Biextensions of Picard stacks and their homological interpretation

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BIEXTENSIONS OF PICARD STACKS AND THEIR HOMOLOGICAL INTERPRETATION

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Abstract. Let $\mathbf{S}$ be a site. We introduce the 2-category of biextensions of strictly commutative Picard $\mathbf{S}$-stacks. We define the pull-back, the push-down, and the sum of such biextensions and we compute their homological interpretation: if $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ are strictly commutative Picard $\mathbf{S}$-stacks, the equivalence classes of biextensions of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$ are parametrized by the cohomology group $\mathrm{Ext}^1([\mathcal{P}]{\otimes}^{\mathbb{L}}[\mathcal{Q}], [\mathcal{G}])$, the isomorphism classes of arrows from such a biextension to itself are parametrized by the cohomology group $\mathrm{Ext}^0([\mathcal{P}]{\otimes}^{\mathbb{L}}[\mathcal{Q}], [\mathcal{G}])$ and the automorphisms of an arrow from such a biextension to itself are parametrized by the cohomology group $\mathrm{Ext}^{-1}([\mathcal{P}]{\otimes}^{\mathbb{L}}[\mathcal{Q}], [\mathcal{G}])$, where $[\mathcal{P}], [\mathcal{Q}]$ and $[\mathcal{G}]$ are the complex associated to $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ respectively.

Contents

Introduction

Let $\mathbf{S}$ be a site. Let P, Q and G be three abelian sheaves on $\mathbf{S}$. In [G] Exposé VII Corollary 3.6.5 Grothendieck proves that the group $\mathrm{Biext}^0(P, Q; G)$ of automorphisms of any biextension of (P, Q) by G and the group $\mathrm{Biext}^1(P, Q; G)$ of isomorphism classes of biextensions of (P, Q) by G, have the following homological

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interpretation:

$$\mathrm{Biext}^i(P,Q;G) \cong \mathrm{Ext}^i(P \overset{\mathbb{L}}{\otimes} Q, G) \qquad (i=0,1) \tag{0.1}$$

where $P \overset{\mathbb{L}}{\otimes} Q$ is the derived functor of the functor $Q \to P \otimes Q$ in the derived category $\mathcal{D}(\mathbf{S})$ of complexes of abelian sheaves on $\mathbf{S}$. The aim of this paper is to find an analogous homological interpretation for biextensions of strictly commutative Picard $\mathbf{S}$-stacks.

Let $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ be three strictly commutative Picard $\mathbf{S}$-stacks. We define a biextension of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$ as a $\mathcal{G}_{\mathcal{P} \times_{\mathbf{1}} \mathcal{Q}}$-torsor $\mathcal{B}$ over $\mathcal{P} \times_{\mathbf{1}} \mathcal{Q}$, endowed with a structure of extension of $\mathcal{Q}_{\mathcal{P}}$ by $\mathcal{G}_{\mathcal{P}}$ and a structure of extension of $\mathcal{P}_{\mathcal{Q}}$ by $\mathcal{G}_{\mathcal{Q}}$, which are compatible one with another. Biextensions of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$ form a 2-category $\mathcal{B}iext(\mathcal{P}, \mathcal{Q}; \mathcal{G})$ where

- the objects are biextensions of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$,
- the 1-arrows are additive functors between biextensions,
- the 2-arrows are morphisms of additive functors.

Equivalence classes of biextensions of strictly commutative Picard $\mathbf{S}$-stacks are endowed with a group law. We denote by $\mathcal{B}iext^1(\mathcal{P}, \mathcal{Q}; \mathcal{G})$ the group of equivalence classes of objects of $\mathcal{B}iext(\mathcal{P}, \mathcal{Q}; \mathcal{G})$, by $\mathcal{B}iext^0(\mathcal{P}, \mathcal{Q}; \mathcal{G})$ the group of isomorphism classes of arrows from an object of $\mathcal{B}iext(\mathcal{P}, \mathcal{Q}; \mathcal{G})$ to itself, and by $\mathcal{B}iext^{-1}(\mathcal{P}, \mathcal{Q}; \mathcal{G})$ the group of automorphisms of an arrow from an object of $\mathcal{B}iext(\mathcal{P}, \mathcal{Q}; \mathcal{G})$ to itself. With these notation our main Theorem is

