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{3})$	$0_{\mathbb{R}},\epsilon_1+\epsilon_2$
$(\bar\chi,\eta)$	$+$	$(\mathrm{adj},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$0_{\mathbb{R}}$
(μ', h')	$-$	$(\square,\overline{\square},\mathbf{1})+\mathrm{h.c.}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$	0
(w_α,μ_α)	$+$	$(\square,\mathbf{1},\overline{\square})+\mathrm{h.c.}$	$(\mathbf{1},\mathbf{1},\mathbf{2})$	$\frac{1}{2}(\epsilon_1+\epsilon_2)$
$(\mu_{\dot a}, h_{\dot a})$	$-$	$(\square,\mathbf{1},\square)+\mathrm{h.c.}$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	$\frac{1}{2}(\epsilon_3-\epsilon_4)$

Table 5. BRST structure and symmetry properties of the D(−1)/D3/D7 moduli. With $(-)^{F_\phi}=\pm$ we denote the statistics, bosonic or fermionic, of the lower component of the doublet. The third and fourth columns report the transformation properties under the symmetry groups. The last column collects the eigenvalues $\vec{\epsilon}\cdot\vec{q}_{\mathrm{SU}(2)^3}$ for the positive weights $\vec{q}$'s specified in the third column.

into one of the doublets. Moreover, the contribution of the $(\bar\chi,\eta)$ multiplet to the super-determinant cancels against an identical contribution coming from a real component in (N_m,d_m) with identical transformation properties and opposite statistics. After discarding these contributions, the super-determinant of Q^2 takes a simple product form in terms of the Ω_q-eigenvalues over complex variables and can be restricted to holomorphic components of the latter, corresponding to the positive weights $\in\mathcal{W}^+(\phi)$. Thus, the k-instanton partition function $Z_k^{(m)}$ of eq. (5.4), now deformed also with the $\mathrm{U}(m)$ parameters b, is given by the localization formula

$$\begin{aligned} Z_k^{(m)}(a,b,\epsilon) &= \int d\mathcal{M}_{k,m}\,\mathrm{e}^{-S^\epsilon_{\mathrm{mod}}(\mathcal{M}_{k,m},a,b)} = \int d\chi \prod_{(\phi,\psi)} d\phi\, d\psi\;\mathrm{e}^{-Q\Xi} \\ &= \int \prod_{i=1}^{k}\frac{d\chi_i}{2\pi\mathrm{i}}\prod_{i<j}^{k}(\chi_i-\chi_j)^2\prod_{\phi}\prod_{q\in\mathcal{W}^+(\phi)}\Omega_q^{-(-)^{F_\phi}}\,. \end{aligned} \tag{5.16}$$

The factor $\prod_{i<j}(\chi_i-\chi_j)^2$, known as Vandermonde determinant, comes from the Jacobian resulting from bringing χ into the diagonal form $\chi=\mathrm{diag}(\chi_1,\chi_2,\ldots\chi_k)$. The integral over χ_i in the second line above has to be thought of as a multiple contour integral, according to the prescription introduced in ref. [67].

The explicit expression for the products appearing in eq. (5.16) can be easily deduced from eq. (5.14) by considering in turn, for each modulus ϕ in table 5, the set of weights corresponding to its symmetry representations. Introducing for notational convenience

$$s_1=\epsilon_2+\epsilon_3\,,\quad s_2=\epsilon_1+\epsilon_3\,,\quad s_3=\epsilon_1+\epsilon_2\,, \tag{5.17}$$

the products for the (–1)/(–1) moduli can be written as

$$
\begin{aligned}
\prod_{\vec{q}\in\mathcal{W}^+(B_\ell)} \Omega_q^{-(-)^{F_\phi}} &= \prod_{\ell=1}^{2}\prod_{i\leq j}^{k}\Big((\chi_i-\chi_j)^2-\epsilon_\ell^2\Big)^{-1}\,,\\
\prod_{\vec{q}\in\mathcal{W}^+(B_{\dot\ell})} \Omega_q^{-(-)^{F_\phi}} &= \prod_{\dot\ell=3}^{4}\prod_{i<j}^{k}\Big((\chi_i+\chi_j)^2-\epsilon_{\dot\ell}^2\Big)^{-1}\prod_{i=1}^{k}\Big(4\chi_i^2-\epsilon_{\dot\ell}^2\Big)^{-1}\,,\\
\prod_{\vec{q}\in\mathcal{W}^+(N_{\dot\alpha\dot a})} \Omega_q^{-(-)^{F_\phi}} &= \prod_{\ell=1}^{2}\prod_{i<j}^{k}\Big((\chi_i+\chi_j)^2-s_\ell^2\Big)\,,\\
\prod_{\vec{q}\in\mathcal{W}^+(N_m,\bar\chi)} \Omega_q^{-(-)^{F_\phi}} &= \prod_{i\leq j}^{k}\Big((\chi_i-\chi_j)^2-s_3^2\Big)\,,
\end{aligned}
\tag{5.18}
$$

while for the (–1)/7 moduli we have

$$
\prod_{\vec{q}\in\mathcal{W}^+(\mu')} \Omega_q^{-(-)^{F_\phi}} = \prod_{i=1}^{k}\prod_{u=1}^{n}\Big(\chi_i-a_u\Big)\,,
\tag{5.19}
$$

and for the (–1)/3 moduli we have

$$
\begin{aligned}
\prod_{\vec{q}\in\mathcal{W}^+(\omega_\alpha)} \Omega_q^{-(-)^{F_\phi}} &= \prod_{i=1}^{k}\prod_{r=1}^{m}\Big((\chi_i-b_r)^2-\frac{(\epsilon_1+\epsilon_2)^2}{4}\Big)^{-1}\,,\\
\prod_{\vec{q}\in\mathcal{W}^+(\mu_{\dot a})} \Omega_q^{-(-)^{F_\phi}} &= \prod_{i=1}^{k}\prod_{r=1}^{m}\Big((\chi_i+b_r)^2-\frac{(\epsilon_3-\epsilon_4)^2}{4}\Big)\,.
\end{aligned}
\tag{5.20}
$$

Putting everything together, the instanton partition function (5.16) is then given by

$$
\begin{aligned}
Z_k^{(m)}(a,b,\epsilon) = &\left(\frac{s_3}{\epsilon_1\epsilon_2}\right)^k\int\prod_{i=1}^{k}\frac{d\chi_i}{2\pi\mathrm{i}}\prod_{i<j}^{k}(\chi_i-\chi_j)^2\Big((\chi_i-\chi_j)^2-s_3^2\Big)\\
&\times\prod_{i<j}^{k}\prod_{\ell=1}^{2}\frac{\Big((\chi_i+\chi_j)^2-s_\ell^2\Big)}{\Big((\chi_i-\chi_j)^2-\epsilon_\ell^2\Big)\Big((\chi_i+\chi_j)^2-\epsilon_{\ell+2}^2\Big)}\\
&\times\prod_{i=1}^{k}\left[\prod_{\ell=1}^{2}\frac{1}{\Big(4\chi_i^2-\epsilon_{\ell+2}^2\Big)}\prod_{r=1}^{m}\frac{\Big((\chi_i+b_r)^2-\frac{(\epsilon_3-\epsilon_4)^2}{4}\Big)}{\Big((\chi_i-b_r)^2-\frac{(\epsilon_1+\epsilon_2)^2}{4}\Big)}\prod_{u=1}^{n}\Big(\chi_i-a_u\Big)\right]\,.
\end{aligned}
\tag{5.21}
$$

This multiple integral should be supplemented by a pole prescription. Here, inspired by the prescription of ref. [67], we take $\mathrm{Im}b_r = 0$ and $\mathrm{Im}\epsilon_1 \gg \mathrm{Im}\epsilon_2 \gg \mathrm{Im}\epsilon_3 \gg \mathrm{Im}\epsilon_4 > 0$, and compute the integrals closing the contours in the upper half-plane $\mathrm{Im}\chi_i > 0$. In appendix C, we provide explicit results of these integrals up to $k = 3$ instantons.

It is interesting to observe that the exotic instantons we have considered here can also be re-interpreted as standard gauge instantons from the D3-brane perspective. Indeed, the D(–1)-instanton partition function (5.21) coincides with that describing gauge instantons in a U(m) gauge theory, with two antisymmetric hyper-multiplets with masses $-\epsilon_3$ and $-\epsilon_4$, and four fundamentals with masses a_u. The cases we are interested in, $m = 0, 1$, correspond however to a rather bizarre choice of the gauge theory where the standard field theory notions tend to lose their meaning. In this sense, our exotic instantons can be thought as a extrapolation of ordinary gauge instanton effects to degenerated limits of quantum field theories.

5.2 Non-perturbative prepotential

In order to obtain the non-perturbative prepotential for the D7-brane gauge theory from the partition function $Z^{(m)}(a, b, \epsilon)$, we first set the vacuum expectation values $b_r = 0$ of the 3/3 scalars to zero, since in our string vacua the D3-branes are fixed at one of the orbifold fixed-points. Thus, from now on we will not consider any more the b-dependence of the instanton partition function. As a second step, we take the limit $\epsilon_I \to 0$ to remove the Lorentz breaking deformations. A simple inspection of the explicit results for $\log Z^{(m)}(a, \epsilon)$ given in eqs. (C.2) and (C.3), shows that this expression diverges as $1/(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)$ in this limit. Such a divergence is typical of interactions in eight dimensions where the $\mathcal{N} = 2$ super-space volume grows like $\int d^8x d^8\theta \sim 1/(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)$. Indeed, although the σ_3-trace components $\hat{x}$ and $\hat{\theta}$ of the moduli $B_{\dot{\ell}}$ and $M_{\dot{\ell}}$ do not in general decouple from the moduli action, they do in some of the fixed points that contribute to the completely localized integral. In these points, $\hat{x}$ and $\hat{\theta}$ effectively represent the super-coordinates of the internal orbifold where the D7-branes are wrapped, and together with the true super-space coordinates x and θ reconstruct an eight-dimensional volume factor. These contributions can then be thought of as coming from regular D(–1)-instantons moving in the full eight-dimensional world-volume of the D7-branes. Moreover, from the explicit results presented in appendix C, we can see that the terms proportional to $1/(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)$ in $\log Z^{(m)}(a, \epsilon)$ are independent of m (and also of b's if we keep these parameters switched on). Thus, they can be associated to a universal quartic prepotential defined as

$$\mathcal{F}_{\rm IV}(a) = \lim_{\epsilon_I \to 0} \epsilon_1\epsilon_2\epsilon_3\epsilon_4 \log Z^{(m)}(a, \epsilon) \ . \tag{5.22}$$

Explicitly we have

$$\mathcal{F}_{\rm IV}(a) = \Big(4a_1a_2a_3a_4\Big)\, q - \Big(\sum_{i<j} a_i^2 a_j^2\Big)\, q^2 + \Big(\frac{16}{3}a_1a_2a_3a_4\Big)\, q^3 + \ldots \tag{5.23}$$

which has indeed quartic mass dimension. Surprisingly, the result (5.23) matches precisely (half of) the non-perturbative quartic prepotential induced by D-instantons on the parent eight-dimensional SO(8) gauge theory living on D7-branes of type I′ [23]. This is consistent with the interpretation of these contributions as coming from bulk or regular instantons, since such instantons are insensitive to the $\mathbb{Z}_2$-orbifold projection.

More interestingly, we can extract a finite quadratic prepotential by subtracting the divergence coming from the eight-dimensional interactions. This quadratic prepotential is defined as

$$\mathcal{F}^{(m)}_{\rm II}(a) = \lim_{\epsilon_I \to 0} \Big(\epsilon_1\epsilon_2 \log Z^{(m)}(a, \epsilon) - \frac{1}{\epsilon_3\epsilon_4}\mathcal{F}_{\rm IV}\Big) \ . \tag{5.24}$$

Since the moduli measure is dimensionless, as shown in eq. (5.10), no dynamically generated scale may appear and the contributions at *all* instanton numbers must be constructed only out of the a's once the ϵ-deformations are switched off. This is what happens. In fact, for $m = 0, 1$ we find the following quadratic prepotentials

$$\begin{aligned}\mathcal{F}_{\rm II}^{(m=0)}(a) &= \Big(-\sum_{i<j} a_i a_j\Big)\, q + \Big(\sum_{i<j} a_i a_j - \frac{1}{4}\sum_i a_i^2\Big)\, q^2 + \Big(-\frac{4}{3}\sum_{i<j} a_i a_j\Big)\, q^3 + \cdots\,,\\ \mathcal{F}_{\rm II}^{(m=1)}(a) &= \Big(3\sum_{i<j} a_i a_j\Big)\, q + \Big(\sum_{i<j} a_i a_j + \frac{7}{4}\sum_i a_i^2\Big)\, q^2 + \Big(4\sum_{i<j} a_i a_j\Big)\, q^3 + \cdots\,.\end{aligned} \tag{5.25}$$

The total quadratic prepotential $\mathcal{F}$n.p.(a), which takes into account the contributions from the various $m = 0, 1$ configurations with their appropriate multiplicity, is obtained inserting eq. (5.25) into eq. (5.2), and reads

$$\mathcal{F}_{\rm n.p.}(a) = 4\Big[2\big(\mathrm{tr}\,a\big)^2 - \mathrm{tr}\,a^2\Big]\, q^2 + O(q^4)\,, \tag{5.26}$$

where we have rewritten in a basis-independent way the sums over the a's. It is important to stress that the terms proportional to q and q^3 cancel when we sum over all possible D-instanton configurations, and that the relative factor inside the square brackets is a consequence of the explicit numerical coefficients we have found in evaluating the instanton integrals using the localization technique.

We can now promote the vacuum expectation values a's to the corresponding dynamical superfield $\Phi(x,\theta)$ and determine the quadratic non-perturbative action according to eq. (5.1). Performing the θ-integration, we then obtain

$$S_{\rm n.p.} \propto \int d^4x\,\Big[2\big(\mathrm{tr}\,F\big)^2 - \mathrm{tr}\,F^2\Big]\, q^2 + O(q^4) \quad + \text{ c.c.} \tag{5.27}$$

From this expression, we can say that the non-perturbative part of the holomorphic couplings f and f' of our $\mathcal{N} = 2$ theory is given by

$$f_{\rm n.p.} = \alpha\, q^2 + O(q^4)\,, \quad f'_{\rm n.p.} = -2\alpha\, q^2 + O(q^4) \tag{5.28}$$

where α is an overall coefficient that accounts for the normalization of the instanton partition function and the numerical factors arising from the θ integrations. We would like to stress again that the vanishing of the contributions at the one and three instanton level is due to the non-trivial cancellations between contributions coming from fixed points with one D3-brane or with none. In the next section we will test this result against a dual heterotic computation that predicts the absence of these odd instanton number contributions to any order and reproduces the relative factor of -2 between the two structures at $k = 2$. This will provide a robust test of our explicit calculations.

6 Heterotic gauge couplings

In this section we exploit the heterotic/type I′ duality to test the results we have found via localization of the integrals on the moduli space of the D(–1)/D3/D7-brane system.

The heterotic dual model can be built from the U(16) compactification of the SO(32) heterotic string on $\mathcal{T}_4/\mathbb{Z}_2$ (with standard embedding of the orbifold curvature into the gauge bundle) [33, 35] and further reduced on $\mathcal{T}_2$ with Wilson lines that break U(16) to $\mathrm{U}(4)^4$. The gauge kinetic terms in this heterotic set-up are corrected at 1-loop by an infinite tower of world-sheet instantons wrapping $\mathcal{T}_2$, which are dual to the D-instantons of the type I′ theory [43, 44]. In this section we will compute the 1-loop heterotic thresholds and, after applying the duality map, show a perfect match against the stringy multi-instanton contributions found in section 5.

6.1 The heterotic orbifold

We first give some details on the heterotic model we will consider. We start from the SO(32) heterotic string with super-string coordinates X^M and ψ^M ($M = 0, \ldots, 9$), and a left-moving SO(32) current algebra realized in terms of 16 complex fermions Λ^I ($I = 1, \ldots, 16$). To find a four-dimensional $\mathcal{N} = 2$ vacuum with gauge group $\mathrm{U}(4)^4$ we compactify the theory on $\mathcal{T}_4/\mathbb{Z}_2 \times \mathcal{T}_2$ with a proper choice of Wilson lines on $\mathcal{T}_2$. More precisely, the $\mathbb{Z}_2$ orbifold group is generated by

$$\hat{g}_0 \;:\qquad X^i \to -X^i\,, \quad \psi^i \to -\psi^i\,, \quad \Lambda^I \to \mathrm{i}\Lambda^I \tag{6.1}$$

where X^i and ψ^i ($i = 4, 5, 6, 7$) are the string coordinates along $\mathcal{T}_4$. This action breaks the gauge group SO(32) down to U(16) corresponding to the 256 massless vectors of the form $\psi^\mu_{-\frac{1}{2}}\Lambda^I_{-\frac{1}{2}}\bar{\Lambda}^{\bar{J}}_{-\frac{1}{2}}|0\rangle$ which are even under $\hat{g}_0$. The further breaking to $\mathrm{U}(4)^4$ is achieved by turning on discrete Wilson lines on $\mathcal{T}_2$. These can be realized in terms of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ freely acting orbifold with each $\mathbb{Z}_2$ acting as a reflection in the U(16) lattice and a half-shift along $\mathcal{T}_2$. More precisely, if we denote by X^8 and X^9 the bosonic coordinates of $\mathcal{T}_2$, and for simplicity take the latter to be a square torus with radii R_8 and R_9, then the two generators $\hat{g}_1$ and $\hat{g}_2$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are defined as

$$\begin{aligned} \hat{g}_1 :\quad & X^8 \to X^8 + \pi R_8\,, \quad \Lambda^{I=1,\ldots,8} \to -\Lambda^{I=1\ldots8}\,, \\ \hat{g}_2 :\quad & X^9 \to X^9 + \pi R_9\,, \quad \Lambda^{I=5,\ldots12} \to -\Lambda^{I=5,\ldots12} \end{aligned} \tag{6.2}$$

This action splits the 16 complex fermions Λ^I into four groups of four, thus realizing the desired breaking from U(16) to $\mathrm{U}(4)^4$.

6.1.1 Partition function and massless spectrum

As a preliminary test, we check that the massless spectrum of this heterotic orbifold is the same as that of the dual type I′ model. To do so, we compute the heterotic partition function

$$\mathcal{Z}_{\text{het}} = \int_{\mathcal{F}} \frac{d^2\tau}{2\tau_2}\, \mathrm{Tr}\left(\frac{1+\hat{g}_0}{2}\,\frac{1+\hat{g}_1}{2}\,\frac{1+\hat{g}_2}{2}\,\frac{1+(-1)^F}{2}\, q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \tag{6.3}$$

where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, $d^2\tau = d\tau_1 d\tau_2$, $\mathcal{F}$ is the fundamental domain of the torus and the GSO projection acts on the right-moving fields. Performing the conformal field theory trace over all sectors, we obtain

$$\mathcal{Z}_{\text{het}} = \frac{1}{2^4}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\,\frac{\nu_4}{\tau_2^2} \sum_{g_i,h_i=0}^{1} \rho\big[{}^{h_0}_{g_0}\big](0;\tau)\; \chi\big[{}^{h_0\,h_1\,h_2}_{g_0\,g_1\,g_2}\big](0;\bar{\tau})\; \Gamma\big[{}^{h_0\,h_1\,h_2}_{g_0\,g_1\,g_2}\big](\tau,\bar{\tau})\,, \tag{6.4}$$

where the factor

$$\frac{\nu_4}{\tau_2^2} \equiv \frac{V_4}{(4\pi^2\alpha')^2\,\tau_2^2} \tag{6.5}$$

arises from the integration over the bosonic zero-modes of the four non-compact directions with a (regularized) volume V_4, and the functions $\rho\big[{}^{h_0}_{g_0}\big](0;\tau)$ and $\chi\big[{}^{h_0\,h_1\,h_2}_{g_0\,g_1\,g_2}\big](0;\bar\tau)$ account for the contributions coming from the trace over the right- and left-moving oscillators, respectively. Their explicit definition and properties can be found in appendix D (see in particular eqs. (D.1)–(D.4)). Finally, $\Gamma\big[{}^{h_0\,h_1\,h_2}_{g_0\,g_1\,g_2}\big](\tau,\bar\tau)$ represents the contribution of the bosonic zero-modes in the internal compact directions and is given explicitly in eqs. (D.7)–(D.9) (see also appendix A).

