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Holographic non-perturbative corrections to gauge couplings

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Abstract: We give a direct microscopic derivation of the F-theory background that corresponds to four D7 branes of type I′ by taking into account the D-instanton contributions to the emission of the axio-dilaton field in the directions transverse to the D7's. The couplings of the axio-dilaton to the D-instanton moduli modify its classical source terms which are shown to be proportional to the elements of the D7 brane chiral ring. Solving the bulk field equations with the non-perturbatively corrected sources yields the full F-theory background.

This solution represents the gravitational dual of the four-dimensional theory living on a probe D3 brane of type I′, namely of the $\mathcal{N} = 2$ Sp(1) SYM theory with $N_f = 4$. Our results provide an explicit microscopic derivation of the non-perturbative gravitational dual of this theory. They also explain the recent observation that the exact coupling for this theory can be entirely reconstructed from its perturbative part plus the knowledge of the chiral ring on the D7 branes supporting its flavor degrees of freedom.

Keywords: D-branes, Gauge-gravity correspondence, F-Theory, Supersymmetric gauge theory

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Contents

1 Introduction

The holographic principle finds its incarnation in string theory typically in the form of a gauge/gravity duality. D branes have been crucial in achieving this progress since, on the one hand, they introduce open string sectors that contain the gauge degrees of freedom, while on the other hand they source closed string fields and produce non-trivial space-times.

Matching the realization of the $\mathcal{N} = 4$ super Yang-Mills theory by means of open strings attached to D3 branes in flat space with its closed string description led Maldacena [1] to conjecture the equivalence of this gauge theory to type IIB string theory on $\mathrm{AdS}_5 \times S_5$; within this framework it was soon understood how to holographically relate gauge theory correlators to bulk amplitudes [2, 3]. This correspondence has been extended to many other conformal situations and has given rise to an entire field of research.

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It is obviously of the greatest interest to study the gauge/gravity duality also in less supersymmetric and non-conformal cases where the gauge theory couplings depend on the energy scale. In a stringy description these couplings correspond to dynamical fields from the closed string sector which assume a non-trivial profile in some extra direction dual to the energy scale, along the ideas put forward by Polyakov [4]. The quantum effective couplings should thus be determined by the bulk equations of motion for the corresponding closed string fields, sourced by the brane system on which the gauge theory lives.

These ideas have been exploited, for instance, by constructing non-trivial gravitational backgrounds that in some regime can be related to a system of branes at a conifold singularity [5] or to wrapped branes inside a Calabi-Yau manifold [6] that support an $\mathcal{N} = 1$ gauge theory in four dimensions. Trusting such solutions in the strong coupling regime accounts for many expected features of the exact vacua of these $\mathcal{N} = 1$ theories. Of course, it would be desirable to test the gauge/gravity duality in some non-conformal case where the exact solution of the gauge theory is known and should be reproduced on the gravitational side, for instance in $\mathcal{N} = 2$ gauge theories whose exact low-energy description is available after the work of Seiberg and Witten [7, 8].

Non-conformal $\mathcal{N} = 2$ SU(N) SYM theories can be engineered through fractional D3 branes at a Kleinian singularity (see, e.g. refs. [9]–[13]). In this set-up the gauge coupling is typically represented by a scalar field from the closed string twisted sector for which the D3 branes at the singularity act as δ-function sources localized in the remaining two transverse directions. Solving the corresponding field equations yields a logarithmic profile that exactly matches the perturbative running of the coupling constant upon identifying the two transverse directions with the (complexified) scale. These perturbative checks have been successfully extended to various $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories realized in orbifold set-ups [14].

However, the exact gauge coupling generically contains also a series of non-perturbative corrections. Are these reproduced on the gravitational side? And how? Clearly, the logarithmic solution described above has to be modified, as a consequence also of the back-reaction of the branes on the geometry of the system. In the $\mathcal{N} = 2$ case it has been speculated that in the large-N limit the correct modification is through an "enhançon" mechanism [15] in which the D3 branes actually expand in the transverse space, forming a ring of radius proportional to the dynamically generated scale where non-perturbative effects become relevant. The resulting profile of the coupling field agrees with the SW solution in this limit [16], but this mechanism lacks an explicit microscopic derivation.

For finite N, the non-perturbative corrections to the gauge coupling contained in the SW solution have been retrieved via multi-instanton calculus and localization techniques by Nekrasov [17, 18]. In the stringy description, they are provided by the inclusion of D-instanton sectors in addition to the the D3/D3 open strings [19, 20]. It is therefore natural to speculate that the corresponding non-perturbative modifications of the dual gravitational solution be provided again by D-instantons which indeed modify closed string interactions. This was in fact the focus of the early interest in these objects. In particular, in refs. [21]–[23] the key ideas and techniques to elucidate the D-instanton contributions to the gravitational effective action were developed. In the meantime, it was recognized that D-instantons represent the stringy counterpart of gauge instantons [24, 25] and the investigation of their effects in the open string sector was pursued in many directions. A key step in extracting multi-instanton corrections is the problematic integration over the moduli space. In supersymmetric cases, though, localization techniques [17, 18, 26] have allowed to overcome this difficulty both in the field-theory [27, 28] and in the string-theory description [29]–[31]. Of course, the integration over moduli space is crucial also to determine D-instanton effects in the closed string sector; and thus being able to rely on localization techniques will be extremely useful.

We expect that D-instantons affect the equation of motion for the closed string field that represents the gauge coupling by modifying the source with non-perturbative terms, so that the new profile of this field coincides with the exact coupling in the gauge theory. We set out to check this expectation by considering a particular case where the $\mathcal{N} = 2$ gauge theory is SU(2) with $N_f = 4$. This theory has vanishing β-function but, when masses for the fundamental hypermultiplets are turned on, the effective coupling at low energies receives non-perturbative corrections; its exact expression is contained in the SW curve for this model [8] and has been recently worked out in ref. [32]. Accounting for this exact coupling from a dual gravitational perspective would represent a valuable step forward.

This conformal theory can be realized on the world-volume of a D3 brane in a local version of type I$'$ superstring theory. The D3 brane (together with its orientifold image) supports an $\mathcal{N} = 2$ Sp(1) $\sim$ SU(2) gauge theory, with fundamental hypermultiplets provided by the open strings stretching between the D3 brane and the four D7 branes (plus the orientifold O7 plane) sitting near one orientifold fixed point. The gauge coupling corresponds to the axio-dilaton field; finding its exact expression amounts thus to determine the consistent F-theory background for this set-up.

This task was tackled long ago by Sen [33] who noticed that the naïve axio-dilaton profile produced by the D7 branes and the orientifold, has logarithmic singularities at the sources' locations. However, such singularities are incompatible with the physical interpretation of the dilaton as the string coupling, so that the exact profile must be modified. Based on the symmetries of the problem, Sen proposed that the F-theory axio-dilaton coincides with the exact gauge coupling encoded in the SW solution for the $\mathcal{N} = 2$ SU(2) SYM theory with $N_f = 4$; this connection was later explained [34] in terms of a D3 brane probing this background. Recently [32], it has been shown that the exact SW solution is retrieved by including D-instanton corrections in the D7/O7/D3 brane system. In this chain of arguments, the gauge/gravity relation is assumed and exploited to express the gravitational background in terms of the known solution of the dual gauge theory. Our purpose is to show that it is possible to compute directly the non-perturbative completion of the gravitational background, and in particular of the axio-dilaton profile, without reference to the gauge theory; getting the same expression of the exact gauge coupling amounts then to a non-trivial check of the gauge/gravity duality at the non-perturbative level.

In this paper, therefore, the D3 brane supporting the $\mathcal{N} = 2$ SYM theory will play no rôle. We will include D-instanton corrections to the local type I$'$ system of D7 branes plus orientifold, and show how the exact F-theory background emerges in this way. We think this is important at the conceptual level and believe that the techniques we develop here to handle the non-perturbative corrections to the profile of fields from the closed string sector could be useful in many situations.

In section 2 we set the stage by considering the effect of displacing the D7 branes from the orientifold by giving a classical expectation value to the adjoint scalar field living on their world-volume and explore the axio-dilaton couplings to the D7 world-volume; this determines how the displaced D7 brane can source this field. In section 3 we add D(-1) branes to the system and consider the couplings of the instanton moduli to the axio-dilaton field. These couplings arise from disks having their boundary attached to the D-instantons with a closed string axio-dilaton vertex in the interior and multiple moduli vertex insertions on the boundary; all diagrams relevant for our purposes turn out (rather surprisingly) to be computable.

We then explain how to extract the non-perturbative corrections to the axio-dilaton sources by suitably integrating these couplings over the instanton moduli space. We demonstrate in section 4 that the relevant integrals coincide with the ones that compute the non-perturbative corrections to the elements of the chiral ring on the D7 branes, namely to the vacuum expectation values of the traces of the powers of the adjoint D7/D7 scalar; these integrals can be computed by means of localization techniques [30]. Thus we explicitly show that the non-perturbative modifications of the source terms replace powers of the classical expectation values of the D7 brane adjoint scalar with the corresponding non-perturbative quantum vacuum expectation values, a fact that we already observed (without proving it) in ref. [32]. Finally, to show that our techniques can be applied in different contexts, and in particular do not require an eight-dimensional sector, we consider in section 5 a slight modification of our set-up in which a $\mathbb{Z}_2$ orbifold projection in enforced in four of the D7 brane world-volume directions. The computation of the axio-dilaton couplings to the D(-1) moduli goes through with suitable modifications (and is in fact slightly simpler), so that also in this case it is possible to obtain the non-perturbative source terms and hence the exact axio-dilaton profile.

The three appendices contain our notations and conventions, technical details and explicit computations that would not fit conveniently in the main text.

2 Axio-dilaton couplings of the D7 action

2.1 The set-up

We consider the so-called type I$'$ superstring theory, that is the projection of the type IIB theory by

$$\Omega = \omega\, \mathbf{I}_2\, (-1)^{F_L}\,, \tag{2.1}$$

where ω is the world-sheet orientation reversal, $\mathbf{I}_2$ is the inversion of two coordinates, say $x^{8,9} \to -x^{8,9}$, and F_L is the target-space left-moving fermion number. If the last two directions are compactified on a torus $\mathcal{T}_2$, the type I$'$ model is T-dual on this torus to the standard type I theory and possesses four O7 planes located at the four fixed points of $\mathcal{T}_2$. Tadpole cancellation requires the presence of sixteen D7 branes (plus their images); if the D7 branes are distributed in groups of four and placed on top of the O7 planes, the cancellation is local and there is no backreaction on the geometry; in particular, the axio-dilaton field

$$\tau = C_0 + \mathrm{i}\,\mathrm{e}^{-\phi} \tag{2.2}$$

is constant also along the directions of $\mathcal{T}_2$, which we will parametrize by the complex coordinate $z = x^8 - \mathrm{i}x^9$.

In the following, we will consider a "local" limit around one of the $\mathcal{T}_2$ fixed points, say $z = 0$, taking thus the transverse space to be simply $\mathbb{C}$, and we will consider moving the four D7 branes out of this fixed point. In this situation, local tadpole cancellation is lost and the axio-dilaton profile becomes non-trivial. We split the field τ into its expectation value $\tau_0 = \mathrm{i}/g_s$, representing the inverse string coupling, and a fluctuation part $\widetilde{\tau}$, namely

$$\tau = \tau_0 + \widetilde{\tau} \tag{2.3}$$

with

$$\widetilde{\tau} = \widetilde{C}_0 + \frac{\mathrm{i}}{g_s}\big(\mathrm{e}^{-\widetilde{\phi}} - 1\big) \sim \widetilde{C}_0 - \frac{\mathrm{i}}{g_s}\,\widetilde{\phi}\,, \tag{2.4}$$

where in the last step we retained only the linear term in the fluctuations. The bulk kinetic term for this field reads[1]

$$S_{\text{bulk}} = -\frac{1}{2\widetilde{\kappa}^2}\int d^{10}x\,\partial_M \widetilde{\overline{\tau}}\,\partial^M \widetilde{\tau}\,, \tag{2.5}$$

where $\widetilde{\overline{\tau}}$ is the complex conjugate of $\widetilde{\tau}$ and $\widetilde{\kappa} = 8\pi^{\frac{7}{2}}\alpha'^2$. Varying in $\widetilde{\overline{\tau}}$, the bulk contribution to the field equation is thus proportional to $\Box\,\widetilde{\tau}$.

As we will review in the next sub-section, the O7 plane and the D7 branes act as sources for the axio-dilaton localized in the two common transverse directions, thus leading to a logarithmic dependence in z. However, this behavior is not acceptable, since the imaginary part of τ, representing the inverse string coupling constant, blows up at the sources' locations. This behavior is in fact modified non-perturbatively, and the correct background corresponds to the particular limit of F-theory considered long ago by Sen in ref. [33]. Here we will show explicitly how the non-perturbative corrections to the axio-dilaton profile are induced by D-instantons.

To do this, we will need to carefully consider the couplings of τ to the D7 branes and to the D(-1) moduli. A crucial rôle in this analysis is played by the fact that the axio-dilaton is the lowest component of a chiral superfield T which contains the massless closed string degrees of freedom of the type I$'$ theory[2]

$$T = \tau_0 + \widetilde{T} = \tau_0 + \widetilde{\tau} + \sqrt{2}\,\theta\widetilde{\lambda} + \ldots + 2\theta^8\left(\frac{\partial^4}{\partial z^4}\widetilde{\overline{\tau}} + \ldots\right). \tag{2.6}$$

[1] It arises from the term

$$-\frac{1}{2\kappa^2}\int d^{10}x\,\frac{\partial_M\overline{\tau}\,\partial^M\tau}{(\mathrm{Im}\,\tau)^2}\,,$$

in the Einstein frame supergravity action, where $\kappa = g_s\widetilde{\kappa}$.

[2] In the context of type IIB superstring in $d = 10$, the analogous organization of the supergravity degrees of freedom in an analytic superfield with lowest component the axio-dilaton [35]–[38] represented a key point in the investigation of the D-instanton effects on the effective action carried out in [19, 21]. We will make use of many of the ideas and techniques of these papers but, since we are interested in the couplings of the axio-dilaton itself, for us the relevant component of the superfield will be the highest one, rather than some of the lower components containing different physical fields.

Here $\widetilde{\lambda}$ is the dilatino, while the remaining supergravity degrees of freedom appear in the omitted terms in the θ-expansion. Our notation (see appendix A) is that the sixteen supersymmetries of type I′ can be arranged in a Majorana-Weyl 10d spinor $\Theta^{\mathcal{A}}$, which in turn, under the $10 \to 8+2$ split of target space intrinsic to the theory, decomposes into a chiral and an anti-chiral 8d spinor: $\Theta^{\mathcal{A}} \to (\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. For our purposes the most important component of the T superfield is the highest one, proportional to θ^8. Its expression involves the fourth holomorphic derivative of the complex conjugate of $\widetilde{\tau}$, as we will check explicitly in appendix B.1 by the computation of a disk diagram with eight θ insertions.

The type I′ theory contains also an open string sector, made by strings whose endpoints are stuck on the D7 branes. The orientifold projection implies that the massless content of this sector is that of an 8d gauge theory. In our local model, the gauge group of this theory is SO(8) and the massless open string degrees of freedom can be arranged in a chiral superfield

$$M = m + \sqrt{2}\,\theta\Lambda + \frac{1}{2}\,\theta\gamma^{\mu\nu}\theta\, F_{\mu\nu} + \dots\,. \tag{2.7}$$

Here m is a complex scalar, Λ is the gaugino and $F_{\mu\nu}$ is the gauge field strength, all in the adjoint representation of SO(8).

2.2 Tree level couplings

Let us now consider the tree-level linear coupling of the axio-dilaton field to the D7 branes.

D7 branes at the origin. As well known [39], Dp branes introduce tadpoles for closed string fields, realized by disks whose boundary lies on the D branes and with a closed string vertex in the interior. These tadpoles can be evaluated taking the inner product of the boundary states representing the branes with the relevant closed string states [39, 40]. Let us momentarily generalize our set-up to the case of N_f D7 branes (instead of four) and an O7 plane placed at the origin of the transverse space spanned by z. The boundary state of the D7 branes, as well as the crosscap state representing the O7 plane, couples to the fluctuations of the dilaton and of the Ramond-Ramond eight-form C_8, which is related to C_0 by bulk Poincaré duality

$$dC_8 = *dC_0\,. \tag{2.8}$$

The profile of these fields can be obtained from the emission amplitude in momentum space, stripped of the polarization, by attaching a free propagator and taking the Fourier transform [40].