Theorem 0.1. *Let $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ be strictly commutative Picard $\boldsymbol{S}$-stacks. Then we have the following isomorphisms of groups*

(a) $\mathcal{B}iext^1(\mathcal{P}, \mathcal{Q}; \mathcal{G}) \cong \mathrm{Ext}^1\big([\mathcal{P}] \overset{\mathbb{L}}{\otimes} [\mathcal{Q}], [\mathcal{G}]\big) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}\big([\mathcal{P}] \overset{\mathbb{L}}{\otimes} [\mathcal{Q}], [\mathcal{G}][1]\big)$,

(b) $\mathcal{B}iext^0(\mathcal{P}, \mathcal{Q}; \mathcal{G}) \cong \mathrm{Ext}^0\big([\mathcal{P}] \overset{\mathbb{L}}{\otimes} [\mathcal{Q}], [\mathcal{G}]\big) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}\big([\mathcal{P}] \overset{\mathbb{L}}{\otimes} [\mathcal{Q}], [\mathcal{G}]\big)$,

(c) $\mathcal{B}iext^{-1}(\mathcal{P}, \mathcal{Q}; \mathcal{G}) \cong \mathrm{Ext}^{-1}\big([\mathcal{P}] \overset{\mathbb{L}}{\otimes} [\mathcal{Q}], [\mathcal{G}]\big) = \mathrm{Hom}_{\mathcal{D}(\boldsymbol{S})}\big([\mathcal{P}] \overset{\mathbb{L}}{\otimes} [\mathcal{Q}], [\mathcal{G}][-1]\big)$,

where $[\mathcal{P}], [\mathcal{Q}]$ and $[\mathcal{G}]$ denote the complex of $\mathcal{D}^{[-1,0]}(\boldsymbol{S})$ corresponding to $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ respectively.

By [D73] §1.4 there is an equivalence of categories between the category of strictly commutative Picard $\mathbf{S}$-stacks and the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ of complexes K of abelian sheaves on $\mathbf{S}$ such that $\mathrm{H}^i(K) = 0$ for $i \neq -1$ or 0. Via this equivalence, the above notion of biextension of strictly commutative Picard $\mathbf{S}$-stacks furnishes a notion of biextension for complexes of abelian sheaves over $\mathbf{S}$ concentrated in degrees -1 and 0 and the above theorem generalizes Grothendieck's result (0.1) to complexes of abelian sheaves concentrated in degrees -1 and 0.

The definitions and results of this paper generalizes those of [Be09]: in fact, in loc.cit. we have defined the notion of biextensions of 1-motives and we have checked Theorem 0.1 for 1-motives (recall that a 1-motive can be seen as a complex of abelian sheaves $[u : A \to B] \in \mathcal{D}^{[-1,0]}(\mathbf{S})$).

The main Theorem 0.1 furnishes also the homological interpretation of extensions of strictly commutative Picard $\mathbf{S}$-stacks which was computed in [Be11]: in fact, it $\mathbf{1}$ is the strictly commutative Picard $\mathbf{S}$-stack such that for any object U of $\mathbf{S}$, $\mathbf{1}(U)$ is the category with one object and one arrow, then

- the 2-category $\mathcal{B}iext(\mathcal{P}, \mathbf{1}; \mathcal{G})$ of biextensions of $(\mathcal{P}, \mathbf{1})$ by $\mathcal{G}$ is equivalent to the 2-category $\mathcal{E}xt(\mathcal{P}, \mathcal{G})$ of extensions of $\mathcal{P}$ by $\mathcal{G}$, and

- in the derived category $\mathrm{Ext}^i([\mathcal{P}]\overset{\mathbb{L}}{\otimes}[\mathbf{1}],[\mathcal{G}]) \cong \mathrm{Ext}^i([\mathcal{P}],[\mathcal{G}])$ for $i=-1,0,1$.