With this information, we can check the massless spectrum. To do so, we first notice that the massless states contribute only to the untwisted amplitudes with $h_1 = h_2 = 0$, since the $\mathbb{Z}_2\times\mathbb{Z}_2$-actions (6.2) have no fixed points. Indeed, expanding the lattice sums for large values of τ, we have

$$\begin{aligned}
&\Gamma\big[{}^{h_0\;0\;\;0}_{g_0\,g_1\,g_2}\big] = 1+\dots\,,\\
&\Gamma\big[{}^{h_0\,h_1\,h_2}_{g_0\,g_1\,g_2}\big] = 0+\dots \quad \text{for } (h_1,h_2)\neq(0,0)\;.
\end{aligned} \tag{6.6}$$

Thus, only the left-moving contributions $\chi\big[{}^{h_0\;0\;\;0}_{g_0\,g_1\,g_2}\big]$ need to be considered. Using the results derived in appendix D (see in particular eq. (D.6)), it is not difficult to obtain their asymptotic expansions for large values of τ, namely

$$\begin{aligned}
&\chi\big[{}^{0\,0\,0}_{0\,0\,0}\big] = \frac{1}{\bar q} + (2_v+502)+\dots\,,\\
&\chi\big[{}^{0\;\,0\;\,0}_{0\,g_1\,g_2}\big] = \frac{1}{\bar q} + (2_v-10)+\dots \qquad \text{for } (g_1,g_2)\neq(0,0)\,,\\
&\chi\big[{}^{0\;\,0\;\,0}_{1\,g_1\,g_2}\big] = \frac{1}{\bar q} + (2_v+14)+\dots\,,\\
&\chi\big[{}^{1\,0\,0}_{0\,0\,0}\big] = \chi\big[{}^{1\,0\,0}_{1\,0\,0}\big] = 128+\dots\quad,\\
&\chi\big[{}^{1\;\,0\;\,0}_{0\,g_1\,g_2}\big] = \chi\big[{}^{1\;\,0\;\,0}_{1\,g_1\,g_2}\big] = 0 \qquad \text{for } (g_1,g_2)\neq(0,0)\;.
\end{aligned} \tag{6.7}$$

Here and below we use the notation $2_{v,c,s}$ to keep track of the transformation properties of the various states with respect to the SO(2) Lorentz little group in four dimensions; thus 2_v stands for two degrees of freedom in the vector representation. Likewise, for the right-moving contributions we find

$$\rho\big[{}^0_0\big] = \mathbf{V}+\mathbf{H}+\dots\,,\quad \rho\big[{}^0_1\big] = \mathbf{V}-\mathbf{H}+\dots\,,\quad \rho\big[{}^1_0\big] = \rho\big[{}^1_1\big] = 2\mathbf{H}+\dots\,, \tag{6.8}$$

where we have denoted by

$$\mathbf{V} = 2_v + 2 - 2_s - 2_c\,,\quad \mathbf{H} = 4 - 2\times 2_s - 2\times 2_c \tag{6.9}$$

the number of physical states in a vector and a hyper-multiplet.

Taking the product of the left- and right-moving contributions, we find that the number of massless states can be written as

$$\Big[\mathbf{V}\times(2_v+2)+4\mathbf{H}\Big]+4\Big[\mathbf{V}\times\mathbf{16}+\mathbf{H}\times(\mathbf{6}+\overline{\mathbf{6}}+4\times\mathbf{4})\Big] \tag{6.10}$$

where the first square bracket corresponds to the gravity multiplet and four hypers containing the geometrical moduli of the $\mathcal{T}_4/\mathbb{Z}_2$ orbifold, and the remaining terms build up four copies of a vector multiplet in the adjoint of U(4) plus one hyper in the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, one in the $\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and four in the $\square$ representations, in perfect agreement with the type I′ dual spectrum of table 1.

6.2 Threshold corrections

The moduli dependence of the gauge kinetics terms in the heterotic model at 1-loop can be extracted from the 2-gluon scattering amplitude on the torus $\langle V_F V_F\rangle$, where

$$V_F=(2\pi\alpha')\,F^I_{\mu\nu}\int d^2z\,\big(X^\mu\partial X^\nu+\psi^\mu\psi^\nu\big)(z)\;J^I_{\rm int}(\bar z) \tag{6.11}$$

is the vertex operator for the emission of a gauge field along the Cartan directions of $\mathrm{U}(4)^4$ for which the corresponding current is $J^I_{\rm int}=\lambda^I\bar\lambda^{\bar I}$. In a theory with eight supercharges, like ours, this is a BPS saturated amplitude and therefore non-trivial contributions come only from the fermionic zero mode part of the string vertices, namely

$$V_F=(2\pi\alpha')\,2\tau_2\,F^I_s\,J^s_{\rm Lor}\,J^I_{\rm int} \tag{6.12}$$

with $J^s_{\rm Lor}=\psi^s\bar\psi^s$, $s=1,2$, being the right-moving fermionic currents along the two Cartan directions of the SO(4) Lorentz group. In the above expression the factor of τ_2 comes from the integration over the position of the vertices over the world-sheet torus.

To compute the 2-point function $\langle V_F V_F\rangle$ it is convenient to exponentiate the string vertices and rewrite the threshold amplitudes as a second derivative of a generating function. Since both the left- and the right-moving parts of the vertices (6.12) are quadratic in free fermions, this generating function is nothing but the weighted partition function[10]

$$\mathcal{Z}_{\rm het}(w,\vec v)=\int_{\mathcal{F}}\frac{d^2\tau}{2\tau_2}{\rm Tr}\Big(\frac{1+\hat g_0}{2}\,\frac{1+\hat g_1}{2}\,\frac{1+\hat g_2}{2}\,\frac{1+(-1)^F}{2}\;q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0-\frac{\bar c}{24}}\,{\rm e}^{2\pi{\rm i}[2\tau_2 w_s J^s_{\rm Lor}+v_I J^I_{\rm int}]}\Big) \tag{6.13}$$

As we know, there two possible tensor structures for the gauge kinetic terms: the single trace ${\rm tr}\,F^2$ term, and the double trace term $({\rm tr}\,F)^2$. The 1-loop thresholds for each structure can be read from appropriate combinations of v_I-derivatives and will be denoted again as Δ and Δ' respectively. To find them, it is enough to specify one Cartan direction along SO(4) and two Cartan directions along one of the four U(4)'s. Thus, we can fix for example $s=1$ and $I=1,2$, and define

$$\Delta_{IJ}=\frac{1}{4!(2\pi)^4}\,\partial^2_w\partial_{v_I}\partial_{v_J}\mathcal{Z}_{\rm het}(w,\vec v)\Big|_{w=\vec v=0} \tag{6.14}$$

[10] For convenience, in the exponentiation we have attached the factor of $2\tau_2$ of the gluon vertex (6.12) to the Lorentz current. We will instead keep track explicitly of the dimensional factors of $(2\pi\alpha')$.

JHEP05(2010)107

with $w = w_1$. Then the quadratic gauge thresholds Δ and Δ' for the two tensor structures can be obtained from the following relations

$$\begin{aligned}\frac{V_4}{16\pi^2}\,\Delta &= (2\pi\alpha')^2\,\big(\Delta_{11} - \Delta_{12}\big)\,,\\ \frac{V_4}{16\pi^2}\,\Delta' &= (2\pi\alpha')^2\,\Delta_{12}\ .\end{aligned} \tag{6.15}$$

We now give some details on the calculation of such quantities.

6.2.1 Calculation

Just like $\mathcal{Z}_{\rm het}$, also the weighted partition function (6.13) can be written as a sum over all projected and twisted sectors. Indeed, using the functions defined in appendix D, we have

$$\mathcal{Z}_{\rm het}(w,\vec{v}) = \frac{\nu_4}{2^4}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^3}\sum_{g_i,h_i=0}^{1}\rho\big[{}^{h_0}_{g_0}\big](w,\tau)\;\chi\big[{}^{h_0\,h_1\,h_2}_{g_0\,g_1\,g_2}\big](\vec{v},\bar\tau)\;\Gamma\big[{}^{h_0\,h_1\,h_2}_{g_0\,g_1\,g_2}\big](\tau,\bar\tau)\,, \tag{6.16}$$

which is a rather obvious generalization of the partition function (6.4). To proceed, it is convenient to organize the various contributions according to the orbits of the modular group. More precisely, exploiting the modular transformation properties of the various building blocks given in appendix D, one can show that the contributions coming from the sectors $\big[{}^{1\,h_1 h_2}_{0\,g_1 g_2}\big]$ and $\big[{}^{1\,h_1 h_2}_{1\,g_1 g_2}\big]$ can be obtained by applying, respectively, the S and TS transformations of the modular group to the amplitudes in the sectors $\big[{}^{0\,h_1 h_2}_{1\,g_1 g_2}\big]$. The latter can therefore be taken as representatives of a modular orbit, and the weighted partition function (6.16) can be rewritten as

$$\begin{aligned}\mathcal{Z}_{\rm het}(w,\vec{v}) = \frac{\nu_4}{2^4}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^3}\sum_{g_i,h_i=0}^{1}\Bigg[&\rho\big[{}^{0}_{0}\big](w,\tau)\;\chi\big[{}^{0\,h_1\,h_2}_{0\,g_1\,g_2}\big](\vec{v},\bar\tau)\;\Gamma\big[{}^{0\,h_1\,h_2}_{0\,g_1\,g_2}\big](\tau,\bar\tau)\\ &+\Big(\rho\big[{}^{0}_{1}\big](w,\tau)\;\chi\big[{}^{0\,h_1\,h_2}_{1\,g_1\,g_2}\big](\vec{v},\bar\tau)\;\Gamma\big[{}^{0\,h_1\,h_2}_{1\,g_1\,g_2}\big](\tau,\bar\tau)\,+\,{\rm orb}\Big)\Bigg]\,.\end{aligned} \tag{6.17}$$

Now we are ready to evaluate eq. (6.14). Using the Riemann identity for the ϑ-functions, we first rewrite the right-moving contributions as follows

$$\begin{aligned}\rho\big[{}^{0}_{0}\big](w,\tau) &= \frac{\vartheta_1(\tau_2 w)^4}{\eta^{12}}\,,\\ \rho\big[{}^{h_0}_{g_0}\big](w,\tau) &= -4\,{\rm e}^{{\rm i}\pi h_0}\,\frac{\vartheta_1(\tau_2 w)^2\,\vartheta\big[{}^{1+h_0}_{1+g_0}\big](\tau_2 w)^2}{\eta^6\,\vartheta\big[{}^{1+h_0}_{1+g_0}\big]^2}\qquad (g_0,h_0)\neq(0,0)\,,\end{aligned} \tag{6.18}$$

from which we easily find

$$\begin{aligned}\partial_w^2\rho\big[{}^{0}_{0}\big](w,\tau)\Big|_{w=0} &= 0\,,\\ \partial_w^2\rho\big[{}^{h_0}_{g_0}\big](w,\tau)\Big|_{w=0} &= -8\,(2\pi)^2\,{\rm e}^{{\rm i}\pi h_0}\,\tau_2^2\qquad (g_0,h_0)\neq(0,0)\ .\end{aligned} \tag{6.19}$$

Thus, only the sectors with $(g_0, h_0) \neq (0,0)$ contribute to the quadratic thresholds. Inserting these results in eq. (6.14) and using eq. (D.7), we have

$$\begin{aligned}\Delta_{IJ} &= -\frac{\nu_4}{4!\,8\pi^2}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{g_i,h_i=0}^{1}\left[\partial_{v_I}\partial_{v_J}\chi\left[{}^{0\,h_1\,h_2}_{1\,g_1\,g_2}\right](\vec{v},\bar{\tau})\Big|_{\vec{v}=0}\Gamma_{2,2}\left[{}^{h_1\,h_2}_{g_1\,g_2}\right](\tau,\bar{\tau};T,U) + \text{orb}\right]\\ &= -\frac{\nu_4}{4!\,8\pi^2}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{h_i=0}^{1}\left[\partial_{v_I}\partial_{v_J}\chi\left[{}^{0\,h_1\,h_2}_{1\;0\;\;,0}\right](\vec{v},\bar{\tau})\Big|_{\vec{v}=0}\sum_{g_i=0}^{1}\Gamma_{2,2}\left[{}^{h_1\,h_2}_{g_1\,g_2}\right](\tau,\bar{\tau};T,U) + \text{orb}\right]\\ &\equiv \sum_{h_i=0}^{1}\Delta_{IJ}^{(h_1\,h_2)}\end{aligned} \tag{6.20}$$

where the second line follows from the fact that the double derivatives of the χ-functions are independent of g_1 and g_2. We would like to stress that, unlike in the case of the partition function where only the untwisted sectors with $(h_1, h_2) = (0,0)$ were relevant to derive the massless spectrum, in the threshold calculation all sectors, including the twisted ones, contribute. In the following we will in turn analyse the two types of contributions, the untwisted one arising from sectors with $(h_1, h_2) = (0,0)$ and the twisted ones arising from sectors with $(h_1, h_2) \neq (0,0)$.

Orbits of $\boldsymbol{\chi}\left[{}^{000}_{100}\right]$. In this case we have

$$\Delta_{IJ}^{(0\,0)} = -\frac{\nu_4}{4!}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left[\chi_{IJ}(\bar{\tau})\sum_{g_i=0}^{1}\Gamma_{2,2}\left[{}^{0\;\;0}_{g_1\,g_2}\right](\tau,\bar{\tau};T,U) + \text{orb}\right] \tag{6.21}$$

where we have introduced the notation

$$\chi_{IJ}(\bar{\tau}) = \frac{1}{8\pi^2}\,\partial_{v_I}\partial_{v_J}\chi\left[{}^{000}_{100}\right](\vec{v},\bar{\tau})\Big|_{\vec{v}=0}\,. \tag{6.22}$$

Using the summation identity (see also eq. (A.16))

$$\sum_{g_i=0}^{1}\Gamma_{2,2}\left[{}^{0\;\;0}_{g_1\,g_2}\right](\tau,\bar{\tau};T,U) = 2\,\Gamma_{2,2}\left(2\tau, 2\bar{\tau}; \frac{T}{2}, U\right)\,, \tag{6.23}$$

and exploiting the modular properties of the right hand side, we have

$$\begin{aligned}\Delta_{IJ}^{(0\,0)} &= -\frac{2\nu_4}{4!}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left[\chi_{IJ}(\bar{\tau}) + \text{orb}\right]\Gamma_{2,2}\left(2\tau, 2\bar{\tau}; \frac{T}{2}, U\right)\\ &= -\frac{2\nu_4}{4!}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,c_{IJ}(\bar{\tau})\,\Gamma_{2,2}\left(\tau, \bar{\tau}; \frac{T}{2}, U\right)\end{aligned} \tag{6.24}$$

where

$$c_{IJ}(\bar{\tau}) = \left[\chi_{IJ}\left(\frac{\bar{\tau}}{2}\right) + \chi_{IJ}\left(\frac{\bar{\tau}+1}{2}\right) + \chi_{IJ}\left(-\frac{1}{2\bar{\tau}}\right)\right] \tag{6.25}$$

is simply twice the Hecke operator $H_{\Gamma^-}(\chi_{IJ})$. It is important to note that c_{IJ} are modular forms of weight zero with no poles and therefore are constants. Indeed, as shown in appendix D (see eq. (D.14)), it turns out that

$$c_{IJ}(\bar{\tau}) = 6\,, \tag{6.26}$$

so that to obtain $\Delta_{IJ}^{(0\,0)}$ we may simply use the general integration formula [50]

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \Gamma_{2,2}(\tau,\bar\tau;T,U) = -\log\left(c\, T_2\, U_2 \left|\eta\,(T)\,\eta(U)\right|^4\right) \tag{6.27}$$

with $c = \frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}}$. Putting everything together and discarding all moduli independent terms, we finally obtain

$$\Delta_{IJ}^{(0\,0)} = \frac{\nu_4}{2}\,\log\left(T_2\, U_2 \left|\eta\left(\frac{T}{2}\right)\eta\,(U)\right|^4\right)\ . \tag{6.28}$$

Orbits of $\chi\left[{0\,h_1\,h_2 \atop 1\;0\;\;0}\right]$ with $(h_1,h_2)\neq(0,0)$. In this case, from eq. (6.20) we have

$$\Delta_{IJ}^{(h_1\,h_2)} = -\frac{\nu_4}{4!}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left[\chi_{IJ}^{(h_1\,h_2)}(\bar\tau)\sum_{g_i=0}^{1}\Gamma_{2,2}\left[{h_1\,h_2 \atop g_1\,g_2}\right](\tau,\bar\tau;T,U)\,+\,\mathrm{orb}\right] \tag{6.29}$$

where

$$\chi_{IJ}^{(h_1\,h_2)}(\bar\tau) = \frac{1}{8\pi^2}\,\partial_{v_I}\partial_{v_J}\chi\left[{0\,h_1\,h_2 \atop 1\;0\;\;0}\right](\vec v,\bar\tau)\Big|_{\vec v=0} \tag{6.30}$$

for any $(h_1,h_2)\neq(0,0)$. Actually, as shown in appendix D, it turns out that these functions are constants, namely

$$\chi_{11}^{(h_1\,h_2)}(\bar\tau) = -\chi_{12}^{(h_1\,h_2)}(\bar\tau) = 2\,, \tag{6.31}$$

so that the calculation of $\Delta_{IJ}^{(h_1\,h_2)}$ drastically simplifies. Furthermore, if we use the summation identity (see also eq. (A.17))

$$\sum_{h_i,g_i=0}^{1}{}'\ \Gamma_{2,2}\left[{h_1\,h_2 \atop g_1\,g_2}\right](\tau,\bar\tau;T,U) = 4\Gamma_{2,2}\left(\tau,\bar\tau;\frac{T}{4},U\right) - 2\Gamma_{2,2}\left(2\tau,2\bar\tau;\frac{T}{2},U\right) \tag{6.32}$$

where the $'$ means that $(h_1,h_2)=(0,0)$ is excluded from the sum, we obtain

$$\begin{aligned}\sum_{h_i,=0}^{1}{}'\ \Delta_{11}^{(h_1\,h_2)} &= -\sum_{h_i,=0}^{1}{}'\ \Delta_{12}^{(h_1\,h_2)} \\ &= -\frac{\nu_4}{2}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left[2\Gamma_{2,2}\left(\tau,\bar\tau;\frac{T}{4},U\right)-\Gamma_{2,2}\left(2\tau,2\bar\tau;\frac{T}{2},U\right)\right] \\ &= \frac{\nu_4}{2}\,\log\left(T_2\,U_2\,|\eta(U)|^4\,\frac{|\eta(\frac{T}{4})|^8}{|\eta(\frac{T}{2})|^4}\right)\end{aligned} \tag{6.33}$$

up to moduli independent terms.