Equivalently, the linear couplings of the D7 branes to the dilaton and axion fluctuations are encoded in their world-volume theory, whose well-known structure is worth recalling briefly here. In the Einstein frame the tree-level Born-Infeld action for N_f D7 branes (and their orientifold images) takes the schematic form

$$S^{\rm BI}_{\rm tree} = -\frac{T_7}{\kappa}\int_{\rm D7} d^8x\left\{2N_f\,{\rm e}^{\widetilde{\phi}} - \frac{(2\pi\alpha')^2}{4}\,{\rm tr}\,F^2 + \frac{(2\pi\alpha')^4}{12}\,{\rm e}^{-\widetilde{\phi}}\,{\rm tr}\,(t_8F^4) + O(F^5)\right\}. \tag{2.9}$$

Here tensor t_8 is the anti-symmetric eight-index tensor appearing in various superstring amplitudes (see e.g. ref. [41]), and $T_7 = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{-4}$ is the D7 brane tension.

The linear part in $\widetilde{\phi}$ in the first term of eq. (2.9) corresponds to the coupling of the dilaton fluctuation to the D7 boundary state, and represents a source localized on the D7 branes given by

$$-2N_f \frac{T_7}{\kappa} \int d^{10}x \, \widetilde{\phi} \, \delta^2(z) \,. \tag{2.10}$$

Taking into account the bulk kinetic term (2.5), and using the fact that $2T_7\kappa = g_s$, the field equation for $\widetilde{\phi}$ reads

$$\Box \, \widetilde{\phi} = 2g_s N_f \, \delta^2(z) \,, \tag{2.11}$$

and its solution is $\widetilde{\phi} = (1/\pi) N_f g_s \log|z|$. If we include the O7 source term, that possesses a negative charge and shifts N_f to $(N_f - 4)$ and properly add the axion $\widetilde{C}_0$ induced by the sources for its Poincaré dual 8-form, we obtain the classical profile for $\widetilde{\tau}$ which has the following logarithmic behavior

$$\widetilde{\tau}_{\mathrm{cl}}(z) = \frac{1}{2\pi \mathrm{i}} (2N_f - 8) \log \frac{z}{z_0} \tag{2.12}$$

where z_0 is a suitable length scale. For $N_f = 4$, τ is constant since the D7 and O7 charges cancel locally.

The remaining terms in the action (2.9) describe the interactions of the dilaton with the gauge field. Note that there is no coupling to the quadratic Yang-Mills Lagrangian, while the dilaton fluctuation couples to the quartic terms in F. If we take into account also the Wess-Zumino action, these quartic terms read

$$S^{(4)}_{\mathrm{tree}} = -\frac{1}{96\pi^3 \, g_s} \int_{\mathrm{D7}} d^8x \, \mathrm{e}^{-\widetilde{\phi}} \, \mathrm{tr}\,(t_8 F^4) - \frac{\mathrm{i}}{192\pi^3} \int_{\mathrm{D7}} \widetilde{C}_0 \, \mathrm{tr}\,(F \wedge F \wedge F \wedge F) \,. \tag{2.13}$$

Using the superfield M introduced in eq. (2.7), they can be rewritten as a superpotential term

$$S^{(4)}_{\mathrm{tree}} = \frac{1}{(2\pi)^4} \int d^8x \, d^8\theta \, F^{(4)}_{\mathrm{tree}} + \mathrm{c.c.} \,, \tag{2.14}$$

with

$$F^{(4)}_{\mathrm{tree}} = \frac{\mathrm{i}\pi}{12} \, \tau \, \mathrm{tr}\, M^4 \,. \tag{2.15}$$

If the SO(8) gauge field gets a non-zero classical expectation value, eq. (2.13) may give rise to source terms for $\widetilde{\phi}$ and $\widetilde{C}_0$. For instance, a constant background field F yields a source term for $\widetilde{\phi}$ proportional to $\mathrm{tr}\,(t_8 F^4)$. On the other hand, in topologically non-trivial sectors with fourth Chern number k, the vacuum is represented by instanton-like configurations[3] that minimize the action (2.13) (at fixed axio-dilaton) yielding simply $S^{(4)}_{\mathrm{tree}} = -2\pi \mathrm{i} \, k \, \tau$. This is the same action that describes the coupling to the axio-dilaton of k D-instantons, which indeed represent these configurations in the string picture [29, 42].

The message we obtain from this analysis is two-fold: on the one side, this suggests that at the non-perturbative level we must take into account the interactions of the axio-dilaton with the D-instantons. On the other side, we see that extra source terms for the

[3] As discussed in ref. [42], for $k = 1$ such a configuration corresponds to the point-like limit of the so-called SO(8) instanton [43, 44].

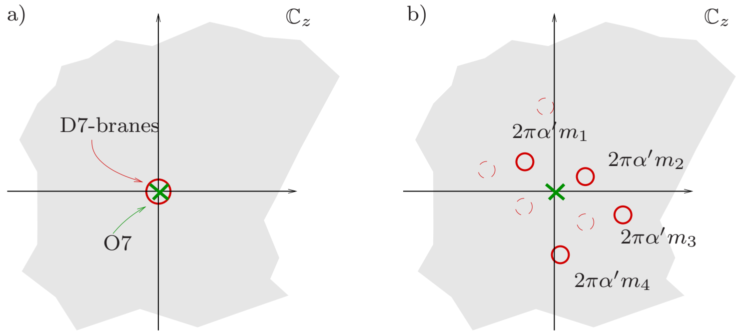

Figure 1. a) The D7 branes and the O7 plane are placed at the origin. b) The D7 branes (and their images) are displaced from the origin; this corresponds to distribution of charges which leads to the profile given in eqs. (2.16)–(2.19) for the axio-dilaton.

axio-dilaton can arise from its interactions with the open string degrees of freedom, if the latter acquire non-trivial expectation values. Such interaction terms are not limited to the one usually considered in the Born-Infeld and Wess-Zumino actions but include other structures, as we will now discuss.

Displaced D7 branes and scalar field couplings. If we modify our set-up by placing the D7 branes at the positions $\pm z_i$, the classical axio-dilaton profile becomes

$$\widetilde{\tau}_{\rm cl}(z) = \frac{1}{2\pi \mathrm{i}}\left\{\sum_{i=1}^{N_f}\left[\log\frac{z-z_i}{z_0}+\log\frac{z+z_i}{z_0}\right]-8\log\frac{z}{z_0}\right\}. \tag{2.16}$$

This is non-trivial even in the case $N_f = 4$.

In the following we concentrate on this case. The dependence from the scale z_0 disappears and we can expand the profile (2.16) for large z, getting

$$\widetilde{\tau}_{\rm cl}(z) = -\frac{1}{2\pi \mathrm{i}}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{\sum_{i=1}^{4} z_i^{2\ell}}{z^{2\ell}}\,. \tag{2.17}$$

Displacing the D7 branes, as depicted in figure 1, corresponds to giving a classical expectation value

$$m_{\rm cl} = \mathrm{diag}\{m_1, m_2, m_3, m_4, -m_1, -m_2, -m_3, -m_4\} \qquad \text{with } m_i = \frac{z_i}{2\pi\alpha'}\,, \tag{2.18}$$

to the SO(8) adjoint scalar field m of the D7 brane world-volume theory, which has canonical dimension of $(\text{length})^{-1}$. In terms of $m_{\rm cl}$, the profile (2.17) reads

$$\widetilde{\tau}_{\rm cl}(z) = -\frac{1}{2\pi \mathrm{i}}\sum_{\ell=1}^{\infty}\frac{(2\pi\alpha')^{2\ell}}{2\ell}\frac{\mathrm{tr}\, m_{\rm cl}^{2\ell}}{z^{2\ell}}\,, \tag{2.19}$$

which solves the following differential equation

$$\Box\,\widetilde{\tau} = -2\mathrm{i}\sum_{\ell=1}^{\infty}\frac{(2\pi\alpha')^{2\ell}\,\mathrm{tr}\, m_{\rm cl}^{2\ell}}{(2\ell)!}\frac{\partial^{2\ell}\delta^2(z)}{\partial z^{2\ell}}\,. \tag{2.20}$$

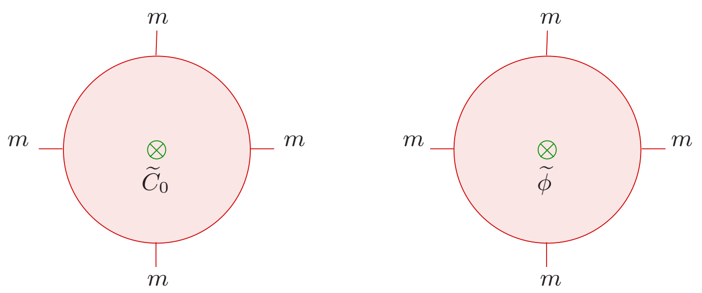

Figure 2. The diagrams describing the coupling of the Ramond-Ramond 0-form and dilaton fluctuations to four scalar fields m of the D7 brane theory. These amplitudes in momentum space turn out to be proportional, as shown in appendix B, to $\operatorname{tr} m^4\,(\bar{p})^4$, which leads, upon insertion of the free propagator and Fourier transform, to a profile $\sim \operatorname{tr} m^4/z^4$.

This is the field equation obtained by varying with respect to $\widetilde{\bar{\tau}}$ an action which, in addition to the bulk term (2.5), contains also a source term localized on the world-volume of the D7 branes

$$S_{\text{source}} = -\frac{T_7}{\widetilde{\kappa}} \int_{\text{D7}} d^8x\, J_{\text{cl}}\, \widetilde{\bar{\tau}} + \text{c.c.} \tag{2.21}$$

Indeed, requiring that

$$\frac{\delta}{\delta\widetilde{\bar{\tau}}}(S_{\text{bulk}} + S_{\text{source}}) = 0\,, \tag{2.22}$$

we obtain

$$\Box\, \widetilde{\tau} = J_{\text{cl}}\, \delta^2(z)\,, \tag{2.23}$$

which coincides with eq. (2.20) if

$$J_{\text{cl}} = -2\mathrm{i} \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell} \operatorname{tr} m_{\text{cl}}^{2\ell}}{(2\ell)!} \frac{\partial^{2\ell}}{\partial z^{2\ell}}\,. \tag{2.24}$$

In momentum space, this current becomes

$$J_{\text{cl}} = -2\mathrm{i} \sum_{\ell=1}^{\infty} (-1)^{\ell} \frac{(2\pi\alpha')^{2\ell} \operatorname{tr} m_{\text{cl}}^{2\ell}}{(2\ell)!}\, \bar{p}^{2\ell}\,, \tag{2.25}$$

where $\bar{p} = (p_8 + \mathrm{i}p_9)/2$ is the momentum conjugate to z.

From a diagrammatic point of view, the source terms (2.25) arise from emission diagrams for τ from disks with multiple insertions of m (which turn out to be proportional to powers of the transverse momentum $\bar{p}$); an example is depicted in figure 2. This means that the D7 action must contain interaction terms linear in $\widetilde{\bar{\tau}}$ of the form (see eq. (2.21))

$$\begin{aligned} &2\mathrm{i}\,\frac{T_7}{\widetilde{\kappa}} \int_{\text{D7}} d^8x \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell} \operatorname{tr} m^{2\ell}}{(2\ell)!} \frac{\partial^{2\ell}\widetilde{\bar{\tau}}}{\partial z^{2\ell}} + \text{c.c.} \\ &= \frac{4\pi\mathrm{i}}{(2\pi)^4} \int_{\text{D7}} d^8x \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell-4} \operatorname{tr} m^{2\ell}}{(2\ell)!} \frac{\partial^{2\ell}\widetilde{\bar{\tau}}}{\partial z^{2\ell}} + \text{c.c.}\,, \end{aligned} \tag{2.26}$$

which, by freezing m to its expectation value m_{cl}, reproduce precisely the source terms of eq. (2.25). We now show that these new terms can be easily incorporated by generalizing the previous analysis.

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Superpotential contributions. Starting from the terms quartic in the adjoint scalar m, these interactions, together with other terms related by supersymmetry, can be expressed as contributions to an $8d$ "superpotential", using the chiral bulk superfield T of eq. (2.6) and of the open string superfield M of eq. (2.7). Indeed, the $\ell = 2$ term in eq. (2.26) is contained in the action

$$S^{(4)}_{\text{tree}} = \frac{1}{(2\pi)^4} \int d^8x\, d^8\theta\, F^{(4)}_{\text{tree}} + \text{c.c.}\,, \tag{2.27}$$

where the superpotential is given by

$$F^{(4)}_{\text{tree}} = \frac{2\pi\mathrm{i}}{4!}\,\mathrm{tr}\, M^4\, T\,, \tag{2.28}$$

and depends on x and θ through the superfields M and T. This is just the expression (2.15) that accounts for the quartic terms in the gauge field in which, however, the axio-dilaton τ has been promoted to the corresponding superfield T. Notice that in eq. (2.27) we can saturate the integration over $d^8\theta$ in different ways. If we pick up all eight θ's from the M^4 factor, we retrieve the quartic gauge action (2.13). At the opposite end, we can take all the eight θ's from the T superfield and, recalling the expansion (2.6), we obtain

$$\frac{4\pi\mathrm{i}}{(2\pi)^4} \int_{\mathrm{D7}} d^8x\, \frac{\mathrm{tr}\, m^4}{4!} \frac{\partial^4 \widetilde{\bar{\tau}}}{\partial z^4} + \text{c.c.}\,, \tag{2.29}$$

which coincides with the $\ell = 2$ term in eq. (2.26). The terms with higher values of ℓ can similarly be written as superpotential terms,[4] so that altogether we have

$$S_{\text{tree}}(M,T) = \frac{1}{(2\pi)^4} \int d^8x\, d^8\theta\, F_{\text{tree}}(M,T) + \text{c.c.}\,, \tag{2.30}$$

with

$$F_{\text{tree}}(M,T) = 2\pi\mathrm{i} \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell-4}\,\mathrm{tr}\, M^{2\ell}}{(2\ell)!} \frac{\partial^{2\ell-4}\, T}{\partial z^{2\ell-4}}\,. \tag{2.31}$$

The action (2.30) reduces to the source action S_{source} if we single out the terms linear in $\widetilde{\bar{\tau}}$, appearing in T accompanied by θ^8, and then set the fields to their classical values. In other words, the current J_{cl} of eq. (2.21) is related to the prepotential by

$$J_{\text{cl}} = -\frac{(2\pi\alpha')^4}{2\pi} \frac{\delta F_{\text{tree}}(M,T)}{\delta\big(\theta^8\, \widetilde{\bar{\tau}}\big)}\bigg|_{T=\tau_0\,,\, M=m_{\text{cl}}} \equiv -\frac{(2\pi\alpha')^4}{2\pi}\, \bar{\delta} F_{\text{tree}}(M,T)\,, \tag{2.32}$$

where in the second step to simplify our notations we have defined the operation $\bar{\delta}$ acting on any object $\star$ as follows:

$$\bar{\delta}\star \equiv \frac{\delta\star}{\delta\big(\theta^8\, \widetilde{\bar{\tau}}\big)}\bigg|_{T=\tau_0\,,\, M=m_{\text{cl}}}. \tag{2.33}$$

[4] Also the term quadratic in the scalar field m, corresponding to $\ell = 1$, can actually be written as a superpotential contribution, at the price of allowing for a non-locality in the transverse directions:

$$F^{(2)}_{\text{tree}} = \frac{2\pi\mathrm{i}}{(2\pi\alpha')^2} \frac{\mathrm{tr}\, M^2}{2} \left(\frac{\partial}{\partial z}\right)^{-2} T\,.$$

It can be shown that this formal writing corresponds to the supersymmetric completion of the Wess-Zumino term that describes the coupling of the D7 brane gauge fields to the Ramond-Ramond form C_4.

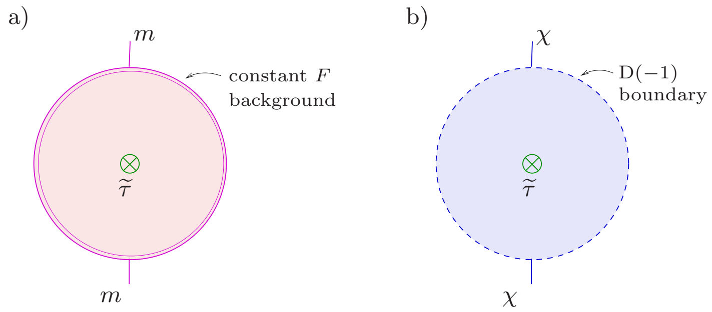

Figure 3. In a constant-F background, indicated in the left figure by a double line for the disk boundary, there is a non-vanishing amplitude with two m scalar fields and the axio-dilaton, proportional to $\bar{p}^2$ (this is the $\ell = 3$ case of eq. (2.34)). In the background represented by a D-instanton, this diagram has a counterpart, represented by the diagram on the right. In this latter, the blue dashed line indicates that the boundary of the disk is attached to D-instantons, and there is the insertions of two χ moduli, whose vertex have the same expression of the m vertices. See section 3 for details.