In [G] Exposé VII Grothendieck states the following geometrical-homological principle: *if an abelian sheaf A on $\boldsymbol{S}$ admits an explicit representation in $\mathcal{D}(\boldsymbol{S})$ by a complex* L. *whose components are direct sums of objects of the kind $\mathbb{Z}[I]$, with I a sheaf on $\boldsymbol{S}$, then the groups* $\mathrm{Ext}^i(A,B)$ *admit an explicit geometrical description for any abelian sheaf B on $\boldsymbol{S}$.*
A first example of this principle is furnished by the geometrical notion of extension of abelian sheaves on $\mathbf{S}$: in fact, if P and G are two abelian sheaves on $\mathbf{S}$, it is a classical result that the group $\mathrm{Ext}^0(P,G)$ is isomorphic to the group of automorphisms of any extension of P by G and the group $\mathrm{Ext}^1(P,G)$ is isomorphic to the group of isomorphism classes of extensions of P by G. The canonical isomorphisms (0.1) are another of this Grothendeick's principle which involves the geometrical notion of biextension of abelian sheaves. Other examples of this Grothendeick's principle are described in [B83]: If P and G are abelian sheaves on $\mathbf{S}$, according to loc.cit. Proposition 8.4, the strictly commutative Picard $\mathbf{S}$-stack of symmetric biextensions of (P,P) by G is equivalent to the strictly commutative Picard $\mathbf{S}$-stack associated to the object $\tau_{\leq 0}\mathbb{R}\mathrm{Hom}(\mathbb{L}\mathrm{Sym}^2(P),G[1])$ of $\mathcal{D}(\mathbf{S})$, and according to loc.cit. Theorem 8.9, the strictly commutative Picard $\mathbf{S}$-stack of the 3-tuple (L,E,α) (resp. the 4-tuple (L,E,α,β)) defining a cubic structure (resp. a Σ-structure) on the G-torsor L is equivalent to the strictly commutative Picard $\mathbf{S}$-stack associated to the object $\tau_{\leq 0}\mathbb{R}\mathrm{Hom}(\mathbb{L}P_2^+(P),G[1])$ (resp. $\tau_{\leq 0}\mathbb{R}\mathrm{Hom}(\mathbb{L}\Gamma_2(P),G[1])$) of $\mathcal{D}(\mathbf{S})$. Our Theorem 0.1 is the first example in the literature where the geometrical-homological principle of Grothendieck is applied to complexes of abelian sheaves.

A strictly commutative Picard $\mathbf{S}$-2-stack is the 2-analog of a strictly commutative Picard $\mathbf{S}$-stack, i.e. it is an $\mathbf{S}$-2-stack in 2-groupoids $\mathbb{P}$ endowed with a morphism of $\mathbf{S}$-2-stacks $+:\mathbb{P}\times_{\mathbf{S}}\mathbb{P}\to\mathbb{P}$ and with associative and commutative constraints (see [T09] Definition 2.3 for more details). As for strictly commutative Picard $\mathbf{S}$-stacks and complexes of abelian sheaves concentrated in degrees -1 and 0, in [T09] Tatar proves that there is a dictionary between strictly commutative Picard $\mathbf{S}$-2-stacks and complexes of abelian sheaves concentrated in degrees -2, -1 and 0. Using this dictionary, we can rewrite Theorem 0.1 as followed: the strictly commutative Picard $\mathbf{S}$-2-stack of biextensions of $(\mathcal{P},\mathcal{G})$ by $\mathcal{Q}$ is equivalent to the strictly commutative Picard $\mathbf{S}$-2-stack associated to the object

$$\tau_{\leq 0}\mathrm{RHom}\big([\mathcal{P}]\overset{\mathbb{L}}{\otimes}[\mathcal{Q}],[\mathcal{G}][1]\big)$$

of $\mathcal{D}^{[-2,0]}(\mathbf{S})$. If $\mathbf{1}$ denotes the strictly commutative Picard $\mathbf{S}$-stack such that for any object U of $\mathbf{S}$, $\mathbf{1}(U)$ is the category with one object and one arrow, biextensions of $(\mathcal{P},\mathbf{1})$ by $\mathcal{G}$ are just extensions of $\mathcal{P}$ by $\mathcal{G}$. According to this remark, Theorem 0.1 furnishes another proof of the main result of [Be11] which states that the strictly commutative Picard $\mathbf{S}$-2-stack of extensions of $\mathcal{P}$ by $\mathcal{G}$ is equivalent to the strictly commutative Picard $\mathbf{S}$-2-stack associated to the object

$$\tau_{\leq 0}\mathrm{RHom}\big([\mathcal{P}],[\mathcal{G}][1]\big)$$

of $\mathcal{D}^{[-2,0]}(\mathbf{S})$.

