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Abstract

The aim of this paper is to investigate, in a bounded domain of $\mathbb{R}^3$, two blending sums of univariate and bivariate C^1 quadratic spline quasi-interpolants.

The main problem consists in constructing the coefficient functionals associated with boundary generators, i.e. generators with supports not entirely inside the domain. In their definition, these functionals involve data points lying inside or on the boundary of the domain. Moreover, the weights of these functionals must be chosen so that the quasi-interpolants have the best approximation order and a reasonable infinite norm.

We give their explicit constructions, infinite norms and error estimates. In order to illustrate the approximation properties of the proposed quasi-interpolants, some numerical examples are presented and compared with those obtained by some other trivariate quasi-interpolants given recently in the literature.

Keywords: Spline approximation, Quasi-interpolation operator, Trivariate splines

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1. Introduction

The reconstruction of volume data is an active area of research, due to its relevance to many applications, such as scientific visualization, medical imaging and computer graphics (see e.g. Nürnberger et al. 2005 and the references therein). These reconstructions require an appropriate non discrete model of the given gridded volume data. Standard approaches use trivariate tensor-product of splines (see e.g. Barthe et al. 2002; Meissner et al. 2000), like trilinear continuous splines and triquadratic and tricubic smooth splines. Other approaches, proposed in the last years, consist in trivariate macro-elements (see e.g. Lai and Le Méhauté 2004, Lai and Schumaker 2007, Chap.18, Schumaker and Sorokina 2005; Schumaker et al. 2009 and the references therein), in the local reconstruction of volume data by using splines given in Bernstein-Bézier form (BB-form) (see e.g. Nürnberger et al. 2005; Rössl et al. 2004; Sorokina and Zeilfelder 2007) or in trivariate box splines (see e.g. Entezari and Möller 2006; Entezari et al. 2008, 2009; Kim et al. 2008; Remogna 2010b,c).

In this paper, following the method proposed in Delvos and Schempp (1989, Chap.6), we investigate blending sums of univariate and bivariate C^1 quadratic spline quasi-interpolants (abbr. QIs) whose data points are inside or on the boundary of the domain Ω of $\mathbb{R}^3$. In Sablonnière (2003a,b) the author proposes a blending sum of univariate and bivariate C^1 quadratic QIs using different types of B-splines, with multiple knots and with supports included in Ω. Instead, here we use generators of one type, which is an advantage from a computational point of view, and with supports overlapping Ω. The main problem consists in finding the coefficient functionals associated with boundary generators involving, in their definition, data points inside or on the boundary of the domain.

First, in Section 2 we define the blending sum of a univariate QI and a bivariate one, and, in Section 3 we present several ways of constructing quasi-interpolants on bounded domains of $\mathbb{R}$ and $\mathbb{R}^2$. Then, in Section 4 and 5 we introduce three different types of quadratic QIs on uniform meshes of a bounded interval of the real line and on a uniform criss-cross triangulation of a bounded rectangle, respectively. In Section 6, we study trivariate blending sums of univariate and bivariate QIs defined in the previous sections. In order to define these trivariate blending sums, special tensor products of univariate and bivariate C^1 quadratic B-splines are used and they yield to C^1 trivariate piecewise polynomials of degree four. Finally, in Section 7, we

present some numerical examples illustrating the approximation properties of the proposed trivariate quasi-interpolants.

2. Construction of trivariate blending sums of quasi-interpolants

In this section we define the blending sum of a univariate QI and a bivariate one.

Let $\Omega^{''} \subset \mathbb{R}$ and $\Omega^{'} \subset \mathbb{R}^2$ be bounded domains. Let $\{B_\alpha(x,y),\ \alpha \in F\}$, $F \subset \mathbb{Z}^2$, be an appropriate set of bivariate B-spline functions spanning a space of bivariate splines of degree ρ defined on a uniform triangulation of $\Omega^{'}$ of mesh size $h > 0$, and let $\{B_\alpha(z),\ \alpha \in \bar{F}\}$, $\bar{F} \subset \mathbb{Z}$, be an appropriate set of B-spline functions in one dimension, spanning a space of splines of degree $\bar{\rho}$ defined on a uniform partition of $\Omega^{''}$ of step-length h.

We consider bivariate and univariate quasi-interpolants (see e.g. de Boor 2001, Chap.12, Lyche and Schumaker 1975, for the univariate case and de Boor et al. 1993, Chap.3, for the bivariate case), $Pf(x,y)$ and $\bar{P}f(z)$, of the form

$$Pf(x,y) = \sum_{\alpha \in F} \mu_\alpha(f) B_\alpha(x,y) = \sum_{(i,j) \in F} \mu_{(i,j)}(f) B_{(i,j)}(x,y), \tag{1}$$

$$\bar{P}f(z) = \sum_{k \in \bar{F}} \bar{\mu}_k(f) B_k(z). \tag{2}$$

where the coefficients $\{\mu_\alpha,\ \alpha \in F\}$ and $\{\bar{\mu}_k,\ k \in \bar{F}\}$ are local linear functionals which are combinations of values of f at some points lying in a neighbourhood of the support of the corresponding B-spline. More specifically, given the sets of data points $\{M_\alpha,\ \alpha \in F_M\}$ and $\{u_k,\ k \in \bar{F}_u\}$ ($F_M \subset \mathbb{Z}^2$, $\bar{F}_u \subset \mathbb{Z}$ finite sets of indices), we consider coefficient functionals of type

$$\mu_\alpha(f) = \sum_{\beta \in F_\alpha} \sigma_\alpha(\beta) f(M_\beta), \quad \sigma_\alpha(\beta) \in \mathbb{R}, \tag{3}$$

$$\bar{\mu}_k(f) = \sum_{\beta \in \bar{F}_k} \bar{\sigma}_k(\beta) f(u_\beta), \quad \bar{\sigma}_k(\beta) \in \mathbb{R}, \tag{4}$$

where $F_\alpha \subset F_M$ and $\bar{F}_k \subset \bar{F}_u$. Furthermore, the QIs P and $\bar{P}$ are exact on some polynomial space $\mathbb{P}_p[x,y]$ and $\mathbb{P}_{\bar{p}}[z]$, respectively.

From the expressions (1) and (2), after some algebra, we get the following expressions for P and $\bar{P}$

$$Pf(x,y) = \sum_{(i,j) \in F_M} f(M_{(i,j)}) L_{(i,j)}(x,y), \qquad \bar{P}f(z) = \sum_{k \in \bar{F}_u} f(u_k)(f) \bar{L}_k(z),$$

where $L_{(i,j)}(x,y)$ and $\bar{L}_k(z)$ are called quasi-Lagrange functions associated with P and $\bar{P}$ respectively, and given by linear combinations of B-splines. This form is more convenient for the definition of the trivariate blending sum and it is useful to compute the infinite norm of the QIs. Indeed, the infinite norms of P and $\bar{P}$ are equal to the Chebyshev norms of their Lebesgue functions respectively defined by:

$$\Lambda = \sum_{(i,j)\in F_M} |L_{i,j}|, \qquad \bar{\Lambda} = \sum_{k\in \bar{F}_u} |L_k|$$

and $\|P\|_\infty = \|\Lambda\|_\infty$, $\|\bar{P}\|_\infty = \left\|\bar{\Lambda}\right\|_\infty$.

Furthermore, according to classical results in approximation theory (see e.g. DeVore and Lorentz 1993, Chap.5, Lyche and Schumaker 1975 for the univariate case and de Boor et al. 1993, Chap.3, Dagnino and Lamberti 2001; Foucher and Sablonnière 2008; Lyche and Schumaker 1975 for the bivariate one) and, in view of the exactness of P on $\mathbb{P}_p[x,y]$ and $\bar{P}$ on $\mathbb{P}_{\bar{p}}[z]$, we have that the rate of convergence is $O(h^p)$ for the two-dimensional case and $O(h^{\bar{p}})$ for the univariate case, i.e.

$$\|f - Pf\|_\infty \leq K\ h^p \left\|f^{(p)}\right\|_\infty, \qquad \left\|f - \bar{P}f\right\|_\infty \leq \bar{K}\ h^{\bar{p}} \left\|f^{(\bar{p})}\right\|_\infty,$$

for some positive constants K, $\bar{K}$ and sufficiently smooth functions f.