6.2.2 Results

Now we can collect all results and obtain the final expressions for the quadratic gauge thresholds in our heterotic model. Plugging the contributions of the two types of orbits (6.28) and (6.33) into eq. (6.20), using eq. (6.15) and recalling that $\nu_4 = V_4/(4\pi^2\alpha')^2$, we find

$$\begin{aligned} \Delta &= 4\log\left(T_2\,U_2\,|\eta(U)|^4\,\frac{|\eta(\frac{T}{4})|^8}{|\eta(\frac{T}{2})|^4}\right), \\ \Delta' &= 4\log\left(\frac{|\eta(\frac{T}{2})|^4}{|\eta(\frac{T}{4})|^4}\right). \end{aligned} \tag{6.34}$$

It is interesting to observe that these expressions are invariant under the following target-space modular transformations

$$\Gamma^0(4)_T \otimes \Gamma_U \tag{6.35}$$

where Γ_U is the standard modular group acting on U, while $\Gamma^0(4)_T$ is the subgroup of the modular transformations on T of the form

$$T \;\to\; \frac{aT+b}{cT+d} \qquad \text{with}\;\; a,b,c,d\in\mathbb{Z}\,,\;\;\; ad-bc=1 \;\;\text{and}\;\; b=0 \bmod 4\,. \tag{6.36}$$

The target-space modular transformations (6.35) are those that are consistent with the Wilson lines which break U(16) to U(4)^4. Any meaningful string amplitude should therefore be invariant under such transformations.

6.3 Holomorphic gauge couplings and duality check

To obtain from the above thresholds the holomorphic gauge couplings of the heterotic model, we can follow the same reasoning described in section 3.2 for the dual type I$'$ theory. We only have to remember that in the heterotic set-up the bulk Kähler potential reads as

$$K = -\log S_2 - \sum_{i=1}^{3}\log\left(T_2^{(i)}\,U_2^{(i)}\right), \tag{6.37}$$

where S_2, related to the four-dimensional dilaton ϕ_4 by $S_2 = \mathrm{e}^{-2\phi_4}$, is the imaginary part of the chiral superfield S which determines the holomorphic coupling function at tree level and plays the same rôle as the t superfield of the type I$'$ theory, and that

$$\widehat{K} = -\log\left(T_2\,U_2\right). \tag{6.38}$$

Then, as shown in appendix E, the 1-loop contributions $f_{(1)}$ and $f'_{(1)}$ are given by the same relations (3.19), now expressed in terms of the heterotic variables [50, 51]. In particular, for the single-trace Yang-Mills term, using eq. (6.34) and recalling that $b=0$, we have

$$\mathrm{Re}\,f_{(1)} = \Delta + \Delta_{\rm univ} + b\,\widehat{K} = 4\log\left(T_2\,U_2\,|\eta(U)|^4\,\frac{|\eta(\frac{T}{4})|^8}{|\eta(\frac{T}{2})|^4}\right) + \Delta_{\rm univ}\,. \tag{6.39}$$

It is important to stress that the universal term $\Delta_{\rm univ}$ is related to the 1-loop corrections of the Kähler potential, which in the heterotic setup is of order $(g_s)^0$ [50, 68], and, like any meaningful amplitude, it must respect all symmetries of the compactification manifold including the target-space modular invariance (6.35). From these considerations we are then led to write

$$\mathrm{Re}\, f_{(1)} = 4 \log \left(\frac{|\eta(\frac{T}{4})|^4}{|\eta(\frac{T}{2})|^4} \right) , \tag{6.40}$$

and

$$\Delta_{\mathrm{univ}} = -4 \log \Big(T_2\, U_2\, |\eta(U)|^4\, |\eta(\frac{T}{4})|^4 \Big) . \tag{6.41}$$

Likewise, for the double-trace coupling we have

$$\mathrm{Re}\, f'_{(1)} = \Delta' + \Delta_{\mathrm{univ}} + b'\, \widehat{K} = -4 \log |\eta(U)|^4 + 4 \log \left(\frac{|\eta(\frac{T}{2})|^4}{|\eta(\frac{T}{4})|^8} \right) \tag{6.42}$$

where the second equality follows upon using $b' = -4$ and eqs. (6.34) and (6.38). Notice that all non-holomorphic terms correctly compensate each other and yield a holomorphic result for $f'_{(1)}$. This is an *a posteriori* confirmation of the universal term (6.41).

The heterotic holomorphic couplings (6.40) and (6.42) are exact and do not receive any kind of corrections beyond 1-loop. Therefore, when translated with the duality map to the type I′ theory, they must contain all information, both perturbative and non-perturbative, on the corresponding type I′ couplings, including the (exotic) instanton corrections computed in section 5. We now show that this is indeed what happens.

Under the heterotic/type I′ duality map, the Kähler modulus of the heterotic theory T is mapped into the axio-dilaton λ of the type I′ model according to (see also ref. [36])

$$\frac{T}{4} \longleftrightarrow \lambda . \tag{6.43}$$

Thus, the weak coupling regime $g_s \sim 1/\lambda_2 \to 0$ in type I′ can be recovered from the large volume expansion $T_2 \to \infty$ of the heterotic theory and *viceversa*. Expanding eqs. (6.40) and (6.42) for large T_2, we find

$$\begin{aligned} \mathrm{Re}\, f_{(1)} &= \frac{\pi}{3} T_2 + 8 \sum_{k=1}^{\infty} \left[\sum_{d|k} \frac{1}{d} \Big(\mathrm{e}^{2\pi \mathrm{i} k \frac{T}{4}} - \mathrm{e}^{2\pi \mathrm{i} k \frac{T}{2}} \Big) + \mathrm{h.c.} \right] , \\ \mathrm{Re}\, f'_{(1)} &= -4 \log |\eta(U)|^4 + 8 \sum_{k=1}^{\infty} \left[\sum_{d|k} \frac{1}{d} \Big(\mathrm{e}^{2\pi \mathrm{i} k \frac{T}{2}} - 2\mathrm{e}^{2\pi \mathrm{i} k \frac{T}{4}} \Big) + \mathrm{h.c.} \right] . \end{aligned} \tag{6.44}$$

After translating into the type I′ variables $q = \mathrm{e}^{\pi \mathrm{i} \lambda} = \mathrm{e}^{\pi \mathrm{i} \frac{T}{4}}$, these formulas predict a tree-level term proportional to

$$\lambda_2 \, \mathrm{tr}\, F^2 , \tag{6.45}$$

a 1-loop contribution

$$-\, 4 \big(\log |\eta(U)|^4 \big) \big(\mathrm{tr}\, F \big)^2 , \tag{6.46}$$

which agrees with the perturbative type I′ result (3.21), as well as a series of instanton-like contributions with even instanton numbers, correcting both the single and the double trace gauge kinetic functions, with a leading term proportional to

$$q^2 \Big(\mathrm{tr}\, F^2 - 2 \big(\mathrm{tr}\, F \big)^2 \Big) . \tag{6.47}$$

The relative coefficient between the two trace structures is in perfect agreement with the multi-instanton calculus of the type I′ theory, as one can see from eq. (5.27). Also the absence of odd instanton corrections is in agreement with the results found in section 5.

The presence of the tree-level term (6.45) proportional to λ_2 may seem puzzling at first sight, since λ is the chiral field accounting for the gauge coupling on a D3-brane and not on a D7-brane. The same type of contribution was found also for the T-dual version of our model in ref. [53] where a convincing explanation for its presence was given. Indeed, it was argued that since the gauge branes are entirely wrapped over the orbifold $\mathcal{T}_4/\mathbb{Z}_2$, their gauge kinetic function, besides the usual "untwisted" contribution, should receive also "twisted" contributions from the exceptional 2-cycles at the orbifold fixed points, where a hidden non-trivial U(1) gauge bundle is localized. In the case of fractional D7-branes this mechanism is responsible for gauge coupling corrections proportional to λ_2. It would be very interesting to explicitly derive this result from disk scattering amplitudes involving twisted fields at the orbifold fixed points.

7 Conclusions

It is by now clear that "exotic" instanton corrections to the effective actions of D-brane worlds can have relevant consequences. Our work has been motivated by the importance of putting on firm grounds the techniques to compute such effects in a four-dimensional context. Let us summarize here our results.

We considered a type I′/heterotic dual pair realizing a $\mathcal{N}=2$ super-conformal gauge theory in four dimensions with gauge group U(4) and a matter content made of four fundamentals plus one antisymmetric hyper-multiplet and its conjugate. The type I′ model is built with D7- and D3-branes in a $\mathcal{T}_4/\mathbb{Z}_2\times\mathcal{T}_2$ background with O7- and O3-planes. In this setup, the U(4) gauge theory lives in the uncompactified part of the world-volume of the D7-branes on top of one of the O7-planes. On the heterotic side this $\mathcal{N}=2$ vacuum is realized starting from the U(16) heterotic model on $\mathcal{T}_4/\mathbb{Z}_2$ after a further compactification on $\mathcal{T}_2$ with non-trivial Wilson lines breaking U(16) down to $\mathrm{U}(4)^4$.

In both settings we studied the terms of the low-energy effective action quadratic in the gauge field strength plus their supersymmetric completion, namely

$$S=\frac{1}{8\pi}\int d^4x\left[\mathrm{Re}f\;\mathrm{tr}\,F^2+\mathrm{Re}f'\left(\mathrm{tr}\,F\right)^2\right]+\cdots \tag{7.1}$$

where f and f' are the Wilsonian couplings. These are *holomorphic* functions of the bulk moduli, which depend on the following set of variables

$$\begin{aligned}\text{type I}' \quad &: \quad \big(t,\lambda,U\big)\,,\\ \text{heterotic} \quad &: \quad \big(S,T,U\big)\,.\end{aligned} \tag{7.2}$$

Here t and S represent the tree-level Yang-Mills coupling in the type I′ and heterotic setups, respectively, λ is the axio-dilaton of type I′, and T and U are the Kähler and complex structures of $\mathcal{T}_2$.

JHEP05(2010)107

In the type I$'$ side, the holomorphic functions f and f' get contributions at the tree-level, at 1-loop and also from D(−1)-branes, which represent exotic instantons for this system. We computed such non-perturbative corrections by means of localization techniques for the integration over the exotic moduli space up to $k = 3$ instantons. Altogether we obtained

$$\begin{aligned} f &= -\mathrm{i}t + \alpha\, q^2 + O(q^4) \ , \\ f' &= -8 \log \eta(U)^2 - 2\alpha\, q^2 + O(q^4) \ . \end{aligned} \tag{7.3}$$

where $q = \exp(\pi \mathrm{i} \lambda)$ and α is a coefficient related to the overall normalization of the measure of the exotic instanton moduli space.

In the heterotic side, instead, the holomorphic couplings f and f' are exact at 1-loop due to their BPS nature. We determined them by computing the 1-loop threshold corrections finding

$$\begin{aligned} f &= -\mathrm{i}S + 8 \log \left(\frac{\eta(\frac{T}{4})^2}{\eta(\frac{T}{2})^2} \right) , \\ f' &= -8 \log \eta(U)^2 + 8 \log \left(\frac{\eta(\frac{T}{2})^2}{\eta(\frac{T}{4})^4} \right) \ . \end{aligned} \tag{7.4}$$

Expanding for large values of T and using the duality map (6.43), these heterotic formulas predict no instanton corrections at $k = 1$ and $k = 3$, and a relative coefficient -2 between the $k = 2$ corrections to f and f', in perfect agreement with the results obtained in the type I$'$ setting. Moreover, the precise match of the 1-loop terms of f' between (7.3) and (7.4) can be taken as a strong evidence that the overall normalization of our coupling functions is the same in the two settings, thus providing an indirect way to fix the numerical factor α. We regard these results as a nice and non-trivial confirmation of the validity of the exotic instanton calculus, which can then be applied with confidence also to four-dimensional theories and to models for which the heterotic dual is not known or does not exist.

We think there are several lessons to be learned from our computations and several new directions which deserve to be explored in the light of our results. In first place, the presence of branes with different world-volume dimensions implies that the standard prescription of localization in four and eight dimensions needs to be changed in a non trivial way in order to extract the corrections for the gauge couplings. In second place, the heterotic computation gives a result for arbitrary instanton numbers, while the explicit integration over the instanton moduli space could be computed only to the order $k = 3$. More extended checks of this duality would be desirable. This could be achievable by noticing that if one considers the D7-branes as non dynamical, then the exotic instanton partition function can be reinterpreted as the ordinary gauge instanton partition function for the four-dimensional theory living on the D3-branes with fundamental matter hyper-multiplets having masses given by the positions of the D7-branes. An analysis to extract some sort of Seiberg-Witten curve in this case seems to be possible even if, as we have discussed, this is a rather unconventional extrapolation of the standard field theory notions. We leave this kind of analysis for future work.

Acknowledgments

We thank P. Camara, E. Dudas, L. Gallot and I. Pesando for several useful discussions. The work of R.P. was partially supported by the European Commission FP7 Programme Marie Curie Grant Agreement PIIF-GA-2008-221571, and the Institutional Partnership grant of the Humboldt Foundation of Germany. The work of F.F. and J.F.M. was partially supported by the ERC Advanced Grant n.226455, “Supersymmetry, Quantum Gravity and Gauge Fields” (SUPERFIELDS) and by the Italian MIUR-PRIN contract 20075ATT78.

A Zero-mode traces and lattice sums

In this appendix we collect some formulas on the traces over the bosonic zero-modes and the lattice sums that are useful for the calculations of the 1-loop threshold corrections, both in the type I′ set-up and in the heterotic model.

Closed strings. In untwisted sectors the bosonic string coordinates have zero-modes $x_{\rm R}$, $x_{\rm L}$, $p_{\rm R}$ and $p_{\rm L}$ that contribute to the Virasoro characters since

$$L_0 = \frac{1}{2}\,p_{\rm R}^2 + \text{osc.}\,, \quad \bar{L}_0 = \frac{1}{2}\,p_{\rm L}^2 + \text{osc.}\ . \tag{A.1}$$

For d real non-compact directions, the right- and left-moving momenta are $(p_L)_\mu = (p_R)_\mu = \sqrt{\frac{\alpha'}{2}}k_\mu$, with k_μ a continuous variable, and thus the zero-mode contribution to $\mathrm{Tr}\, q^{L_0}\bar{q}^{\bar{L}_0}$ is simply

$$\int d^dx \int \frac{d^dk}{(2\pi)^d}\ \mathrm{e}^{-\pi\alpha'\tau_2\, k^2} = \frac{V_d}{(4\pi^2\alpha'\tau_2)^{\frac{d}{2}}} \tag{A.2}$$

where we have set $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ with $\tau = \tau_1 + \mathrm{i}\tau_2$.

Now let us consider two directions compactified on a 2-torus $\mathcal{T}_2$. They can be described by two periodic coordinates $x^i \in [0, \sqrt{\alpha'}]$, a complex structure U and a (complexified) Kähler parameter T which are encoded in the metric G_{ij} and in the Kalb-Ramond field B_{ij}, according to

$$G = \frac{T_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}\,, \quad B = \begin{pmatrix} 0 & -T_1 \\ T_1 & 0 \end{pmatrix}\ . \tag{A.3}$$

In the following we will denote as G^{ij} the components of the inverse metric

$$G^{-1} = \frac{1}{T_2U_2}\begin{pmatrix} |U|^2 & -U_1 \\ -U_1 & 1 \end{pmatrix}\ . \tag{A.4}$$

In this case, the right and left bosonic zero-modes are given by

$$(p_L)_i = \frac{1}{\sqrt{2}}\big(n_i - (G-B)_{ij}w^j\big)\,, \quad (p_R)_i = \frac{1}{\sqrt{2}}\big(n_i + (G+B)_{ij}w^j\big) \tag{A.5}$$

with $n_i, w^i \in \mathbb{Z}$, and eq. (A.1) should actually read

$$L_0 = \frac{1}{2}(p_R)_iG^{ij}(p_R)_j + \text{osc.}\,, \quad \bar{L}_0 = \frac{1}{2}(p_L)_iG^{ij}(p_L)_j + \text{osc.} \tag{A.6}$$

Thus, the zero-mode contribution to the Virasoro character $\mathrm{Tr}\, q^{L_0}\bar{q}^{\bar{L}_0}$ becomes

$$\sum_{(\vec{n},\vec{w})\in\mathbb{Z}^4} \mathrm{e}^{-\pi\tau_2 n_i G^{ij} n_j + 2\pi\mathrm{i}\tau_1 w^i n_i + 2\pi\tau_2 w^i (BG^{-1})_i{}^j n_j - \pi\tau_2 w^i (G-BG^{-1}B)_{ij} w^j} \,. \tag{A.7}$$

Utilizing the Poisson resummation formula

$$\sum_{\vec{n}\in\mathbb{Z}^2} \mathrm{e}^{-\pi\, n^T X n + 2\pi\mathrm{i} Y^T n} = \frac{1}{\sqrt{\det X}} \sum_{\vec{m}\in\mathbb{Z}^2} \mathrm{e}^{-\pi(m-Y)^T\cdot X^{-1}(m-Y)} \tag{A.8}$$

and the explicit form of the torus metric and B-field, we can rewrite eq. (A.7) as

$$\frac{1}{\tau_2}\,\Gamma_{2,2}(\tau,\bar{\tau};T,U)\,, \tag{A.9}$$

where we have defined

$$\begin{aligned}\Gamma_{2,2}(\tau,\bar{\tau};T,U) &= T_2 \sum_{(\vec{m},\vec{w})\in\mathbb{Z}^4} \mathrm{e}^{-\frac{\pi}{\tau_2}(\tau w^i - m^i)(G-B)_{ij}(\bar{\tau} w^j - m^j)} \\ &= T_2 \sum_M \mathrm{e}^{2\pi\mathrm{i} T \det M}\, \mathrm{e}^{-\frac{\pi T_2}{\tau_2 U_2}|(1\, U)\, M\, \binom{\tau}{-1}|^2}\end{aligned} \tag{A.10}$$

with

$$M = \begin{pmatrix} w^1 & m^1 \\ w^2 & m^2 \end{pmatrix} \,. \tag{A.11}$$

In the right-hand sides of eq. (A.10) the prefactor is just the volume of the torus (in units of α'), since $\int d^2x\sqrt{\det G} = \alpha'\, T_2$.

By suitably reshuffling the summation variables, it is easy to check that $\Gamma_{2,2}$ is invariant under the modular group acting on the world-sheet parameter τ; indeed

$$\begin{aligned}\Gamma_{2,2}(\tau+1,\bar{\tau}+1;T,U) &= \Gamma_{2,2}(\tau,\bar{\tau};T,U)\,, \\ \Gamma_{2,2}(-1/\tau,-1/\bar{\tau};T,U) &= \Gamma_{2,2}(\tau,\bar{\tau};T,U)\,.\end{aligned} \tag{A.12}$$

In presence of Wilson lines and/or insertions of projection operators, the lattice sum corresponding to the trace over bosonic zero-modes is formally identical to eq. (A.10), but with

$$M = \begin{pmatrix} w^1 + \frac{h_1}{2} & m^1 + \frac{g_1}{2} \\ w^2 + \frac{h_2}{2} & m^2 + \frac{g_2}{2} \end{pmatrix}\,, \qquad (m^i, w^i)\in\mathbb{Z}^2 \tag{A.13}$$

where the parameters g_i and h_i depend on the type of Wilson lines or projection operators. To explicitly exhibit such a dependence we introduce the notation

$$\Gamma_{2,2}\big[{}^{h_1\, h_2}_{g_1\, g_2}\big](\tau,\bar{\tau};T,U) \tag{A.14}$$

to denote the lattice sum on the torus. Of course, we have $\Gamma_{2,2}\big[{}^{0\,0}_{0\,0}\big] \equiv \Gamma_{2,2}$.

Under the world-sheet modular group, the lattice sum (A.14) has the following transformation properties

$$\begin{aligned}\Gamma_{2,2}\big[{}^{h_1\, h_2}_{g_1\, g_2}\big](\tau+1,\bar{\tau}+1;T,U) &= \Gamma_{2,2}\big[{}^{\;\;h_1\;\;\;\;\;\; h_2}_{g_1+h_1\; g_2+h_2}\big](\tau,\bar{\tau};T,U)\,, \\ \Gamma_{2,2}\big[{}^{h_1\, h_2}_{g_1\, g_2}\big](-1/\tau,-1/\bar{\tau};T,U) &= \Gamma_{2,2}\big[{}^{g_1\, g_2}_{h_1\, h_2}\big](\tau,\bar{\tau};T,U)\,,\end{aligned} \tag{A.15}$$

which are a generalization of those in eq. (A.12).