Notice that the current $J_{\rm cl}$ defined by eq. (2.32) is dimensionless since $\bar{\delta}F_{\rm tree}$ has dimensions of (length)$^{-8}$; indeed, if $\star$ has scaling dimensions of (length)$^{\nu}$, then $\bar{\delta}\star$ has scaling dimensions of (length)$^{\nu-4}$. Using this definition, it is easy to check that by applying the operation $\bar{\delta}$ to the prepotential (2.31), the current (2.24) is correctly reproduced.

If in eqs. (2.30) and (2.31) we take all θ's from the superfield T, we obtain the source terms (2.26). These, however, are linked by supersymmetry to terms, among others, with the schematic structure

$$F^4\, m^{2\ell-4}\frac{\partial^{2\ell-4}\,\tau}{\partial z^{2\ell-4}} \tag{2.34}$$

arising when we select all θ's from the $M^{2\ell}$ part. The presence of such interaction terms indicates that, should we turn on a background gauge field F on the D7 branes, we would find a non-vanishing disk amplitude for the emission of $(2\ell-4)$ open vertices for m and one closed τ vertex, proportional to $\bar{p}^{2\ell-4}$, as represented in figure 3a). Instead of a constant background, we could consider an instanton solution of the quartic theory, which, as discussed in [42], is represented by a D-instanton. For a disk with its boundary attached to a D-instanton we expect therefore to find a non-vanishing amplitude, proportional to $\bar{p}^{2\ell-4}$, with the insertion of $(2\ell-4)$ moduli whose vertex formally has the same structure of the vertex of m and of one closed string vertex for τ, as represented in figure 3b). As we will see in the next section, such amplitudes, as well as the analogous ones with higher number of insertions, are indeed present and play a key rôle in the computation of the non-perturbative corrections to the axio-dilaton profile.

The quantum-corrected source terms. Let us summarize the situation. We are interested in the source terms for the axio-dilaton field τ induced by the presence of the D7 branes. At tree-level, the effective action on the D7 branes for the open and closed string massless sector is given (up to non-linear terms in the closed string fluctuations) by $S_{\rm tree}(M,T)$ of eq. (2.30).

Integrating out the open string sector one obtains the contribution of the D7 branes to the effective action for the closed string degrees of freedom:

$$\mathrm{e}^{-\Gamma(T)} = \int \mathcal{D}M\, \mathrm{e}^{-S_{\text{tree}}(M,T)}\,, \tag{2.35}$$

where $\mathcal{D}M$ denotes the functional integration measure over all the massless open degrees of freedom appearing in the superfield M. With $\Gamma(T)$ we indicate, with a slight abuse of notation, the effective action for the closed string degrees of freedom, even if we do not want to claim that they only appear arranged in the superfield T. The effective action $\Gamma(T)$ in general depends on the classical values of the open string fields, like for example m_{cl}, and contains source terms linear in the closed string fluctuations, and in particular in $\widetilde{\overline{\tau}}$. From eq. (2.35) it follows that such effective source terms can be expressed as the quantum expectation value of the interaction terms $S_{\text{tree}}(M, \widetilde{\overline{\tau}})$ linear in $\widetilde{\overline{\tau}}$, given in eq. (2.26), with respect to the usual D7 effective action for the open string modes M at fixed τ_0, namely

$$\Gamma_{\text{source}} = \int \mathcal{D}M\, S_{\text{tree}}(M, \widetilde{\overline{\tau}})\, \mathrm{e}^{-S_{\text{tree}}(M,\tau_0)} + \text{c.c.} = \langle S_{\text{tree}}(M, \widetilde{\overline{\tau}})\rangle + \text{c.c.} \tag{2.36}$$

Writing

$$\Gamma_{\text{source}} = -\frac{T_7}{\tilde{\kappa}} \int d^8x\, J\, \widetilde{\overline{\tau}} + \text{c.c.} \tag{2.37}$$

in analogy to what we did in eq. (2.21) at the classical level, and using the explicit expression (2.26), we deduce that the quantum current J (in momentum space) is

$$J = -2\mathrm{i} \sum_{\ell=1}^{\infty} (-1)^{\ell}\, \frac{(2\pi\alpha')^{2\ell}\, \langle \mathrm{tr}\, m^{2\ell}\rangle}{(2\ell)!}\, \bar{p}^{2\ell}. \tag{2.38}$$

The quantum current J is therefore obtained from the classical one given in eq. (2.25) by promoting the classical values $m_{\text{cl}}^{2\ell}$ of the field m to the corresponding quantum expectation values $\langle \mathrm{tr}\, m^{2\ell}\rangle$, which are the elements of the so-called chiral ring of the D7 brane theory.

Another way to express the quantum current is by writing the effective action Γ in terms of a superpotential:

$$\Gamma = \frac{1}{(2\pi)^4} \int d^8x\, d^8\theta\, F + \text{c.c.} \tag{2.39}$$

where F can have classical, perturbative and non-perturbative parts:

$$F = F_{\text{tree}} + F_{\text{pert}} + F_{\text{n.p.}}\,. \tag{2.40}$$

The classical term F_{tree} is just the one given in eq. (2.31) with M frozen to m_{cl}. Due to the high degree of supersymmetry, the perturbative contributions to the superpotential vanish. Non-perturbative terms correspond to the contributions of topologically non-trivial sectors in the functional integral (2.35) and are represented in the string picture by the inclusion of D-instantons in the background.

The source terms Γ_{source} are obtained by selecting in Γ the terms linear in $\widetilde{\overline{\tau}}$, and then setting the closed string fields to their classical value, i.e. $T = \tau_0$. If we start from the superpotential, we must also single out the terms proportional to θ^8. Therefore to get

$\Gamma_{\rm source}$ only the combination $\theta^8\widetilde{\bar{\tau}}$ is relevant. This is precisely the combination appearing inside the superfield T, and thus we can ignore all other possible dependencies on $\widetilde{\bar{\tau}}$ that may arise in the effective action. Therefore we can write

$$J = -\frac{(2\pi\alpha')^4}{2\pi}\,\bar{\delta}F\,, \tag{2.41}$$

where $\bar{\delta}$ is the variation operator defined in eq. (2.33).

In the next sections we will compute the non-perturbative prepotential $F_{\rm n.p.}$ induced by the presence of D-instantons, including the couplings to the closed string fields, and will use eq. (2.41) to derive the corresponding source terms for the axio-dilaton, from which the non-perturbative corrections to its profile follow. In doing so, we expect to find that the effective source is obtained by taking into account the instanton corrections to the chiral ring elements, thus verifying the heuristic discussion that led us to eq. (2.38).

3 Axio-dilaton couplings in the D-instanton action

The non-perturbative sectors of the SO(8) field theory living on the D7 branes can be described by adding k D-instantons in the same orientifold fixed point occupied by the D7's [29, 42]. These D-instantons are sources for the Ramond-Ramond scalar $\widetilde{C}_0$. Thus, from the Wess-Zumino part of the D7 action (2.13), we see that k D(-1) branes correspond to a gauge field configuration with fourth Chern number k. Moreover, for this configuration the classical quartic action reduces to k times the D-instanton action.

The physical excitations of the open strings with at least one end-point on the D(-1) branes account for the moduli of such instanton configurations which will be collectively denoted $\mathcal{M}_{(k)}$. They comprise the neutral sector, corresponding to D(-1)/D(-1) open strings, which contains those moduli that do not transform under the SO(8) gauge group, and the charged sector, arising from D(-1)/D7 open strings, which includes those moduli that transform in the fundamental representation of SO(8).

The neutral moduli are a vector a_μ and a complex scalar χ (plus its conjugate $\bar{\chi}$) in the Neveu-Schwarz sector, and a chiral fermion η_α and an anti-chiral fermion $\lambda_{\dot{\alpha}}$ in the Ramond sector. The bosonic moduli have canonical dimensions of (length)$^{-1}$, while the fermionic ones have canonical dimensions of (length)$^{-\frac{3}{2}}$. All these neutral moduli are $k\times k$ matrices, but the consistency with the orientifold projection on the D7 branes requires that χ, $\bar{\chi}$ and $\lambda_{\dot{\alpha}}$ transform in the anti-symmetric (or adjoint) representation of SO(k), while a_μ and η_α are in the symmetric one. The diagonal parts of a_μ and η_α represent the bosonic and fermionic Goldstone modes of the (super)translations of the D7 branes world-volume that are broken by the D-instantons and thus can be identified, respectively, with the bosonic and fermionic coordinates x_μ and θ_α of the $8d$ superspace. More explicitly, we have

$$x_\mu \sim \ell_s^2\,{\rm tr}\,(a_\mu)\,, \qquad \theta_\alpha \sim \ell_s^2\,{\rm tr}\,(\eta_\alpha)\,, \tag{3.1}$$

where the factors of the string length $\ell_s=\sqrt{\alpha'}$ have been introduced to give x_μ and θ_α their standard dimensions.

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	SO(k)	SO(8)	dimensions	vertex
a_μ	symm	$\mathbf{1}$	$(\text{length})^{-1}$	$V_a = \ell_s\, a_\mu\, \psi^\mu\, \mathrm{e}^{-\varphi}$
η_α	symm	$\mathbf{1}$	$(\text{length})^{-3/2}$	$V_\eta = \ell_s^{3/2}\, \eta_\alpha\, S^{\alpha -}\, \mathrm{e}^{-\frac{1}{2}\varphi}$
$\lambda_{\dot\alpha}$	adj	$\mathbf{1}$	$(\text{length})^{-3/2}$	$V_\lambda = \ell_s^{3/2}\, \lambda_{\dot\alpha}\, S^{\dot\alpha +}\, \mathrm{e}^{-\frac{1}{2}\varphi}$
$\bar\chi$	adj	$\mathbf{1}$	$(\text{length})^{-1}$	$V_{\bar\chi} = \ell_s\, \bar\chi\, \Psi\, \mathrm{e}^{-\varphi}$
χ	adj	$\mathbf{1}$	$(\text{length})^{-1}$	$V_\chi = \ell_s\, \chi\, \overline{\Psi}\, \mathrm{e}^{-\varphi}$
μ	$\mathbf{k}$	$\mathbf{8}_v$	$(\text{length})^{-3/2}$	$V_\mu = \ell_s^{3/2}\, \mu\, \Delta\, S^{+}\, \mathrm{e}^{-\frac{1}{2}\varphi}$

Table 1. Transformation properties, scaling dimensions and vertices of the moduli in the D(-1)/D7 system. In the last column, φ is the bosonic field appearing in the bosonization of the superghost, the S's are spin fields with chiral (anti-chiral) indices $\alpha(\dot\alpha)$ and $+(-)$ in the first eight and last two directions respectively, while Δ is the bosonic twist field in the first eight directions. The factors of the string length $\ell_s = \sqrt{\alpha'}$ are inserted in order to make the vertex operators dimensionless.

Since the D(-1)/D7 open strings have eight directions with mixed Dirichlet-Neumann boundary conditions, there are no bosonic charged excitations that satisfy the physical state condition in the Neveu-Schwarz sector. On the other hand, the fermionic Ramond sector of the D(-1)/D7 system contains physical moduli, denoted as μ and $\bar\mu$ depending on the orientation. They are, respectively, $k \times N$ and $N \times k$ matrices (with $N = 8$ in our specific case). Since the orientifold parity (2.1) exchanges the two orientations, μ and $\bar\mu$ are related according to $\bar\mu = -\,{}^{\mathrm{t}}\mu$. For all the physical moduli listed above, it is possible to write vertex operators of conformal dimension 1 that are collected in table 1.

The D-instanton action can be derived by computing correlation functions on disks with at least part of their boundary lying on the D(-1) branes; for details in the type I$'$ model we refer to refs. [29, 42] and here we only recall the results that are most significant for our present purposes.

The first contribution to the moduli action is the classical part

$$S_{\rm cl} = \frac{2\pi k}{g_s} = -2\pi\mathrm{i}\, k\, \tau_0 \tag{3.2}$$

corresponding to the (topological) normalization of disk amplitudes with D(-1) boundary conditions. This term accounts for disk amplitudes on k D-instantons with no moduli insertions [20, 45].

The second type of contributions originates from disks amplitudes involving the vertex operators of the D(-1)/D(-1) or D(-1)/D7 strings. They lead to the following moduli action [29, 42]

$$\begin{aligned} S_{\rm mod}(\mathcal{M}_{(k)}) = \frac{1}{g_0^2}\,\mathrm{tr}\,\Big\{ &\mathrm{i}\,\lambda_{\dot\alpha}\gamma_\mu^{\dot\alpha\beta}[a^\mu,\eta_\beta] - \frac{\mathrm{i}}{\sqrt2}\,\lambda_{\dot\alpha}[\chi,\lambda^{\dot\alpha}] - \frac{\mathrm{i}}{\sqrt2}\,\eta^\alpha[\bar\chi,\eta_\alpha] - \mathrm{i}\,\sqrt2\,{}^{\mathrm{t}}\mu\,\chi\,\mu \\ &- \frac14[a_\mu,a_\nu][a^\mu,a^\nu] - [a_\mu,\chi][a^\mu,\bar\chi] - \frac12[\chi,\bar\chi]^2\Big\} \end{aligned} \tag{3.3}$$

where $g_0^2 = g_s\,(4\pi^3\alpha'^2)^{-1}$ is the Yang-Mills coupling constant of the zero-dimensional gauge theory defined on the D-instantons. Notice that this action does not depend on the diagonal

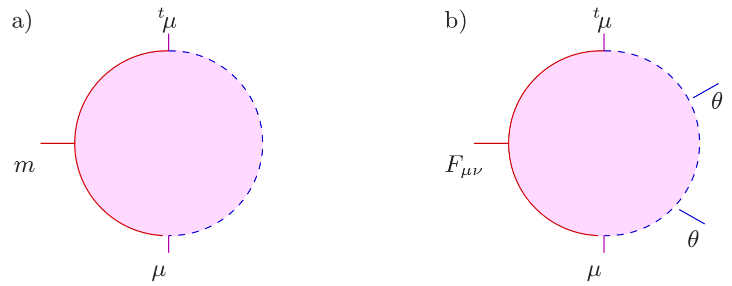

Figure 4. a) The disk diagram describing the interaction of the D-instantons with the adjoint scalar field m living on the D7 branes. b) The susy broken by the instanton relates the previous diagram to a diagram involving the field-strength $F_{\mu\nu}$ with two extra insertions of the vertices for the modulus θ, effectively promoting $m(x)$ to the superfield $M(x,\theta)$.

components x_μ and θ_α, defined in eq. (3.1), which are true zero-modes of the instanton system. The quartic interactions $[a_\mu, a_\nu][a^\mu, a^\nu]$ can be disentangled by introducing seven auxiliary fields D_m ($m = 1, \cdots, 7$) with canonical dimension of (length)$^{-2}$, and replacing $S_{\rm mod}(\mathcal{M}_{(k)})$ with

$$\begin{aligned} S'_{\rm mod}(\mathcal{M}_{(k)}) = \frac{1}{g_0^2}\,{\rm tr}\Big\{ &{\rm i}\,\lambda_{\dot\alpha}\gamma_\mu^{\dot\alpha\beta}[a^\mu,\eta_\beta] - \frac{\rm i}{\sqrt2}\,\lambda_{\dot\alpha}[\chi,\lambda^{\dot\alpha}] - \frac{\rm i}{\sqrt2}\,\eta^\alpha[\bar\chi,\eta_\alpha] - {\rm i}\,\sqrt2\,{}^{\rm t}\mu\,\chi\,\mu \\ &+\frac12 D_m D^m - \frac12 D_m(\tau^m)_{\mu\nu}\,[a^\mu,a^\nu] - [a_\mu,\chi][a^\mu,\bar\chi] - \frac12[\chi,\bar\chi]^2\Big\}\,. \end{aligned} \tag{3.4}$$

Here $(\tau^m)_{\mu\nu}$ are the seven γ-matrices of SO(7), implying that the eight-dimensional vector indices $\mu, \nu, \cdots$ of SO(8) can be interpreted also as spinorial indices of SO(7). By eliminating the auxiliary fields D_m through their equations of motion and exploiting the properties of the τ^m matrices, one can easily check that $S'_{\rm mod}(\mathcal{M}_{(k)})$ and $S_{\rm mod}(\mathcal{M}_{(k)})$ are equivalent. From now on, we will work with the action (3.4) and the enlarged set of moduli that includes also the auxiliary fields D_m.