We define the trivariate extensions (Delvos and Schempp, 1989, Chap.6) of the QIs P and $\bar{P}$ in the following way

$$\begin{aligned} Pf(x,y,z) &= \sum_{(i,j)\in F_M} f(M_{(i,j)},z)L_{(i,j)}(x,y), \\ \bar{P}f(x,y,z) &= \sum_{k\in \bar{F}_u} f(x,y,u_k)(f)\bar{L}_k(z). \end{aligned}$$

Now we construct the trivariate blending sum. Let $\Omega = \Omega' \times \Omega''$, and let $B_{i,j,k}(x,y,z) = B_{i,j}(x,y)B_k(z)$, $(i,j) \in F$, $k \in \bar{F}$ be the tensor product B-splines of degree $\rho + \bar{\rho}$ spanning the tensor product spline space defined on Ω on a partition obtained from the univariate and the bivariate ones.

We consider two bivariate QIs P_1 and P_2 exact on the spaces $\mathbb{P}_{p_1}[x,y]$ and $\mathbb{P}_{p_2}[x,y]$, respectively

$$\begin{aligned} P_1 f(x,y,z) &= \sum_{(i,j)\in F_M} f(M_{(i,j)},z)L^{(1)}_{(i,j)}(x,y), \\ P_2 f(x,y,z) &= \sum_{(i,j)\in F_M} f(M_{(i,j)},z)L^{(2)}_{(i,j)}(x,y), \end{aligned}$$

with $p_1 < p_2$ and two univariate QIs $\bar{P}_1$ and $\bar{P}_2$ exact on the spaces $\mathbb{P}_{\bar{p}_1}[z]$ and $\mathbb{P}_{\bar{p}_2}[z]$, respectively

$$\bar{P}_1 f(x,y,z) = \sum_{k \in \bar{F}_u} f(x,y,u_k)\bar{L}_k^{(1)}(z), \quad \bar{P}_2 f(x,y,z) = \sum_{k \in \bar{F}_u} f(x,y,u_k)\bar{L}_k^{(2)}(z),$$

with $\bar{p}_1 < \bar{p}_2$ and we define (Delvos and Schempp, 1989) the trivariate blending sum

$$R = P_1\bar{P}_2 + P_2\bar{P}_1 - P_1\bar{P}_1.$$

It is a piecewise polynomial of degree $\rho + \bar{\rho}$ and it can be written in the following form

$$Rf(x,y,z) = \sum_{(i,j,k) \in F_M \times \bar{F}_u} f(M_{(i,j)}, u_k) L_{(i,j,k)}(x,y,z) \tag{5}$$

where the trivariate quasi-Lagrange functions are defined by

$$L_{i,j,k}(x,y,z) = L_{i,j}^{(1)}(x,y)\bar{L}_k^{(2)}(z) + L_{i,j}^{(2)}(x,y)\bar{L}_k^{(1)}(z) - L_{i,j}^{(1)}(x,y)\bar{L}_k^{(1)}(z).$$

We recall that the operator R can be also expressed in terms of tensor product B-splines $B_{i,j,k}(x,y,z)$.

We can state the following important properties of R.

Theorem 1. *The operator R is exact on the subspace $\mathbb{P}_R = (\mathbb{P}_{p_1}[x,y] \otimes \mathbb{P}_{\bar{p}_2}[z]) \oplus (\mathbb{P}_{p_2}[x,y] \otimes \mathbb{P}_{\bar{p}_1}[z])$ of the space $\mathbb{P}_{p_2}[x,y] \otimes \mathbb{P}_{\bar{p}_2}[z]$. Moreover the infinite norm of R is bounded by*

$$\|R\|_\infty \leq \|P_1\|_\infty \left\|\bar{P}_2\right\|_\infty + \|P_2\|_\infty \left\|\bar{P}_1\right\|_\infty + \|P_1\|_\infty \left\|\bar{P}_1\right\|_\infty.$$

Proof. Following the same logical scheme used in Theorem 4.1 of Sablonnière (2003a), the exactness of R on the space $\mathbb{P}_R$ follows from the definition (5) and from the exactness properties of each operator P_1, P_2, $\bar{P}_1$, $\bar{P}_2$. □

Moreover, since $\mathbb{P}_R \subset \mathbb{P}_q[x,y,z]$, with $q = \min\{p_2, \bar{p}_2\}$, standard results in approximation theory (see also Demichelis and Sablonnière 2007, 2010 for the case of trivariate blending sums of univariate and bivariate quadratic QIs) allow us to deduce the following theorem, where

- $\omega(\varphi, t) = \max\{|\varphi(x) - \varphi(y)|;\ x, y \in \Omega,\ \|x - y\| \leq t\}$ is the usual modulus of continuity of $\varphi \in C(\Omega)$;

- $D^\beta = D^{\beta_1\beta_2\beta_3} = \frac{\partial^{|\beta|}}{\partial x^{\beta_1}\partial y^{\beta_2}\partial z^{\beta_3}}$, with $\beta_1+\beta_2+\beta_3 = |\beta|$;

- $\omega(D^r f, h) = \max\{\omega(D^\beta f, h),\ |\beta| = r\}$.

Theorem 2. *Let $f \in C^r(\Omega)$, $r = 0, 1, \ldots, q$. Then there exist constants $K_r > 0$ such that*

$$\|f - Rf\|_\infty \le K_r h^r \omega(D^r f, h).$$

If in addition $f \in C^{q+1}(\Omega)$ then there exists a constant $K_{q+1} > 0$ such that

$$\|f - Rf\|_\infty \le K_{q+1} h^{q+1} \max_{|\beta|=q+1} \left\|D^\beta f\right\|_\infty .$$

We also introduce the concept of superconvergence. If we have a quasi-interpolant P ($\bar{P}$ for the univariate case) exact on $\mathbb{P}_p[x,y]$ ($\mathbb{P}_{\bar{p}}[z]$), then $(f - Pf)$ is $O(h^p)$ ($(f - \bar{P}f)$ is $O(h^{\bar{p}})$) for sufficiently smooth functions. We say that the QI satisfies superconvergence properties at a specific point M if $(f - Pf)(M) = O(h^{p+1})$ ($(f - \bar{P}f)$ is $O(h^{\bar{p}+1})$).

Therefore, if we suppose that the operators P_2 and $\bar{P}_2$ show superconvergence properties respectively at the sets of points $\{M_\alpha^s\} \subset \{M_\alpha,\ \alpha \in F_M\}$ and $\{u_k^s\} \subset \{u_k,\ k \in \bar{F}_u\}$, then the operator R presents superconvergence properties on the tensor product grid of the points considered in the univariate case and the points of the bivariate case, i.e. the set of points $\{(M_\alpha^s, u_k^s)\}$, and we have the following result.

Theorem 3. *At the points $\{(M_\alpha^s, u_k^s)\}$, the operators R is exact on the subspace $(\mathbb{P}_{p_1}[x,y] \otimes \mathbb{P}_{\bar{p}_2+1}[z]) \oplus (\mathbb{P}_{p_2+1}[x,y] \otimes \mathbb{P}_{\bar{p}_1}[z])$ of the space $\mathbb{P}_{p_2+1}[x,y] \otimes \mathbb{P}_{\bar{p}_2+1}[z]$.*

In Section 6, we apply the technique proposed in this section for the construction of trivariate blending sums to the univariate and bivariate C^1 quadratic QIs introduced in Sections 4 and 5, respectively, and constructed using the methods proposed in Section 3.

3. On the construction of quadratic spline quasi-interpolants

In this section we introduce general methods that will be applied in Sections 4 and 5 for the construction of univariate and bivariate quadratic QIs, respectively (for the bivariate case see Remogna 2010a).