In the heterotic threshold computation of section 6, the shifts h_i and g_i take the values 0 or 1 only, as they arise from a $\mathbb{Z}_2$ (freely-acting) orbifold procedure. In particular, $h_i = 0$ and $h_i = 1$ correspond, respectively, to untwisted and twisted sectors, while $g_i = 0$ and $g_i = 1$ indicate to the absence or the presence of the projection operator. In this case there are some useful summation identities; in particular we have

$$\begin{aligned}
&\sum_{g_i=0}^{1} \Gamma_{2,2}\big[{}^{h_1\, h_2}_{g_1\, g_2}\big](\tau,\bar\tau;T,U) = 2\,\Gamma_{2,2}\big[{}^{h_1\, h_2}_{0\ \ 0}\big]\left(2\tau,2\bar\tau;\frac{T}{2},U\right)\,,\\
&\sum_{h_i=0}^{1} \Gamma_{2,2}\big[{}^{h_1\, h_2}_{g_1\, g_2}\big](\tau,\bar\tau;T,U) = 2\,\Gamma_{2,2}\big[{}^{0\ \ 0}_{g_1\, g_2}\big]\left(\frac{\tau}{2},\frac{\bar\tau}{2};\frac{T}{2},U\right)\,.
\end{aligned} \tag{A.16}$$

Using these identities, we also find

$$\begin{aligned}
&\sum_{g_i=0}^{1}\Big(\Gamma_{2,2}\big[{}^{0\ \ 1}_{g_1\, g_2}\big](\tau,\bar\tau;T,U)+\Gamma_{2,2}\big[{}^{1\ \ 0}_{g_1\, g_2}\big](\tau,\bar\tau;T,U)+\Gamma_{2,2}\big[{}^{1\ \ 1}_{g_1\, g_2}\big](\tau,\bar\tau;T,U)\Big)\\
&\qquad = 2\sum_{h_i=0}^{1}\Gamma_{2,2}\big[{}^{h_1\, h_2}_{0\ \ 0}\big]\left(2\tau,2\bar\tau;\frac{T}{2},U\right) - 2\,\Gamma_{2,2}\big[{}^{0\,0}_{0\,0}\big]\left(2\tau,2\bar\tau;\frac{T}{2},U\right)\\
&\qquad = 4\,\Gamma_{2,2}\left(\tau,\bar\tau;\frac{T}{4},U\right) - 2\,\Gamma_{2,2}\left(2\tau,2\bar\tau;\frac{T}{2},U\right)\,.
\end{aligned} \tag{A.17}$$

Open strings. Also the bosonic zero-modes x and p of the open string coordinates contribute to the Virasoro characters since

$$L_0 = \frac{1}{2}\,p^2 + \text{osc.}\,. \tag{A.18}$$

In the case of d real non-compact directions with Neumann-Neumann (NN) boundary conditions, we have $p_\mu = \sqrt{2\alpha'}k_\mu$, with k_μ a continuous variable, and thus the zero-mode contribution to $\mathrm{Tr}\,q^{L_0}$ is simply

$$\int d^d x \int \frac{d^d k}{(2\pi)^d}\ \mathrm{e}^{-\pi\alpha'\tau_2\, k^2} = \frac{V_d}{(4\pi^2\alpha'\tau_2)^{\frac{d}{2}}} \equiv \frac{\nu_d}{(\tau_2)^{\frac{d}{2}}} \tag{A.19}$$

where we have set $q = \mathrm{e}^{-\pi\tau_2}$ with τ_2 being the real modular parameter of an annulus. This expression is formally identical to the closed string one in eq. (A.2).

Now consider a pair of directions compactified on a 2-torus $\mathcal{T}_2$. If these directions have NN boundary conditions, the open strings carry a quantized momentum given by $p_i = \sqrt{2}\,n_i$, with $n_i \in \mathbb{Z}$, and the Virasoro generator becomes

$$L_0 = n_i\,\mathcal{G}^{ij}\,n_j + \text{osc.}\,, \tag{A.20}$$

where $\mathcal{G}^{ij}$ is the inverse of the open string metric [69]

$$\mathcal{G}_{ij} = (G_{ik}+B_{ik})G^{kl}(G_{lj}-B_{lj})\,. \tag{A.21}$$

In matrix form we have

$$\mathcal{G} = \frac{|T|^2}{T_2 U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} , \quad \mathcal{G}^{-1} = \frac{T_2}{|T|^2 U_2} \begin{pmatrix} |U|^2 & -U_1 \\ -U_1 & 1 \end{pmatrix} . \tag{A.22}$$

Then, the zero-mode contribution to the partition function reads

$$P(\tau_2; T, U) = \sum_{\vec{n} \in \mathbb{Z}^2} \mathrm{e}^{-\pi\tau_2\, n_i \mathcal{G}^{ij} n_j} = \sum_{\vec{n} \in \mathbb{Z}^2} \mathrm{e}^{-\pi\tau_2 \frac{|n_1 U - n_2|^2 T_2}{|T|^2 U_2}} . \tag{A.23}$$

In the case of open strings with Dirichlet-Dirichlet (DD) boundary conditions, the bosonic zero-modes account for the integer windings w^i around the torus, and the Virasoro operator becomes

$$L_0 = w^i G_{ij} w^j + \text{osc.} . \tag{A.24}$$

Thus, the zero-mode trace for two compact DD directions is

$$W(\tau_2; T_2, U) = \sum_{\vec{w} \in \mathbb{Z}^2} \mathrm{e}^{-\pi\tau_2\, w^i G_{ij} w^j} = \sum_{\vec{w} \in \mathbb{Z}^2} \mathrm{e}^{-\pi\tau_2 \frac{|w^1 + w^2 U|^2 T_2}{U_2}} . \tag{A.25}$$

Notice that W does not depend on the B field and hence on T_1, as opposed to what happens for P. Notice also that the two functions P and W are related to each other by T-duality. Indeed, performing a T-duality along the directions of $\mathcal{T}_2$, one exchanges NN with DD boundary conditions and makes the following replacements

$$U \longrightarrow -\frac{1}{U} , \quad T \longrightarrow -\frac{1}{T} , \tag{A.26}$$

under which P and W are mapped to each other as one can easily check from the explicit expressions given above.

If the DD string endpoints are separated by a distance $\vec{v}$ along $\mathcal{T}_2$, the trace (A.25) generalizes to

$$W_{\vec{v}}(\tau_2; T_2, U) = \sum_{\vec{w} \in \mathbb{Z}^2} \mathrm{e}^{-\pi\tau_2\, (w^i - v^i) G_{ij} (w^j - v^j)} . \tag{A.27}$$

Furthermore, all these formulas can be easily generalized to higher dimensional factorized tori. For example, for a 4-torus $\mathcal{T}_2^{(1)} \times \mathcal{T}_2^{(2)}$ the lattice sums over momentum and winding modes become

$$P_4(\tau_2) = \prod_{i=1}^{2} P(\tau_2; T^{(i)}, U^{(i)}) \quad \text{and} \quad W_4(\tau_2) = \prod_{i=1}^{2} W(\tau_2; T_2^{(i)}, U^{(i)}) . \tag{A.28}$$

B Details on the type I′ computations

In this appendix we provide some details on the perturbative computations performed in the type I′ model to recover the tadpole cancellation conditions and the 1-loop corrections to the D7-brane effective action.

JHEP05(2010)107

Preliminarly, we note that the 1-loop amplitudes can be conveniently written in terms of the SO(4) level-one characters defined by

$$O_4 = \frac{\vartheta_3^2 + \vartheta_4^2}{2\eta^2}\,, \quad V_4 = \frac{\vartheta_3^2 - \vartheta_4^2}{2\eta^2}\,, \quad S_4 = \frac{\vartheta_2^2 - \vartheta_1^2}{2\eta^2}\,, \quad C_4 = \frac{\vartheta_2^2 + \vartheta_1^2}{2\eta^2}\,, \tag{B.1}$$

where the ϑ-functions and their properties are collected in appendix F. On these characters the generators T and S of the modular group are represented by the following matrices

$$T = \mathrm{e}^{-\frac{\pi \mathrm{i}}{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix}\,, \quad S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\,. \tag{B.2}$$

Consequently, the transformation $P = TST^2S$, which relates the Möbius amplitudes in the direct and transverse channels according to

$$\frac{\mathrm{i}\tau_2}{2} + \frac{1}{2} = \frac{i}{4\ell} + \frac{1}{2} = P\left(i\ell + \frac{1}{2}\right)\,, \tag{B.3}$$

is represented on the characters by the matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\,. \tag{B.4}$$

In addition we introduce the supersymmetric combinations of characters

$$\begin{aligned} Q_o &= V_4 O_4 - C_4 C_4\,, \quad Q_v = O_4 V_4 - S_4 S_4\,, \\ Q_s &= O_4 C_4 - S_4 O_4\,, \quad Q_c = V_4 S_4 - C_4 V_4\,, \end{aligned} \tag{B.5}$$

which have a definite parity under the $\mathbb{Z}_2$-orbifold generators. These characters transform under S and P in the same way as O_4, V_4, S_4 and C_4 respectively. Their expression in terms of ϑ-functions is

$$\begin{aligned} Q_o + Q_v &= \frac{\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4}{2\eta^4} = -\frac{\vartheta_1^4}{\eta^4}\,, \\ Q_o - Q_v &= \frac{\vartheta_3^2\vartheta_4^2 - \vartheta_4^2\vartheta_3^2}{2\eta^4} = -\frac{\vartheta_1^2\,\vartheta_2^2}{\eta^4}\,, \\ Q_s + Q_c &= \frac{\vartheta_3^2\vartheta_2^2 - \vartheta_2^2\vartheta_3^2}{2\eta^4} = -\frac{\vartheta_1^2\,\vartheta_4^2}{\eta^4}\,, \\ Q_s - Q_c &= \frac{\vartheta_4^2\vartheta_2^2 - \vartheta_2^2\vartheta_4^2}{2\eta^4} = -\frac{\vartheta_1^2\,\vartheta_3^2}{\eta^4} \end{aligned} \tag{B.6}$$

where the second equalities in the right-hand sides follow from the Riemann identity (F.3). Finally, it is also useful to recall the modular transformations of the Dedekin η-function

$$\begin{aligned} \eta(\tau+1) &= \mathrm{e}^{\frac{\mathrm{i}\pi}{12}}\,\eta(\tau)\,, \\ \eta\left(-1/\tau\right) &= \sqrt{-\mathrm{i}\tau}\;\eta(\tau)\,, \\ \eta\left(\frac{\mathrm{i}\tau_2}{2} + \frac{1}{2}\right) &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}}\,\sqrt{\mathrm{i}/\tau_2}\;\eta\left(\frac{\mathrm{i}}{2\tau_2} + \frac{1}{2}\right)\,. \end{aligned} \tag{B.7}$$

B.1 Tadpole cancellation

The tadpole constraints arise from the analysis of the Klein bottle amplitude and of the annuli and Möbius diagrams with boundaries on D7- and/or on D3-branes. For each boundary these 1-loop diagrams contain a trace over the corresponding CP indices and hence, when the orbifold/orientifold generators are inserted, also a trace on the corresponding γ matrices. All such amplitudes can be constructed in a rather straightforward manner by collecting such CP factors and the traces over zero- and non-zero-modes. Here we give some details for the various amplitudes in the direct (open string) channel, and then perform a modular transformation to obtain their expression in the transverse (closed string) channel and determine the massless tadpoles. To simplify a bit the calculation, but without any loss of generality, we switch off the B field in the internal space so that the momentum sum and the winding sum become related to each other under the world-sheet modular transformations in a simple way. Indeed, applying the Poisson resummation formula (A.8) to eq. (A.27), we find

$$W_{\vec{v}}(\tau_2;T_2,U) = \frac{1}{\tau_2 T_2}\, P_{\vec{v}}\left(\frac{1}{\tau_2};T_2,U\right)\,, \tag{B.8}$$

where we have defined

$$P_{\vec{v}}(\tau_2;T_2,U) = \sum_{\vec{n}\in\mathbb{Z}^2} \mathrm{e}^{-\pi\tau_2\, n_i G^{ij} n_j - 2\pi \mathrm{i} n_i v^i}\ . \tag{B.9}$$

Note that for $v^i = 0$, we have $P_{\vec{0}}(\tau_2;T_2,U) = P(\tau_2;T_2,U)$. Similar transformation properties can be obtained also for the lattice sums P_4 and W_4 on a 4-torus. In the following we will understand the dependence on the Kähler and complex structures of these functions to simplify the notation.

1-loop amplitudes in the direct channel. In our model the annulus amplitude is defined by

$$\mathcal{A} = \int_0^\infty \frac{d\tau_2}{2\tau_2}\,\mathrm{Tr}\left[\frac{1}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,\mathrm{e}^{-\pi\tau_2(L_0-\frac{c}{24})}\right] \tag{B.10}$$

where the trace is taken over all types of open strings, i.e. 7/7, 3/3, 7/3 and 3/7, as well as over their CP indices and sectors. For the 7/7 strings we find

$$\begin{aligned}\mathcal{A}_{7/7} = \frac{\nu_4}{4}\int_0^\infty \frac{d\tau_2}{\tau_2^3}\sum_{\vec{\alpha},\vec{\alpha}'}\Bigg\{ & N_{\vec{\alpha}}\, N_{\vec{\alpha}'}\,\frac{Q_o+Q_v}{\eta^8}\left(\frac{\mathrm{i}\tau_2}{2}\right)\,P_4(\tau_2)\\ & + 4\,\mathrm{tr}\,\gamma_{\vec{\alpha}}(\hat{g})\,\mathrm{tr}\,\gamma_{\vec{\alpha}'}(\hat{g})\,\frac{Q_o-Q_v}{\eta^2\theta_2^2}\left(\frac{\mathrm{i}\tau_2}{2}\right)\Bigg\}\,W_{\vec{\alpha}-\vec{\alpha}'}(\tau_2)\end{aligned} \tag{B.11}$$

where ν_4 is the dimensionless volume introduced in eq. (A.19). The two lines in (B.11) correspond, respectively, to the two contributions with 1 and $\hat{g}$ inserted in the traces, and the sum is over all pairs of D7-brane fixed points.

Let us now consider the 3/3 annuli. The location of the D3-branes is identified by a 6-vector $\vec{\xi}$ that we write as $\vec{\xi} = (\vec{\xi}_4, \vec{\xi}_2)$ to exhibit the position along $\mathcal{T}_4$ and $\mathcal{T}_2$. Then, proceeding similarly as for the 7/7 annuli, we find

$$\mathcal{A}_{3/3} = \frac{\nu_4}{4}\int_0^\infty \frac{d\tau_2}{\tau_2^3}\left\{\sum_{\vec{\xi},\vec{\xi}'} M_{\vec{\xi}} M_{\vec{\xi}'}\,\frac{Q_o+Q_v}{\eta^8}\left(\frac{\mathrm{i}\tau_2}{2}\right)\, W_{\vec{\xi}-\vec{\xi}'}(\tau_2)\right.$$
$$\left. + 4\sum_{\vec{\xi},\vec{\xi}'}{}' \mathrm{tr}\,\gamma_{\vec{\xi}}(\hat{g})\,\mathrm{tr}\,\gamma_{\vec{\xi}'}(\hat{g})\,\frac{Q_o-Q_v}{\eta^2\vartheta_2^2}\left(\frac{\mathrm{i}\tau_2}{2}\right)\, W_{\vec{\xi}_2-\vec{\xi}'_2}(\tau_2)\right\} \tag{B.12}$$

where $W_{\vec{\xi}-\vec{\xi}'}$ is the obvious generalization to $\mathcal{T}_4\times\mathcal{T}_2$ of the winding sum (A.27), and in the second line the $'$ means that the sum is restricted to couples of fixed points $\vec{\xi}$ and $\vec{\xi}'$ lying on top of each other on $\mathcal{T}_4$ and separated by a distance $\vec{\xi}_2-\vec{\xi}'_2$ on $\mathcal{T}_2$. More precisely this sum is over fixed-point pairs satisfying $\vec{\xi}_4=\vec{\xi}'_4$. This constraint arises because only states with zero winding number along $\mathcal{T}_4$ can contribute to the trace when $\hat{g}$ is inserted.

Finally, the annulus contribution from 7/3 and 3/7 strings turns out to be

$$\mathcal{A}_{7/3}+\mathcal{A}_{3/7} = -\frac{\nu_4}{4}\int_0^\infty \frac{d\tau_2}{\tau_2^3}\sum_{\vec{\alpha},\vec{\xi}}\left\{2\,N_{\vec{\alpha}}\,M_{\vec{\xi}}\,\frac{Q_s+Q_c}{\eta^2\vartheta_4^2}\left(\frac{\mathrm{i}\tau_2}{2}\right)\right.$$
$$\left. + 2\,\mathrm{tr}\,\gamma_{\vec{\alpha}}(\hat{g})\,\mathrm{tr}\,\gamma_{\vec{\xi}}(\hat{g})\,\frac{Q_s-Q_c}{\eta^2\vartheta_3^2}\left(\frac{\mathrm{i}\tau_2}{2}\right)\right\} W_{\vec{\alpha}-\vec{\xi}_2}(\tau_2)\,. \tag{B.13}$$

Let us now turn to the Möbius amplitudes, which in our model are given by

$$\mathcal{M} = \int_0^\infty \frac{d\tau_2}{2\tau_2}\,\mathrm{Tr}\left[\frac{\Omega'}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,\mathrm{e}^{-\pi\tau_2(L_0-\frac{c}{24})}\right] \tag{B.14}$$

with the trace computed over open strings of type 7/7 and 3/3, and their CP indices. Of course the 7/3 and 3/7 strings do not contribute to the Möbius amplitudes.

For the 7/7 strings we have

$$\mathcal{M}_{7/7} = -\frac{\nu_4}{4}\int_0^\infty \frac{d\tau_2}{\tau_2^3}\sum_{\vec{\alpha}}\left\{\mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega')\,\gamma_{\vec{\alpha}}^T(\Omega')\right)\frac{Q_o+Q_v}{\eta^8}\left(\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right)\,P_4(\tau_2)\right.$$
$$\left. + 4\,\mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega'\hat{g})\,\gamma_{\vec{\alpha}}^T(\Omega'\hat{g})\right)\frac{Q_o-Q_v}{\eta^2\vartheta_2^2}\left(\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right)\right\} W(\tau_2)\,, \tag{B.15}$$

while for Möbius diagrams with their boundary on D3-branes we find

$$\mathcal{M}_{3/3} = -\frac{\nu_4}{4}\int_0^\infty \frac{d\tau_2}{\tau_2^3}\sum_{\vec{\xi}}\left\{4\,\mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega')\,\gamma_{\vec{\xi}}^T(\Omega')\right)\frac{Q_o-Q_v}{\eta^2\vartheta_2^2}\left(\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right)\right.$$
$$\left. + \mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega'\hat{g})\,\gamma_{\vec{\xi}}^T(\Omega'\hat{g})\right)\frac{Q_o+Q_v}{\eta^8}\left(\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right)\,W_4(\tau_2)\right\} W(\tau_2)\,. \tag{B.16}$$

JHEP05(2010)107

The last type of contribution which is relevant is that corresponding to a Klein bottle. This is a closed string amplitude which in our model is given by

$$\begin{aligned}\mathcal{K} &= \int_0^\infty \frac{d\tau_2}{2\tau_2}\,\mathrm{Tr}\left[\frac{\Omega'}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,\frac{1+(-1)^{\bar{F}}}{2}\,q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}}\right]\\ &= \int_0^\infty \frac{d\tau_2}{2\tau_2}\,\mathrm{Tr}\left[\frac{\Omega'}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,\mathrm{e}^{-4\pi\tau_2(L_0-\frac{c}{24})}\right]\end{aligned} \tag{B.17}$$

where $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$ with $\tau=\tau_1+\mathrm{i}\tau_2$ being the modular paramenter, and the trace taken over both the untwisted and twisted closed string spectra. Since Ω' exchanges left- and right-movers, only those states with $L_0=\bar{L}_0$ and $F=\bar{F}$ contribute, thus explaining the espression in the second line. Evaluating the traces, we find

$$\mathcal{K} = \frac{\nu_4}{4}\int_0^\infty \frac{d\tau_2}{\tau_2^3}\left\{\frac{Q_o+Q_v}{\eta^8}(2\mathrm{i}\tau_2)\left[P_4(\tau_2)+W_4(\tau_2)\right]-32\,\frac{Q_s+Q_c}{\eta^2\vartheta_4^2}(2\mathrm{i}\tau_2)\right\}W(\tau_2)\ . \tag{B.18}$$

The terms proportional to (Q_o+Q_v) come from $\mathrm{Tr}\,1$ and $\mathrm{Tr}\,\hat{g}$ in the untwisted closed string sectors, while the terms proportional to (Q_s+Q_c) account for two identical contributions from the twisted sectors.