A further type of contributions to the instanton action comes from the interactions among the instanton moduli and the gauge fields propagating on the world-volume of the D7 branes, which we have combined into the super-field (2.7). Such interactions can be obtained by computing mixed disk amplitudes involving the vertex operators for the charged moduli and the vertex operators for the dynamical fields. For example, in figure 4a) we have represented the mixed D(-1)/D7 diagram describing the coupling between the complex scalar m with two charged moduli μ, and yielding a term in the action proportional to ${\rm tr}\left\{{}^{\rm t}\mu\,\mu\,m\right\}$. By exploiting the identification (3.1) between the fermionic superspace coordinates θ_α and the trace part of η_α, it is immediate to realize that the couplings with the higher components of the superfield $M(x,\theta)$ can be obtained from disk diagrams with extra fermionic insertions, like for instance the one involving the gauge field strength $F_{\mu\nu}$ and two θ's represented in figure 4b). Equivalently, these interactions can be obtained by acting with the broken supersymmetry transformations on the lower terms; all in all this amounts to replace the moduli action (3.4) with

$$S'_{\rm mod}(\mathcal{M}_{(k)}, M) = S'_{\rm mod}(\mathcal{M}_{(k)}) + \frac{{\rm i}\sqrt2}{g_0^2}{\rm tr}\left\{{}^{\rm t}\mu\,\mu\,M(x,\theta)\right\}. \tag{3.5}$$

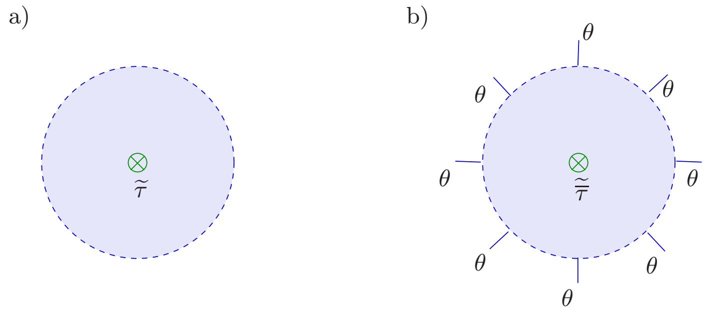

Figure 5. a) The disk diagram describing the interaction of the D-instantons with the axio-dilaton vertex. b) The diagram, linked by supersymmetry to the previous one, which comprises the insertion of eight θ-vertices and a vertex for $\bar{\tau}$.

The last type of contributions to the instanton action comes from the interactions among the moduli and the closed string excitations describing the gravitational degrees of freedom propagating in the bulk. The simplest of such contributions corresponds to the coupling of the axio-dilaton fluctuations to the D-instantons which can be obtained, for example, by considering the tree-level Born-Infeld and Wess-Zumino actions of k D-instantons, or equivalently by computing the overlap between the D(-1) boundary state and the vertex operator of the axio-dilaton field represented by the disk diagram of figure 5a). This amounts to simply replace the classical instanton action (3.2) with

$$S_{\rm cl} = -2\pi {\rm i}\, k\, \tau \tag{3.6}$$

i.e. to promote the expectation value τ_0 to the entire axio-dilaton field τ. Actually, by exploiting the broken supersymmetries we can do more and further promote τ to the full-fledged superfield T introduced in eq. (2.6). Thus, if we include all couplings among the D-instantons and the closed string degrees of freedom, the classical action (3.6) becomes

$$S'_{\rm cl} = -2\pi {\rm i}\, k\, T = -2\pi {\rm i}\, k\, \tau_0 - 2\pi {\rm i}\, k\, \widetilde{T}\,. \tag{3.7}$$

Notice that, among others, this action contains an interaction term with the following structure

$$\theta^8\, \bar{p}^4\, \widetilde{\bar{\tau}} \tag{3.8}$$

which accounts for the coupling among eight superspace coordinates and the (complex conjugate) axio-dilaton. This term is represented by the diagram in figure 5b) and can be regarded as the highest supersymmetric partner of the simple emission diagram in figure 5a). In particular the coupling (3.8) implies the existence of a non-trivial amplitude involving eight vertices for the θ's and one vertex for the axion field $\widetilde{C}_0$. In appendix B we compute this amplitude using conformal field theory techniques, thus finding an explicit confirmation of the above structure.

As argued in section 2.2, in the presence of a non-constant axio-dilaton profile we expect further interaction terms in the moduli action involving the derivatives of τ. In particular, in the present configuration we expect to find couplings with the following schematic structure

$$\mathrm{tr}\,(\chi^{2\ell})\, \frac{\partial^{2\ell}\widetilde{\tau}}{\partial z^{2\ell}} \qquad \text{or} \qquad \mathrm{tr}\,(\chi^{2\ell})\, \bar{p}^{2\ell}\, \widetilde{\tau}\,, \tag{3.9}$$

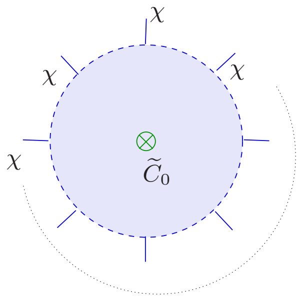

Figure 6. The disk diagram describing the interaction between any even number of χ moduli and the axion vertex.

which are the D-instanton analogues of the interactions (2.34) that are present on the D7 branes (see also figure 3). To check this idea we can compute the mixed open/closed disk amplitude with 2ℓ vertices for χ and one vertex for the axion $\widetilde{C}_0$. Since we are interested in effects depending on the z-derivatives of τ, it is easier to work with the vertex for the emission of the Ramond-Ramond field strength $F_z = \partial_z \widetilde{C}_0 \equiv \mathrm{i}\, \bar{p}\, \widetilde{C}_0$, given by

$$V_{\widetilde{C}_0}(w,\bar{w}) = \frac{2\pi g_s \ell_s}{8}\, \mathrm{i}\, \bar{p}\, \widetilde{C}_0\, S^{\dot{\alpha}+}(w)\, \mathrm{e}^{\mathrm{i}\,\ell_s \bar{p} Z(w)}\, \mathrm{e}^{-\frac{1}{2}\varphi(w)}\, \widetilde{S}_{\dot{\alpha}}{}^{+}(\bar{w})\, \mathrm{e}^{\mathrm{i}\,\ell_s \bar{p} \widetilde{Z}(\bar{w})}\, \mathrm{e}^{-\frac{1}{2}\widetilde{\varphi}(\bar{w})}\,. \tag{3.10}$$

Here we have included the normalization prefactor $(2\pi g_s \ell_s)/8$ in order to be consistent with the bulk action (2.5) and used the charge conjugation matrix $C_{\dot{\alpha}\dot{\beta}}$ to contract the antichiral spinor indices of the left and right spin fields S and $\widetilde{S}$ (see appendix A for some details). Moreover, we have assumed that the axion depends only on z and thus have used only the momentum $\bar{p}$ in its emission vertex. This is clearly a simplification since in general we could allow also for a dependence on $\bar{z}$ (and hence also on p); however, this is enough for our purposes since, as discussed in detail in section 2, we are after the terms with holomorphic derivatives of the axio-dilaton field.[5]

A quick inspection shows that the only non-trivial couplings are among $\widetilde{C}_0$ and an even number of χ moduli due to the Chan-Paton structure of the latter. Usually, the calculation of amplitudes involving a large number of vertex operators is a daunting task, but in this particular case it can be carried out explicitly for an arbitrary number of χ insertions: indeed, in order to soak up the superghost number anomaly, all but one vertices V_χ must be chosen in the 0-superghost picture where they are extremely simple. The details of these calculations are provided in appendix B.2, while here we simply quote the final result which, for the amplitude corresponding to the diagram in figure 6, is

$$\Big\langle\!\Big\langle \underbrace{V_\chi \cdots V_\chi}_{2\ell}\, V_{\widetilde{C}_0} \Big\rangle\!\Big\rangle = 2\pi\mathrm{i}\, \frac{(2\pi\alpha')^{2\ell}}{(2\ell)!}\, \mathrm{tr}\,(\chi^{2\ell})\, \bar{p}^{2\ell}\, \widetilde{C}_0\,. \tag{3.11}$$

[5]Notice that we do not consider either any dependence of the closed string background on the momenta p_μ along the D7 brane world-volume in order not to generate interactions in the instanton action that would lead to terms breaking the eight-dimensional Lorentz invariance.

After including the contribution from the dilaton fluctuations, we can conclude that the instanton action contains also the following terms[6]

$$- 2\pi \mathrm{i} \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell}}{(2\ell)!} \,\mathrm{tr}\,(\chi^{2\ell})\, \bar{p}^{2\ell}\, \widetilde{\tau} \tag{3.12}$$

in agreement with our expectations. By exploiting the broken supersymmetries or, equivalently, by inserting vertices for the superspace coordinates θ_α, we can promote the axion to the complete superfield (2.6), obtaining

$$- 2\pi \mathrm{i} \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell}}{(2\ell)!} \,\mathrm{tr}\,(\chi^{2\ell})\, \bar{p}^{2\ell}\, \widetilde{T}\,. \tag{3.13}$$

This implies that the instanton moduli action contains, among others, the following terms

$$- 4\pi \mathrm{i} \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell}}{(2\ell)!} \,\mathrm{tr}\,(\chi^{2\ell})\, \bar{p}^{2\ell+4}\, \theta^8\, \widetilde{\widetilde{\tau}}\,, \tag{3.14}$$

which, as we will discuss in the next section, are responsible for the non-perturbative corrections in the τ profile.

Collecting all contributions we have found, we conclude that the action for k D-instantons of type I$'$ in the presence of a non-trivial holomorphic axio-dilaton background is given by

$$S_{\text{inst}} = -2\pi \mathrm{i}\, k\, \tau_0 + \widetilde{S}_{\text{inst}} \tag{3.15}$$

with

$$\widetilde{S}_{\text{inst}} = S'_{\text{mod}}(\mathcal{M}_{(k)}, M) - 2\pi \mathrm{i} \sum_{\ell=0}^{\infty} \frac{(2\pi\alpha')^{2\ell}}{(2\ell)!} \,\mathrm{tr}\,(\chi^{2\ell})\, \bar{p}^{2\ell}\, \widetilde{T}\,. \tag{3.16}$$

Notice that for $\ell = 0$ we have $\mathrm{tr}\,(\chi^{2\ell}) = k$; thus the $\ell = 0$ term in the above expression is just $-2\pi \mathrm{i}\, k\, \widetilde{T}$, which is precisely the fluctuation part of the classical instanton action as indicated in eq. (3.7).

In order to perform the integral over the instanton moduli space and obtain explicit results at instanton number k, we have to exploit the localization formulas and adopt Nekrasov's approach to the multi-instanton calculus [17, 18] (see also refs. [26]–[28]). In our stringy context this corresponds to use a deformed instanton action that is obtained by coupling the instanton moduli to a non-trivial "graviphoton" background in space-time [46, 47]. Since the case we are considering has already been discussed in great detail in ref. [29], we do not repeat the analysis of the deformation effects here and simply recall that the graviphoton field-strength $\mathcal{F}_{\mu\nu}$ can be aligned along the SO(8) Cartan generators and taken to be of the following form

$$\mathcal{F}_{\mu\nu} = -2\mathrm{i} \begin{pmatrix} \varepsilon_1\sigma_2 & 0 & 0 & 0 \\ 0 & \varepsilon_2\sigma_2 & 0 & 0 \\ 0 & 0 & \varepsilon_3\sigma_2 & 0 \\ 0 & 0 & 0 & \varepsilon_4\sigma_2 \end{pmatrix} \qquad \text{where} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{3.17}$$

[6] Recall that, with Euclidean signature, scattering amplitudes and couplings in the action differ by a minus sign.

with $\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4=0$. This closed string background modifies the instanton action (3.16) inducing new couplings that can be explicitly computed from string amplitudes. In particular the term $S'_{\rm mod}$ acquires a dependence on the deformation parameters ε_i's; however, the explicit expression of this modified moduli action is not necessary in the following, and thus we refer the interested reader to ref. [29], and specifically to sections 4 and 5 of that paper, where all details can be found. In the next section we will use this approach to show how D-instantons produce the non-perturbative corrections to the axio-dilaton profile.

4 The non-perturbative axio-dilaton profile

The effects produced by D-instantons are encoded in the non-perturbative low-energy effective action of the open and closed string degrees of freedom supported by the D7 branes. To obtain this action we first introduce the k-instanton canonical partition function $\widetilde{Z}_k$, which is given by the following integral over the moduli space

$$\widetilde{Z}_k=\int d\mathcal{M}_{(k)}\,\mathrm{e}^{-\widetilde{S}_{\rm inst}}\,. \tag{4.1}$$

Here $\widetilde{S}_{\rm inst}$ is the instanton action (3.16) suitably deformed by the graviphoton background (3.17), i.e. with $S'_{\rm mod}(\mathcal{M}_{(k)},M)$ replaced by $S'_{\rm mod}(\mathcal{M}_{(k)},M;\varepsilon_i)$. When the fluctuations of the open and closed string fields are switched off, $\widetilde{Z}_k$ reduces to the ordinary D-instanton partition function Z_k; indeed

$$\widetilde{Z}_k\big|_{T=\tau_0\,,\,M=m_{\rm cl}}=\int d\mathcal{M}_{(k)}\,\mathrm{e}^{-S'_{\rm mod}(\mathcal{M}_{(k)},m_{\rm cl};\varepsilon_i)}\equiv Z_k\,. \tag{4.2}$$

The above integral over moduli space can be explicitly computed via localization methods using Nekrasov's prescription as done in refs. [29, 30] for our brane system.

Summing over all instanton numbers, we obtain the gran-canonical instanton partition function

$$\widetilde{\mathcal{Z}}=\sum_{k=0}^{\infty}q^k\,\widetilde{Z}_k \tag{4.3}$$

where we have set $\widetilde{Z}_0=1$ and $q=\mathrm{e}^{2\pi\mathrm{i}\tau_0}$. Then, following Nekrasov [17, 18], we define the non-perturbative prepotential induced by the D-instantons on the D7 branes as follows

$$\widetilde{F}_{\rm n.p.}=\lim_{\mathcal{E}\to 0}\,\mathcal{E}\log\widetilde{\mathcal{Z}} \tag{4.4}$$

where $\mathcal{E}=\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$. Expanding in powers of q leads to

$$\widetilde{F}_{\rm n.p.}=\sum_{k=1}^{\infty}q^k\,\widetilde{F}_k=\sum_{k=1}^{\infty}q^k\int d\widehat{\mathcal{M}}_{(k)}\,\mathrm{e}^{-\widetilde{S}_{\rm inst}} \tag{4.5}$$

where in the last term we have exhibited the fact that $\widetilde{F}_k$ is really an integral over the centered moduli $\widehat{\mathcal{M}}_{(k)}$, i.e. all moduli but the "center of mass" coordinates x_μ and their superpartners θ_α defined in (3.1). The explicit expressions of the first four $\widetilde{F}_k$'s in terms of the partition functions $\widetilde{Z}_k$ are

$$
\begin{aligned}
\widetilde{F}_1 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\, \widetilde{Z}_1\,,\\
\widetilde{F}_2 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\Big(\widetilde{Z}_2 - \frac{1}{2}\,\widetilde{Z}_1^2\Big)\,,\\
\widetilde{F}_3 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\Big(\widetilde{Z}_3 - \widetilde{Z}_2\widetilde{Z}_1 + \frac{1}{3}\,\widetilde{Z}_1^3\Big)\,,\\
\widetilde{F}_4 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\Big(\widetilde{Z}_4 - \widetilde{Z}_3\widetilde{Z}_1 - \frac{1}{2}\,\widetilde{Z}_2^2 + \widetilde{Z}_2\widetilde{Z}_1^2 - \frac{1}{4}\,\widetilde{Z}_1^4\Big)\,.
\end{aligned}
\tag{4.6}
$$

Given the prepotential (4.5) we obtain the corresponding non-perturbative source current for the axio-dilaton from eq. (2.41), namely

$$
J_{\text{n.p.}} = -\frac{(2\pi\alpha')^4}{2\pi}\,\bar\delta\widetilde{F}_{\text{n.p.}} = -\frac{(2\pi\alpha')^4}{2\pi}\sum_{k=1}^{\infty} q^k\,\bar\delta\widetilde{F}_k\,,
\tag{4.7}
$$

where $\bar\delta$ is the variation operator defined in eq. (2.33). Exploiting the Leibniz rule and eq. (4.2), we easily obtain

$$
\begin{aligned}
\bar\delta\widetilde{F}_1 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\,\bar\delta\widetilde{Z}_1\,,\\
\bar\delta\widetilde{F}_2 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\big(\bar\delta\widetilde{Z}_2 - Z_1\,\bar\delta\widetilde{Z}_1\big)\,,\\
\bar\delta\widetilde{F}_3 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\big(\bar\delta\widetilde{Z}_3 - Z_1\,\bar\delta\widetilde{Z}_2 - Z_2\,\bar\delta\widetilde{Z}_1 + Z_1^2\,\bar\delta\widetilde{Z}_1\big)\,,\\
\bar\delta\widetilde{F}_4 &= \lim_{\mathcal{E}\to 0} \mathcal{E}\big(\bar\delta\widetilde{Z}_4 - Z_1\,\bar\delta\widetilde{Z}_3 - Z_3\,\bar\delta\widetilde{Z}_1 - Z_2\,\bar\delta\widetilde{Z}_2 + Z_1^2\,\bar\delta\widetilde{Z}_2 + 2Z_2Z_1\,\bar\delta\widetilde{Z}_1 - Z_1^3\,\bar\delta\widetilde{Z}_1\big)\,.
\end{aligned}
\tag{4.8}
$$

Using the explicit form of the instanton action $\widetilde{S}_{\text{inst}}$ given in eq. (3.16), it is rather straightforward to show that

$$
\bar\delta\widetilde{Z}_k = 4\pi\mathrm{i}\sum_{\ell=0}^{\infty}(2\pi\alpha')^{2\ell}\,\bar{p}^{2\ell+4}\,Z_k^{(2\ell)}
\tag{4.9}
$$

where we have defined

$$
Z_k^{(2\ell)} = \frac{1}{(2\ell)!}\int d\mathcal{M}_{(k)}\,\mathrm{tr}\,(\chi^{2\ell})\,\mathrm{e}^{-\widetilde{S}_{\text{inst}}}\Big|_{T=\tau_0\,,\,M=m_{\text{cl}}}\,.
\tag{4.10}
$$

Notice that

$$
Z_k^{(0)} = k\,Z_k\,.
\tag{4.11}
$$

Furthermore, since for $k=1$ there are no χ moduli, we simply have

$$
Z_1^{(2\ell)} = 0 \quad \forall\,\ell\neq 0\,,
\tag{4.12}
$$

so that the following relation holds: $\bar\delta\widetilde{Z}_1 = 4\pi\mathrm{i}\,\bar{p}^4\,Z_1$.