This paper is organized as followed: in Section 1 we recall some basic results on the 2-category of strictly commutative Picard $\mathbf{S}$-stacks. Let $\mathcal{G}$ be a gr-$\mathbf{S}$-stack. In Section 2 we define the notions of $\mathcal{G}$-torsor, morphism of $\mathcal{G}$-torsors and morphism of morphisms of $\mathcal{G}$-torsors, getting the 2-category of $\mathcal{G}$-torsors. In Section 3 we recall some basic results on the 2-category of extensions of strictly commutative Picard $\mathbf{S}$-stacks. Let $\mathcal{P}$ and $\mathcal{G}$ be two strictly commutative Picard $\mathbf{S}$-stacks. In Section 4 we prove that it exists an equivalence of 2-categories between the 2-category of extensions of $\mathcal{P}$ by $\mathcal{G}$ and the 2-category consisting of the data $(\mathcal{E}, I, M, \alpha, \chi)$, where $\mathcal{E}$ is a $\mathcal{G}$-torsor over $\mathcal{P}$, I is a trivialization of its pull-back via the additive functor $\mathbf{1} : \mathbf{e} \to \mathcal{P}$, $M : p_1^* \; \mathcal{E} \wedge p_2^* \; \mathcal{E} \longrightarrow +^* \; \mathcal{E}$ is a morphism of $\mathcal{G}$-torsors (where $+$ is the group law of $\mathcal{P}$ and $p_i : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ are the projections), and α and χ are two isomorphisms of morphisms of $\mathcal{G}$-torsors involving the morphism of $\mathcal{G}$-torsors M (Theorem 4.1). This generalizes to strictly commutative Picard $\mathbf{S}$-stacks the following result of Grothendieck ([G] Exposé VII 1.1.6 and 1.2): if P and Q are two abelian sheaves, to have an extension of P by G is the same thing as to have the 4-tuple (P, G, E, φ), where E is a G_P-torsor over P, and $\varphi : pr_1^*E \; pr_2^*E \to +^*E$ is an isomorphism of torsors over $P \times P$ satisfying some associativity and commutativity conditions. Let $\mathcal{P}, \mathcal{Q}, \mathcal{G}$ be three strictly commutative Picard $\mathbf{S}$-stacks. In Section 5 we define the notions of biextension of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$, morphism of such biextensions and morphism of morphisms of such biextensions, getting the 2-category of biextensions of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$. In Section 6 we introduce the notions of pull-back and push-down of biextensions of strictly commutative Picard $\mathbf{S}$-stacks. This will allow us to define a group law for equivalence classes of biextensions of strictly commutative Picard $\mathbf{S}$-stacks. In Section 7 we prove the cases (b) and (c) of Theorem 0.1. In order to prove the case (a) we need to introduce an intermediate 2-category $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ that we construct using a strictly commutative Picard $\mathbf{S}$-stack $\mathcal{G}$ and a complex $\mathcal{L}^{\cdot}$ of strictly commutative Picard $\mathbf{S}$-stacks (Section 8). This 2-category $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ has the following *homological description*:

$$\Psi^i_{\mathcal{L}^{\cdot}}(\mathcal{G}) \cong \mathrm{Ext}^i(\mathrm{Tot}([\mathcal{L}^{\cdot}]), [\mathcal{G}]) \qquad (i = 0, 1, 2) \tag{0.2}$$