The construction of such coefficient functionals is related to the differential quasi-interpolants (abbr. DQIs) $\hat{Q}$, $\tilde{Q}$ exact on $\mathbb{P}_2[x,y]$ and $\mathbb{P}_2[z]$,

respectively, defined by (Sablonnière 2003a, Schoenberg 1973, Chap.2 for the univariate case and Sablonnière 1982, Chap.6, for the bivariate case)

$$\hat{Q}f(x,y) = \sum_{\alpha\in\mathbb{Z}^2} \left(f(\alpha) - \frac{h^2}{8}\Delta f(\alpha) \right) B_\alpha(x,y), \tag{6}$$

$$\tilde{Q}f(z) = \sum_{k\in\mathbb{Z}} \left(f(k) - \frac{h^2}{8} f''(k) \right) B_k(z). \tag{7}$$

By convenient discretisations of (partial) derivatives in (6) and (7), some quasi-interpolants based only on function values are proposed in Sablonnière (2003a,b).

If we use these QIs in a bounded domain, the coefficient functionals associated with boundary generators (i.e. B-splines with supports not entirely inside Ω) make use of data points outside Ω (see Foucher and Sablonnière 2008 for $d = 2$). In order to use only data points inside or on the boundary of Ω we have to define new coefficient functionals for boundary B-splines.

Thus, we propose two different methods of constructing boundary functionals: one method leads to an operator called near-best and the other leads to an operator that exhibits superconvergence properties at some specific points of the domain.

3.1. Near-best quasi-interpolation operators

The near-best quasi-interpolation operators are obtained by minimizing upper bounds of their infinite norms. This method is described and used in Barrera et al. (2003, 2008, 2009); Ibáñez (2003); Remogna (2010a,c).

From (3) (similarly for the univariate case (4)), it is clear that, for $\|f\|_\infty \leq 1$ and $\alpha \in F$, $|\mu_\alpha(f)| \leq \|\sigma_\alpha\|_1$ where σ_α is the vector with components $\sigma_\alpha(\beta)$, from which we deduce immediately

$$|Pf| \leq \sum_{\alpha\in F} |\mu_\alpha(f)|\, B_\alpha \leq \max_{\alpha\in F} |\mu_\alpha(f)| \leq \max_{\alpha\in F} \|\sigma_\alpha\|_1,$$

therefore

$$\|P\|_\infty \leq \max_{\alpha\in F} \|\sigma_\alpha\|_1.$$

Now we try to find a solution $\sigma_\alpha^* \in \mathbb{R}^{\mathrm{card}(F_\alpha)}$ of the minimization problem (see e.g. Barrera et al. 2003, Ibáñez 2003, Chap.3)

$$\|\sigma_\alpha^*\|_1 = \min \left\{ \|\sigma_\alpha\|_1; \sigma_\alpha \in \mathbb{R}^{\mathrm{card}(F_\alpha)}, V_\alpha \sigma_\alpha = b_\alpha \right\},$$

where $V_\alpha \sigma_\alpha = b_\alpha$ is a linear system expressing that P is exact on $\mathbb{P}_2[x, y]$. In our case we require that the coefficient functional coincides with the differential one for $f \in \mathbb{P}_2[x, y]$. This problem is a l_1-minimization problem and there are many well-known techniques for approximating the solutions, not unique in general (see e.g. Watson 1980, Chap.6). Since the minimization problem is equivalent to a linear programming one, here we use the simplex method.

3.2. Quasi-interpolation operators with superconvergence properties

These quasi-interpolants are constructed in such a way that a phenomenon of superconvergence occurs at some specific points of the domain.

If we construct an operator P exact on $\mathbb{P}_2[x, y]$, then $f - Pf$ is $O(h^3)$ for sufficiently smooth functions. If we want superconvergence, i.e. $f - Pf = O(h^4)$ at some specific points, we have to require that, for $f \in \mathbb{P}_3[x, y]$, the quasi-interpolant Pf interpolates the function f at those points. So we impose $Pf(M) = f(M)$ for $f \in \mathbb{P}_3[x, y] \backslash \mathbb{P}_2[x, y]$ and M specific point of the domain.

This leads to a system of linear equations with free parameters which are determined by minimizing the infinite norm of the functional and solving the corresponding l_1-minimization problem.

The same technique is used for a univariate QI $\bar{P}$ exact on $\mathbb{P}_2[z]$.

4. Quadratic spline quasi-interpolants on a bounded interval

Let $\Omega'' = [0, m_3 h]$ be a bounded interval of the real line endowed with a partition $\mathcal{T}_{m_3}$ into m_3 subintervals of length h. We denote by $\mathcal{S}_2(\Omega'', \mathcal{T}_{m_3})$ the space of C^1 quadratic splines on this uniform partition. This space is generated by the $(m_3 + 2)$ quadratic B-splines $\{B_k, k \in A''\}$ (de Boor, 2001, Chap.9), (Chui, 1988, Chap.1), where $A'' = \{k,\ 0 \leq k \leq m_3 + 1\}$, with simple knots $\{kh,\ -2 \leq k \leq m_3 + 2\}$.

We use the set of $(m_3 + 2)$ data points $\{u_k,\ k \in A''\}$ defined by

$$u_0 = 0, \quad u_k = \left(k - \frac{1}{2}\right) h, \ k = 1, \ldots, m_3, \quad u_{m_3+1} = m_3 h, \tag{8}$$

and we denote the values of a function f at these points by $f_k = f(u_k)$.

In this space we consider three distinct univariate operators. The first one, $\bar{S}_1$, is related to the well-known Schoenberg-Marsden operator (see e.g.

Marsden 1975). The other two operators $\bar{Q}_1$ and $\bar{Q}_2$ are related to the univariate operator S_2 proposed in Sablonnière (2003a), exact on $\mathbb{P}_2[z]$ and deduced from (7) by replacing the second derivative by a second order divided difference based on three consecutive points. It has the form

$$Q^{**}f = \sum_{k\in\mathbb{Z}} \left(\frac{5}{4} f_k - \frac{1}{8}(f_{k-1} + f_{k+1}) \right) B_k. \tag{9}$$

Using the two different strategies introduced in Section 3, we construct boundary functionals associated with the B-splines whose supports are not entirely inside the interval Ω''.

In the interior of the domain, i.e. for $k = 2, \ldots, m_3 - 1$, the quasi-interpolants $\bar{Q}_1$ and $\bar{Q}_2$ make use of the same inner functionals given in (9).

4.1. The Schoenberg-Marsden-like operator $\bar{S}_1$

The Schoenberg-Marsden-like operator is defined by

$$\bar{S}_1 f = \sum_{k\in A''} \bar{\gamma}_k(f) B_k = \sum_{k\in A''} f_k L_k,$$

with coefficient functionals

$$\begin{array}{l} \bar{\gamma}_0(f) = 2f_0 - f_1,\ \|\bar{\gamma}_0\|_\infty = 3, \quad \bar{\gamma}_{m_3+1}(f) = 2f_{m_3+1} - f_{m_3},\ \|\bar{\gamma}_{m_3+1}\|_\infty = 3 \\ \bar{\gamma}_k(f) = f_k, \quad \|\bar{\gamma}_k\|_\infty = 1, \qquad k = 1, \ldots, m_3, \end{array}$$

ensuring the exactness on $\mathbb{P}_1[z]$ and quasi-Lagrange functions given by

$$\begin{array}{l} L_0 = 2B_0, \qquad L_1 = B_1 - B_0, \\ L_k = B_k, \ \ k = 2, \ldots m_3 - 1, \\ L_{m_3} = B_{m_3} - B_{m_3+1}, \qquad L_{m_3+1} = 2B_{m_3+1}. \end{array}$$