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Equipped with this information, we can rewrite eq. (4.8) as follows

$$
\begin{aligned}
\bar{\delta}\widetilde{F}_1 &= 4\pi\mathrm{i}\,\bar{p}^4\Big\{\lim_{\mathcal{E}\to 0}\,\mathcal{E}Z_1\Big\}\,,\\
\bar{\delta}\widetilde{F}_2 &= 4\pi\mathrm{i}\sum_{\ell=0}^{\infty}(2\pi\alpha')^{2\ell}\,\bar{p}^{2\ell+4}\Big\{\lim_{\mathcal{E}\to 0}\,\mathcal{E}\big(Z_2^{(2\ell)}-Z_1Z_1^{(2\ell)}\big)\Big\}\,,\\
\bar{\delta}\widetilde{F}_3 &= 4\pi\mathrm{i}\sum_{\ell=0}^{\infty}(2\pi\alpha')^{2\ell}\,\bar{p}^{2\ell+4}\Big\{\lim_{\mathcal{E}\to 0}\,\mathcal{E}\big(Z_3^{(2\ell)}-Z_1Z_2^{(2\ell)}-Z_2Z_1^{(2\ell)}+Z_1^2Z_1^{(2\ell)}\big)\Big\}\,,\\
\bar{\delta}\widetilde{F}_4 &= 4\pi\mathrm{i}\sum_{\ell=0}^{\infty}(2\pi\alpha')^{2\ell}\,\bar{p}^{2\ell+4}\Big\{\lim_{\mathcal{E}\to 0}\,\mathcal{E}\big(Z_4^{(2\ell)}-Z_1Z_3^{(2\ell)}-Z_3Z_1^{(2\ell)}-Z_2Z_2^{(2\ell)}\\
&\qquad\qquad +Z_1^2Z_2^{(2\ell)}+2Z_2Z_1Z_1^{(2\ell)}-Z_1^3Z_1^{(2\ell)}\big)\Big\}\,.
\end{aligned}
\tag{4.13}
$$

We now show that the expressions in braces are directly related to the elements of the chiral ring of the SO(8) gauge theory living on the D7 brane world-volume.

4.1 The SO(8) chiral ring

The elements of the chiral ring of the SO(8) gauge theory are the vacuum expectation values of the traces of powers of the adjoint scalar field from the D7/D7 strings, i.e. $\langle \mathrm{tr}\, m^J\rangle$. The non-perturbative contributions to these correlators can be computed via localization methods by inserting suitable operators in the instanton partition function as shown in ref. [30]. Specifically we have

$$
\begin{aligned}
\langle \mathrm{tr}\, m^J\rangle_{\text{n.p.}} &= \sum_{k=1}^{\infty} q^k\langle \mathrm{tr}\, m^J\rangle_k\\
&= \lim_{\mathcal{E}\to 0}\Big\{\frac{1}{\widetilde{\mathcal{Z}}}\sum_{k=1}^{\infty}q^k\int d\mathcal{M}_{(k)}\,\mathrm{e}^{-\widetilde{S}_{\text{inst}}}\,\mathcal{O}_{(k,J)}\Big\}\Big|_{T=\tau_0\,,\,M=m_{\text{cl}}}
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
\mathcal{O}_{(k,J)} = \sum_{I=1}^{k}\Big[\chi_I^J &- \sum_i(\chi_I+\varepsilon_i)^J+\sum_{i<j}(\chi_I+\varepsilon_i+\varepsilon_j)^J\\
&-\sum_{i<j<\ell}(\chi_I+\varepsilon_i+\varepsilon_j+\varepsilon_\ell)^J+(\chi_I+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)^J\Big]\,.
\end{aligned}
\tag{4.15}
$$

Here the χ_I's are the elements of the χ matrix when it is written in a skew-diagonalized form, i.e. in the Cartan basis of SO(k). More explicitly we have

- for $k=2n$
$$\big\{\chi_I\big\}=\big\{\chi_1,\cdots,\chi_n,-\chi_1,\cdots,-\chi_n\big\}\,, \tag{4.16}$$

- for $k=2n+1$
$$\big\{\chi_I\big\}=\big\{\chi_1,\cdots,\chi_n,-\chi_1,\cdots,-\chi_n,\,0\big\}\,, \tag{4.17}$$

and

$$\mathrm{tr}\,(\chi^{2\ell}) = \sum_{I=1}^{k} (-1)^\ell\, \chi_I^{2\ell}\,. \tag{4.18}$$

Due to the symmetry of the theory, all odd elements of the chiral ring vanish, i.e.

$$\langle \mathrm{tr}\, m^J \rangle_{\text{n.p.}} = 0 \quad \text{for } J \text{ odd}\,. \tag{4.19}$$

Furthermore, it is immediate to check that $\mathcal{O}_{(k,2)} = 0$ for all k, so that

$$\langle \mathrm{tr}\, m^2 \rangle_{\text{n.p.}} = 0\,. \tag{4.20}$$

Therefore, the first non-trivial element of the SO(8) chiral ring is $\langle \mathrm{tr}\, m^4 \rangle_{\text{n.p.}}$, which now we are going to analyze in some detail.

Setting $J = 4$ in eq. (4.15), we easily find

$$\mathcal{O}_{(k,4)} = 24\, \mathcal{E}\, k\,; \tag{4.21}$$

from this it follows that

$$\begin{aligned}
\langle \mathrm{tr}\, m^4 \rangle_{\text{n.p.}} &= 24 \lim_{\mathcal{E}\to 0} \left\{ \frac{\mathcal{E}}{\widetilde{\mathcal{Z}}} \sum_{k=1}^{\infty} k\, q^k\, \widetilde{\mathcal{Z}}_k \right\}\Big|_{T=\tau_0\,,\, M=m_{\text{cl}}} \\
&= 24\, q\, \frac{\partial}{\partial q} \lim_{\mathcal{E}\to 0} \left\{ \mathcal{E} \log \widetilde{\mathcal{Z}} \right\}\Big|_{T=\tau_0\,,\, M=m_{\text{cl}}} \\
&= 24\, q\, \frac{\partial}{\partial q}\, \widetilde{F}_{\text{n.p.}}\Big|_{T=\tau_0\,,\, M=m_{\text{cl}}}
\end{aligned} \tag{4.22}$$

where in the last step we introduced the prepotential according to eq. (4.4). Expanding in powers of q and using eq. (4.6), at the first few instanton numbers we find

$$\begin{aligned}
\langle \mathrm{tr}\, m^4 \rangle_1 &= 24 \widetilde{F}_1\big|_{T=\tau_0\,,\, M=m_{\text{cl}}} = 24 \lim_{\mathcal{E}\to 0} \mathcal{E} Z_1\,, \\
\langle \mathrm{tr}\, m^4 \rangle_2 &= 48 \widetilde{F}_2\big|_{T=\tau_0\,,\, M=m_{\text{cl}}} = 24 \lim_{\mathcal{E}\to 0} \mathcal{E}\big(2Z_2 - Z_1^2\big)\,, \\
\langle \mathrm{tr}\, m^4 \rangle_3 &= 72 \widetilde{F}_3\big|_{T=\tau_0\,,\, M=m_{\text{cl}}} = 24 \lim_{\mathcal{E}\to 0} \mathcal{E}\big(3Z_3 - 3Z_2 Z_1 + Z_1^3\big)\,, \\
\langle \mathrm{tr}\, m^4 \rangle_4 &= 96 \widetilde{F}_4\big|_{T=\tau_0\,,\, M=m_{\text{cl}}} = 24 \lim_{\mathcal{E}\to 0} \mathcal{E}\big(4Z_4 - 4Z_3 Z_1 - 2Z_2^2 + 4Z_2 Z_1^2 - Z_1^4\big)\,.
\end{aligned} \tag{4.23}$$

Recalling that $Z_k^{(0)} = k\, Z_k$, in the right hand sides above we recognize the same expressions of the $\ell = 0$ terms of the quantities appearing inside braces in eq. (4.13).

These results can be generalized order by order in the instanton expansion to the higher elements of the SO(8) chiral ring. We refer to appendix C for the details while here we simply quote the results for the first few instanton numbers. For any integer $\ell > 0$ we find

$$\begin{aligned}
\langle \mathrm{tr}\, m^{2\ell+4} \rangle_1 &= 0\,, \\
\langle \mathrm{tr}\, m^{2\ell+4} \rangle_2 &= (-1)^\ell\, (2\ell+4)! \lim_{\mathcal{E}\to 0} \mathcal{E} Z_2^{(2\ell)}\,, \\
\langle \mathrm{tr}\, m^{2\ell+4} \rangle_3 &= (-1)^\ell\, (2\ell+4)! \lim_{\mathcal{E}\to 0} \mathcal{E}\big(Z_3^{(2\ell)} - Z_1 Z_2^{(2\ell)}\big)\,, \\
\langle \mathrm{tr}\, m^{2\ell+4} \rangle_4 &= (-1)^\ell\, (2\ell+4)! \lim_{\mathcal{E}\to 0} \mathcal{E}\big(Z_4^{(2\ell)} - Z_1 Z_3^{(2\ell)} - Z_2 Z_2^{(2\ell)} + Z_1^2 Z_2^{(2\ell)}\big)\,.
\end{aligned} \tag{4.24}$$

Since $Z_1^{(2\ell)} = 0$ for $\ell > 0$, the right hand sides have precisely the same structures appearing in eq. (4.13).

The non-perturbative correlators $\langle \mathrm{tr}\, m^J \rangle_k$ have been computed for the first few values of J and k in ref. [30]; here, for completeness but also as an illustration of how explicit these computations are, we report their results in our current conventions,[7] namely

$$\begin{aligned}
\langle \mathrm{tr}\, m^2 \rangle_{\text{n.p.}} &= 0\,, \\
\langle \mathrm{tr}\, m^4 \rangle_{\text{n.p.}} &= -192\,\mathrm{Pf}m\, q - 96 \sum_{i<j} m_i^2 m_j^2\, q^2 - 768\,\mathrm{Pf}m\, q^3 + \ldots\,, \\
\langle \mathrm{tr}\, m^6 \rangle_{\text{n.p.}} &= 1440 \sum_{i<j<k} m_i^2 m_j^2 m_k^2\, q^2 + 7680\,\mathrm{Pf}m \sum_i m_i^2\, q^3 + \ldots\,, \\
\langle \mathrm{tr}\, m^8 \rangle_{\text{n.p.}} &= -6720\,(\mathrm{Pf}m)^2\, q^2 - 35840\,\mathrm{Pf}m \sum_{i<j} m_i^2 m_j^2\, q^3 + \ldots\,,
\end{aligned} \tag{4.25}$$

where $\mathrm{Pf}m = m_1 m_2 m_3 m_4$.

4.2 The τ profile

Using the results (4.23) and (4.24), the coefficients $\bar{\delta}\widetilde{F}_k$ of prepotential can be rewritten in a compact way as follows

$$\bar{\delta}\widetilde{F}_k = 4\pi\mathrm{i} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(2\pi\alpha')^{2\ell}}{(2\ell+4)!}\, \bar{p}^{2\ell+4}\, \langle \mathrm{tr}\, m^{2\ell+4} \rangle_k\,. \tag{4.26}$$

Thus, after summing over all instanton numbers k and suitably relabeling the index ℓ, the non-perturbative source current (4.7) becomes

$$J_{\text{n.p.}} = -2\mathrm{i} \sum_{\ell=1}^{\infty} (-1)^\ell \frac{(2\pi\alpha')^{2\ell}\, \langle \mathrm{tr}\, m^{2\ell} \rangle_{\text{n.p.}}}{(2\ell)!}\, \bar{p}^{2\ell}\,. \tag{4.27}$$

Here we have used the fact that $\langle \mathrm{tr}\, m^2 \rangle_{\text{n.p.}} = 0$ to set the starting value of the summation index at $\ell = 1$. This is exactly of the form (2.38) expected from the general analysis of section 2. From the current (4.27) we deduce the following non-perturbative axio-dilaton profile

$$\widetilde{\tau}_{\text{n.p.}}(z) = -\frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{\infty} \frac{(2\pi\alpha')^{2\ell}}{2\ell} \frac{\langle \mathrm{tr}\, m^{2\ell} \rangle_{\text{n.p.}}}{z^{2\ell}}\,. \tag{4.28}$$

This analysis shows the existence of a direct connection between the elements of the SO(8) chiral ring and the non-perturbative source terms for the axion-dilaton field that was already observed in ref. [32].

Introducing the quantity

$$a = \frac{z}{2\pi\alpha'} \tag{4.29}$$

[7] Notice that the instanton expansion parameter q of ref. [30] is mapped to our $-q$. Notice also that our current mass eigenvalues m_i are rescaled with a factor of $1/\sqrt{2}$ with respect to those used in ref. [32], i.e. $m_i^{\text{here}} = m_i^{\text{there}}/\sqrt{2}$.

which in the holographic correspondence parametrizes the Coulomb branch of the dual $\mathcal{N}=2$ SU(2) $\sim$ Sp(1) gauge theory, and adding to $\widetilde{\tau}_{\rm n.p.}$ the classical contribution (2.19), we can exhibit the resulting axio-dilaton τ as an expansion in inverse powers of a, namely

$$\begin{aligned}2\pi\mathrm{i}\tau \,=\, 2\pi\mathrm{i}\tau_0 - \frac{\sum_i m_i^2}{a^2} + \frac{1}{a^4}\Big[-\frac{1}{2}\sum_i m_i^4 + 48\,\mathrm{Pf}m\,q + 24\sum_{i<j} m_i^2 m_j^2\,q^2 + 192\,\mathrm{Pf}m\,q^3 + \ldots\Big]\\ +\frac{1}{a^6}\Big[-\frac{1}{3}\sum_i m_i^6 - 240\sum_{i<j<k} m_i^2 m_j^2 m_k^2\,q^2 - 1280\,\mathrm{Pf}m\sum_i m_i^2\,q^3 + \ldots\Big]\\ +\frac{1}{a^8}\Big[-\frac{1}{4}\sum_i m_i^8 + 840\,(\mathrm{Pf}m)^2\,q^2 + 4480\,\mathrm{Pf}m\sum_{i<j} m_i^2 m_j^2\,q^3 + \ldots\Big] + \ldots\,.\end{aligned} \tag{4.30}$$

As shown in ref. [32], this is in perfect agreement with the effective coupling derived from the SW curve for the SU(2) $N_f=4$ theory [8].

5 An orbifold example

The techniques described in the previous sections are very flexible and can be applied in different contexts; in particular they do not require an eight-dimensional set-up. To show this we now briefly analyze an orbifold model that yields results directly in four-dimensions.

5.1 Moduli spectrum and effective action

We consider the type I′ theory described in section 2 and mod it out with a $\mathbb{Z}_2$ projection generated by

$$g \;:\quad x^m \to -x^m \quad \text{for } m = 4,\ldots,7\,. \tag{5.1}$$

The "Lorentz" group SO(8) considered so far is then broken to SO(4) $\times$ SO(4), and all representations split correspondingly. For example, the eight-dimensional vector index decomposes into two four-dimensional indices, which we denote respectively by $\mu=0,\ldots,3$ and $m=4,\ldots,7$ (as already indicated in eq. (5.1)), one for each SO(4) factor, and so on. In the spinor representation g can be identified by the chirality operator of the second SO(4), so that objects carrying a chiral (or anti-chiral) spinor index a (or $\dot{a}$) of this SO(4) are even (or odd) under the orbifold.

In this model tadpole cancellation requires the presence of two fractional D7 branes extending both along the four space-time directions x^μ and along the orbifolded ones x^m. If the Chan-Paton factors of the D7 branes do not transform under g (that is, if the D7 branes are in the trivial $\mathbb{Z}_2$ representation), the low-energy open string excitations on the D7 branes give rise to a theory with gauge group SO(4) as shown in ref. [30].

