On the Stability Criterion in a Saturated Atmosphere

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ABSTRACT

The expression of the moist buoyancy frequency indicates that it is not completely correct to use the moist adiabatic lapse rate as a static stability parameter of a saturated atmosphere, because the contribution of both the water vapor and condensed water gradients is ignored. This paper shows that the vapor gradient effect is usually larger than the condensed water one, and can be taken into account by introducing a new temperature lapse rate as a stability parameter. The new lapse rate is always greater than the moist adiabatic one, implying an increase of the atmospheric stability estimate at any temperature and for both liquid water and ice saturation conditions.

1. Introduction

It has been observed in the past (Lalas and Einaudi 1974; Durran and Klemp 1982) that the replacement of the dry adiabatic by the saturated adiabatic lapse rate in studying the static stability of a saturated atmosphere is inadequate, especially when the stability is small. The reason comes from the expression of the moist buoyancy frequency (Durran and Klemp 1982), that by splitting the total water mixing ratio gradient into vapor and liquid components gives

\[ N^2_w = \frac{g}{T} \left( 1 + \frac{L_v r_w}{R_d T} \left( \frac{dT}{dz} + \Gamma_w^* \right) \right) - \frac{g}{1 + r_{tw}} \frac{dr_w}{dz} \]

Thus \( \Gamma_w^* \) is the correct lapse to assess the stability of a saturated atmosphere for which the liquid water gradient is insignificant (a reasonable approximation in many realistic situations). The new stability parameter is derived in section 2 and is compared with \( \Gamma_w \) in section 3. Section 4 focuses then on the effect of the liquid water gradient.

2. Derivation of the stability parameter \( \Gamma_w^* \)

The following notation has been used in this work: subscripts \( d, v, w, l, tw \) indicate that the concerned quantities refer, respectively, to dry air, vapor, saturated vapor, condensed phase (liquid or solid), total water (saturated vapor and condensed phase); \( \rho \) denotes the density; \( r \) the mixing ratio; \( T \) the temperature; \( g \) the gravity acceleration; \( z \) the height; \( p_d \) the dry air partial pressure; \( e_w \) the saturation pressure of vapor over liquid water (replaced by \( e_i \) if saturation is over ice); \( p = p_d + e_w \) the total pressure. Moreover \( \varepsilon = R_d / R_v \) is the ratio between the gas constants of dry air and water vapor; \( L_v \) is the specific latent heat of vaporization (replaced by the sublimation one if saturation is over ice); \( c_p \) is the specific heat at constant pressure; and \( \Gamma_d \) and \( \Gamma_w \) have the usual meaning of the dry and saturated adiabatic lapse rates, respectively.

In (1), with \( r_{tw} = \varepsilon e_w / p_d \), the gradient of the saturation mixing ratio of water vapor can be derived as...
\[
\frac{dr_w}{dz} = r_w \left( \frac{1}{e_w} \frac{de_w}{dz} - \frac{1}{p_d} \frac{dp_d}{dz} \right). 
\]

By multiplying now by \(1/p_d r_w = r_w/(\alpha e e_w)\) the hydrostatic equation \(d(p_d + e_w)/dz = -g p_d(1 + r_w)\) and by taking into account \(p_d = R_d p_d T\), the last term of (3) can be expressed as
\[
\frac{1}{p_d} \frac{dp_d}{dz} = - \frac{r_w}{e_w} \frac{de_w}{dz} - \frac{g}{R_d} (1 + r_w).
\]

By substituting it into (3) and using the Clapeyron equation \((1/e_w)\) \(de_w/dz = L R_o^T T^{-2} dT/dz\) we obtain
\[
\frac{dr_w}{dz} = r_w \left[ \frac{L_w}{T} (\epsilon + r_w) \frac{dT}{dz} + g(1 + r_w) \right]. 
\]

If (5) is inserted in (1), the expression for the square of the moist buoyancy frequency in a saturated atmosphere becomes
\[
N_w^2 = \frac{g}{T} \left[ \frac{1 + L_w r_w}{R_d T} \left( 1 - \frac{\epsilon + r_w}{1 + r_w} \right) \right] \frac{dT}{dz} 
+ \left[ 1 + \frac{L_w r_w}{R_d T} \right] \Gamma_w = \frac{g}{T} \frac{dr_w}{dz} + \frac{g}{1 + r_w} \frac{dr_w}{dz}. 
\]

Then (2) is obtained by defining
\[
\Gamma_w = \left( 1 + \frac{L_w r_w}{R_d T} \right) \left[ \frac{\epsilon + r_w}{R_d} \right] \frac{1}{1 + \frac{L_w r_w}{R_d T} \left( 1 - \frac{\epsilon + r_w}{1 + r_w} \right)}. \]

In this way (2) is equivalent to (1) and correctly reproduces the results of Durran and Klemp (1982) when \(r_i = 0\). In this form the atmospheric lapse rate appears only explicitly, making it easier to evaluate the moist buoyancy frequency.