4.2. The near-best operator $\bar{Q}_1$

Now we define a QI exact on $\mathbb{P}_2[z]$

$$\bar{Q}_1 f = \sum_{k\in A''} \bar{\mu}_k(f) B_k = \sum_{k\in A''} f_k L_k^{(1)},$$

where the boundary functionals are functionals of near-best type (see Section 3.1). For example, in order to construct the coefficient functional $\bar{\mu}_0$, we consider a 4-point linear functional of the form

$$\bar{\mu}_0(f) = a_1 f_0 + a_2 f_1 + a_3 f_2 + a_4 f_3.$$

We impose $\bar{\mu}_0(f) \equiv (f - \frac{h^2}{8} f'')(-\frac{h}{2})$, for $f \equiv 1, z, z^2$, obtaining the exactness on the space $\mathbb{P}_2[z]$. This leads to the system:

$$a_1 + a_2 + a_3 + a_4 = 1, \quad a_2 + 3a_3 + 5a_4 = -1, \quad a_2 + 9a_3 + 25a_4 = 0,$$

whose solution depends on the parameter a_4

$$a_1 = \frac{7}{3} - \frac{8}{3}a_4, \quad a_2 = -\frac{3}{2} + 5a_4, \quad a_3 = \frac{1}{6} - \frac{10}{3}a_4.$$

Minimizing the norm $\|\bar{\mu}_0\|_\infty$ we obtain

$$a_1 = \frac{23}{15}, \quad a_2 = 0, \quad a_3 = -\frac{5}{6}, \quad a_4 = \frac{3}{10},$$

with $\|\bar{\mu}_0\|_\infty \approx 2.67$.

Following the same logical scheme we construct the other coefficient functionals:

$$\begin{array}{l}
\bar{\mu}_0(f) = \frac{23}{15} f_0 - \frac{5}{6} f_2 + \frac{3}{10} f_3, \quad \|\bar{\mu}_0\|_\infty = \frac{8}{3} \approx 2.67 \\
\bar{\mu}_1(f) = \frac{7}{8} f_1 + \frac{1}{4} f_2 - \frac{1}{8} f_3, \quad \|\bar{\mu}_1\|_\infty = \frac{5}{4} = 1.25 \\
\bar{\mu}_{m_3}(f) = \frac{7}{8} f_{m_3} + \frac{1}{4} f_{m_3-1} - \frac{1}{8} f_{m_3-2}, \quad \|\bar{\mu}_{m_3}\|_\infty = \frac{5}{4} = 1.25 \\
\bar{\mu}_{m_3+1}(f) = \frac{23}{15} f_{m_3+1} - \frac{5}{6} f_{m_3-1} + \frac{3}{10} f_{m_3-2}, \quad \|\bar{\mu}_{m_3+1}\|_\infty = \frac{8}{3} \approx 2.67.
\end{array}$$

For $k = 2, \ldots, m_3 - 1$, the $\bar{\mu}_k$'s are defined by (9), i.e.

$$\bar{\mu}_k(f) = \frac{5}{4} f_k - \frac{1}{8}(f_{k-1} + f_{k+1}), \quad \|\bar{\mu}_k\|_\infty = \frac{3}{2} = 1.5.$$

The quasi-Lagrange functions $L_k^{(1)}$ are

$$\begin{array}{l}
L_0^{(1)} = \frac{23}{15} B_0, \qquad L_1^{(1)} = \frac{7}{8} B_1 - \frac{1}{8} B_2, \qquad L_2^{(1)} = -\frac{1}{8} B_3 + \frac{5}{4} B_2 + \frac{1}{4} B_1 - \frac{5}{6} B_0, \\
L_3^{(1)} = -\frac{1}{8} B_4 + \frac{5}{4} B_3 - \frac{1}{8} B_2 - \frac{1}{8} B_1 + \frac{3}{10} B_0, \\
L_k^{(1)} = \frac{5}{4} B_k - \frac{1}{8}(B_{k-1} + B_{k+1}), \qquad k = 4, \ldots, m_3 - 3, \\
L_{m_3-2}^{(1)} = -\frac{1}{8} B_{m_3-3} + \frac{5}{4} B_{m_3-2} - \frac{1}{8} B_{m_3-1} - \frac{1}{8} B_{m_3} + \frac{3}{10} B_{m_3+1}, \\
L_{m_3-1}^{(1)} = -\frac{1}{8} B_{m_3-2} + \frac{5}{4} B_{m_3-1} + \frac{1}{4} B_{m_3} - \frac{5}{6} B_{m_3+1}, \\
L_{m_3}^{(1)} = \frac{7}{8} B_{m_3} - \frac{1}{8} B_{m_3-1}, \qquad L_{m_3+1}^{(1)} = \frac{23}{15} B_{m_3+1}.
\end{array}$$

4.3. The operator $\bar{Q}_2$ with superconvergence properties

The second QI exact on $\mathbb{P}_2[z]$, is defined by Foucher and Sablonnière (2009)

$$\bar{Q}_2 f = \sum_{k\in A''} \bar{\lambda}_k(f) B_k = \sum_{k\in A''} f_k L_k^{(2)},$$

where the boundary functionals are choosing to induce superconvergence at the knots $\mathcal{T}_{m_3}$ and the midpoints $\{u_k,\, k \in A''\}$ defined by (8) (see Section 3.2). The corresponding quasi-Lagrange functions $L_k^{(2)}$ are

$$\begin{array}{l}
L_0^{(2)} = -\frac{2}{5}B_1 + \frac{12}{5}B_0, \qquad L_1^{(2)} = -\frac{1}{8}B_2 + \frac{13}{8}B_1 - \frac{13}{8}B_0, \\
L_2^{(2)} = -\frac{1}{8}B_3 + \frac{5}{4}B_2 - \frac{1}{4}B_1 + \frac{1}{4}B_0, \;\; L_3^{(2)} = -\frac{1}{8}B_4 + \frac{5}{4}B_3 - \frac{1}{8}B_2 + \frac{1}{40}B_1 - \frac{1}{40}B_0, \\
L_k^{(1)} = \frac{5}{4}B_k - \frac{1}{8}(B_{k-1} + B_{k+1}), \qquad k = 4, \ldots, m_3 - 3, \\
L_{m_3-2}^{(2)} = -\frac{1}{8}B_{m_3-3} + \frac{5}{4}B_{m_3-2} - \frac{1}{8}B_{m_3-1} + \frac{1}{40}B_{m_3} - \frac{1}{40}B_{m_3+1}, \\
L_{m_3-1}^{(2)} = -\frac{1}{8}B_{m_3-2} + \frac{5}{4}B_{m_3-1} - \frac{1}{4}B_{m_3} + \frac{1}{4}B_{m_3+1}, \\
L_{m_3}^{(2)} = -\frac{1}{8}B_{m_3-1} + \frac{13}{8}B_{m_3} - \frac{13}{8}B_{m_3+1}, \quad L_{m_3+1}^{(2)} = -\frac{2}{5}B_{m_3} + \frac{12}{5}B_{m_3+1}.
\end{array}$$

For the operators above introduced, the following results are immediate:

- the infinite norms of $\bar{S}_1$, $\bar{Q}_1$ and $\bar{Q}_2$ satisfy

$$\left\|\bar{S}_1\right\|_\infty = 1, \;\; \left\|\bar{Q}_1\right\|_\infty = \frac{19}{12} \approx 1.58, \;\; \left\|\bar{Q}_2\right\|_\infty = \frac{73}{48} \approx 1.52;$$

- in view of the exactness of $\bar{Q}_v$, $v = 1, 2$, on $\mathbb{P}_2[z]$, we have that the rate of convergence is $O(h^3)$, i.e. for a function $f \in C^3(\Omega'')$ then there exist positive constants $\bar{K}_v$, $v = 1, 2$, such that

$$\left\|f - \bar{Q}_v f\right\|_\infty \le \bar{K}_v h^3 \left\|f^{(3)}\right\|_\infty, \quad v = 1, 2.$$

5. Quadratic spline quasi-interpolants on a bounded rectangle

Let $\Omega' = [0, m_1 h] \times [0, m_2 h]$ be a rectangular domain divided into $m_1 m_2$ squares, each of them subdivided into 4 triangles by its diagonals. We denote by $\mathcal{S}_2(\Omega', \mathcal{T}_{m_1 m_2})$ the space of C^1 quadratic splines on the triangulation $\mathcal{T}_{m_1 m_2}$ of Ω' obtained in this way. This space is generated by the $(m_1 + 2)(m_2 + 2)$ scaled translates of the classical Zwart-Powell quadratic box spline (ZP-element) $\{B_{i,j}, (i, j) \in A'\}$ (de Boor et al. 1993, Chap.1, Chui and Wang 1984, Chui 1988, Chap.3), where $A' = \{(i, j),\, 0 \le i \le m_1 + 1,\, 0 \le j \le m_2 + 1\}$.