The non-perturbative sectors of this theory are obtained by adding fractional D(−1) branes. The corresponding instanton moduli are easily obtained by projecting the spectrum described in section 3 onto the sub-space of moduli that are even under the $\mathbb{Z}_2$ action. In the neutral Neveu-Schwarz sector we find the vector a^μ and the two scalars χ and $\bar{\chi}$, since the a^m's are projected out. In the neutral Ramond sector the surviving moduli are $\eta_{\alpha a}$ and $\lambda_{\dot{\alpha}a}$, where the index α ($\dot{\alpha}$) labels the chiral (anti-chiral) spinor representation of the first SO(4), while $\eta_{\dot\alpha\dot a}$ and $\lambda_{\alpha\dot a}$ are projected out. Also in this orbifold model, the consistency with the orientifold projection requires that a^μ and $\eta_{\alpha a}$ transform in the symmetric representation of the instanton symmetry group SO(k), while $\chi, \bar\chi$ and $\lambda_{\dot\alpha a}$ must transform in the antisymmetric representation. Again the abelian part of a^μ and $\eta_{\alpha a}$ can be identified with the supercoordinates of a chiral superspace, this time in four dimensions, namely

$$x_\mu \sim \ell_s^2 \,\mathrm{tr}\,(a_\mu)\,, \qquad \theta_{\alpha a} \sim \ell_s^2 \,\mathrm{tr}\,(\eta_{\alpha a}) \tag{5.2}$$

in complete analogy with eq. (3.1). In the charged sector, the fermionic moduli μ, transforming in the bi-fundamental representation of SO(k) $\times$ SO(4), are unaffected by the orbifold projection and thus remain in the spectrum.

The instanton action in this orbifold model can be simply obtained by projecting the one described in section 3 on the even moduli. Since this projection does not change the formal structure of the action, we do not write it again. However it is worth mentioning that now there is a non-vanishing coupling between the axion $\widetilde C_0$ and four θ's instead of eight, as it is explicitly shown in appendix B.1 where the disk amplitude among the axion vertex and four θ vertices is computed. This result gives a direct evidence that the axio-dilaton field τ can be promoted to a full $\mathcal{N}=2$ chiral superfield T as follows

$$T = \tau_0 + \widetilde T = \tau_0 + \widetilde\tau + \sqrt{2}\,\theta\widetilde\lambda + \ldots + \theta^4 \frac{\partial^2}{\partial z^2}\widetilde{\bar\tau}\,. \tag{5.3}$$

Therefore, we can conclude that the complete moduli action has the same form given in eqs. (3.15) and (3.16) with $\widetilde T$ given by eq. (5.3) instead of eq. (2.6).

5.2 The non-perturbative axio-dilaton source

In this orbifold model the D-instantons induce a non-perturbative four-dimensional effective action given by

$$S_{\text{n.p.}} = \frac{1}{(2\pi)^2}\int d^4x\, d^4\theta\, \widetilde F_{\text{n.p.}}\,, \tag{5.4}$$

where the prepotential $\widetilde F_{\text{n.p.}}$ is defined as in eq. (4.5) but with the integrals performed over the centered moduli described in the previous subsection. Correspondingly, the non-perturbative source term for the axio-dilaton is

$$J_{\text{n.p.}} = -\frac{(2\pi\alpha')^2}{2\pi}\,\bar\delta\widetilde F_{\text{n.p.}} = -\frac{(2\pi\alpha')^2}{2\pi}\,\frac{\delta\widetilde F_{\text{n.p.}}}{\delta\big(\theta^4\,\widetilde{\bar\tau}\big)}\bigg|_{T=\tau_0\,,\,M=m_{\text{cl}}} \tag{5.5}$$

where now $m_{\text{cl}} = \mathrm{diag}\{m_1, m_2, -m_1, -m_2\}$. Expanding in powers of q, one finds that that the coefficients $\bar\delta F_k$ are formally equal to the those written in eq. (4.8), but with $\bar\delta\widetilde Z_k$ given by

$$\bar\delta\widetilde Z_k = -2\pi\mathrm{i}\sum_{\ell=0}^{\infty}(2\pi\alpha')^{2\ell}\bar p^{2\ell+2}Z_k^{(2\ell)}\,. \tag{5.6}$$

In particular the terms up to $k = 4$ are

$$
\begin{aligned}
\bar{\delta}\widetilde{F}_1 &= -2\pi \mathrm{i}\, \bar{p}^2 \Big\{ \lim_{\mathcal{E}\to 0} \mathcal{E} Z_1 \Big\}\,, \\
\bar{\delta}\widetilde{F}_2 &= -2\pi \mathrm{i} \sum_{\ell=0}^{\infty} (2\pi\alpha')^{2\ell}\, \bar{p}^{2\ell+2} \Big\{ \lim_{\mathcal{E}\to 0} \mathcal{E}\big(Z_2^{(2\ell)} - Z_1 Z_1^{(2\ell)}\big) \Big\}\,, \\
\bar{\delta}\widetilde{F}_3 &= -2\pi \mathrm{i} \sum_{\ell=0}^{\infty} (2\pi\alpha')^{2\ell}\, \bar{p}^{2\ell+2} \Big\{ \lim_{\mathcal{E}\to 0} \mathcal{E}\big(Z_3^{(2\ell)} - Z_1 Z_2^{(2\ell)} - Z_2 Z_1^{(2\ell)} + Z_1^2 Z_1^{(2\ell)}\big) \Big\}\,, \\
\bar{\delta}\widetilde{F}_4 &= -2\pi \mathrm{i} \sum_{\ell=0}^{\infty} (2\pi\alpha')^{2\ell}\, \bar{p}^{2\ell+2} \Big\{ \lim_{\mathcal{E}\to 0} \mathcal{E}\big(Z_4^{(2\ell)} - Z_1 Z_3^{(2\ell)} - Z_3 Z_1^{(2\ell)} - Z_2 Z_2^{(2\ell)} \\
&\qquad\qquad + Z_1^2 Z_2^{(2\ell)} + 2 Z_2 Z_1 Z_1^{(2\ell)} - Z_1^3 Z_1^{(2\ell)}\big) \Big\}\,.
\end{aligned}
\tag{5.7}
$$

The right hand sides can be rewritten in terms of the chiral ring of the SO(4) theory that has been computed using Nekrasov's prescription and localization techniques in ref. [30]. The non-perturbative SO(4) correlators $\langle \mathrm{tr}\, m^J \rangle_{\text{n.p.}}$ are as in eq. (4.14) but with $\mathcal{O}_{(k,J)}$ given by

$$
\mathcal{O}_{(k,J)} = \sum_{I=1}^{k} \Big[\chi_I^J - (\chi_I + \varepsilon_1)^J - (\chi_I + \varepsilon_2)^J + (\chi_I + \varepsilon_1 + \varepsilon_2)^J \Big]\,. \tag{5.8}
$$

Explicit computations show that non-trivial contributions arise only for J even due to the symmetry of the theory. The first non-trivial element, corresponding to $J = 2$, can be expressed in a closed form for any instanton number k since $\mathcal{O}_{(k,2)} = 2\,\mathcal{E}\,k$, where $\mathcal{E} = \varepsilon_1\varepsilon_2$. Therefore we have

$$
\begin{aligned}
\langle \mathrm{tr}\, m^2 \rangle_{\text{n.p.}} &= \sum_{k=1}^{\infty} q^k \langle \mathrm{tr}\, m^2 \rangle_k = 2 \lim_{\mathcal{E}\to 0} \left\{ \frac{\mathcal{E}}{\widetilde{\mathcal{Z}}} \sum_{k=1}^{\infty} k\, q^k\, \widetilde{\mathcal{Z}}_k \right\}\Bigg|_{T=\tau_0\,,\, M=m_{\text{cl}}} \\
&= 2\, q\, \frac{\partial}{\partial q} \lim_{\mathcal{E}\to 0} \Big\{ \mathcal{E} \log \mathcal{Z} \Big\} = 2\, q\, \frac{\partial}{\partial q} F_{\text{n.p.}}\,.
\end{aligned}
\tag{5.9}
$$

The subsequent elements of the chiral ring for $J > 2$ can be explicitly computed order by order in the instanton expansion with exactly the same methods described in section 4 and appendix C. The results for the first few instanton numbers and $\ell > 0$ are

$$
\begin{aligned}
\langle \mathrm{tr}\, m^{2\ell+2} \rangle_1 &= 0\,, \\
\langle \mathrm{tr}\, m^{2\ell+2} \rangle_2 &= (-1)^\ell\, (2\ell+2)!\, \lim_{\mathcal{E}\to 0} \mathcal{E} Z_2^{(2\ell)}\,, \\
\langle \mathrm{tr}\, m^{2\ell+2} \rangle_3 &= (-1)^\ell\, (2\ell+2)!\, \lim_{\mathcal{E}\to 0} \mathcal{E}\big(Z_3^{(2\ell)} - Z_1 Z_2^{(2\ell)}\big)\,, \\
\langle \mathrm{tr}\, m^{2\ell+2} \rangle_4 &= (-1)^\ell\, (2\ell+2)!\, \lim_{\mathcal{E}\to 0} \mathcal{E}\big(Z_4^{(2\ell)} - Z_1 Z_3^{(2\ell)} - Z_2 Z_2^{(2\ell)} + Z_1^2 Z_2^{(2\ell)}\big)\,.
\end{aligned}
\tag{5.10}
$$

Since $Z_1^{(2\ell)} = 0$ for $\ell > 0$, the right hand sides have precisely the same structures appearing in eq. (5.7). We thus can write

$$
\bar{\delta}\widetilde{F}_k = -2\pi \mathrm{i} \sum_{\ell=0}^{\infty} (-1)^\ell\, \frac{(2\pi\alpha')^{2\ell}}{(2\ell+2)!}\, \bar{p}^{2\ell+2}\, \langle \mathrm{tr}\, m^{2\ell+2} \rangle_k\,, \tag{5.11}
$$

so that eq. (5.5) becomes

$$J_{\rm n.p.} = {\rm i} \sum_{\ell=0}^{\infty} (-1)^{\ell} \, \frac{(2\pi\alpha')^{2\ell+2}}{(2\ell+2)!} \, \bar{p}^{2\ell+2} \, \langle {\rm tr}\, m^{2\ell+2} \rangle_{\rm n.p.} \, . \tag{5.12}$$

For completeness we report, in our conventions, the explicit expressions of $\langle {\rm tr}\, m^J \rangle_{\rm n.p.}$ for the first few values of J and the first few instanton numbers, that have been computed in ref. [30], namely

$$\begin{aligned}
\langle {\rm tr}\, m^2 \rangle_{\rm n.p.} &= 4\, {\rm Pf} m\, q - 2 \sum_i m_i^2\, q^2 + 4\, {\rm Pf} m\, q^3 - 2 \sum_i m_i^2\, q^4 + \dots \, , \\
\langle {\rm tr}\, m^4 \rangle_{\rm n.p.} &= -12\, ({\rm Pf} m)^2\, q^2 + 16\, {\rm Pf} m \sum_i m_i^2\, q^3 - 6 \Big(\Big(\sum_i m_i^2 \Big)^2 + 6\, ({\rm Pf} m)^2 \Big) q^4 + \dots \, , \\
\langle {\rm tr}\, m^6 \rangle_{\rm n.p.} &= 40\, ({\rm Pf} m)^3\, q^3 - 90\, ({\rm Pf} m)^2 \sum_i m_i^2\, q^4 + \dots \, , \\
\langle {\rm tr}\, m^8 \rangle_{\rm n.p.} &= -140\, ({\rm Pf} m)^4\, q^4 + \dots
\end{aligned} \tag{5.13}$$

where ${\rm Pf} m = m_1 m_2$.

6 Conclusions

The conclusion we can draw is that the non-perturbative corrections to the profile of closed string fields representing the gravitational dual of gauge couplings can be microscopically derived by including the modification of the source terms in their equations of motion induced by instantonic branes. We have explicitly shown this non-perturbative holographic correspondence in the particular case of the axio-dilaton in Sen's local limit of the type I$'$ theory recovering the complete F-theory background, and also in an orbifolded version of it. Remarkably, the non-perturbative corrections to the gauge couplings can be entirely reconstructed from the knowledge of the chiral ring of the flavor degrees of freedom.

It would be extremely interesting, and we think it should be possible, to generalize this derivation to other cases and to other gravitational fields.

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A Notations and conventions

We denote the $10d$ space coordinates by x^M, $M = 0, \dots 9$, those along the world-volume of the D7 branes by x^μ, while for the last two directions we introduce the complex coordinates

$$z = x^8 - {\rm i}\, x^9 \, , \qquad \bar{z} = x^8 + {\rm i}\, x^9 \, . \tag{A.1}$$

Thus, our brane set-up breaks the $10d$ "Lorentz" group SO(10) into SO(8) $\times$ SO(2). We will indicate the bosonic string fields corresponding to the coordinates (A.1) as Z and $\overline{Z}$. In the complex basis, the non-vanishing metric components read $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$; of course then $g^{z\bar{z}} = g^{\bar{z}z} = 2$. We indicate by

$$\partial = \frac{1}{2}\big(\partial_8 + \mathrm{i}\,\partial_9\big)\,, \qquad \bar{\partial} = \frac{1}{2}\big(\partial_8 - \mathrm{i}\,\partial_9\big) \tag{A.2}$$

the holomorphic, respectively anti-holomorphic, derivatives. We also introduce the complex combination of momenta

$$p = \frac{1}{2}\big(p_8 - \mathrm{i}p_9\big)\,, \qquad \bar{p} = \frac{1}{2}\big(p_8 + \mathrm{i}p_9\big)\,. \tag{A.3}$$

The scalar product $\vec{p}\cdot\vec{x}$ in the last two directions can then be expressed as $\bar{p}z + p\bar{z}$.

The Laplace operator in these directions can be written as $\Box = 4\bar{\partial}\partial$. We define the two-dimensional δ-function $\delta^2(x^8, x^9)$, which we sometimes loosely denote as $\delta^2(z)$, with respect to the integration measure $dx^8\,dx^9$, so that the logarithm satisfies the Laplace equation with the following normalization

$$\Box \log\big(|z|^2/|z_0|^2\big) = 4\pi\,\delta^2(z)\,. \tag{A.4}$$

At the level of the Clifford algebra, if we denote by $\widehat{\Gamma}^M$ the 32-dimensional γ-matrices for SO(10), we have the decomposition

$$\widehat{\Gamma}^\mu = \Gamma^\mu \otimes \mathbb{1}\,, \qquad \widehat{\Gamma}^8 = \Gamma \otimes \sigma^1\,, \qquad \widehat{\Gamma}^9 = \Gamma \otimes \sigma^2\,, \tag{A.5}$$

where Γ^μ are the 16-dimensional γ-matrices for SO(8), and Γ the corresponding chirality matrix. They are tensored with matrices acting on the 2-dimensional spinor space for SO(2). In the complex basis for the last two directions we have in particular

$$\widehat{\Gamma}^z = \widehat{\Gamma}^8 - \mathrm{i}\,\widehat{\Gamma}^9 = 2\Gamma \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \widehat{\Gamma}^{\bar{z}} = \widehat{\Gamma}^8 + \mathrm{i}\,\widehat{\Gamma}^9 = 2\Gamma \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \tag{A.6}$$

using the metric in the complex basis to lower the indices, we also have $\hat{\Gamma}_z = \frac{1}{2}\hat{\Gamma}^{\bar{z}}$ and $\hat{\Gamma}_{\bar{z}} = \frac{1}{2}\hat{\Gamma}^z$.

The $10d$ chirality matrix $\hat{\Gamma}$ decomposes as

$$\widehat{\Gamma} = \Gamma \otimes \sigma^3\,. \tag{A.7}$$

If we denote the chiral (or antichiral) spinor indices of SO(10) by $\mathcal{A}$ (or $\dot{\mathcal{A}}$), those of SO(8) by α (or $\dot{\alpha}$) and those of SO(2) by $+$ (or $-$), this decomposition corresponds to splitting these indices as follows

$$\mathcal{A} = (\alpha, +) \cup (\dot{\alpha}, -)\,, \qquad \dot{\mathcal{A}} = (\dot{\alpha}, +) \cup (\alpha, -)\,. \tag{A.8}$$

Thus, if $\Theta^{\dot{\mathcal{A}}}$ is a $10d$ anti-chiral Majorana-Weyl spinor, it decomposes as

$$\Theta^{\dot{\mathcal{A}}} \to (\bar{\theta}^{\dot{\alpha}+}, \theta^{\alpha-})\,; \tag{A.9}$$

if the $10d$ anti-chiral nature of these spinor is known, the $\pm$ indices become redundant, and we often do not write them.