1-loop amplitudes in the transverse channel. In order to obtain the tadpole condition, we have to rewrite the above amplitudes in the transverse channel. For the annulus this is achieved by writing $\tau_2=\frac{2}{\ell}$, and then using the modular transformation properties of the lattice sums and the supersymmetric characters. In particular, with the help of eq. (B.8) we have

$$\begin{aligned}W_{\vec{v}}\big(\tau_2;T_2^{(i)},U^{(i)}\big) &= \frac{\ell}{2T_2^{(i)}}\,P_{\vec{v}}\left(\frac{\ell}{2};T_2^{(i)},U^{(i)}\right)\ ,\\ P_{\vec{v}}\big(\tau_2;T_2^{(i)},U^{(i)}\big) &= \frac{T_2^{(i)}\,\ell}{2}\,W_{\vec{v}}\left(\frac{\ell}{2};T_2^{(i)},U^{(i)}\right)\ .\end{aligned} \tag{B.19}$$

Using these relations, we find that in the transverse channel the amplitudes (B.11), (B.12) and (B.13) become

$$\begin{aligned}\mathcal{A}_{7/7} = \frac{\nu_4}{4}\int_0^\infty d\ell\,\sum_{\vec{\alpha},\vec{\alpha}'}\Bigg\{&\frac{T_2^{(1)}T_2^{(2)}}{32T_2}\,N_{\vec{\alpha}}\,N_{\vec{\alpha}'}\,\frac{Q_o+Q_v}{\eta^8}(\mathrm{i}\ell)\,W_4\left(\frac{\ell}{2}\right)\\ &-\frac{1}{2T_2}\mathrm{tr}\,\gamma_{\vec{\alpha}}(\hat{g})\,\mathrm{tr}\,\gamma_{\vec{\alpha}'}(\hat{g})\,\frac{Q_s+Q_c}{\eta^2\theta_4^2}(\mathrm{i}\ell)\Bigg\}\,P_{\vec{\alpha}-\vec{\alpha}'}\left(\frac{\ell}{2}\right)\ ,\end{aligned} \tag{B.20}$$

$$\begin{aligned}\mathcal{A}_{3/3} = \frac{\nu_4}{4}\int_0^\infty d\ell\,\Bigg\{&\frac{1}{32T_2^{(1)}T_2^{(2)}T_2}\sum_{\vec{\xi},\vec{\xi}'}M_{\vec{\xi}}M_{\vec{\xi}'}\,\frac{Q_o+Q_v}{\eta^8}(\mathrm{i}\ell)\,P_{\vec{\xi}-\vec{\xi}'}\left(\frac{\ell}{2}\right)\\ &-\frac{1}{2T_2}\sum_{\vec{\xi},\vec{\xi}'}{}'\mathrm{tr}\,\gamma_{\vec{\xi}}(\hat{g})\,\mathrm{tr}\,\gamma_{\vec{\xi}'}(\hat{g})\,\frac{Q_s+Q_c}{\eta^2\vartheta_4^2}(\mathrm{i}\ell)\,P_{\vec{\xi}_2-\vec{\xi}_2'}\left(\frac{\ell}{2}\right)\Bigg\}\ ,\end{aligned} \tag{B.21}$$

and

$$\mathcal{A}_{7/3}+\mathcal{A}_{3/7}=\frac{\nu_4}{4}\int_0^\infty d\ell\,\sum_{\vec{\alpha},\vec{\xi}}\left\{\frac{1}{4T_2}\,N_{\vec{\alpha}}\,M_{\vec{\xi}}\,\frac{Q_o-Q_v}{\eta^2\vartheta_2^2}(\mathrm{i}\ell)\right.$$
$$\left.-\frac{1}{4T_2}\,\mathrm{tr}\,\gamma_{\vec{\alpha}}(\hat{g})\,\mathrm{tr}\,\gamma_{\vec{\xi}}(\hat{g})\,\frac{Q_s-Q_c}{\eta^2\vartheta_3^2}(\mathrm{i}\ell)\right\}P_{\vec{\alpha}-\vec{\xi}_2}\left(\frac{\ell}{2}\right)\,. \tag{B.22}$$

The transverse channel for the Möbius diagrams is reached by means of the transformation (B.3) which on the lattice sums implies in particular the following relations

$$\begin{aligned} P\big(\tau_2;T_2^{(i)},U^{(i)}\big)&=2\ell\,T_2^{(i)}\,W\big(2\ell;T_2^{(i)},U^{(i)}\big)\,,\\ W\big(\tau_2;T_2^{(i)},U^{(i)}\big)&=\frac{2\ell}{T_2^{(i)}}\,P\big(2\ell;T_2^{(i)},U^{(i)}\big)\,. \end{aligned}\tag{B.23}$$

Then, in the transverse channel the Möbius amplitudes (B.15) and (B.16) become

$$\mathcal{M}_{7/7}=-\frac{\nu_4}{4}\int_0^\infty d\ell\,\sum_{\vec{\alpha}}\left\{\frac{2T_2^{(1)}T_2^{(2)}}{T_2}\,\mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega')\,\gamma_{\vec{\alpha}}^{T}(\Omega')\right)\frac{Q_o+Q_v}{\eta^8}\left(\mathrm{i}\ell+\frac{1}{2}\right)W_4(2\ell)\right.$$
$$\left.+\frac{8}{T_2}\,\mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega'\hat{g})\,\gamma_{\vec{\alpha}}^{T}(\Omega'\hat{g})\right)\frac{Q_o-Q_v}{\eta^2\vartheta_2^2}\left(\mathrm{i}\ell+\frac{1}{2}\right)\right\}P(2\ell)\,, \tag{B.24}$$

and

$$\mathcal{M}_{3/3}=-\frac{\nu_4}{4}\int_0^\infty d\ell\sum_{\vec{\xi}}\left\{\frac{8}{T_2}\mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega')\,\gamma_{\vec{\xi}}^{T}(\Omega')\right)\frac{Q_o-Q_v}{\eta^2\vartheta_2^2}\left(\mathrm{i}\ell+\frac{1}{2}\right)\right. \tag{B.25}$$
$$\left.+\frac{2}{T_2^{(1)}T_2^{(2)}T_2}\mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega'\hat{g})\gamma_{\vec{\xi}}^{T}(\Omega'\hat{g})\right)\frac{Q_o+Q_v}{\eta^8}\left(\mathrm{i}\ell+\frac{1}{2}\right)P_4(2\ell)\right\}P(2\ell).$$

Finally, the exchange channel for the Klein bottle is reached by the transformation $\tau_2=\frac{1}{2\ell}$, so that using eqs. (B.23) and (B.7) and the modular properties of the characters, we get

$$\mathcal{K}=\frac{\nu_4}{4}\int_0^\infty d\ell\,\left\{\frac{Q_o+Q_v}{\eta^8}(\mathrm{i}\ell)\left[\frac{32T_2^{(1)}T_2^{(2)}}{\alpha' T_2}W_4\big(2\ell\big)+\frac{32}{T_2^{(1)}T_2^{(2)}T_2}P_4\big(2\ell\big)\right]\right.$$
$$\left.+\frac{256}{T_2}\,\frac{Q_o-Q_v}{\eta^2\vartheta_2^2}(\mathrm{i}\ell)\right\}P\big(2\ell\big)\,. \tag{B.26}$$

The annulus, Möbius and Klein bottle amplitudes exhibit divergences for $\ell\to\infty$ which are due to the exchange of massless closed string states. The exchanged states can be identified from the corresponding character Q_o, Q_v, Q_s and Q_c, while the $T_2^{(i)}$-dependence specifies the volume of the D-brane/O-plane source. Since only massless states contribute to the divergences we can discard the massive character Q_c. Contributions proportional to Q_s, which correspond to the exchange of a twisted state, should cancel identically since they appear only in the annulus amplitudes. From eqs. (B.20)–(B.22), we see that this requires

$$\mathrm{tr}\,\gamma_{\vec{\alpha}}(\hat{g}) = \mathrm{tr}\,\gamma_{\vec{\xi}}(\hat{g}) = 0 \tag{B.27}$$

Defining $\mathcal{V}_4 = T_2^{(1)} T_2^{(2)}$, we see that the contributions proportional to Q_o and Q_v can be put in the form

$$\begin{aligned}
\mathcal{A}_{o,v} &= \frac{\mathcal{V}_4}{32 T_2}\left[\sum_{\vec{\alpha}} N_{\vec{\alpha}} \pm \frac{1}{\mathcal{V}_4}\sum_{\vec{\xi}} M_{\vec{\xi}}\right]^2 , \\
\mathcal{M}_{o,v} &= -\frac{2\mathcal{V}_4}{T_2}\Bigg[\sum_{\vec{\alpha}} \mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega')\gamma_{\vec{\alpha}}^{T}(\Omega')\right) \pm \frac{1}{\mathcal{V}_4}\sum_{\vec{\alpha}} \mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega'\hat{g})\,\gamma_{\vec{\alpha}}^{T}(\Omega'\hat{g})\right) \\
&\qquad \pm \frac{1}{\mathcal{V}_4}\sum_{\vec{\xi}} \mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega')\,\gamma_{\vec{\xi}}^{T}(\Omega')\right) + \frac{1}{\mathcal{V}_4^2}\sum_{\vec{\xi}} \mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega'\hat{g})\,\gamma_{\vec{\xi}}^{T}(\Omega'\hat{g})\right)\Bigg] , \\
\mathcal{K}_{o,v} &= \frac{32\mathcal{V}_4}{T_2}\left[1 \pm \frac{1}{\mathcal{V}_4}\right]^2 ,
\end{aligned} \tag{B.28}$$

with the upper sign referring to Q_o and the lower one to Q_v. The open/close string consistency requires that the sum of these three amplitudes should form a complete squares. This condition implies the following constraints on the CP traces

$$\begin{aligned}
N_{\vec{\alpha}} =&\ \mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega')\gamma_{\vec{\alpha}}^{T}(\Omega')\right) = \mathrm{tr}\left(\gamma_{\vec{\alpha}}^{-1}(\Omega'\hat{g})\gamma_{\vec{\alpha}}^{T}(\Omega'\hat{g})\right) , \\
M_{\vec{\xi}} =&\ \mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega')\,\gamma_{\vec{\xi}}^{T}(\Omega')\right) = \mathrm{tr}\left(\gamma_{\vec{\xi}}^{-1}(\Omega'\hat{g})\gamma_{\vec{\xi}}^{T}(\Omega'\hat{g})\right) ,
\end{aligned} \tag{B.29}$$

which are satisfied with the matrices given in eqs. (2.9) and (2.10). Plugging them into eq. (B.28), we then find

$$\mathcal{A}_{o,v} + \mathcal{M}_{o,v} + \mathcal{K}_{o,v} = \frac{\mathcal{V}_4}{32 T_2}\left[\Big(\sum_{\vec{\alpha}} N_{\vec{\alpha}} - 32\Big) \pm \frac{1}{\mathcal{V}_4}\Big(\sum_{\vec{\xi}} M_{\vec{\xi}} - 32\Big)\right]^2 , \tag{B.30}$$

so that the cancellation of the tadpoles is globally achieved if

$$\sum_{\vec{\alpha}} N_{\vec{\alpha}} = 32 \quad \text{and} \quad \sum_{\vec{\xi}} M_{\vec{\xi}} = 32 \,. \tag{B.31}$$

However, for any $\vec{\alpha}$ we can also impose the stronger conditions

$$N_{\vec{\alpha}} = 8 \quad \text{and} \quad \sum_{\vec{\xi}_4} M_{\vec{\xi}} = 8 \tag{B.32}$$

where the sum runs over all 6-vectors $\vec{\xi}$ of the form $(\vec{\xi}_4, \vec{\xi}_2)$ for any fixed $\vec{\xi}_2$, which ensure local cancelation of the tadpoles along the last torus $\mathcal{T}_2^{(3)}$.

B.2 1-loop magnetized diagrams

Here we discuss in turn the various 1-loop diagrams for open strings (partly) attached to magnetized D7-branes that were considered in section 3.1.

As a preliminary step, using the Cayley matrix $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i}\end{pmatrix}$ we transform in the complex basis the matrices γ acting on the D7-brane CP indices in order to be consistent with what is done on the magnetization (see eq. (3.5)). Denoting by $\tilde{\gamma} = S\gamma S^{-1}$ these transformed matrices, from eq. (2.9) we easily find

$$\tilde{\gamma}(\Omega') = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \tilde{\gamma}(\hat{g}) = \tilde{\gamma}(\Omega'\hat{g}) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{B.33}$$

where we have omitted the label $\vec{\alpha}$ since we are focusing on a given fixed-point.

In presence of magnetic fluxes h_i open strings satisfy twisted boundary conditions. Taking the magnetic fluxes oriented along, say, the first complex direction and denoting by ν the open string twist, the contribution of the worldsheet fermions to the partition function becomes

$$\begin{aligned}
(Q_o+Q_v)(\nu) &= \frac{\vartheta_3(\nu)\vartheta_3^3 - \vartheta_4(\nu)\vartheta_4^3 - \vartheta_2(\nu)\vartheta_2^3}{2\eta^4} = -\frac{\vartheta_1(\frac{\nu}{2})^4}{\eta^4}\,,\\
(Q_o-Q_v)(\nu) &= \frac{\vartheta_3(\nu)\vartheta_3\vartheta_4^2 - \vartheta_4(\nu)\vartheta_4\vartheta_3^2}{2\eta^4} = -\frac{\vartheta_1(\frac{\nu}{2})^2\,\vartheta_2(\frac{\nu}{2})^2}{\eta^4}\,,\\
(Q_s+Q_c)(\nu) &= \frac{\vartheta_3(\nu)\vartheta_3\vartheta_2^2 - \vartheta_2(\nu)\vartheta_2\vartheta_3^2}{2\eta^4} = -\frac{\vartheta_1(\frac{\nu}{2})^2\,\vartheta_4(\frac{\nu}{2})^2}{\eta^4}\,,\\
(Q_s-Q_c)(\nu) &= \frac{\vartheta_4(\nu)\vartheta_4\vartheta_2^2 - \vartheta_2(\nu)\vartheta_2\vartheta_4^2}{2\eta^4} = -\frac{\vartheta_1(\frac{\nu}{2})^2\,\vartheta_3(\frac{\nu}{2})^2}{\eta^4}\,,
\end{aligned} \tag{B.34}$$

where the right hand sides follow from the Riemann identity (F.3). In particular for an open string stretching between the i-th and j-th D7-brane, the twist is given by $\nu = \frac{\nu_{ij}\tau_2}{2}$ with ν_{ij} related to the magnetic fluxes h_i and h_j at the two endpoints via eq. (3.7). On the other hand, the contribution of a twisted complex bosonic coordinate is

$$-\frac{\mathrm{i}(h_i-h_j)}{4\pi^2\alpha'}\,\frac{\eta}{\vartheta_1\left(\frac{\mathrm{i}\nu_{ij}\tau_2}{2}\right)}\,. \tag{B.35}$$

The 1-loop amplitudes for magnetized D7-branes can be read from the formulas written in the last subsection after replacing the characters Q_o, Q_v, Q_s and Q_c by their twisted versions (B.34) and the contribution of one complex bosonic direction by (B.35). The quadratic structures in the background field are then extracted from the h^2-terms in the expansion of these string amplitudes. Notice that at this order the contributions proportional to (Q_o+Q_v) can be neglected since they are of order h^4. Thus, the magnetized version of the annulus amplitude (B.11) is

$$\begin{aligned}
\mathcal{A}_{7/7}(h) &= \int_0^\infty \frac{d\tau_2}{2\tau_2}\sum_{i,j}\mathrm{Tr}_{\,(h_i,h_j)}\left(\frac{1}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,\mathrm{e}^{-\pi\tau_2(L_0-\frac{c}{24})}\right) \\
&= \mathrm{i}\,\nu_4\sum_{i,j}\left(\tilde{\gamma}(\hat{g})\right)^i_{\;i}\left(\tilde{\gamma}(\hat{g})\right)^j_{\;j}(h_i-h_j)\int_0^\infty\frac{d\tau_2}{2\tau_2^2}\,\frac{\vartheta_1\left(\frac{\mathrm{i}\nu_{ij}\tau_2}{4}\right)^2\vartheta_2\left(\frac{\mathrm{i}\nu_{ij}\tau_2}{4}\right)^2}{\vartheta_1\left(\frac{\mathrm{i}\nu_{ij}\tau_2}{2}\right)\eta^3\,\vartheta_2^2}\,W(\tau_2)
\end{aligned} \tag{B.36}$$

where in the second line we have understood that the second argument of the ϑ-functions is $\frac{\mathrm{i}\tau_2}{2}$ and have written only those terms that can contribute to the quadratic action. Expanding up to $O(h^2)$, we find

$$\begin{aligned}\mathcal{A}_{7/7}(h) &= \frac{V_4}{8\pi^2}\left(\sum_{I,J}\left(\frac{h_I-h_J}{2\pi\alpha'}\right)^2-\sum_{I,J}\left(\frac{h_I+h_J}{2\pi\alpha'}\right)^2\right)\int_0^\infty\frac{d\tau_2}{2\tau_2}\,W(\tau_2)+O(h^3)\\ &= \frac{V_4}{8\pi^2}\,4\,(\mathrm{tr}\,\mathcal{H})^2\int_0^\infty\frac{d\tau_2}{2\tau_2}\,W(\tau_2)+O(h^3)\end{aligned}\tag{B.37}$$

Notice that the contributions of oscillator modes completely cancel at this order. This is a consequence of the fact that the quadratic terms in $\mathcal{N}=2$ gauge theories receive contributions only from BPS states [56].