The set of data points $\{M_{i,j},\ (i,j) \in A'\}$ is formed by the centers of the squares ($m_1 m_2$ points), the midpoints of boundary segments ($2(m_1+m_2)$ points) and the four vertices of Ω'. These points are defined by $M_{i,j} = (s_i, t_j)$ where

$$\begin{array}{lll} s_0 = 0, & s_i = (i-\frac{1}{2})h,\ 1 \le i \le m_1, & s_{m_1+1} = m_1 h, \\ t_0 = 0, & t_j = (j-\frac{1}{2})h,\ 1 \le j \le m_2, & t_{m_2+1} = m_2 h. \end{array} \tag{10}$$

The values of the function f at those points are denoted by $f_{i,j} = f(M_{i,j})$.

As in the univariate case, we define three different quasi-interpolants. The simplest one, S_1, is a bivariate extension of the Schoenberg-Marsden operator (Chui and Wang, 1984; Sablonnière, 2003a,b). The other two kinds of operators, constructed in Remogna (2010a), are related to the operator S_2, proposed in Sablonnière (2003a,b), exact on $\mathbb{P}_2[x,y]$ and obtained from (6) by replacing the Laplacian Δ by its five points discretisation. The operator S_2 has the form

$$Q^* f = \sum_{(i,j)\in\mathbb{Z}^2} \left(\frac{3}{2} f_{i,j} - \frac{1}{8}(f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1}) \right) B_{i,j}. \tag{11}$$

Using the two different strategies introduced in Section 3, we have constructed in Remogna (2010a) the boundary functionals, associated with the box splines whose supports overlap with Ω'.

5.1. The Schoenberg-Marsden-like operator S_1

The first operator is defined by

$$S_1 f = \sum_{(i,j)\in A'} \gamma_{i,j}(f) B_{i,j} = \sum_{(i,j)\in A'} f_{i,j} L_{i,j},$$

with $\gamma_{i,j}(f) = f_{i,j}$, $\|\gamma_{i,j}\|_\infty = 1$ for $i = 1,\dots,m_1$, $j = 1,\dots,m_2$ and

$$\begin{array}{ll} \gamma_{0,0}(f) = & 4f_{0,0} - 2(f_{1,0} + f_{0,1}) + f_{1,1}, \quad \|\gamma_{0,0}\|_\infty = 9 \\ \gamma_{i,0}(f) = & 2f_{i,0} - f_{i,1}, \quad \|\gamma_{i,0}\|_\infty = 3 \ \ i = 1,\dots,m_1. \end{array}$$

For the three other edges and vertices of Ω' we have analogous formulas. These coefficient functionals are constructed in order to ensure the exactness on the space of bilinear polynomials, $\mathbb{P}_{11}[x,y]$ spanned by $\{1,\ x,\ y,\ xy\}$. We can express the quasi-interpolant $S_1 f$ by means of the quasi-Lagrange functions L_α. After some algebra, we get in the neighbourhood of the origin

$$L_{0,0} = 4B_{0,0}, \quad L_{1,0} = 2B_{1,0} - 2B_{0,0}, \quad L_{1,1} = B_{1,1} + B_{0,0} - B_{1,0} - B_{0,1}.$$

Along the lower edge, for $i = 2, \ldots, m_1 - 1$, we get:

$$L_{i,0} = 2B_{i,0}, \quad L_{i,1} = B_{i,1} - B_{i,0},$$

and analogous formulas for the three other edges and vertices of Ω'. For the pairs (i, j) with $i = 2, \ldots, m_1 - 1$ and $j = 2, \ldots, m_2 - 1$ we have $L_{i,j} = B_{i,j}$.

Studying directly the Chebyshev norm of its Lebesgue function defined by $\Lambda = \sum_{(i,j)\in A'} |L_{i,j}|$, we obtain

$$\|S_1\|_\infty = \|\Lambda\|_\infty = 1.$$

5.2. The near-best operator Q_1

Now we define a QI exact on $\mathbb{P}_2[x, y]$

$$Q_1 f = \sum_{(i,j)\in A'} \mu_{i,j}(f) B_{i,j} = \sum_{(i,j)\in A'} f_{i,j} L^{(1)}_{i,j},$$

where the boundary functionals are of near-best type (see Remogna 2010a for detail). By the functional definitions, after some algebra, we get the quasi-Lagrange functions $L^{(1)}_\alpha$, with

$$L^{(1)}_{i,j} = \frac{3}{2} B_{i,j} - \frac{1}{8}(B_{i,j-1} + B_{i,j+1} + B_{i-1,j} + B_{i+1,j})$$

for the pairs (i, j) with $i = 4, \ldots, m_1 - 3$ and $j = 4, \ldots, m_2 - 3$. The other $L^{(1)}_{i,j}$-splines have particular definitions. In the neighbourhood of the origin we have

$$\begin{aligned}
&L^{(1)}_{0,0} = \tfrac{22}{9} B_{0,0} + \tfrac{38}{45}(B_{1,0} + B_{0,1}), \quad L^{(1)}_{1,0} = 0,\\
&L^{(1)}_{2,0} = -\tfrac{1}{9} B_{0,0} + \tfrac{23}{15} B_{2,0} - \tfrac{1}{2} B_{0,1} + \tfrac{19}{18} B_{1,0}, \quad L^{(1)}_{3,0} = \tfrac{23}{15} B_{3,0} + \tfrac{3}{10} B_{0,1} - \tfrac{11}{30} B_{1,0},\\
&L^{(1)}_{1,1} = \tfrac{3}{4} B_{1,1} - \tfrac{3}{2} B_{0,0}, \quad L^{(1)}_{2,1} = -\tfrac{1}{8} B_{0,2} + \tfrac{7}{8} B_{2,1} + \tfrac{1}{4} B_{1,1} - \tfrac{1}{8} B_{2,2} - \tfrac{1}{16} B_{1,2} - \tfrac{1}{3} B_{0,1},\\
&L^{(1)}_{3,1} = -\tfrac{1}{8} B_{1,1} + \tfrac{7}{8} B_{3,1} - \tfrac{1}{8} B_{3,2} - \tfrac{1}{16} B_{1,2},
\end{aligned}$$

$$\begin{aligned}
L^{(1)}_{2,2} = \;& -\tfrac{7}{12}(B_{0,2} + B_{2,0}) - \tfrac{1}{16}(B_{3,1} + B_{1,3}) + \tfrac{3}{8}(B_{1,2} + B_{2,1}) + \tfrac{5}{18} B_{0,0}\\
& -\tfrac{1}{8}(B_{0,3} + B_{3,0}) - \tfrac{1}{8}(B_{2,3} + B_{3,2}) + \tfrac{3}{2} B_{2,2},\\
L^{(1)}_{3,2} = \;& \tfrac{3}{2} B_{3,2} - \tfrac{1}{8} B_{3,3} - \tfrac{1}{8} B_{4,0} - \tfrac{1}{8} B_{2,0} - \tfrac{1}{16} B_{2,1} + \tfrac{3}{8} B_{3,1} - \tfrac{1}{16} B_{1,3} + \tfrac{3}{10} B_{0,2}\\
& -\tfrac{1}{8} B_{4,2} - \tfrac{7}{12} B_{3,0} - \tfrac{1}{16} B_{4,1} - \tfrac{1}{8} B_{2,2},\\
L^{(1)}_{3,3} = \;& -\tfrac{1}{8}(B_{3,2} + B_{2,3}) + \tfrac{3}{10}(B_{3,0} + B_{0,3}) - \tfrac{1}{16}(B_{1,2} + B_{2,1}) + \tfrac{3}{2} B_{3,3}\\
& -\tfrac{1}{16}(B_{1,4} + B_{4,1}) - \tfrac{1}{8}(B_{3,4} + B_{4,3}).
\end{aligned}$$