### 3. Lapse rates intercomparison

The saturated adiabatic lapse rate is given (Lalas and Einaudi 1974; Durran and Klemp 1982) by
\[
\Gamma_w = \frac{\left( 1 + \frac{L_w r_w}{R_d T} \right) \left( 1 + r_w + r_i \right)}{1 + \frac{c_{pd} r_w + c_i r_i + L^2 \epsilon (\epsilon + r_w)}{c_{pd} R_d T^2}}, \quad (8)
\]

By neglecting the contribution of the liquid phase, and with \(c_{pd} \ll L^2 \epsilon/(R_d T^2)\), it can be simplified to the first order in \(r_w\) as
\[
\Gamma_w = \Gamma_{w1} = \Gamma_w \left( 1 + \frac{L_w r_w}{R_d T} \right), \quad (9)
\]

a formula that is often found (sometimes using \(r_w \approx \epsilon e_w/p\)) in meteorological books (Tverskoi 1965; Bluestein 1992).

By substituting (8) into (7) the new stability parameter can be written as
\[
\frac{\left( 1 + \frac{L_w r_w}{R_d T} \right) \left( 1 + r_w + r_i \right) - \frac{r_w}{c_{pd} r_w + c_i r_i + L^2 \epsilon (\epsilon + r_w)} \frac{L^2 r_w}{R_d T^2}}{1 + \frac{L_w r_w}{R_d T} \left( 1 + r_w + r_i \right) - \frac{L_w r_w \epsilon}{R_d T^2}} \leq \frac{L_w r_w}{R_d T}, \quad (10)
\]

This expression is of the fifth order in \(r_w\). It can be truncated at the first order giving a formula similar to (9) but with the term \((L_w r_w)/(R_d T)\) at the numerator multiplied by 2. This is, however, a bad approximation, because the error that is introduced is higher than using \(\Gamma_w\) instead of \(\Gamma_{w1}\).

A better, empirical, approximation is given by
\[
\Gamma_w = \Gamma_{w1} = \frac{1 + \frac{L_w r_w}{R_d T} \left( 1 + \frac{c_{pd} r_w + c_i r_i + L^2 \epsilon (\epsilon + r_w)}{c_{pd} R_d T^2} \right)}{1 + \frac{L_w r_w}{R_d T}}, \quad (11)
\]

where the values \(\alpha = 1.18\) and \(\beta = 0.93\) have been chosen with an optimization procedure in order to minimize the errors in the range of pressures and temperatures typical of the standard atmosphere.

From (10) it can be observed that \(\Gamma_{w1} > \Gamma_w\) when
\[
\frac{c_{pd} r_w + c_i r_i + L^2 \epsilon (\epsilon + r_w)}{c_{pd} R_d T^2} \leq \frac{L_w r_w}{R_d T}, \quad (12)
\]
that is to say when \( c_{pd} + c_{pe} r_e + c_r r_e < L_e (\varepsilon + r_e) / T \). This inequality is always satisfied, and so a stability criterion based on \( \Gamma^* \) always gives a greater stability than by using \( \Gamma_e \). This is the consequence of the stabilizing effect of the vapor profile, as it will be showed now by means of the parcel theory.

Following Durran and Klemp (1982), \( N_w^2 \) can be expressed as \( \frac{g d(\ln p) / d\zeta}{\rho} \), where \( \rho \) is the total density and the subscripts \( p \) and \( e \) denote values for the parcel and the environment. By writing \( \rho \) as \( \rho_0 (1 + r_w) \), and with

\[
\rho_0 = \frac{p}{R_0 T (1 + r_w / \varepsilon)},
\]

\( N_w^2 \) becomes

\[
N_w^2 = \left[ \frac{g d \rho_0 / d\zeta}{\rho_0} \right]_e + \left[ \frac{g dr_w / d\zeta}{1 + r_w / \varepsilon} \right]_e.
\]

By now taking into account that the parcel is subject to a saturated adiabatic process in pressure equilibrium with the environment, and by using (3) and the Clapeyron equation, the first term in (13) can be transformed as

\[
\left[ \frac{g d \rho_0 / d\zeta}{\rho_0} \right]_e = \frac{g}{T} \left( 1 + \frac{L_e r_e}{R_0 T} \right) \frac{dT}{d\zeta} \frac{\rho}{\rho_0}.
\]

that is to say the first term of (1). The total water content being conserved in the parcel, the second term in (13) takes the form of the second and third terms in (1).

The use of \( \Gamma_e \) as a stability parameter of a saturated atmosphere is therefore incorrect because it relies uniquely on the density variations that happen both in the parcel and in the environment and are proportional to \( (dT / dz)_e = -\Gamma_e \) and \( (dT / dz)_w = -\Gamma_w \) respectively. The terms that are neglected rely, on the contrary, on the density variations that occur only in the environment, being proportional to \( dr_w / dz \), which is null in the parcel if there is no precipitation.