Now let us consider a Möbius strip with its boundary on the magnetized D7-branes. Taking into account that, due to the presence of Ω' inside the trace, only the configurations with $h_j=-h_i$ give a non-vanishing contribution, from (B.15) we have

$$\begin{aligned}\mathcal{M}_{7/7}(h) &= \int_0^\infty\frac{d\tau_2}{2\tau_2}\sum_i\mathrm{Tr}_{\,(h_i,-h_i)}\left(\frac{\Omega'}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,\mathrm{e}^{-\pi\tau_2(L_0-\frac{c}{24})}\right)\\ &= \mathrm{i}\,\nu_4\sum_i\left(\tilde{\gamma}^{-1}(\Omega'\hat{g})\tilde{\gamma}^\dagger(\Omega'\hat{g})\right)^i_{\ i}\ (2h_i)\int_0^\infty\frac{d\tau_2}{2\tau_2^2}\,\frac{\vartheta_1\big(\frac{\mathrm{i}\nu_i\tau_2}{2}\big)^2\,\vartheta_2\big(\frac{\mathrm{i}\nu_i\tau_2}{2}\big)^2}{\vartheta_1\big(\mathrm{i}\nu_i\tau_2\big)\,\eta^3\,\vartheta_2^2}\,W(\tau_2)\end{aligned}\tag{B.38}$$

where now the second argument of all modular functions is $\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}$, and again only those structures contributing to the quadratic terms have been written. Expanding to order h^2, as before we find a complete cancelation between the modular forms in the numerator and denominator with the result

$$\begin{aligned}\mathcal{M}_{7/7}(h) &= \frac{V_4}{8\pi^2}\,4\sum_I\left(\frac{h_I}{2\pi\alpha'}\right)^2\int_0^\infty\frac{d\tau_2}{2\tau_2}\,W(\tau_2)+O(h^3)\\ &= -\frac{V_4}{8\pi^2}\,4\,\mathrm{tr}\,\mathcal{H}^2\int_0^\infty\frac{d\tau_2}{2\tau_2}\,W(\tau_2)+O(h^3)\ .\end{aligned}\tag{B.39}$$

Finally, we consider the annulus amplitudes with mixed 7/3 boundary conditions. First we observe that since the magnetic fluxes are turned on only on D7-branes, these amplitudes are proportional either to $\mathrm{tr}_{\,\mathrm{D3}}(1)$ or to $\mathrm{tr}_{\,\mathrm{D3}}\tilde{\gamma}(\hat{g})=0$, and therefore only the unprojected part contributes to the result. Indeed, we find

$$\begin{aligned}\mathcal{A}_{7/3}(h)+\mathcal{A}_{3/7}(h) &= \int_0^\infty\frac{d\tau_2}{2\tau_2}\sum_{i,a}\mathrm{Tr}_{\,(h_i,a)}\left(\frac{1}{2}\,\frac{1+\hat{g}}{2}\,\frac{1+(-1)^F}{2}\,q^{L_0-\frac{c}{24}}\right)\\ &= -\frac{\mathrm{i}\,\nu_4}{2}\sum_{i,a}h_i\int_0^\infty\frac{d\tau_2}{2\tau_2^2}\,\frac{\vartheta_1\big(\frac{\mathrm{i}\nu_i\tau_2}{4}\big)^2\,\vartheta_4\big(\frac{\mathrm{i}\nu_i\tau_2}{4}\big)^2}{\vartheta_1\big(\frac{\mathrm{i}\nu_i\tau_2}{2}\big)\,\eta^3\,\vartheta_4^2}\,W(\tau_2)\ .\end{aligned}\tag{B.40}$$

Expanding to second order in h, we obtain

$$\begin{aligned}\mathcal{A}_{7/3}(h)+\mathcal{A}_{3/7}(h) &= -\frac{V_4}{16\pi^2}\sum_{I,a}\left(\frac{h_I}{2\pi\alpha'}\right)^2\int_0^\infty\frac{d\tau_2}{2\tau_2}\,W(\tau_2)+O(h^3)\\ &= \frac{V_4}{8\pi^2}\,m\,\mathrm{tr}\,\mathcal{H}^2\int_0^\infty\frac{d\tau_2}{2\tau_2}\,W(\tau_2)+O(h^3)\ .\end{aligned}\tag{B.41}$$

JHEP05(2010)107

Collecting eqs. (B.37), (B.39) and (B.41), the total 1-loop effective action is

$$
\begin{aligned}
S_{1-\text{loop}} &= \mathcal{A}_{7/7}(h)+\mathcal{M}_{7/7}(h)+\mathcal{A}_{7/3}(h)+\mathcal{A}_{3/7}(h) \\
&= -\frac{V_4}{8\pi^2}\Big[(4-m)\,\mathrm{tr}\,\mathcal{H}^2-4\,(\mathrm{tr}\,\mathcal{H})^2\Big]\int_0^\infty \frac{d\tau_2}{2\tau_2}\,W(\tau_2)+O(h^3)
\end{aligned}
\tag{B.42}
$$

as reported in eq. (3.10) of the main text.

C D-instanton sums: explicit results up to 3 instantons

In this appendix we present the results of calculations up to $k=3$ instantons, including finite $\epsilon_{1,2}$ (gravitational) and $\epsilon_{3,4}$ (anti-symmetric hyper-multiplet mass) corrections.

According to eq. (5.21), the 1-instanton partition function is given by

$$
Z_1=\frac{\epsilon_1+\epsilon_2}{\epsilon_1\epsilon_2}\int\frac{d\chi_1}{2\pi\mathrm{i}}\,\frac{1}{(4\chi_1^2-\epsilon_3^2)(4\chi_1^2-\epsilon_4^2)}\prod_{r=1}^{m}\frac{(\chi_1+b_r)^2-\frac{(\epsilon_3-\epsilon_4)^2}{4}}{(\chi_1-b_r)^2-\frac{(\epsilon_1+\epsilon_2)^2}{4}}\prod_{u=1}^{4}(\chi_1-a_u)\,. \tag{C.1}
$$

The pole prescription is specified by $\mathrm{Im}b_r=0$, and $\mathrm{Im}\epsilon_1\gg\mathrm{Im}\epsilon_2\gg\mathrm{Im}\epsilon_3\gg\mathrm{Im}\epsilon_4>0$, and the integral is computed by closing the contour in the upper half-plane, $\mathrm{Im}\chi_1>0$. The poles contributing to the integral (C.1) are located then at $\chi_1=b_r+\frac{\epsilon_1+\epsilon_2}{2}$ $(r=1,\dots,m)$, $\chi_1=\frac{\epsilon_3}{2}$ and $\chi_1=\frac{\epsilon_4}{2}$. Z_1 is then the sum of residues at these points. A simple inspection of this formula shows that no dependence on b_r arises at the leading $\frac{1}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}$ order. This can be seen by noticing that the b-dependent factors cancel between numerator and denominator at $\chi_1=\frac{\epsilon_3}{2},\frac{\epsilon_4}{2}\approx 0$, and that the b-dependent poles $\chi_1\approx b_r$ contributes only to the $\frac{1}{\epsilon_1\epsilon_2}$-order.

For higher instanton numbers one should perform the integrations over $\chi_1,\dots,\chi_k$ one after the other, subsequently evaluating the residues at the poles satisfying the above mentioned rules. Unfortunately, the problem becomes algebraically more and more complicated as k increases, and we have been able to explicitly perform the integrations up to $k=3$ only. Since our main interest is the pure U(4) gauge theory living on the D7 world-volume, when there are also D3-branes, we only present the result of the calculations for $b=0$.

The result for $m=0$ can be written as

$$
\begin{aligned}
\epsilon_1\epsilon_2\log Z^{(m=0)}(a,\epsilon) &= \left(\frac{4a_1a_2a_3a_4}{\epsilon_3\epsilon_4}-\sum_{i<j}a_ia_j-\frac{\epsilon_1+\epsilon_2}{2}\sum_i a_i-\frac{\epsilon_3^2+\epsilon_3\epsilon_4+\epsilon_4^2}{4}\right)q \\
&+\Bigg(-\frac{1}{\epsilon_3\epsilon_4}\sum_{i<j}{a_i}^2{a_j}^2+\frac{\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2}{4\epsilon_3\epsilon_4}\sum_i a_i^2-\frac{\left(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2\right)^2}{16\epsilon_3\epsilon_4} \\
&\quad+\sum_{i<j}a_ia_j-\frac14\sum_i a_i^2+\frac{\epsilon_1+\epsilon_2}{4}\sum_i a_i+\frac{3\left(\epsilon_3^2+\epsilon_3\epsilon_4+\epsilon_4^2\right)}{8}\Bigg)q^2 \\
&+\left(\frac{16a_1a_2a_3a_4}{3\epsilon_3\epsilon_4}-\frac43\sum_{i<j}a_ia_j-\frac{\epsilon_1+\epsilon_2}{3}\sum_i a_i-\frac{\epsilon_3^2+\epsilon_3\epsilon_4+\epsilon_4^2}{3}\right)q^3+\cdots
\end{aligned}
\tag{C.2}
$$

while for $m=1$ and $b=0$ we find

$$\epsilon_1\epsilon_2 \log Z^{(m=1)}(a,\epsilon)=\left(\frac{4a_1a_2a_3a_4}{\epsilon_3\epsilon_4}+3\sum_{i<j}a_ia_j-\frac{\epsilon_1+\epsilon_2}{2}\sum_i a_i+\frac{3\left(\epsilon_3^2+\epsilon_3\epsilon_4+\epsilon_4^2\right)}{4}\right)q$$
$$+\left(-\frac{1}{\epsilon_3\epsilon_4}\sum_{i<j}{a_i}^2{a_j}^2+\frac{\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2}{4\epsilon_3\epsilon_4}\sum_i a_i^2-\frac{\left(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2\right)^2}{16\epsilon_3\epsilon_4}\right.$$
$$\left.+\sum_{i<j}a_ia_j+\frac{7}{4}\sum_i a_i^2+\frac{\epsilon_1+\epsilon_2}{4}\sum_i a_i-\frac{\epsilon_3^2+\epsilon_3\epsilon_4+\epsilon_4^2}{8}-\frac{\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2}{2}\right)q^2$$
$$+\left(\frac{16a_1a_2a_3a_4}{3\epsilon_3\epsilon_4}+4\sum_{i<j}a_ia_j-\frac{\epsilon_1+\epsilon_2}{3}\sum_i a_i+\left(\epsilon_3^2+\epsilon_3\epsilon_4+\epsilon_4^2\right)\right)q^3+\cdots \tag{C.3}$$

It is interesting to notice that the two formulas (C.2) and (C.3) share their divergent $\frac{1}{\epsilon_3\epsilon_4}$ term. A more extensive inspection for finite non-zero b shows that indeed the full b-dependence cancels in this term, so that in any case $\mathcal{F}_{\mathrm{IV}}$ is given by

$$\mathcal{F}_{\mathrm{IV}}=4\,a_1a_2a_3a_4\,q+\left(-\sum_{i<j}{a_i}^2{a_j}^2+\frac{1}{4}(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2)-\frac{1}{16}(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2)^2\right)q^2$$
$$+\frac{16}{3}\,a_1a_2a_3a_4\,q^3+\cdots \tag{C.4}$$

This expression matches precisely with $\frac{1}{2}\mathcal{F}_{\mathrm{SO}(8)}$, including its gravitational corrections. After the subtraction of the quartic terms, the resulting quadratic prepotentials are then given by the formulas (5.25) in the main text.

D Details on the heterotic computation

In this appendix we provide some details on the calculations presented in section 6 for the heterotic model.

In the partition function (6.4) and its weighted version (6.16), the trace over the right-moving oscillators of the heterotic string can be written as a spin-structure sum as follows

$$\rho\left[{0\atop 0}\right](w,\tau)=\frac{1}{2}\sum_{a,b=0}^{1}\frac{\mathrm{e}^{\mathrm{i}\pi(a+b+ab)}}{\eta^{12}}\,\vartheta\left[{a\atop b}\right](2\tau_2 w)\,\vartheta\left[{a\atop b}\right]^3=\frac{\vartheta_1(\tau_2 w)^4}{\eta^{12}}\,, \tag{D.1}$$

and

$$\rho\left[{h_0\atop g_0}\right](w,\tau)=\frac{4}{2}\sum_{a,b=0}^{1}\frac{\mathrm{e}^{\mathrm{i}\pi(a+b+ab+bh_0)}}{\eta^6\,\vartheta\left[{1+h_0\atop 1+g_0}\right]^2}\,\vartheta\left[{a\atop b}\right](2\tau_2 w)\,\vartheta\left[{a\atop b}\right]\,\vartheta\left[{a+h_0\atop b+g_0}\right]^2$$
$$=-4\,\mathrm{e}^{\mathrm{i}\pi h_0}\,\frac{\vartheta_1(\tau_2 w)^2\,\vartheta\left[{1+h_0\atop 1+g_0}\right](\tau_2 w)^2}{\eta^6\,\vartheta\left[{1+h_0\atop 1+g_0}\right]^2} \tag{D.2}$$

for $(g_0,h_0)\neq(0,0)$. Here we have used the standard conventions for the ϑ-functions summarised in appendix F, and in particular the Riemann identity to perform the summation over a and b. The trace over the left-moving modes, instead, is given by

$$\chi\left[\begin{smallmatrix}0 & h_1 & h_2\\ 0 & g_1 & g_2\end{smallmatrix}\right](\vec{v},\bar{\tau}) = \frac{1}{2}\sum_{a,b=0}^{1}\frac{1}{\bar{\eta}^{24}}\,\bar{\vartheta}\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](v_1)\,\bar{\vartheta}\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](v_2)\,\bar{\vartheta}\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right]^2\,\bar{\vartheta}\left[\begin{smallmatrix}a+h_1\\ b+g_1\end{smallmatrix}\right]^4\,\bar{\vartheta}\left[\begin{smallmatrix}a+h_2\\ b+g_2\end{smallmatrix}\right]^4\,\bar{\vartheta}\left[\begin{smallmatrix}a+h_1+h_2\\ b+g_1+g_2\end{smallmatrix}\right]^4, \quad \text{(D.3)}$$

and

$$\begin{aligned}\chi\left[\begin{smallmatrix}h_0 & h_1 & h_2\\ g_0 & g_1 & g_2\end{smallmatrix}\right](\vec{v},\bar{\tau}) = \frac{4}{2}\sum_{a,b=0}^{1}\frac{\mathrm{e}^{\mathrm{i}\pi(ag_0+bh_0-\frac{1}{2}g_0h_0)}}{\bar{\eta}^{18}\,\bar{\vartheta}\left[\begin{smallmatrix}1+h_0\\ 1+g_0\end{smallmatrix}\right]^2}\,&\bar{\vartheta}\left[\begin{smallmatrix}a+\frac{h_0}{2}\\ b+\frac{g_0}{2}\end{smallmatrix}\right](v_1)\,\bar{\vartheta}\left[\begin{smallmatrix}a+\frac{h_0}{2}\\ b+\frac{g_0}{2}\end{smallmatrix}\right](v_2)\,\bar{\vartheta}\left[\begin{smallmatrix}a+\frac{h_0}{2}\\ b+\frac{g_0}{2}\end{smallmatrix}\right]^2\\ \times\,\bar{\vartheta}\left[\begin{smallmatrix}a+\frac{h_0}{2}+h_1\\ b+\frac{g_0}{2}+g_1\end{smallmatrix}\right]^4\,\bar{\vartheta}\left[\begin{smallmatrix}a+\frac{h_0}{2}+h_2\\ b+\frac{g_0}{2}+g_2\end{smallmatrix}\right]^4\,&\bar{\vartheta}\left[\begin{smallmatrix}a+\frac{h_0}{2}+h_1+h_2\\ b+\frac{g_0}{2}+g_1+g_2\end{smallmatrix}\right]^4\end{aligned} \quad \text{(D.4)}$$

for $(g_0,h_0)\neq(0,0)$. The factors of 4 in eqs. (D.2) and (D.4) account for the 16 fixed points of the $\mathbb{Z}_2$ orbifold action in our heterotic model. Notice also that for $(g_0,h_0)\neq(0,0)$ the left-moving amplitudes $\chi\left[\begin{smallmatrix}h_0 & h_1 & h_2\\ g_0 & g_1 & g_2\end{smallmatrix}\right]$ involve $\mathbb{Z}_4$ ϑ-functions with half-integer characteristics but, as we will see, they can always be rewritten in terms of standard ϑ-functions, as expected for a $\mathbb{Z}_2$-orbifold. Furthermore, the relative phases between the different structures in all these formulas are fixed by modular invariance up to discrete torsions that we have chosen to be zero for simplicity. In particular, one can show the following modular transformation properties

$$\begin{aligned}\rho\left[\begin{smallmatrix}h_0\\ g_0\end{smallmatrix}\right](w,\tau+1)\,\chi\left[\begin{smallmatrix}h_0 & h_1 & h_2\\ g_0 & g_1 & g_2\end{smallmatrix}\right](\vec{v},\bar{\tau}+1) &= \rho\left[\begin{smallmatrix}h_0\\ g_0+h_0\end{smallmatrix}\right](w,\tau)\,\chi\left[\begin{smallmatrix}h_0 & h_1 & h_2\\ g_0+h_0 & g_1+h_1 & g_2+h_2\end{smallmatrix}\right](\vec{v},\bar{\tau}),\\ \rho\left[\begin{smallmatrix}h_0\\ g_0\end{smallmatrix}\right](w,-1/\tau)\,\chi\left[\begin{smallmatrix}h_0 & h_1 & h_2\\ g_0 & g_1 & g_2\end{smallmatrix}\right](\vec{v},-1/\bar{\tau}) &= |\tau|^{-8+4(g_0+h_0-h_0g_0)}\,\rho\left[\begin{smallmatrix}g_0\\ h_0\end{smallmatrix}\right](w,\tau)\,\chi\left[\begin{smallmatrix}g_0 & g_1 & g_2\\ h_0 & h_1 & h_2\end{smallmatrix}\right](\vec{v},\bar{\tau})\,.\end{aligned} \quad \text{(D.5)}$$

In the calculation of the massless spectrum of this heterotic compactification, one needs the explicit expressions for the left-moving functions of the type $\chi\left[\begin{smallmatrix}h_0 & 0 & 0\\ g_0 & g_1 & g_2\end{smallmatrix}\right](\vec{0},\bar{\tau})$. They are given by

$$\begin{aligned}\chi\left[\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right](\vec{0},\bar{\tau}) &= \frac{\bar{\vartheta}_3^{16}+\bar{\vartheta}_4^{16}+\bar{\vartheta}_2^{8}}{2\bar{\eta}^{24}}\,,\\ \chi\left[\begin{smallmatrix}0&0&0\\0&g_1&g_2\end{smallmatrix}\right](\vec{0},\bar{\tau}) &= \frac{\bar{\vartheta}_3^{8}\,\bar{\vartheta}_4^{8}}{\bar{\eta}^{24}} \qquad \text{for } (g_1,g_2)\neq(0,0)\,,\\ \chi\left[\begin{smallmatrix}0&0&0\\1&g_1&g_2\end{smallmatrix}\right](\vec{0},\bar{\tau}) &= \frac{\bar{\vartheta}_3^{2}\,\bar{\vartheta}_4^{2}}{\bar{\eta}^{24}}\left(\bar{\vartheta}\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]^{16}-\bar{\vartheta}\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]^{16}\right) = \frac{\bar{\vartheta}_3^{8}\,\bar{\vartheta}_4^{8}}{2\bar{\eta}^{24}}\left(\bar{\vartheta}_3^4+\bar{\vartheta}_4^4\right),\\ \chi\left[\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right](\vec{0},\bar{\tau}) &= \frac{\bar{\vartheta}_2^{2}\,\bar{\vartheta}_3^{2}}{\bar{\eta}^{24}}\left(\bar{\vartheta}\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]^{16}-\bar{\vartheta}\left[\begin{smallmatrix}1/2\\1\end{smallmatrix}\right]^{16}\right) = \frac{\bar{\vartheta}_2^{8}\,\bar{\vartheta}_3^{8}}{2\bar{\eta}^{24}}\left(\bar{\vartheta}_2^4+\bar{\vartheta}_3^4\right),\\ \chi\left[\begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix}\right](\vec{0},\bar{\tau}) &= \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}\,\frac{\bar{\vartheta}_2^{2}\,\bar{\vartheta}_4^{2}}{\bar{\eta}^{24}}\left(\bar{\vartheta}\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]^{16}-\bar{\vartheta}\left[\begin{smallmatrix}1/2\\-1/2\end{smallmatrix}\right]^{16}\right) = \frac{\bar{\vartheta}_2^{8}\,\bar{\vartheta}_4^{8}}{2\bar{\eta}^{24}}\left(\bar{\vartheta}_2^4+\bar{\vartheta}_4^4\right),\\ \chi\left[\begin{smallmatrix}1&0&0\\0&g_1&g_2\end{smallmatrix}\right](\vec{0},\bar{\tau}) &= \chi\left[\begin{smallmatrix}1&0&0\\1&g_1&g_2\end{smallmatrix}\right](\vec{0},\bar{\tau}) = 0 \qquad \text{for } (g_1,g_2)\neq(0,0)\,,\end{aligned} \quad \text{(D.6)}$$

where we have repeatedly used the identities (F.9) to rewrite the $\mathbb{Z}_4$ ϑ-functions in terms of the standard Jacobi ϑ-functions.