Along the lower edge, for $i = 4, \ldots, m_1 - 3$, we have:

$$L^{(1)}_{i,0} = \tfrac{23}{15} B_{i,0}, \quad L^{(1)}_{i,1} = \tfrac{7}{8} B_{i,1} - \tfrac{1}{8} B_{i,2},$$

$$\begin{aligned} L^{(1)}_{i,2} = \; & -\tfrac{7}{12} B_{i,0} - \tfrac{1}{8}(B_{i-1,0} + B_{i+1,0}) + \tfrac{3}{8} B_{i,1} - \tfrac{1}{16}(B_{i-1,1} + B_{i+1,1}) \\ & + \tfrac{3}{2} B_{i,2} - \tfrac{1}{8}(B_{i-1,2} + B_{i+1,2}) - \tfrac{1}{8} B_{i3}, \\ L^{(1)}_{i,3} = \; & \tfrac{3}{10} B_{i,0} - \tfrac{1}{16}(B_{i-1,1} + B_{i+1,1}) - \tfrac{1}{8} B_{i,2} + \tfrac{3}{2} B_{i,3} \\ & - \tfrac{1}{8}(B_{i-1,3} + B_{i+1,3}) - \tfrac{1}{8} B_{i,4}, \end{aligned}$$

and analogous formulas for the three other edges and vertices.

5.3. The operator Q_2 with superconvergence properties

The third QI, exact on $\mathbb{P}_2[x, y]$, is defined by

$$Q_2 f = \sum_{(i,j) \in A'} \lambda_{i,j}(f) B_{i,j} = \sum_{(i,j) \in A'} f_{i,j} L^{(2)}_{i,j},$$

where we choose boundary functionals inducing superconvergence at some specific points (see Remogna 2010a for detail). Using the notations given in (10) these specific points are (see Fig. 1): the vertices of squares $V_{r,l} = (rh, lh)$, $r = 0, \ldots, m_1$, $l = 0, \ldots, m_2$, the centers of squares $M_{r,l} = (s_r, t_l)$, $r = 1, \ldots, m_1$, $l = 1, \ldots, m_2$, the midpoints $C_{r,l} = (s_r, lh)$ of horizontal edges $V_{r-1,l} V_{r,l}$, $r = 1, \ldots, m_1$, $l = 0, \ldots, m_2$, the midpoints $D_{r,l} = (rh, t_l)$ of vertical edges $V_{r,l-1} V_{r,l}$, $r = 0, \ldots, m_1$, $l = 1, \ldots, m_2$. We remark that if $A' = \mathbb{Z}^2$ the quasi-interpolant defined by (11) is superconvergent at these points (Foucher and Sablonnière, 2007).

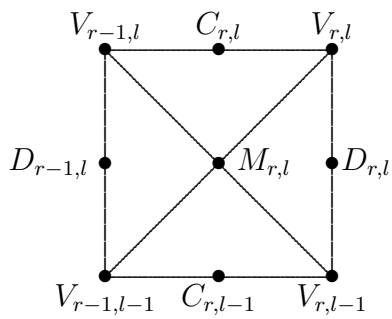

Figure 1: Points where the operator Q_2 shows superconvergence properties.

We can express the quasi-interpolant $Q_2 f$ by means of the quasi-Lagrange functions $L_\alpha^{(2)}$ with

$$L_{i,j}^{(2)} = \frac{3}{2} B_{i,j} - \frac{1}{8}(B_{i,j-1} + B_{i,j+1} + B_{i-1,j} + B_{i+1,j})$$

for the pairs (i,j) with $i = 4, \ldots, m_1 - 3$ and $j = 4, \ldots, m_2 - 3$. The other $L_{i,j}^{(2)}$-splines have particular definitions. In the neighbourhood of the origin we have

$$\begin{aligned}
L_{0,0}^{(2)} &= \tfrac{1403}{504} B_{0,0} - \tfrac{4}{15} B_{1,1}, \quad L_{1,0}^{(2)} = \tfrac{131}{60} B_{1,0} - \tfrac{173}{300} B_{0,1} - \tfrac{1}{12} B_{2,1}, \\
L_{2,0}^{(2)} &= -\tfrac{397}{1440} B_{0,0} - \tfrac{2}{15} B_{1,1} + \tfrac{12}{5} B_{2,0} - \tfrac{1}{12} B_{3,1} - \tfrac{7}{30} B_{2,1} + \tfrac{9}{40} B_{1,0}, \\
L_{3,0}^{(2)} &= -\tfrac{1}{12} B_{4,1} + \tfrac{3}{20} B_{0,1} + \tfrac{12}{5} B_{3,0} - \tfrac{1}{12} B_{2,1} - \tfrac{7}{30} B_{3,1}, \\
L_{4,0}^{(2)} &= \tfrac{11}{224} B_{0,0} + \tfrac{12}{5} B_{4,0} - \tfrac{1}{120} B_{1,0} - \tfrac{7}{30} B_{4,1} - \tfrac{1}{12} B_{3,1} - \tfrac{1}{12} B_{5,1}, \\
L_{1,1}^{(2)} &= -\tfrac{63}{32} B_{0,0} - \tfrac{13}{40}(B_{1,0} + B_{0,1}) + \tfrac{33}{20} B_{1,1} - \tfrac{1}{4}(B_{2,0} + B_{0,2}), \\
L_{2,1}^{(2)} &= -\tfrac{47}{60} B_{1,0} - \tfrac{9}{8} B_{2,0} - \tfrac{1}{4} B_{3,0} - \tfrac{1}{20} B_{1,1} + \tfrac{13}{8} B_{2,1} + \tfrac{1}{8} B_{0,2} - \tfrac{1}{24} B_{1,2} - \tfrac{1}{8} B_{2,2}, \\
L_{3,1}^{(2)} &= \tfrac{3}{50} B_{1,0} - \tfrac{1}{4} B_{2,0} - \tfrac{9}{8} B_{3,0} - \tfrac{1}{4} B_{4,0} - \tfrac{7}{40} B_{0,1} + \tfrac{1}{40} B_{1,1} + \tfrac{13}{8} B_{3,1} - \tfrac{1}{8} B_{3,2},
\end{aligned}$$

$$\begin{aligned}
L_{2,2}^{(2)} = \;& \tfrac{317}{288} B_{0,0} + \tfrac{1}{4}(B_{0,1} + B_{1,0}) + \tfrac{1}{8}(B_{3,0} + B_{0,3}) - \tfrac{1}{15} B_{1,1} - \tfrac{1}{6}(B_{1,2} + B_{2,1}) \\
& -\tfrac{1}{24}(B_{1,3} + B_{3,1}) - \tfrac{1}{8}(B_{2,3} + B_{3,2}) + \tfrac{3}{2} B_{2,2}, \\
L_{3,2}^{(2)} = \;& -\tfrac{37}{160} B_{0,0} + \tfrac{1}{8} B_{2,0} + \tfrac{1}{8} B_{4,0} - \tfrac{1}{24} B_{2,1} - \tfrac{1}{24} B_{4,1} - \tfrac{1}{6} B_{3,1} - \tfrac{1}{40} B_{0,2} \\
& +\tfrac{1}{40} B_{1,2} - \tfrac{1}{8} B_{2,2} - \tfrac{1}{8} B_{4,2} + \tfrac{3}{2} B_{3,2} - \tfrac{1}{8} B_{3,3}, \\
L_{3,3}^{(2)} = \;& -\tfrac{1}{40}(B_{3,0} + B_{0,3}) + \tfrac{1}{40}(B_{3,1} + B_{1,3}) - \tfrac{1}{8}(B_{3,2} + B_{2,3}) + \tfrac{3}{2} B_{3,3} \\
& -\tfrac{1}{8}(B_{3,4} + B_{4,3}).
\end{aligned}$$