In a saturated atmosphere the water vapor mixing ratio \( r_w \) usually decreases with the height, due to the dominant influence of temperature on the saturation vapor pressure. The water vapor profile has therefore a stabilizing effect, implying \( \Gamma^* > \Gamma_e \). The liquid water profile, on the contrary, can either increase or decrease the stability, depending on whether \( dr / dz \) is negative or positive.

The values of \( \Gamma^* \) and the errors that one makes by using \( \Gamma_e \) or the approximation \( \Gamma^*_w \) are listed in Tables 1 and 2 for various values of pressure and temperature, neglecting the contribution of the condensed water mixing ratio. The tables of Iribarne and Godson (1981, appendix) have been employed for the thermodynamic properties of air and vapor, and the standard value \( g = 9.80665 \) m s\(^{-2} \) has been used.

Numerical values of the differences \( \left( \Gamma^*_w - \Gamma^*_e \right) \) in Tables 1 and 2 confirm that a stability criterion based on \( \Gamma^*_w \) always gives a greater stability than the one obtained by using \( \Gamma_e \). At a given pressure the upper values of \( \left| \Gamma^*_w - \Gamma^*_e \right| \) are of about 0.6°C km\(^{-1} \) over water and 0.5°C km\(^{-1} \) over ice. Over water the temperature at which this upper value occurs decreases with pressure; over ice it occurs almost always at 0°C. Also \( \Gamma^*_w \) decreases with the pressure, and therefore the maximum value of the relative error that one makes by using \( \Gamma_e \) instead of \( \Gamma^*_w \) remains almost constant, around 17%.

From a practical point of view an error of 0.6°C km\(^{-1} \) is negligible if compared with the uncertainties of measurements, but from a theoretical point of view and for modeling purposes it is more correct to use \( \Gamma^*_w \).
whether increase or decrease the stability, depending on

4. Influence of the liquid water profile

As mentioned above, the liquid water profile can either increase or decrease the stability, depending on whether \( \text{dr}_l/\text{dz} \) is negative or positive.

In clouds at any given level the water content varies considerably over short distances in a manner that is closely related to the variation of the vertical air velocity; typically, on the macroscale, the cloud water content increases with height above the cloud base, has a maximum somewhere in the upper half of the cloud, and then decreases again toward the cloud top (Pruppacher and Klett 1980). On the local scale the effect of the \( \text{dr}_l/\text{dz} \) term can therefore be as large as the effect of the \( \text{dr}_w/\text{dz} \) term retained here by introducing \( \Gamma^*_w \), or even larger, and is quite unpredictable. On a wider scale, however, a rough estimate of their mutual importance can be given.

Figure 1 shows the vertical profiles of \( \text{dr}_l/\text{dz} \) and \( \text{dr}_w/\text{dz} \) calculated in a standard atmosphere assuming that the liquid content \( \rho_l \) increases linearly from 0 to 1 g

![Diagram](image-url)

**FIG. 1.** Vertical profiles of \( \text{dr}_l/\text{dz} \) (dotted line), \( \text{dr}_w/\text{dz} \) (dashed line), and \( \text{dr}_w/\text{dz} = \text{dr}_l/\text{dz} + \text{dr}_w/\text{dz} \) (continuous line) in an idealized situation [standard atmosphere with a cumulus, assuming that the liquid content increases linearly from 0 to 1 g m\(^{-3}\) from 2.5 to 5.5 km (zone A), keeps constant up to 7.5 km (zone B), and then decreases linearly to zero between 7.5 and 9 km (zone C)]. The line \( \text{dr}_l/\text{dz} = 0 \) has been plotted as a reference.
m$^{-3}$ from 2.5 to 5.5 km (zone A), keeps constant up to 7.5 km (zone B), and then decreases linearly to zero between 7.5 and 9 km (zone C); this schematic profile has been derived from an example concerning the cumulus clouds (Pruppacher and Klett 1980, their Fig. 2-13b). In Fig. 1 the $dr/dz$ curve is almost always closer to the $dwl/dz$ one than $drl/dz = 0$; therefore, in this example, the estimate of the atmospheric stability by using $\Gamma^*_w$ (i.e., implicitly assuming $dr_w/dz = dr/dz$) would be almost always more accurate than by using $\Gamma_w$ (i.e., assuming $dr_w/dz = 0$). Even if $dp/dz$ were doubled and the cloud extended to the ground this result would not change dramatically. Only at higher altitudes would $\Gamma_w$ give a better prediction, but if one considers that when the liquid water gradient is negligible $\Gamma^*_w$ gives the correct prediction and that the differences between $\Gamma^*_w$ and $\Gamma_w$ decrease with the height, the use of $\Gamma^*_w$ instead of $\Gamma_w$ can in general be suggested.

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