The $10d$ charge conjugation matrix $\widehat{\mathcal{C}}$, such that $\widehat{\mathcal{C}}\,\widehat{\Gamma}^M\,\widehat{\mathcal{C}}^{-1} = -\,{}^t\widehat{\Gamma}^M$, becomes

$$\widehat{\mathcal{C}} = -\mathrm{i}\,\mathcal{C}\otimes\sigma^2\,, \tag{A.10}$$

where $\mathcal{C}$ is the charge conjugation matrix in $8d$, which satisfies $\mathcal{C}\,\Gamma^M\,\mathcal{C}^{-1} = -\,{}^t\Gamma^M$, while the matrix σ^2 represents charge conjugation in the last two dimensions.

For the SO(8) Clifford algebra it is possible to choose a chiral basis in which

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \mathbb{1}_8 & 0 \\ 0 & -\mathbb{1}_8 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \tag{A.11}$$

with $C = \sigma^2\otimes\sigma^3\otimes\sigma^2$ when written as a tensor product of two-dimensional factors. In this basis the spinor indices α and $\dot\alpha$ enumerate a specific ordering of the spinor weights. This is particularly well suited to deal with world-sheet computations involving spin fields, whose charges in the bosonized description are exactly the components of the spinor weights.

The massless vertex in the Ramond-Ramond sector of the closed superstring, in the $(-\frac{1}{2},-\frac{1}{2})$ picture, reads

$$\pi g_s\ell_s\,F_{\dot{\mathcal{A}}\dot{\mathcal{B}}}\,S^{\dot{\mathcal{A}}}(w)\,\mathrm{e}^{\mathrm{i}\,\ell_s p_L\cdot X(w)}\,\mathrm{e}^{-\frac{1}{2}\varphi(w)}\,\widetilde{S}^{\dot{\mathcal{B}}}(\bar w)\,\mathrm{e}^{\mathrm{i}\,\ell_s p_R\cdot\widetilde{X}(\bar w)}\,\mathrm{e}^{-\frac{1}{2}\widetilde{\varphi}(\bar w)}\,. \tag{A.12}$$

Here $S^{\dot{\mathcal{A}}}$ and $\widetilde{S}^{\dot{\mathcal{B}}}$ are $10d$ left and right spin-fields while $\mathrm{e}^{-\frac{1}{2}\varphi}$ and $\mathrm{e}^{-\frac{1}{2}\widetilde{\varphi}}$ are the superghost factors; our conventions are that the type IIB GSO projection singles out antichiral spinors in both left- and right-moving sectors. The bispinor polarization $F_{\dot{\mathcal{A}}\dot{\mathcal{B}}}$, when expanded in p-forms, reads

$$F_{\dot{\mathcal{A}}\dot{\mathcal{B}}} = \frac{1}{8}\sum_{p\ \mathrm{odd}} F_{M_1\ldots M_p}\left(\mathcal{C}\widehat{\Gamma}^{M_1\ldots M_p}\right)_{\dot{\mathcal{A}}\dot{\mathcal{B}}}\,, \tag{A.13}$$

where $F_{M_1\ldots M_p}$ is the field-strength of the Ramond-Ramond $(p-1)$-form potential.

Consider the Ramond-Ramond scalar $\widetilde{C}_0$, depending only on the last two directions; its field strength is a 1-form has the non-vanishing component $F_z = \partial_z\widetilde{C}_0$ (i.e. $\mathrm{i}\,\bar p\,\widetilde{C}_0$ in momentum space). It follows from eqs. (A.12) and (A.13) that the corresponding vertex contains the matrix

$$\widehat{\mathcal{C}}\,\widehat{\Gamma}^z = -2\begin{pmatrix} \mathcal{C}\Gamma & 0 \\ 0 & 0 \end{pmatrix} \tag{A.14}$$

which, using eq. (A.11), has the only non-vanishing components

$$(\widehat{\mathcal{C}}\widehat{\Gamma}^z)_{\alpha+\,,\,\beta+} = -2C_{\alpha\beta}\,, \qquad (\widehat{\mathcal{C}}\widehat{\Gamma}^z)_{\dot\alpha+\,,\,\dot\beta+} = 2C_{\dot\alpha\dot\beta}\,. \tag{A.15}$$

The second set of these components is antichiral in $10d$ and appears thus in axion vertex (3.10).

The notations for vector and spinor indices in the orbifold case considered in section 5 are given in the main text below eq. (5.1). Let us just remark that in this case we replace the spinor index α of SO(8) with a couple (α, a) or $(\dot\alpha, \dot a)$, where α and a are chiral spinor indices for the first and the second factors in the decomposition SO(8) $\to$ SO(4) $\times$ SO(4); similarly for the antichiral case.

B Relevant string diagrams

This appendix contains some details about the calculation of disk diagrams that produce the couplings of the axio-dilaton to the D-instantons moduli considered in the main text.

B.1 Interaction among $\widetilde{C}_0$ and the θ moduli

Here we describe the explicit computation of the coupling among the Ramond-Ramond scalar $\widetilde{C}_0$ and the θ-moduli, corresponding to the higher term in the closed superfield T.

Orbifold case. Let us first consider the set-up described in section 5, where this coupling is technically easier to derive, with respect to the type I' set-up, since it contains only four θ insertions.

We consider thus a disk diagram having D(-1) boundary conditions and insert an axion vertex in the interior and four vertices for the fermionic moduli η on the boundary

$$A = \langle\!\langle V_\eta V_\eta V_\eta V_\eta V_{\widetilde{C}_0} \rangle\!\rangle \,. \tag{B.1}$$

Actually, we are interested in the components of the fermionic moduli along the identity (see eq. (3.1)), namely

$$\eta_{\alpha a} = \mathcal{N}\, \ell_s^{-2}\, \theta_{\alpha a}\, \mathbb{1}_k \,, \tag{B.2}$$

where the notation for spinor indices is the one introduced in section 5.1, and $\mathcal{N}$ is a purely numerical factor that we will fix soon. We take the axion vertex in the $(-1/2, -1/2)$ picture, so it has the form of eq. (3.10). Since on the disk the total (right + left) superghost charge must equal -2, three open string vertices, say those with polarizations $\eta_{\alpha_i a_i}$ located at positions x_i with $i = 1, 2, 3$, can be taken in the $(-1/2)$ picture, as in table 1, while the fourth one has to be taken in the $(1/2)$ picture; the latter has the form

$$\ell_s^{3/2}\, \eta_{\alpha_4 a_4}\, \partial \bar{Z}(x_4)\, S^{\alpha_4 a_4 +}(x_4)\, \mathrm{e}^{+\frac{1}{2}\varphi(x_4)} \,. \tag{B.3}$$

Since we are inserting the maximum number of θ's, the spinor indices of the four fermionic vertices have to be all different. We can thus choose a fixed value for them, and then sum over all cyclically inequivalent orderings. For instance, we can make the choice indicated below:

	$\phi_1 \ldots \phi_4$	ϕ_5	φ
$\alpha_1 a_1 -$	$++++$	$-$	$-$
$\alpha_2 a_2 -$	$----$	$-$	$-$
$\alpha_3 a_3 -$	$++--$	$-$	$-$
$\alpha_4 a_4 +$	$--++$	$+$	$+$
$\dot{\mathcal{A}}$	$+---$	$+$	$-$
$\dot{\mathcal{B}}$	$-+++$	$+$	$-$

(B.4)

The last column reports (twice) the superghost charge of the various vertices. Moreover, $\phi_1, \ldots \phi_5$ are the world-sheet scalars that bosonize the spin fields in the five couples of directions. The spinor indices are associated to (twice) their charges with respect to these fields, i.e. to twice the corresponding weight vectors. We have also taken into account the

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fact that the spinor indices $\dot{\mathcal{A}}$ and $\dot{\mathcal{B}}$ of the spin fields of the axion vertex must be such as to saturate all charges; their last component must thus be of type $+$, while their first four components must take values opposite to each other. Different choices of these values (we have altogether 8 possibilities) lead to contributions that would arise from diagrams with different orderings of the θ's, so we can fix them as in the table above and multiply the result by 8.

Taking into account the above considerations, the disk amplitude (B.1) becomes

$$A = \frac{2\pi}{g_s}\,2\pi g_s \ell_s\,\mathrm{i}\,\bar{p}\,\widetilde{C}_0\,\mathcal{N}^4\,\ell_s^{-2}\,k\,\theta^4 \int \frac{\prod_{i=1}^4 dx_i\,dw\,d\bar{w}}{dV_{\mathrm{CKG}}} C_{\mathrm{tot}}\,. \tag{B.5}$$

Here the factor $\frac{2\pi}{g_s}$ is the topological normalization of the disk, the remaining overall constants arise from the normalizations and polarizations of the vertices, while dV_{CKG} stands for the volume of the conformal Killing group. With C_{tot} we have denoted the sum over the inequivalent ordering of the four open string vertices, that is

$$C_{\mathrm{tot}} = C_{1234} + C_{2134} + C_{3214} + C_{1324} + C_{2314} + C_{3124} \tag{B.6}$$

where

$$C_{1234} = S_{1234}(x_i, w, \bar{w}) X(x_4, w, \bar{w})\,. \tag{B.7}$$

Here S, X are the relevant correlators of spin fields and bosonic world-sheet fields, computed with the choice of indices in eq. (B.4). After identifying the left and right components of the closed string vertex ($\widetilde{S}^{\dot{\mathcal{B}}} = S^{\dot{\mathcal{B}}}$ and $\widetilde{Z} = -Z$), one finds

$$\begin{aligned} S_{1234}(x_i, w, \bar{w}) &= \left\langle S^{\alpha_1 a_1 -}(x_1)\,S^{\alpha_2 a_2 -}(x_2)\,S^{\alpha_3 a_3 -}(x_3)\,S^{\alpha_4 a_4 +}(x_4)\,S^{\dot{\mathcal{A}}}(w)\,S^{\dot{\mathcal{B}}}(\bar{w})\right\rangle \\ &\quad \left\langle \mathrm{e}^{-\frac{1}{2}\varphi(x_1)}\,\mathrm{e}^{-\frac{1}{2}\varphi(x_2)}\,\mathrm{e}^{-\frac{1}{2}\varphi(x_3)}\,\mathrm{e}^{+\frac{1}{2}\varphi(x_4)}\,\mathrm{e}^{-\frac{1}{2}\varphi(w)}\,\mathrm{e}^{-\frac{1}{2}\varphi(\bar{w})}\right\rangle \\ &= \big[(x_1-x_2)(x_1-w)(x_2-\bar{w})(x_3-x_4)(x_3-\bar{w})(w-\bar{w})\big]^{-1}(x_4-\bar{w}) \end{aligned} \tag{B.8}$$

and

$$X(x_4, w, \bar{w}) = \left\langle \partial\overline{Z}(x_4)\,\mathrm{e}^{\mathrm{i}\,\ell_s \bar{p} Z(w)}\,\mathrm{e}^{-\mathrm{i}\,\ell_s \bar{p} Z(\bar{w})}\right\rangle = \mathrm{i}\ell_s\,\bar{p}\,\frac{(w-\bar{w})}{(x_4-w)(x_4-\bar{w})}\,, \tag{B.9}$$

so that altogether

$$C_{1234} = \mathrm{i}\ell_s\bar{p}\,\big[(x_1-x_2)(x_1-w)(x_2-\bar{w})(x_3-x_4)(x_3-\bar{w})(x_4-w)\big]^{-1}. \tag{B.10}$$

The other inequivalent correlators contributing to C_{tot} can be computed by inserting the vertices with indices $\alpha_i a_i$ at permuted locations. Each of them can be equivalently represented in four different ways obtained by cyclically permuting the four open string vertices, which is guaranteed to lead to the same result after integration. It turns out to be convenient, rather than using eq. (B.6), to rewrite C_{tot} as one-half the sum of twelve terms, two in each equivalence class, chosen so that some partial sums algebraically simplify. For instance, one has

$$\begin{aligned} C_{1234} &= (\mathrm{i}\ell_s\bar{p})\,\big[(x_1-x_2)(x_1-w)(x_2-\bar{w})(x_3-x_4)(x_3-\bar{w})(x_4-w)\big]^{-1}, \\ C_{1243} &= (-\mathrm{i}\ell_s\bar{p})\,\big[(x_1-x_2)(x_1-w)(x_2-\bar{w})(x_3-x_4)(x_4-\bar{w})(x_3-w)\big]^{-1}, \\ C_{2143} &= (\mathrm{i}\ell_s\bar{p})\,\big[(x_1-x_2)(x_2-w)(x_1-\bar{w})(x_3-x_4)(x_3-\bar{w})(x_4-w)\big]^{-1}, \\ C_{2134} &= (-\mathrm{i}\ell_s\bar{p})\,\big[(x_1-x_2)(x_2-w)(x_1-\bar{w})(x_3-x_4)(x_4-\bar{w})(x_3-w)\big]^{-1}, \end{aligned} \tag{B.11}$$

so that

$$C_{(1)} = C_{1234} + C_{1243} + C_{2143} + C_{2134} = (-\mathrm{i}\ell_s \bar{p}) \frac{(w-\bar{w})^2}{\prod_{i=1}^4 |x_i - w|^2} \,. \tag{B.12}$$

Similarly, one finds

$$C_{(2)} = C_{1324} + C_{1423} + C_{2314} + C_{2413} = C_{(1)} \tag{B.13}$$

and

$$C_{(3)} = C_{1342} + C_{1432} + C_{2341} + C_{2431} = C_{(1)} \,. \tag{B.14}$$

The total correlator arising from the sum over inequivalent orderings is thus

$$C_{\mathrm{tot}} = \frac{1}{2}\big(C_{(1)} + C_{(2)} + C_{(3)}\big) = \frac{3}{2}(-\mathrm{i}\ell_s \bar{p}) \frac{(w-\bar{w})^2}{\prod_{i=1}^4 |x_i - w|^2} \,. \tag{B.15}$$

We can now insert this expression into eq. (B.5) and proceed to evaluate the integral.

Fixing $x_1 \to \infty, w = \mathrm{i}, \bar{w} = -\mathrm{i}$, the integrand gets multiplied by $|x_1 - w|^2 (w - \bar{w})$ and the integral becomes

$$\begin{aligned} &-\frac{3\mathrm{i}\ell_s \bar{p}}{2}(2\mathrm{i})^3 \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{x_2} dx_3 \int_{-\infty}^{x_3} dx_4 \prod_{i=2}^{4} (x_i^2 + 1)^{-1} \\ &\qquad = -\frac{3\mathrm{i}\ell_s \bar{p}}{2}(2\mathrm{i})^3 \frac{1}{3!} \int_{-\infty}^{+\infty} \prod_{i=2}^{4} dx_i \prod_{i=2}^{4} (x_i^2 + 1)^{-1} = -\frac{(2\pi)^3}{4} \ell_s \bar{p} \,. \end{aligned} \tag{B.16}$$

Substituting this result into eq. (B.5) we get finally the amplitude

$$A = -2\pi \mathrm{i}\, k\, \frac{(2\pi)^4 \mathcal{N}^4}{4} \bar{p}^2 \widetilde{C}_0\, \theta^4 \,. \tag{B.17}$$

If we choose

$$\mathcal{N} = \frac{\sqrt{2}}{2\pi} \,, \tag{B.18}$$

this amplitude corresponds to the following term in the instanton moduli action:

$$-\,2\pi \mathrm{i}\, k\, \theta^4\, \partial^2 \widetilde{C}_0 \,. \tag{B.19}$$

This coupling promotes the Ramond-Ramond part of $-2\pi \mathrm{i} k \widetilde{\tau}$ appearing in the moduli action (see eq. (3.6)) to the highest Ramond-Ramond part of $-2\pi \mathrm{i} k \widetilde{T}$ where

$$\widetilde{T} = \widetilde{\tau} + \sqrt{2}\,\theta \widetilde{\lambda} + \ldots + \theta^4 \frac{\partial^2}{\partial z^2} \widetilde{\bar{\tau}} \,, \tag{B.20}$$

represents the analogous of the type I′ closed string superfield of eq. (2.6) in the orbifold case.

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The flat case. The previous calculation easily generalizes to the local limit of type I′ theory considered in sections 2–4, where the disk diagram describing the interaction of the axion $\widetilde{C}_0$ with the maximum number of θs, that in this case is eight, can again be computed by exactly the same kind of techniques, namely by fixing explicit values for the spinor indices for one ordering and summing over non-equivalent re-orderings. In this case, five of the θ vertices are in the $(-1/2)$ picture and three in the $(+1/2)$ picture, so that from the bosonic correlator we get two extra powers of $\bar{p}$ with respect to the previous computation. All in all, taking into account the different combinatorics, the result is that

$$A = \langle\!\langle V_\eta V_\eta V_\eta V_\eta V_\eta V_\eta V_\eta V_\eta V_{\widetilde{C}_0} \rangle\!\rangle = 2\pi \mathrm{i}\, k\, \frac{(2\pi)^8\, \mathcal{N}^8}{8} \bar{p}^4 \widetilde{C}_0\, \theta^8 \,. \tag{B.21}$$

Using eq. (B.18), this corresponds to a term in the moduli action of the form

$$-\,2\pi \mathrm{i}\, k\, 2\theta^8\, \partial^4 \widetilde{C}_0 \,, \tag{B.22}$$

which is indeed part of the interaction term $-2\pi \mathrm{i} k \widetilde{T}$ involving the closed string superfield of eq. (2.6) (see eqs. (3.7) and (3.8).