Along the lower edge, for $i = 5, \ldots, m_1 - 4$, we have:

$$L_{i,0}^{(2)} = \tfrac{12}{5} B_{i,0} - \tfrac{7}{30} B_{i,1} - \tfrac{1}{12}(B_{i-1,1} + B_{i+1,1}),$$

and for $i = 4, \ldots, m_1 - 3$:

$$\begin{aligned}
L_{i,1}^{(2)} = \;& -\tfrac{9}{8} B_{i,0} - \tfrac{1}{4}(B_{i-1,0} + B_{i+1,0}) + \tfrac{13}{8} B_{i,1} - \tfrac{1}{8} B_{i,2}, \\
L_{i,2}^{(2)} = \;& \tfrac{1}{8}(B_{i-1,0} + B_{i+1,0}) - \tfrac{1}{6} B_{i,1} - \tfrac{1}{24}(B_{i-1,1} + B_{i+1,1}) + \tfrac{3}{2} B_{i,2} \\
& -\tfrac{1}{8}(B_{i-1,2} + B_{i+1,2}) - \tfrac{1}{8} B_{i,3}, \\
L_{i,3}^{(2)} = \;& -\tfrac{1}{40} B_{i,0} + \tfrac{1}{40} B_{i,1} - \tfrac{1}{8} B_{i,2} + \tfrac{3}{2} B_{i,3} - \tfrac{1}{8}(B_{i-1,3} + B_{i+1,3}) - \tfrac{1}{8} B_{i,4}.
\end{aligned}$$

Analogous formulas exist for the three other edges and vertices.

For the operators above introduced, the following results are valid:

- the value of the infinite norm of Q_1 is less than 2, and that of Q_2 is less than 3 (see Remogna 2010a);
- in view of the exactness of Q_v, $v = 1, 2$, on $\mathbb{P}_2[x, y]$, we have that the rate of convergence is $O(h^3)$, i.e. for a function $f \in C^3(\Omega')$ then there exist positive constants K_v, $v = 1, 2$, such that

$$\|f - Q_v f\|_\infty \leq K_v h^3 \max_{|\beta|=3} \left\|D^\beta f\right\|_\infty, \quad v = 1, 2.$$

6. Trivariate blending sums of C^1 quadratic spline quasi-interpolants

In this section, we apply the technique and the results of Section 2 to study trivariate QIs on a parallelepiped $\Omega = [0, m_1 h] \times [0, m_2 h] \times [0, m_3 h]$ which are blending sums of trivariate extensions of univariate and bivariate QIs defined in Sections 4 and 5.

We divide Ω into $m_1 m_2 m_3$ equal cubes (see Fig. 2(a)) and each cube is subdivided into 4 vertical prisms with triangular horizontal sections (see Fig. 2(b)). Thus, we obtain the partition $\mathcal{P}_{\mathbf{m}}$, $\mathbf{m} = (m_1, m_2, m_3)$, that is the tensor product of a uniform criss-cross triangulation of $[0, m_1 h] \times [0, m_2 h]$ and a uniform partition of the segment $[0, m_3 h]$.

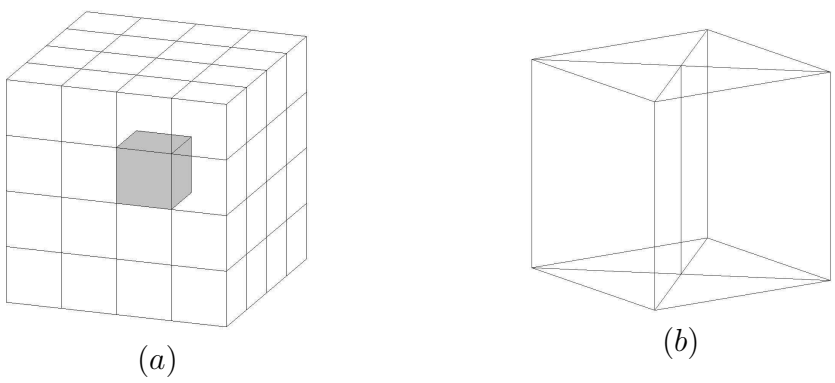

Figure 2: The parallelepiped Ω divided into equal cubes (a) and the subdivision of a cube into 4 vertical prisms with triangular horizontal sections (b).

For the projection $\Omega' = [0, m_1 h] \times [0, m_2 h]$ of Ω on the $xy-$plane we use the notations of Section 5, and for the projection $\Omega'' = [0, m_3 h]$ of Ω on the $z-$axis we use the notations of Section 4.

The set of data points is given by $\{(M_{i,j}, u_k)\,(i, j) \in A',\ k \in A''\}$, where $(M_{i,j}, u_k) = (s_i, t_j, u_k)$, with s_i, t_j defined by (10) and u_k by (8).

We consider the two families of univariate B-splines $\{B_k, k \in A''\}$, defined in Section 4, and of bivariate box splines $\{B_{i,j}, (i,j) \in A'\}$, defined in Section 5. Therefore the spline space $\mathcal{S}_{2,2}(\Omega, \mathcal{P}_{\mathbf{m}})$ is generated by the $(m_1+2)(m_2+2)(m_3+2)$ tensor product B-splines

$$B_{i,j,k}(x,y,z) = B_{i,j}(x,y)B_k(z), \quad \nu \in A,$$

with $A = \{(i,j,k),\, 0 \le i \le m_1+1,\, 0 \le j \le m_2+1,\, 0 \le k \le m_3+1\}$. Their properties are immediate consequences of properties of bivariate and univariate B-splines; in particular they are positive and form a partition of unity on Ω.

Given the trivariate extensions of bivariate and univariate QIs

$$\begin{aligned}
\bar{S}_1 f(x,y,z) &= \sum_{k \in A''} f(x,y,u_k) L_k(z), \\
\bar{Q}_v f(x,y,z) &= \sum_{k \in A''} f(x,y,u_k) L_k^{(v)}(z), \quad v = 1,2 \\
S_1 f(x,y,z) &= \sum_{(i,j) \in A'} f(s_i,t_j,z) L_{i,j}(x,y), \\
Q_v f(x,y,z) &= \sum_{(i,j) \in A'} f(s_i,t_j,z) L_{i,j}^{(v)}(x,y), \quad v = 1,2
\end{aligned}$$

we define the trivariate blending sums

$$R_v = S_1 \bar{Q}_v + Q_v \bar{S}_1 - S_1 \bar{S}_1, \quad v = 1,2$$

which are trivariate piecewise polynomials of degree four. Setting

$$L_{i,j,k}^{(v)}(x,y,z) = L_{i,j}(x,y) L_k^{(v)}(z) + L_{i,j}^{(v)}(x,y) L_k(z) - L_{i,j}(x,y) L_k(z)$$

we obtain

$$R_v f = \sum_{(i,j,k) \in A} f(s_i,t_j,u_k) L_{i,j,k}^{(v)}.$$

From Theorems 1, 3 and 2, we can state the following results

Theorem 4. *The operators R_v, $v = 1,2$, are exact on the 16-dimensional subspace $(\mathbb{P}_{11}[x,y] \otimes \mathbb{P}_2[z]) \oplus (\mathbb{P}_2[x,y] \otimes \mathbb{P}_1[z])$, spanned by monomials $\{1$, x, y, z, x^2, y^2, z^2, xy, xz, yz, x^2z, xz^2, y^2z, yz^2, xyz, $xyz^2\}$, of the 18-dimensional space $\mathbb{P}_2[x,y] \otimes \mathbb{P}_2[z]$. Moreover their infinite norms satisfy*

$$\|R_1\|_\infty \le \frac{55}{12} \approx 4.58 \quad \textit{and} \quad \|R_2\|_\infty \le \frac{265}{48} \approx 5.52.$$