Finally, in the partition function (6.4) and its weighted version (6.16) the function $\Gamma\left[\begin{smallmatrix}h_0 & h_1 & h_2\\ g_0 & g_1 & g_2\end{smallmatrix}\right](\tau,\bar{\tau})$ represents the contribution of the bosonic zero-modes in the internal compact directions, which is given by

$$
\begin{aligned}
\Gamma\left[\begin{smallmatrix}0\,h_1\,h_2\\0\,g_1\,g_2\end{smallmatrix}\right] &= \frac{1}{\tau_2^3}\,\Gamma_{4,4}\;\Gamma_{2,2}\left[\begin{smallmatrix}h_1\,h_2\\g_1\,g_2\end{smallmatrix}\right]\,,\\
\Gamma\left[\begin{smallmatrix}h_0\,h_1\,h_2\\g_0\,g_1\,g_2\end{smallmatrix}\right] &= \frac{1}{\tau_2}\Gamma_{2,2}\left[\begin{smallmatrix}h_1\,h_2\\g_1\,g_2\end{smallmatrix}\right] \quad \text{for } (g_0,h_0)\neq(0,0)\ .
\end{aligned}
\tag{D.7}
$$

Here $\Gamma_{4,4}$ is the standard lattice sum over $\mathcal{T}_4$, while $\Gamma_{2,2}\left[\begin{smallmatrix}h_1\,h_2\\g_1\,g_2\end{smallmatrix}\right]$ is the lattice sum for the various sectors of the $\mathbb{Z}_2\times\mathbb{Z}_2$ freely acting orbifold over $\mathcal{T}_2$, namely

$$
\Gamma_{2,2}\left[\begin{smallmatrix}h_1\,h_2\\g_1\,g_2\end{smallmatrix}\right](\tau,\bar\tau;T,U) = T_2\sum_M \mathrm{e}^{2\pi\mathrm{i}T\,\det M}\,\mathrm{e}^{-\frac{\pi T_2}{\alpha'\tau_2 U_2}\left|(1\,U)\,M\,\binom{\tau}{-1}\right|^2}
\tag{D.8}
$$

with the sum running over four integers specifying the wrapping numbers over $\mathcal{T}_2$ according to

$$
M = \begin{pmatrix} w^1+\frac{h_1}{2} & m^1+\frac{g_1}{2}\\ w^2+\frac{h_2}{2} & m^2+\frac{g_2}{2}\end{pmatrix}\,,\qquad (m^i,w^i)\in\mathbb{Z}^2\ .
\tag{D.9}
$$

In the calculation of the 1-loop thresholds, the second derivatives of the $\chi\left[\begin{smallmatrix}h_0\,h_1\,h_2\\g_0\,g_1\,g_2\end{smallmatrix}\right]$ functions are needed. In particular, in eq. (6.21) we have used the following expressions

$$
\begin{aligned}
\chi_{11}(\bar\tau) &= \frac{1}{8\pi^2}\,\partial_{v_1}^2\chi\left[\begin{smallmatrix}0\,0\,0\\1\,0\,0\end{smallmatrix}\right](\vec 0,\bar\tau) = \frac{1}{8\pi^2}\frac{\bar\vartheta_3^2\bar\vartheta_4^2}{\bar\eta^{24}}\left(\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]''\,\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]^{15}-\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]''\,\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]^{15}\right),\\
\chi_{12}(\bar\tau) &= \frac{1}{8\pi^2}\,\partial_{v_1}\partial_{v_2}\chi\left[\begin{smallmatrix}0\,0\,0\\1\,0\,0\end{smallmatrix}\right](\vec 0,\bar\tau) = \frac{1}{8\pi^2}\frac{\bar\vartheta_3^2\bar\vartheta_4^2}{\bar\eta^{24}}\left(\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]'^2\,\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]^{14}-\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]'^2\,\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]^{14}\right),
\end{aligned}
\tag{D.10}
$$

while in eq. (6.29) we have introduced

$$
\begin{aligned}
\chi_{11}^{(h_1\,h_2)}(\bar\tau) &= \frac{1}{8\pi^2}\,\partial_{v_1}^2\chi\left[\begin{smallmatrix}0\,h_1\,h_2\\1\;0\;\;0\end{smallmatrix}\right](0,\bar\tau)\\
&= \frac{1}{8\pi^2}\frac{\bar\vartheta_3^2\bar\vartheta_4^2}{\bar\eta^{24}}\,\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]^7\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]^7\left(\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]''\,\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]-\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]''\,\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]\right),\\
\chi_{12}^{(h_1\,h_2)}(\bar\tau) &= \frac{1}{8\pi^2}\,\partial_{v_1}\partial_{v_2}\chi\left[\begin{smallmatrix}0\,h_1\,h_2\\1\;0\;\;,0\end{smallmatrix}\right](0,\bar\tau)\\
&= \frac{1}{8\pi^2}\frac{\bar\vartheta_3^2\bar\vartheta_4^2}{\bar\eta^{24}}\,\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]^6\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]^6\left(\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]'^2\,\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]^2-\bar\vartheta\left[\begin{smallmatrix}1\\1/2\end{smallmatrix}\right]'^2\,\bar\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]^2\right)
\end{aligned}
\tag{D.11}
$$

for any $(h_1,h_2)\neq(0,0)$.

Using the duplication formulas together with the modular properties of the ϑ-functions, and the relations collected in appendix F, one can show that

$$
\begin{aligned}
\chi_{11}\left(\frac{\bar\tau}{2}\right) &= \frac{1}{24\,\bar\eta^{12}}\left[\bar\vartheta_3^8(\bar\vartheta_3^4+\bar\vartheta_4^4-2\hat E_2)-\bar\vartheta_2^8(\bar\vartheta_2^4-\bar\vartheta_4^4-2\hat E_2)\right],\\
\chi_{11}\left(\frac{\bar\tau+1}{2}\right) &= -\frac{1}{24\,\bar\eta^{12}}\left[\bar\vartheta_4^8(\bar\vartheta_4^4+\bar\vartheta_3^4-2\hat E_2)+\bar\vartheta_2^8(\bar\vartheta_2^4+\bar\vartheta_3^4+2\hat E_2)\right],\\
\chi_{11}\left(\frac{-1}{2\bar\tau}\right) &= \frac{1}{24\,\bar\eta^{12}}\left[\bar\vartheta_3^8(\bar\vartheta_3^4+\bar\vartheta_2^4+2\hat E_2)-\bar\vartheta_4^8(\bar\vartheta_4^4-\bar\vartheta_2^4+2\hat E_2)\right].
\end{aligned}
\tag{D.12}
$$

Here, in the right-hand sides the ϑ-functions are evaluated at $\bar{\tau}$, and as customary [34, 39] we have replaced the second Eisenstein series E_2 with the modular form $\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2}$. Likewise, we have

$$\chi_{12}\left(\frac{\bar{\tau}}{2}\right) = \chi_{12}\left(\frac{\bar{\tau}+1}{2}\right) = \chi_{12}\left(\frac{-1}{2\bar{\tau}}\right) = \frac{1}{8\,\bar{\eta}^{12}}\,\bar{\vartheta}_2^4\,\bar{\vartheta}_3^4\,\bar{\vartheta}_4^4 = 2\,. \tag{D.13}$$

From these expressions, the Hecke transforms of χ_{IJ} read

$$\begin{aligned} c_{11}(\bar{\tau}) &= \left[\chi_{11}\left(\frac{\bar{\tau}}{2}\right) + \chi_{11}\left(\frac{\bar{\tau}+1}{2}\right) + \chi_{11}\left(\frac{-1}{2\bar{\tau}}\right)\right] = 6\,,\\ c_{12}(\bar{\tau}) &= \left[\chi_{12}\left(\frac{\bar{\tau}}{2}\right) + \chi_{12}\left(\frac{\bar{\tau}+1}{2}\right) + \chi_{12}\left(\frac{-1}{2\bar{\tau}}\right)\right] = 6\,. \end{aligned} \tag{D.14}$$

Finally, from eq. (D.11) and the ϑ-functions properties we have

$$\begin{aligned} \chi_{11}^{(h_1\,h_2)}(\bar{\tau}) &= \frac{1}{128\,\bar{\eta}^{24}}\,\bar{\vartheta}_2^8\,\bar{\vartheta}_3^8\,\bar{\vartheta}_4^8 = 2\,,\\ \chi_{12}^{(h_1\,h_2)}(\bar{\tau}) &= -\frac{1}{128\,\bar{\eta}^{24}}\,\bar{\vartheta}_2^8\,\bar{\vartheta}_3^8\,\bar{\vartheta}_4^8 = -2\,, \end{aligned} \tag{D.15}$$

as reported in eq. (6.31) of the main text.

E Holomorphic couplings

Here we briefly review the relation between string thresholds and holomorphic gauge couplings required by $\mathcal{N} \geq 1$ supersymmetry in four dimensions following refs. [50, 51]; our model exhibits $\mathcal{N} = 2$ supersymmetry, and in the main text we focused directly on this case, but we find convenient to recap here how the $\mathcal{N} = 2$ relations arise from the general $\mathcal{N} = 1$ case.

In a supersymmetric string vacuum, the 1-loop gauge coupling constant $g(\mu)$ for a gauge group G is given by

$$\left.\frac{4\pi}{g^2(\mu)}\right|_{1-\text{loop}} = \frac{4\pi}{g_0^2} + \frac{1}{4\pi}\left[b\,\log\left(\frac{4\pi\mu^2}{g_0^2 M_{\text{Pl}}^2}\right) + \Delta\right] \tag{E.1}$$

where g_0 is the bare tree-level coupling, Δ is the threshold correction and b is the 1-loop β-function coefficient given by

$$b = 3\,T(G) - \sum_r \widetilde{n}_r\,T(r)\,. \tag{E.2}$$

Here, $\widetilde{n}_r$ is the number of $\mathcal{N} = 1$ chiral multiplets transforming in the representation r of G whose index is denoted by $T(r)$, with $T(G)$ standing for $T(\text{adj})$ (see eq. (2.20)). Formula (E.1) applies to any supersymmetric string vacuum, either heterotic or type II or type I.

On the other hand, as explained for example in ref. [51], the general form of the 1-loop coupling constant required by supersymmetry in the four-dimensional effective quantum field theory is

$$\begin{aligned}\frac{4\pi}{g^2(\mu)}\bigg|_{1-\text{loop}} =& \mathrm{Re}\, f_{(0)} + \frac{1}{4\pi}\Bigg[\mathrm{Re}\, f_{(1)} + b\log\left(\frac{\mu^2}{M_{\mathrm{Pl}}^2}\right) + 2T(G)\log\left(\mathrm{Re}\, f_{(0)}\right) \\ & - c\,K - 2\sum_r \widetilde{n}_r\, T(r)\, \log Z_r\Bigg]\end{aligned} \tag{E.3}$$

where $f_{(0)}$ and $f_{(1)}$ are, respectively, the tree-level and 1-loop contributions to the Wilsonian holomorphic gauge coupling function f, K is the tree-level Kähler potential, and Z_r is the tree-level Kähler metric for the chiral superfield in the representation r. Finally, the coefficient c is defined as

$$c = T(G) - \sum_r \widetilde{n}_r\, T(r) = b - 2\, T(G)\ . \tag{E.4}$$

In comparing eqs. (E.1) and (E.3) we must take into account the fact that the tree-level coupling obtained from string theory does not necessarily coincide with $\mathrm{Re}\, f_{(0)}$ since there may be 1-loop effects that spoil this identification [51]. Indeed, in general we have

$$\frac{4\pi}{g_0^2} = \mathrm{Re}\, f_{(0)} + \frac{1}{4\pi}\Delta_{\mathrm{univ}}(M,\bar{M}) \tag{E.5}$$

where $\Delta_{\mathrm{univ}}(M,\bar{M})$ is a universal term related to the 1-loop correction of the Kähler potential, which is a (real) function of the compactification moduli M (different from those parametrizing the gauge coupling) that can mix with the dilaton. Using this information and equating the string theory expression (E.1) with the field theory one (E.3), we easily obtain

$$\mathrm{Re}\, f_{(1)} = \Delta + \Delta_{\mathrm{univ}} + c\,\log\left(\mathrm{Re}\, f_{(0)}\right) + c\,K + 2\sum_r \widetilde{n}_r\, T(r)\,\log Z_r\ . \tag{E.6}$$

The terms in the right hand side contain non-holomorphic pieces that, for consistency, must compensate each other in order to yield a holomorphic result for $f_{(1)}$.

When the $\mathcal{N}=1$ chiral multiplets organize into $\mathcal{N}=2$ hyper-multiplets, like in our case, it is possible to show that [52]

$$c\,\log\left(\mathrm{Re}\, f_{(0)}\right) + c\,K + 2\sum_r \widetilde{n}_r\, T(r)\,\log Z_r = b\,\widehat{K}\,, \tag{E.7}$$

where $\widehat{K}$ is related to the Kähler metric of the adjoint chiral multiplet which becomes part of the $\mathcal{N}=2$ vector multiplet according to $\widehat{K} = \log Z_{\mathrm{adj}}$. Thus, for $\mathcal{N}=2$ vacua the relation (E.6) reduces to

$$\mathrm{Re}\, f_{(1)} = \Delta + \Delta_{\mathrm{univ}} + b\,\widehat{K}\ . \tag{E.8}$$

eq. (E.7) can be explicitly verified in our specific type I$'$ and heterotic setups. To this purpose let us recall that the various quantities to be used are:

- in type I′

$$
\begin{aligned}
&\mathrm{Re}\, f_{(0)} = t_2\,, && \widehat{K} = -\log(\lambda_2 U_2)\,, && c = b - 8 = -8\,,\\
&Z_{\Box} = (t_2^{(1)} U_2^{(1)} t_2^{(2)} U_2^{(2)})^{-1/2}\,, && Z_{\rm adj} = (\lambda_2 U_2)^{-1}\,, &&\\
&Z_{{\substack{\Box\\\Box}}_1} = Z_{\overline{\substack{\Box\\\Box}}_1} = (t_2^{(1)} U_2^{(1)})^{-1}\,, && Z_{{\substack{\Box\\\Box}}_2} = Z_{\overline{\substack{\Box\\\Box}}_2} = (t_2^{(2)} U_2^{(2)})^{-1}\,, &&\\
&\widetilde{n}_{\rm adj} = \widetilde{n}_{{\substack{\Box\\\Box}}_{1,2}} = \widetilde{n}_{\overline{\substack{\Box\\\Box}}_{1,2}} = 1\,, && \widetilde{n}_{\Box} = 8\,; &&
\end{aligned}
\tag{E.9}
$$

- in heterotic

$$
\begin{aligned}
&\mathrm{Re}\, f_{(0)} = S_2\,, && \widehat{K} = -\log(T_2 U_2)\,, && c = b - 8 = -8\,,\\
&Z_{\Box} = (T_2^{(1)} U_2^{(1)} T_2^{(2)} U_2^{(2)})^{-1/2}\,, && Z_{\rm adj} = (T_2 U_2)^{-1}\,, &&\\
&Z_{{\substack{\Box\\\Box}}_1} = Z_{\overline{\substack{\Box\\\Box}}_1} = (T_2^{(1)} U_2^{(1)})^{-1}\,, && Z_{{\substack{\Box\\\Box}}_2} = Z_{\overline{\substack{\Box\\\Box}}_2} = (T_2^{(2)} U_2^{(2)})^{-1}\,, &&\\
&\widetilde{n}_{\rm adj} = \widetilde{n}_{{\substack{\Box\\\Box}}_{1,2}} = \widetilde{n}_{\overline{\substack{\Box\\\Box}}_{1,2}} = 1\,, && \widetilde{n}_{\Box} = 8\,. &&
\end{aligned}
\tag{E.10}
$$

The Kähler metrics for the various multiplets in the type I′ model written in eq. (E.9) can be deduced from those reported for example in refs. [55, 57, 60]; those for the heterotic model written in eq. (E.10) can be obtained upon replacing the type I′ variables with the corresponding heterotic ones, or can be deduced from results existing in the literature, for example in refs. [50, 51, 68].

F Theta functions

In this appendix we collect some useful formulas on the Jacobi ϑ-functions and the Dedekind η-function. We adopt the standard definitions

$$
\vartheta\left[{a \atop b}\right](v|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{a}{2})^2}\, \mathrm{e}^{2\pi\mathrm{i}(n-\frac{a}{2})(v-\frac{b}{2})}\,, \qquad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \tag{F.1}
$$

with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. We also take $\vartheta_1 \equiv \vartheta\left[{1 \atop 1}\right]$, $\vartheta_2 \equiv \vartheta\left[{1 \atop 0}\right]$, $\vartheta_3 \equiv \vartheta\left[{0 \atop 0}\right]$, $\vartheta_4 \equiv \vartheta\left[{0 \atop 1}\right]$. Most of the times, when we do not write explicitly their arguments, we understand that the ϑ-functions are evaluated at $v = 0$ with modular parameter τ. Sometimes, we indicate only its first argument v. When ambiguities may arise, we write explicitly both arguments. In the calculations we also need the properties of the second Eisenstein series E_2 defined by

$$
E_2(\tau) = \frac{12}{\mathrm{i}\pi}\,\partial_\tau \log\eta(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\,. \tag{F.2}
$$

E_2 does not enjoy nice modular properties, but $\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2}$ is a modular form of weight 2, even if it is not holomorphic.