B.2 Interaction of $\widetilde{C}_0$ with many χ-moduli

Here we consider the interaction among the axion $\widetilde{C}_0$ and any (even) number n of χ moduli, namely

$$A = \langle\!\langle \underbrace{V_\chi \cdots V_\chi}_{n=2\ell}\, V_{\widetilde{C}_0} \rangle\!\rangle \,, \tag{B.23}$$

and sketch the derivation of its expression, reported in eq. (3.11), which was crucial in our discussion. For this computation no substantial difference arises in considering the flat or the orbifolded case, so we proceed directly in the former case.

Again the axion vertex is taken in the picture $(-\frac{1}{2}, -\frac{1}{2})$, and is given in eq. (3.10). One of the χ vertices, which we place in position x_1, is in the superghost picture (-1) and is thus given in table 1, while all the others are in the picture (0), namely

$$V_\chi^{(0)}(x_i) = \ell_s\, \chi\, \partial \overline{Z}(x_i) \,. \tag{B.24}$$

Then the amplitude (B.23) takes the form

$$\begin{aligned} A = {} & \frac{2\pi}{g_s}\, \frac{2\pi g_s \ell_s}{8}\, (\ell_s)^n\, \mathrm{tr}\,(\chi^n)\, \mathrm{i}\, \bar{p}\, \widetilde{C}_0\, \frac{(n-1)!}{n!} \\ & \times \int \frac{\prod_{i=1}^n dx_i\, dw\, d\bar{w}}{dV_{\mathrm{CKG}}} A_1(x_1, w, \bar{w})\, A_2(x_1, w, \bar{w})\, A_3(x_i, w, \bar{w}) \,. \end{aligned} \tag{B.25}$$

This expression includes the disk normalization factor $2\pi/g_s$ and polarizations and normalizations of the vertices; there is an $1/n!$ symmetry factor but a $(n-1)!$ arises from the sum over cyclically inequivalent orderings of the vertices, since it easy to see that all of them lead to the same answer. Finally, $A_{1,2,3}$ are respectively the superghost, spin fields and bosonic correlators:

$$A_1(x_1,w,\bar{w}) = \left\langle \mathrm{e}^{-\varphi(x_1)}\,\mathrm{e}^{-\frac{1}{2}\varphi(w)}\,\mathrm{e}^{-\frac{1}{2}\varphi(\bar{w})}\right\rangle, \tag{B.26}$$

$$A_2(x_1,w,\bar{w}) = \mathcal{C}_{\dot{\alpha}\dot{\beta}}\left\langle \overline{\Psi}(x_1)\,S^{\dot{\alpha}+}(w)\,S^{\dot{\alpha}+}(\bar{w})\right\rangle, \tag{B.27}$$

$$A_3(x_i,w,\bar{w}) = \left\langle \partial\bar{Z}(x_2)\dots\partial\bar{Z}(x_n)\mathrm{e}^{\mathrm{i}\,\ell_s\bar{p}Z(w)}\,\mathrm{e}^{-\mathrm{i}\,\ell_s\bar{p}Z(\bar{w})}\right\rangle. \tag{B.28}$$

Using standard methods we obtain

$$A_1(x_1,w,\bar{w})A_2(x_1,w,\bar{w}) = 8\left[(x_1-w)(x_1-\bar{w})(w-\bar{w})\right]^{-1}, \tag{B.29}$$

$$A_3(x_i,w,\bar{w}) = (-\mathrm{i}\ell_s\bar{p})^{n-1}\frac{(w-\bar{w})^{n-1}}{\prod_{i=2}^{n}|x_i-w|^2}. \tag{B.30}$$

Fixing $x_1\to\infty, w=\mathrm{i}, \bar{w}=-\mathrm{i}$, the integrand gets multiplied by $|x_1-w|^2(w-\bar{w})$ and the integral becomes

$$\begin{aligned}
&(-\mathrm{i}\ell_s\bar{p})^{n-1}(2\mathrm{i})^{n-1}\int_{-\infty}^{+\infty}dx_2\int_{-\infty}^{x_2}dx_3\dots\int_{-\infty}^{x_{n-1}}dx_n\prod_{i=2}^{n}(x_i^2+1)^{-1}\\
&\qquad = 2^{n-1}\ell_s^{n-1}\bar{p}^{n-1}\frac{1}{(n-1)!}\int_{-\infty}^{+\infty}\prod_{i=2}^{n}dx_i\prod_{i=2}^{n}(x_i^2+1)^{-1}\\
&\qquad = \frac{(2\pi\ell_s)^{n-1}}{(n-1)!}\bar{p}^{n-1}.
\end{aligned} \tag{B.31}$$

The amplitude reads therefore

$$A = 2\pi\mathrm{i}\,(2\pi\alpha')^n\,\frac{\mathrm{tr}\,(\chi^n)}{n!}\,\bar{p}^n\,\widetilde{C}_0\,, \tag{B.32}$$

where we have taken into account that $\ell_s=\sqrt{\alpha'}$. This is just eq. (3.11) used in the main text.

C D7 brane chiral ring and non-perturbative axio-dilaton sources

In this appendix we give some further details on the D-instanton induced source terms for the axio-dilaton and their relation with the elements of the chiral ring on the D7 branes. The non-perturbative current $J_{\text{n.p.}}$ is given in eq. (4.7) in terms of the quantities $\widetilde{\delta F}_k$, which in turn are expressed in eq. (4.13) in terms of the instanton partition functions Z_k and of the quantities $Z_k^{(2\ell)}$ defined in eq. (4.10).

It is possible to show that the expressions appearing in eq. (4.13) are directly related to the non-perturbative contributions to the elements $\langle\mathrm{tr}\,m^J\rangle_{\text{n.p.}}$ of the D7 brane chiral ring. This relation, which represents one of the main points of section 4, has been explicitly shown there in the case of $\langle\mathrm{tr}\,m^4\rangle_{\text{n.p.}}$. Here we give further details on how this relation arises.

As shown in ref. [30], $\langle\mathrm{tr}\,m^J\rangle_{\text{n.p.}}$ can be computed via localization techniques through the formula given in eq. (4.14), which we rewrite here follows:

$$\langle\mathrm{tr}\,m^J\rangle_{\text{n.p.}} = \sum_{k=1}^{\infty}q^k\langle\mathrm{tr}\,m^J\rangle_k = \lim_{\mathcal{E}\to 0}\left\{\frac{1}{\mathcal{Z}}\sum_{k=1}^{\infty}q^k Z_{(k,J)}\right\}, \tag{C.1}$$

where

$$Z_{(k,J)} = \int d\mathcal{M}_{(k)}\, \mathrm{e}^{-S_{\text{inst}}}\, \mathcal{O}_{(k,J)}\,, \tag{C.2}$$

with $\mathcal{O}_{(k,J)}$ given in eq. (4.15). Notice that the right hand side contains the moduli action evaluated at $T = \tau_0\,,\ M = m_{\text{cl}}$, which we distinguish by stripping off the tilde sign. Analogously, $\mathcal{Z}$ is the instanton partition function of eq. (4.3) evaluated at $T = \tau_0\,,$ $M = m_{\text{cl}}$.

Comparing the coefficient of q^k in eq. (C.1) we have, for the first few instanton levels,

$$\begin{aligned}
\langle \mathrm{tr}\, m^J \rangle_1 &= \lim_{\mathcal{E}\to 0} Z_{(1,J)}\,, \\
\langle \mathrm{tr}\, m^J \rangle_2 &= \lim_{\mathcal{E}\to 0} \big(Z_{(2,J)} - Z_1\, Z_{(1,J)}\big)\,, \\
\langle \mathrm{tr}\, m^J \rangle_3 &= \lim_{\mathcal{E}\to 0} \big(Z_{(3,J)} - Z_1\, Z_{(2,J)} - Z_2\, Z_{(1,J)} + Z_1^2\, Z_{(1,J)}\big)\,, \\
\langle \mathrm{tr}\, m^J \rangle_4 &= \lim_{\mathcal{E}\to 0} \big(Z_{(4,J)} - Z_1\, Z_{(3,J)} - Z_2\, Z_{(2,J)} + Z_1^2\, Z_{(2,J)} \\
&\qquad + 2Z_2\, Z_1\, Z_{(1,J)} - Z_3\, Z_{(1,J)} - Z_1^3\, Z_{(1,J)}\big)\,,
\end{aligned} \tag{C.3}$$

and so on. In section 4 we argued that $\langle \mathrm{tr}\, m^J \rangle_{\text{n.p.}} = 0$ for J odd, so the next non-trivial case to be considered after the cases $J = 2, 4$ given in the main text, is $J = 6$. From eq. (4.15) it is possible to show that

$$\mathcal{O}_{(k,6)} = 60\, k\, \mathcal{E}\, h_2(\varepsilon_i) - 360\, \mathcal{E}\, \mathrm{tr}\,(\chi^2)\,, \tag{C.4}$$

where we have defined

$$h_2(\varepsilon_i) = 2 \sum_i \varepsilon_i^2 + 3 \sum_{i<j} \varepsilon_i \varepsilon_j\,. \tag{C.5}$$

Therefore, it follows that the integrals $Z_{(k,J)}$ are related to the partition functions Z_k and to the integrals with χ insertions $Z_k^{(2\ell)}$ defined in eq. (4.10), according to

$$Z_{(k,6)} = 60\, k\, \mathcal{E}\, h_2(\varepsilon_i)\, Z_k - 720\, \mathcal{E}\, Z_k^{(2)}\,. \tag{C.6}$$

Substituting this result into eq. (C.3), we obtain (taking into account that $Z_1^{(2\ell)} = 0$ for all ℓ)

$$\langle \mathrm{tr}\, m^6 \rangle_1 = \lim_{\mathcal{E}\to 0} 60\, h_2(\varepsilon_i)\, \mathcal{E}\, Z_1 = \lim_{\mathcal{E}\to 0} 60\, h_2(\varepsilon_i)\, F_1\,, \tag{C.7}$$

$$\begin{aligned}
\langle \mathrm{tr}\, m^6 \rangle_2 &= \lim_{\mathcal{E}\to 0} \Big\{ - 720\, \mathcal{E}\, Z_2^{(2)} + 120\, h_2(\varepsilon_i)\, \mathcal{E} \Big(Z_2 - \frac{1}{2}\, Z_1^2 \Big) \Big\} \\
&= \lim_{\mathcal{E}\to 0} \Big\{ - 720\, \mathcal{E}\, Z_2^{(2)} + 120\, h_2(\varepsilon_i)\, \mathcal{E} \big(F_2 + \mathcal{O}(\varepsilon_i) \big) \Big\}\,,
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
\langle \mathrm{tr}\, m^6 \rangle_3 &= \lim_{\mathcal{E}\to 0} \Big\{ - 720\, \mathcal{E}\, \big(Z_3^{(2)} - Z_1\, Z_2^{(2)}\big) + 180\, h_2(\varepsilon_i)\, \mathcal{E} \Big(Z_3 - Z_2 Z_1 + \frac{1}{3}\, Z_1^3 \Big) \Big\} \\
&= \lim_{\mathcal{E}\to 0} \Big\{ - 720\, \mathcal{E}\, \big(Z_3^{(2)} - Z_1\, Z_2^{(2)}\big) + 180\, h_2(\varepsilon_i)\, \mathcal{E} \big(F_3 + \mathcal{O}(\varepsilon_i) \big) \Big\}\,,
\end{aligned} \tag{C.9}$$

$$\begin{aligned}\langle \mathrm{tr}\, m^6\rangle_4 &= \lim_{\mathcal{E}\to 0}\Big\{-720\,\mathcal{E}\,\big(Z_4^{(2)}-Z_1\,Z_3^{(2)}-Z_2\,Z_2^{(2)}+Z_1^2\,Z_2^{(2)}\big)\\ &\qquad +240\,h_2(\varepsilon_i)\,\mathcal{E}\Big(Z_4-Z_3\,Z_1-\frac{1}{2}\,Z_2^2+Z_2\,Z_1^2-\frac{1}{4}\,Z_1^4\Big)\Big\}\\ &= \lim_{\mathcal{E}\to 0}\Big\{-720\,\mathcal{E}\,\big(Z_4^{(2)}-Z_1\,Z_3^{(2)}-Z_2\,Z_2^{(2)}+Z_1^2\,Z_2^{(2)}\big)\\ &\qquad +240\,h_2(\varepsilon_i)\,\mathcal{E}\,\big(F_4+\mathcal{O}(\varepsilon_i)\big)\Big\}\,. \end{aligned} \tag{C.10}$$

We see that all terms in these expressions which contain only the ordinary instanton partition functions Z_k arrange themselves in the combinations F_k. These are the coefficients in the instanton expansion of the non-perturbative prepotential introduced in eqs. (4.4)–(4.6), evaluated at $T=\tau_0\,,\ M=m_{\mathrm{cl}}$. The prepotential is finite in the $\mathcal{E}\to 0$ limit, so that the above equations reduce to

$$\begin{aligned}\langle \mathrm{tr}\, m^6\rangle_2 &= -6!\,\lim_{\mathcal{E}\to 0}\mathcal{E}\,Z_2^{(2)}\,,\\ \langle \mathrm{tr}\, m^6\rangle_3 &= -6!\,\lim_{\mathcal{E}\to 0}\mathcal{E}\,\big(Z_3^{(2)}-Z_1\,Z_2^{(2)}\big)\,,\\ \langle \mathrm{tr}\, m^6\rangle_4 &= -6!\,\lim_{\mathcal{E}\to 0}\mathcal{E}\,\big(Z_4^{(2)}-Z_1\,Z_3^{(2)}-Z_2\,Z_2^{(2)}+Z_1^2\,Z_2^{(2)}\big)\,,\end{aligned} \tag{C.11}$$

and so on. We recognize in the right hand sides above exactly the same expressions appearing at the order $\ell=3$ in the variations $\delta\widetilde{F}_k$ as given in eq. (4.13).

Let us now consider the third non-trivial element of the chiral ring, namely $\langle \mathrm{tr}\, m^8\rangle_{\text{n.p.}}$. From eq. (4.15), one can show with a bit of work that

$$\mathcal{O}_{(k,8)} = 56\,k\,\mathcal{E}\,h_4(\varepsilon_i) - 3360\,\mathcal{E}\,h_2(\epsilon_i)\,\mathrm{tr}\,(\chi^2) + 8!\,\mathcal{E}\,\mathrm{tr}\,(\chi^4)\,, \tag{C.12}$$

where we have defined

$$h_4(\varepsilon_i) = 6\sum_i \varepsilon_i^2 + 3\sum_{i\neq j}\varepsilon_i^3\varepsilon_j + 20\sum_{i<j}\varepsilon_i^2\varepsilon_j^2 + 30\sum_{i<j}\sum_{l\neq i,j}\varepsilon_i\varepsilon_j\varepsilon_l^2 + 45\,\mathcal{E}\,. \tag{C.13}$$

It follows that

$$Z_{(k,8)} = 56\,k\,\mathcal{E}\,h_4(\varepsilon_i)\,Z_k - 3360\,\mathcal{E}\,h_2(\varepsilon_i)\,Z_k^{(2)} + 8!\,\mathcal{E}\,Z_k^{(4)}\,. \tag{C.14}$$

Inserting this expression into eq. (C.3) we again find that all terms containing only Z_k's arrange themselves in the prepotential coefficients F_k and disappear in the limit $\mathcal{E}\to 0$; moreover also the terms containing $Z_k^{(2)}$ vanish in this limit so that we remain only with

$$\begin{aligned}\langle \mathrm{tr}\, m^8\rangle_2 &= 8!\,\lim_{\mathcal{E}\to 0}\mathcal{E}\,Z_2^{(4)}\,,\\ \langle \mathrm{tr}\, m^8\rangle_3 &= 8!\,\lim_{\mathcal{E}\to 0}\mathcal{E}\,\big(Z_3^{(4)}-Z_1\,Z_2^{(4)}\big)\,,\\ \langle \mathrm{tr}\, m^8\rangle_4 &= 8!\,\lim_{\mathcal{E}\to 0}\mathcal{E}\,\big(Z_4^{(4)}-Z_1\,Z_3^{(4)}-Z_2\,Z_2^{(4)}+Z_1^2\,Z_2^{(4)}\big)\,,\end{aligned} \tag{C.15}$$

and so on. This corresponds to the $\ell=4$ terms in the right hand side of eq. (4.13).

We infer that this correspondence holds for every value of ℓ, as written in eq. (4.24) in the main text, and links the non-perturbative source terms for the axio-dilaton, hence its profile, to the elements of the chiral ring on the D7 branes.

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