According to the superconvergence properties of the univariate and bivariate operators $\bar{Q}_2$ and Q_2, the operator R_2 presents superconvergence properties on the tensor product grid of the three points of each interval considered in the univariate case (see Section 4) and the nine points of each square of the bivariate case (see Section 5), i.e. the operator R_2 presents superconvergence properties at the points

$$\begin{aligned}
&(V_{i,j}, kh) = (ih, jh, kh), \quad 0 \le i \le m_1, \quad 0 \le j \le m_2, \quad 0 \le k \le m_3;\\
&(V_{i,j}, u_k) = (ih, jh, u_k), \quad 0 \le i \le m_1, \quad 0 \le j \le m_2, \quad 1 \le k \le m_3;\\
&(M_{i,j}, kh) = (s_i, t_j, kh), \quad 1 \le i \le m_1, \quad 1 \le j \le m_2, \quad 0 \le k \le m_3;\\
&(M_{i,j}, u_k) = (s_i, t_j, u_k), \quad 1 \le i \le m_1, \quad 1 \le j \le m_2, \quad 1 \le k \le m_3;\\
&(C_{i,j}, kh) = (s_i, jh, kh), \quad 1 \le i \le m_1, \quad 0 \le j \le m_2, \quad 0 \le k \le m_3;\\
&(C_{i,j}, u_k) = (s_i, jh, u_k), \quad 1 \le i \le m_1, \quad 0 \le j \le m_2, \quad 1 \le k \le m_3;\\
&(D_{i,j}, kh) = (ih, t_j, kh), \quad 0 \le i \le m_1, \quad 1 \le j \le m_2, \quad 0 \le k \le m_3;\\
&(D_{i,j}, u_k) = (ih, t_j, u_k), \quad 0 \le i \le m_1, \quad 1 \le j \le m_2, \quad 1 \le k \le m_3.
\end{aligned} \tag{12}$$

Theorem 5. *At the points defined by (12), the operators R_2 is exact on the 28-dimensional subspace $(\mathbb{P}_{11}[x,y] \otimes \mathbb{P}_3[z]) \oplus (\mathbb{P}_3[x,y] \otimes \mathbb{P}_1[z])$ spanned by $\{1, x, y, z, x^2, y^2, z^2, xy, xz, yz, x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz, x^3z, xz^3, y^3z, yz^3, x^2yz, xy^2z, xyz^2, xyz^3\}$ of the 40-dimensional space $\mathbb{P}_3[x,y] \otimes \mathbb{P}_3[z]$.*

Moreover we deduce the following theorem.

Theorem 6. *Let $f \in C^r(\Omega)$, $r = 0, 1, 2$. Then there exist constants $K_{v,r} > 0$, $v = 1, 2$, such that*

$$\|f - R_v f\|_\infty \le K_{v,r} h^r \omega(D^r f, h).$$

If in addition $f \in C^3(\Omega)$ then there exist constants $K_{v,3} > 0$, $v = 1, 2$, such that

$$\|f - R_v f\|_\infty \le K_{v,3} h^3 \max_{|\beta|=3} \left\|D^\beta f\right\|_\infty .$$

7. Numerical Results

In this section we present some numerical results obtained by a computational procedure developed in a Matlab environment. These procedures are constructed by extending those proposed in The MathWorks (2002), Dagnino and Lamberti (2000a,b). We approximate the following functions:

1. the smooth trivariate test function of Franke type

$$f_1(x,y,z) = \frac{1}{2}e^{-10((x-\frac{1}{4})^2+(y-\frac{1}{4})^2)} + \frac{3}{4}e^{-16((x-\frac{1}{2})^2+(y-\frac{1}{4})^2+(z-\frac{1}{4})^2)} + \frac{1}{2}e^{-10((x-\frac{3}{4})^2+(y-\frac{1}{8})^2+(z-\frac{1}{2})^2)} - \frac{1}{4}e^{-20((x-\frac{3}{4})^2+(y-\frac{3}{4})^2)},$$

   on the cube $[-\frac{1}{2},\frac{1}{2}]^3$,
2. $f_2(x,y,z) = \frac{1}{9}\tanh(9(z-x-y)+1)$, on the cube $[-\frac{1}{2},\frac{1}{2}]^3$,
3. the Marschner-Lobb function (Marschner and Lobb, 1994)

$$f_3(x,y,z) = \frac{1}{2(1+\beta_1)}\left(1-\sin\frac{\pi z}{2}+\beta_1\left(1+\cos\left(2\pi\beta_2\cos\left(\frac{\pi\sqrt{x^2+y^2}}{2}\right)\right)\right)\right)$$

   $\beta_1 = \frac{1}{4}$ and $\beta_2 = 6$, on the cube $[-1,1]^3$,
4. $f_4(x,y,z) = \dfrac{\pi y e^{xy}}{40(e-2)}\sin\pi z$, on the cube $[0,1]^3$.

For each test function, using a $130\times130\times130$ uniform three-dimensional grid G of points in the domain, we compute the maximum absolute error $Ef = \max_{(u,v,w)\in G}|f(u,v,w)-Qf(u,v,w)|$, for $Q = R_1, R_2$, for increasing values of m_1, m_2 and m_3, see Table 1. In the table we also report an estimate of the approximation order, rf, obtained by the logarithm to base 2 of the ratio between two consecutive errors.

In Nürnberger et al. (2005) the authors propose a quasi-interpolation method for quadratic piecewise polynomials in three variables in BB-form and give some numerical results using the test functions f_1, f_2, f_3. We denote their quasi-interpolating spline by sq_f and, in the fourth column of Table 1, we report the corresponding maximum absolute error and the approximation order estimate.

Furthermore, in Sorokina and Zeilfelder (2007) the authors propose a local quasi-interpolation method based on cubic C^1 splines on type-6 tetrahedral partition in three variables in BB-form and give some numerical results with the test functions f_1 and f_3. We denote their quasi-interpolating spline by sc_f and, in the fifth column of Table 1, we report the corresponding maximum absolute error and the approximation order estimate.

We can notice that the overall smallest error is obtained with the operator R_2, although the bound for its infinite norm (and maybe also the infinite

$m_1 = m_2 = m_3$	$R_1 f$ Ef	rf	$R_2 f$ Ef	rf	sq_f Ef	rf	sc_f Ef	rf
f_1								
16	6.5(-3)		3.3(-3)		4.3(-2)		4.3(-2)	
32	8.1(-4)	3.0	2.3(-4)	3.8	1.1(-2)	2.0	1.1(-2)	2.0
64	9.5(-5)	3.1	1.8(-5)	3.7	2.8(-3)	2.0	2.8(-3)	2.0
128	8.4(-6)	3.5	1.9(-6)	3.2	6.9(-4)	2.0	6.9(-4)	2.0
f_2								
16	6.2(-3)		2.8(-3)		8.8(-3)			
32	1.1(-3)	2.5	3.0(-4)	3.2	2.4(-3)	1.9		
64	1.7(-4)	2.7	2.7(-5)	3.5	6.3(-4)	1.9		
128	1.7(-5)	3.3	1.9(-6)	3.8	1.6(-4)	2.0		
f_3								
16	1.9(-1)		2.1(-1)					
32	1.5(-1)	0.4	1.3(-1)	0.7	1.8(-1)		1.8(-1)	
64	3.2(-2)	2.2	2.0(-2)	2.7	1.2(-1)	0.5	1.2(-1)	0.5
128	4.6(-3)	2.8	1.5(-3)	3.7	4.0(-2)	1.6	4.0(-2)	1.6
f_4								
16	1.6(-4)		2.2(-5)					
32	2.0(-5)	3.0	2.3(-6)	3.3				
64	2.6(-6)	3.0	2.1(-7)	3.4				
128	3.2(-7)	3.0	3.1(-8)	2.8				

Table 1: Maximum absolute errors and numerical convergence orders.

norm itself) is greater than that of R_1. Furthermore, we observe that for these quasi-interpolants, the error decreases faster than for the quadratic and cubic C^1 piecewise polynomials proposed in Nürnberger et al. (2005), Sorokina and Zeilfelder (2007), respectively, but the computation time is larger.

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