JHEP05(2010)107

- Jacobi/Riemann identities and other relations:

$$\vartheta_3^4-\vartheta_2^4-\vartheta_4^2=0\,,\quad \vartheta_3^{12}-\vartheta_2^{12}-\vartheta_4^{12}=48\,\eta^{12}\,,\quad \vartheta_2\,\vartheta_3\,\vartheta_4=2\,\eta^3\,,$$
$$\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab}\vartheta\left[{a \atop b}\right](w)\,\vartheta\left[{a \atop b}\right]\vartheta\left[{a+h \atop b+g}\right](w)^2=-\vartheta_1\left(\frac{w}{2}\right)^2\vartheta\left[{1+h \atop 1+g}\right]\left(\frac{w}{2}\right)^2\,. \tag{F.3}$$

- Duplication formulas:

$$\begin{aligned}
\vartheta_2(0|2\tau)&=\sqrt{\frac{\vartheta_3^2-\vartheta_4^2}{2}}\,, & \vartheta_4(0|2\tau)&=\sqrt{\vartheta_3\,\vartheta_4}\,,\\
\vartheta_3(0|2\tau)&=\sqrt{\frac{\vartheta_3^2+\vartheta_4^2}{2}}\,, & \eta(2\tau)&=\sqrt{\frac{\vartheta_3\,\eta}{2}}\,,\\
E_2(2\tau)&=\frac{1}{4}\left(2E_2+\vartheta_3^4+\vartheta_4^4\right)\,,\\
\vartheta_3\left(0|\frac{\tau}{2}\right)&=\sqrt{\vartheta_3^2+\vartheta_2^2}\,, & \vartheta_2\left(0|\frac{\tau}{2}\right)&=\sqrt{2\,\vartheta_2\,\vartheta_3}\,,\\
\vartheta_4\left(0|\frac{\tau}{2}\right)&=\sqrt{\vartheta_3^2-\vartheta_2^2}\,, & \eta\left(\frac{\tau}{2}\right)&=\sqrt{\vartheta_4\,\eta}\,,\\
E_2\left(\frac{\tau}{2}\right)&=2E_2-\vartheta_2^4-\vartheta_3^4\,.
\end{aligned} \tag{F.4}$$

- Modular transformations:

$$\begin{aligned}
T\ &:\quad \vartheta\left[{a \atop b}\right](v|\tau+1)=e^{\frac{i\pi}{4}a(2-a)}\vartheta\left[{a \atop a+b-1}\right](v|\tau)\,,\quad \eta(\tau+1)=e^{\frac{i\pi}{12}}\,\eta(\tau)\,,\\
S\ &:\quad \vartheta\left[{a \atop b}\right]\left(\frac{v}{\tau}\Big|\frac{-1}{\tau}\right)=\sqrt{-i\tau}\,e^{\frac{i\pi}{2}(ab+\frac{2v^2}{\tau})}\,\vartheta\left[{b \atop -a}\right](v|\tau)\,,\quad \eta\left(\frac{-1}{\tau}\right)=\sqrt{-i\tau}\,\eta(\tau)\,,
\end{aligned} \tag{F.5}$$

which imply

$$\begin{aligned}
T\ &:\quad \vartheta_3\leftrightarrow\theta_4\,,\quad \vartheta_2\to e^{\frac{i\pi}{4}}\,\vartheta_2\,,\quad \hat{E}_2\to\hat{E}_2\,,\\
S\ &:\quad \frac{\vartheta_2}{\eta}\leftrightarrow\frac{\vartheta_4}{\eta}\,,\quad \frac{\vartheta_3}{\eta}\to\frac{\vartheta_3}{\eta}\,,\quad \hat{E}_2\to\tau^2\,\hat{E}_2\,.
\end{aligned} \tag{F.6}$$

- $\mathbb{Z}_4$ ϑ-functions:

$$\begin{aligned}
\vartheta\left[{0 \atop \pm 1/2}\right](v|\tau)&=\vartheta_1(2v|4\tau)+\vartheta_4(2v|4\tau)\,,\\
\vartheta\left[{1 \atop \pm 1/2}\right](v|\tau)&=\frac{e^{-\frac{i\pi}{8}}}{\sqrt{2}}\left[\vartheta_1\left(v|\tau+\frac{1}{2}\right)+\vartheta_2\left(v|\tau+\frac{1}{2}\right)\right]\,.
\end{aligned} \tag{F.7}$$

- Derivatives of ϑ-functions (the $'$ denotes a derivative with respect to the first argument of the ϑ-functions):

$$\begin{aligned}
&\frac{\vartheta_2''}{\vartheta_2}=-\frac{\pi^2}{3}\left(E_2+\vartheta_3^4+\vartheta_4^4\right)\,,\quad \frac{\vartheta_3''}{\vartheta_3}=-\frac{\pi^2}{3}\left(E_2+\vartheta_2^4-\vartheta_4^4\right)\,,\\
&\frac{\vartheta_4''}{\vartheta_4}=-\frac{\pi^2}{3}\left(E_2-\vartheta_2^4-\vartheta_3^4\right)\,,\quad \vartheta_1'=2\pi\eta^3\,,\qquad \frac{\vartheta_1'''}{\vartheta_1'}=-\pi^2E_2\,.
\end{aligned} \tag{F.8}$$

- Useful identities (all ϑ-functions have vanishing first argument):

$$\vartheta\left[{0 \atop \pm 1/2}\right]^4 = \frac{1}{2}\,\vartheta_3\,\vartheta_4\left(\vartheta_3^2+\vartheta_4^2\right), \qquad \vartheta\left[{1 \atop \pm 1/2}\right]^4 = \frac{1}{2}\,\vartheta_3\,\vartheta_4\left(\vartheta_3^2-\vartheta_4^2\right),$$

$$\frac{\vartheta\left[{0 \atop \pm 1/2}\right]''}{\vartheta\left[{0 \atop \pm 1/2}\right]} = -\frac{\pi^2}{12}\left(\vartheta_3^4 - 6\vartheta_3^2\,\vartheta_4^2 + \vartheta_4^4 + 4\hat{E}_2\right),$$

$$\frac{\vartheta\left[{1 \atop \pm 1/2}\right]''}{\vartheta\left[{1 \atop \pm 1/2}\right]} = -\frac{\pi^2}{12}\left(\vartheta_3^4 + 6\vartheta_3^2\,\vartheta_4^2 + \vartheta_4^4 + 4\hat{E}_2\right),$$

$$\frac{\vartheta\left[{0 \atop \pm 1/2}\right]'}{\vartheta\left[{0 \atop \pm 1/2}\right]} = \frac{\pi}{2}\left(\vartheta_3^2+\vartheta_4^2\right), \qquad \frac{\vartheta\left[{1 \atop \pm 1/2}\right]'}{\vartheta\left[{0 \atop \pm 1/2}\right]} = \frac{\pi}{2}\left(\vartheta_3^2-\vartheta_4^2\right),$$

$$\vartheta\left[{0 \atop \pm 1/2}\right]^4\!\left(\frac{\tau}{2}\right) = \vartheta_4^2\,\vartheta_3^2\,, \qquad \vartheta\left[{1 \atop \pm 1/2}\right]^4\left(\frac{\tau}{2}\right) = \vartheta_4^2\,\vartheta_2^2\,,$$

$$\frac{\vartheta\left[{0 \atop \pm 1/2}\right]''\left(\frac{\tau}{2}\right)}{\vartheta\left[{0 \atop \pm 1/2}\right]\left(\frac{\tau}{2}\right)} = 4\,\frac{\vartheta_4''(2\tau)}{\vartheta_4(2\tau)} = \frac{\pi^2}{3}\left(\vartheta_3^4+\vartheta_4^4-2E_2\right),$$

$$\frac{\vartheta\left[{1 \atop \pm 1/2}\right]''\left(\frac{\tau}{2}\right)}{\vartheta\left[{1 \atop \pm 1/2}\right]\left(\frac{\tau}{2}\right)} = \frac{\vartheta_2''\left(\frac{\tau+1}{2}\right)}{\vartheta_2\left(\frac{\tau+1}{2}\right)} = \frac{\pi^2}{3}\left(\vartheta_2^4-\vartheta_4^4-2E_2\right),$$

$$\frac{\vartheta\left[{0 \atop \pm 1/2}\right]'\left(\frac{\tau}{2}\right)}{\vartheta\left[{0 \atop \pm 1/2}\right]\left(\frac{\tau}{2}\right)} = 2\,\frac{\vartheta_1'(2\tau)}{\vartheta_4(2\tau)} = \pi\vartheta_2^2\,, \qquad \frac{\vartheta\left[{1 \atop \pm 1/2}\right]'\left(\frac{\tau}{2}\right)}{\vartheta\left[{1 \atop \pm 1/2}\right]\left(\frac{\tau}{2}\right)} = \frac{\vartheta_1'\left(\frac{\tau+1}{2}\right)}{\vartheta_2\left(\frac{\tau+1}{2}\right)} = \pi\vartheta_3^2\,. \tag{F.9}$$

References

[1] R. Blumenhagen, M. Cvetič and T. Weigand, *Spacetime instanton corrections in 4D string vacua — the seesaw mechanism for D-brane models*, *Nucl. Phys.* **B 771** (2007) 113 [hep-th/0609191] [SPIRES].

[2] L.E. Ibáñez and A.M. Uranga, *Neutrino Majorana masses from string theory instanton effects*, *JHEP* **03** (2007) 052 [hep-th/0609213] [SPIRES].

[3] B. Florea, S. Kachru, J. McGreevy and N. Saulina, *Stringy Instantons and Quiver Gauge Theories*, *JHEP* **05** (2007) 024 [hep-th/0610003] [SPIRES].

[4] R. Argurio, M. Bertolini, G. Ferretti, A. Lerda and C. Petersson, *Stringy Instantons at Orbifold Singularities*, *JHEP* **06** (2007) 067 [arXiv:0704.0262] [SPIRES].

[5] M. Bianchi, F. Fucito and J.F. Morales, *D-brane Instantons on the T^6/Z_3 orientifold*, *JHEP* **07** (2007) 038 [arXiv:0704.0784] [SPIRES].

[6] L.E. Ibáñez, A.N. Schellekens and A.M. Uranga, *Instanton Induced Neutrino Majorana Masses in CFT Orientifolds with MSSM-like spectra*, *JHEP* **06** (2007) 011 [arXiv:0704.1079] [SPIRES].

[7] R. Blumenhagen, M. Cvetič, D. Lüst, R. Richter, and T. Weigand, *Non-perturbative Yukawa Couplings from String Instantons*, *Phys. Rev. Lett.* **100** (2008) 061602 [arXiv:0707.1871] [SPIRES].

[8] O. Aharony and S. Kachru, *Stringy Instantons and Cascading Quivers*, *JHEP* **09** (2007) 060 [arXiv:0707.3126] [SPIRES].

[9] R. Blumenhagen, M. Cvetič, R. Richter, and T. Weigand, *Lifting D-Instanton Zero Modes by Recombination and Background Fluxes*, *JHEP* **10** (2007) 098 [arXiv:0708.0403] [SPIRES].

[10] P.G. Camara, E. Dudas, T. Maillard and G. Pradisi, *String instantons, fluxes and moduli stabilization*, *Nucl. Phys.* **B 795** (2008) 453 [arXiv:0710.3080] [SPIRES].

[11] L.E. Ibáñez and A.M. Uranga, *Instanton Induced Open String Superpotentials and Branes at Singularities*, *JHEP* **02** (2008) 103 [arXiv:0711.1316] [SPIRES].

[12] C. Petersson, *Superpotentials From Stringy Instantons Without Orientifolds*, *JHEP* **05** (2008) 078 [arXiv:0711.1837] [SPIRES].

[13] R. Blumenhagen, S. Moster and E. Plauschinn, *Moduli Stabilisation versus Chirality for MSSM like Type IIB Orientifolds*, *JHEP* **01** (2008) 058 [arXiv:0711.3389] [SPIRES].

[14] M. Cvetič, R. Richter, 2 and T. Weigand, *(Non-)BPS bound states and D-brane instantons*, *JHEP* **07** (2008) 012 [arXiv:0803.2513] [SPIRES].

[15] M. Billò et al., *Flux interactions on D-branes and instantons*, *JHEP* **10** (2008) 112 [arXiv:0807.1666] [SPIRES].

[16] M. Billò et al., *Non-perturbative effective interactions from fluxes*, *JHEP* **12** (2008) 102 [arXiv:0807.4098] [SPIRES].

[17] L.E. Ibáñez and R. Richter, *Stringy Instantons and Yukawa Couplings in MSSM-like Orientifold Models*, *JHEP* **03** (2009) 090 [arXiv:0811.1583] [SPIRES].

[18] R. Argurio, G. Ferretti and C. Petersson, *Instantons and Toric Quiver Gauge Theories*, *JHEP* **07** (2008) 123 [arXiv:0803.2041] [SPIRES].

[19] C. Angelantonj, C. Condeescu, E. Dudas and M. Lennek, *Stringy Instanton Effects in Models with Rigid Magnetised D-branes*, *Nucl. Phys.* **B 818** (2009) 52 [arXiv:0902.1694] [SPIRES].

[20] M. Bianchi, F. Fucito and J.F. Morales, *Dynamical supersymmetry breaking from unoriented D-brane instantons*, *JHEP* **08** (2009) 040 [arXiv:0904.2156] [SPIRES].

[21] M. Buican and S. Franco, *SUSY breaking mediation by D-brane instantons*, *JHEP* **12** (2008) 030 [arXiv:0806.1964] [SPIRES].

[22] M. Billò, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Classical solutions for exotic instantons?*, *JHEP* **03** (2009) 056 [arXiv:0901.1666] [SPIRES].

[23] M. Billò et al., *Exotic instanton counting and heterotic/type-I' duality*, *JHEP* **07** (2009) 092 [arXiv:0905.4586] [SPIRES].

[24] F. Fucito, J.F. Morales and R. Poghossian, *Exotic prepotentials from D(-1)D7 dynamics*, *JHEP* **10** (2009) 041 [arXiv:0906.3802] [SPIRES].

[25] C. Petersson, P. Soler and A.M. Uranga, *D-instanton and polyinstanton effects from type-I' D0-brane loops*, arXiv:1001.3390 [SPIRES].

[26] R. Blumenhagen, M. Cvetič, S. Kachru and T. Weigand, *D-Brane Instantons in Type II Orientifolds*, *Ann. Rev. Nucl. Part. Sci.* **59** (2009) 269 [arXiv:0902.3251] [SPIRES].

[27] E. Witten, *Small Instantons in String Theory*, *Nucl. Phys.* **B 460** (1996) 541 [hep-th/9511030] [SPIRES].

[28] M.R. Douglas, *Branes within branes*, hep-th/9512077 [SPIRES].

[29] M.B. Green and M. Gutperle, *Effects of D-instantons*, *Nucl. Phys.* **B 498** (1997) 195 [hep-th/9701093] [SPIRES].

[30] M.B. Green and M. Gutperle, *D-instanton partition functions*, *Phys. Rev.* **D 58** (1998) 046007 [hep-th/9804123] [SPIRES].

[31] M.B. Green and M. Gutperle, *D-instanton induced interactions on a D3-brane*, *JHEP* **02** (2000) 014 [hep-th/0002011] [SPIRES].

[32] M. Billò et al., *Classical gauge instantons from open strings*, *JHEP* **02** (2003) 045 [hep-th/0211250] [SPIRES].

[33] M. Berkooz et al., *Anomalies, Dualities and Topology of $D = 6$ $N = 1$ Superstring Vacua*, *Nucl. Phys.* **B 475** (1996) 115 [hep-th/9605184] [SPIRES].

[34] E. Kiritsis and N.A. Obers, *Heterotic/type-I duality in $D < 10$ dimensions, threshold corrections and D-instantons*, *JHEP* **10** (1997) 004 [hep-th/9709058] [SPIRES].

[35] G. Aldazabal, A. Font, L.E. Ibáñez, A.M. Uranga and G. Violero, *Non-perturbative heterotic $D = 6, 4$, $N = 1$ orbifold vacua*, *Nucl. Phys.* **B 519** (1998) 239 [hep-th/9706158] [SPIRES].

[36] W. Lerche and S. Stieberger, *Prepotential, mirror map and F-theory on $K3$*, *Adv. Theor. Math. Phys.* **2** (1998) 1105 [hep-th/9804176] [SPIRES].

[37] M. Gutperle, *Heterotic/type-I duality, D-instantons and a $N = 2$ AdS/CFT correspondence*, *Phys. Rev.* **D 60** (1999) 126001 [hep-th/9905173] [SPIRES].

[38] E. Gava, K.S. Narain and M.H. Sarmadi, *Instantons in $N = 2$ Sp(N) superconformal gauge theories and the AdS/CFT correspondence*, *Nucl. Phys.* **B 569** (2000) 183 [hep-th/9908125] [SPIRES].

[39] E. Kiritsis, N.A. Obers and B. Pioline, *Heterotic/type-II triality and instantons on $K3$*, *JHEP* **01** (2000) 029 [hep-th/0001083] [SPIRES].

[40] B. Grossman, T.W. Kephart and J.D. Stasheff, *Solutions to Yang-Mills field equations in eight-dimensions and the last Hopf map*, *Commun. Math. Phys.* **96** (1984) 431 [*Erratum ibid.* **100** (1985) 311] [SPIRES].

[41] B. Grossman, T.W. Kephart and J.D. Stasheff, *Solutions to gauge field equations in eight-dimensions: conformal invariance and the last Hopf map*, *Phys. Lett.* **B 220** (1989) 431 [SPIRES].

[42] N.A. Nekrasov, *Seiberg-Witten Prepotential From Instanton Counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [hep-th/0206161] [SPIRES].

[43] C. Bachas, C. Fabre, E. Kiritsis, N.A. Obers and P. Vanhove, *Heterotic/type-I duality and D-brane instantons*, *Nucl. Phys.* **B 509** (1998) 33 [hep-th/9707126] [SPIRES].

[44] C. Bachas, *Heterotic versus type-I*, *Nucl. Phys. Proc. Suppl.* **68** (1998) 348 [hep-th/9710102] [SPIRES].

[45] A. Sen, *F-theory and Orientifolds*, *Nucl. Phys.* **B 475** (1996) 562 [hep-th/9605150] [SPIRES].

[46] T. Banks, M.R. Douglas and N. Seiberg, *Probing F-theory with branes*, *Phys. Lett.* **B 387** (1996) 278 [hep-th/9605199] [SPIRES].

[47] M. Bianchi and J.F. Morales, *Unoriented D-brane Instantons vs Heterotic worldsheet Instantons*, *JHEP* **02** (2008) 073 [arXiv:0712.1895] [SPIRES].

[48] M. Bianchi and A. Sagnotti, *Twist symmetry and open string Wilson lines*, *Nucl. Phys.* **B 361** (1991) 519 [SPIRES].

[49] E.G. Gimon and J. Polchinski, *Consistency Conditions for Orientifolds and D-Manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038] [SPIRES].

[50] L.J. Dixon, V. Kaplunovsky and J. Louis, *Moduli dependence of string loop corrections to gauge coupling constants*, *Nucl. Phys.* **B 355** (1991) 649 [SPIRES].

[51] V. Kaplunovsky and J. Louis, *On Gauge couplings in string theory*, *Nucl. Phys.* **B 444** (1995) 191 [hep-th/9502077] [SPIRES].

[52] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Perturbative couplings of vector multiplets in $N = 2$ heterotic string vacua*, *Nucl. Phys.* **B 451** (1995) 53 [hep-th/9504006] [SPIRES].

[53] P.G. Camara and E. Dudas, *Multi-instanton and string loop corrections in toroidal orbifold models*, *JHEP* **08** (2008) 069 [arXiv:0806.3102] [SPIRES].

[54] T.W. Grimm and J. Louis, *The effective action of $N = 1$ Calabi-Yau orientifolds*, *Nucl. Phys.* **B 699** (2004) 387 [hep-th/0403067] [SPIRES].

[55] D. Lüst, S. Reffert and S. Stieberger, *Flux-induced Soft Supersymmetry Breaking in Chiral Type IIB Orientifolds with D3/D7-branes*, *Nucl. Phys.* **B 706** (2005) 3 [hep-th/0406092] [SPIRES].

[56] M. Bianchi and J.F. Morales, *RG flows and open/closed string duality*, *JHEP* **08** (2000) 035 [hep-th/0006176] [SPIRES].

[57] M. Billò et al., *Instantons in $N = 2$ magnetized D-brane worlds*, *JHEP* **10** (2007) 091 [arXiv:0708.3806] [SPIRES].

[58] M. Berg, M. Haack and B. Körs, *String loop corrections to Kähler potentials in orientifolds*, *JHEP* **11** (2005) 030 [hep-th/0508043] [SPIRES].

[59] M. Billò et al., *Instanton effects in $N = 1$ brane models and the Kähler metric of twisted matter*, *JHEP* **12** (2007) 051 [arXiv:0709.0245] [SPIRES].

[60] M. Bertolini, M. Billó, A. Lerda, J.F. Morales and R. Russo, *Brane world effective actions for D-branes with fluxes*, *Nucl. Phys.* **B 743** (2006) 1 [hep-th/0512067] [SPIRES].

[61] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [hep-th/0211108] [SPIRES].

[62] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238 [SPIRES].

[63] M. Mariño and N. Wyllard, *A note on instanton counting for $N = 2$ gauge theories with classical gauge groups*, *JHEP* **05** (2004) 021 [hep-th/0404125] [SPIRES].

[64] M. Billó, M. Frau, F. Fucito and A. Lerda, *Instanton calculus in R-R background and the topological string*, *JHEP* **11** (2006) 012 [hep-th/0606013] [SPIRES].

[65] F. Fucito, J.F. Morales and A. Tanzini, *D-instanton probes of non-conformal geometries*, *JHEP* **07** (2001) 012 [hep-th/0106061] [SPIRES].

[66] U. Bruzzo and F. Fucito, *Superlocalization formulas and supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 678** (2004) 638 [math-ph/0310036] [SPIRES].

[67] G.W. Moore, N. Nekrasov and S. Shatashvili, *D-particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [hep-th/9803265] [SPIRES].

[68] I. Antoniadis, C. Bachas, C. Fabre, H. Partouche and T.R. Taylor, *Aspects of type I-type II-heterotic triality in four dimensions*, *Nucl. Phys.* **B 489** (1997) 160 [hep-th/9608012] [SPIRES].

[69] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [SPIRES].