where $\Psi^1_{\mathcal{L}^{\cdot}}(\mathcal{G})$ is the group of equivalence classes of objects of $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$, $\Psi^0_{\mathcal{L}^{\cdot}}(\mathcal{G})$ is the group of isomorphism classes of arrows from an object of $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ to itself, and $\Psi^{-1}_{\mathcal{L}^{\cdot}}(\mathcal{G})$ is the group of automorphisms of an arrow from an object of $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ to itself. In section 9, to any complex of the kind $[\mathcal{P}] = [d^P : P^{-1} \to P^0]$ we associate a canonical flat partial resolution $[\mathcal{L}^{\cdot}(\mathcal{P})]$ whose components are direct sums of objects of the kind $\mathbb{Z}[I]$ with I an abelian sheaf on $\mathbf{S}$. Here "partial resolution" means that we have an isomorphism between the cohomology groups of $[\mathcal{P}]$ and of this partial resolution only in degree 1, 0 and -1. This is enough for our goal since only the groups $\mathrm{Ext}^1, \mathrm{Ext}^0$ and Ext^{-1} are involved in the statement of Theorem 0.1. The category $\Psi_{\mathcal{L}^{\cdot}(\mathcal{P}) \otimes \mathcal{L}^{\cdot}(\mathcal{G})}(\mathcal{G})$ admit the following *geometrical description*:

$$\Psi_{\mathcal{L}^{\cdot}(\mathcal{P}) \otimes \mathcal{L}^{\cdot}(\mathcal{G})}(\mathcal{G}) \cong \mathcal{B}iext([\mathcal{P}], [\mathcal{Q}]; [\mathcal{G}]) \tag{0.3}$$

Putting together this geometrical description (0.3) with the homological description (0.2), in Section 10 we finally prove Theorem 0.1.

Notation

Let $\mathbf{S}$ be a site. Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site $\mathbf{S}$: all complexes that we consider in this paper are cochain complexes (excepted in Section 9 and 10 where we switch to homological notation). Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K = (K^i)_i$ such that $K^i = 0$ for $i \neq -1$ or 0. The good truncation $\tau_{\leq n} K$ of a complex K of $\mathcal{K}(\mathbf{S})$ is the following complex: $(\tau_{\leq n} K)^i = K^i$ for $i < n, (\tau_{\leq n} K)^n = \ker(d^n)$ and $(\tau_{\leq n} K)^i = 0$ for $i > n$. For any $i \in \mathbb{Z}$, the shift functor $[i] : \mathcal{K}(\mathbf{S}) \to \mathcal{K}(\mathbf{S})$ acts on a complex

$K=(K^n)_n$ as $(K[i])^n = K^{i+n}$ and $d^n_{K[i]} = (-1)^i d^{n+i}_K$. If $\mathrm{L}^{\cdot\cdot}$ is a bicomplex of abelian sheaves on the site $\mathbf{S}$, we denote by $\mathrm{Tot}(\mathrm{L}^{\cdot\cdot})$ the total complex of $\mathrm{L}^{\cdot\cdot}$: it is the cochain complex whose component of degree n is $\mathrm{Tot}(\mathrm{L}^{\cdot\cdot})^n = \sum_{i+j=n} \mathrm{L}^{ij}$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes K such that $\mathrm{H}^i(K)=0$ for $i \neq -1$ or 0. If K and K' are complexes of $\mathcal{D}(\mathbf{S})$, the group $\mathrm{Ext}^i(K,K')$ is by definition $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K,K'[i])$ for any $i \in \mathbb{Z}$. Let $\mathrm{RHom}(-,-)$ be the derived functor of the bifunctor $\mathrm{Hom}(-,-)$. The cohomology groups $\mathrm{H}^i\big(\mathrm{RHom}(K,K')\big)$ of $\mathrm{RHom}(K,K')$ are isomorphic to $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K,K'[i])$.

A **2-category** $\mathcal{A} = (A, C(a,b), K_{a,b,c}, U_a)_{a,b,c \in A}$ is given by the following data:

- a set A of objects $a,b,c,...$;
- for each ordered pair (a,b) of objects of A, a category $C(a,b)$;
- for each ordered triple (a,b,c) of objects A, a functor $K_{a,b,c} : C(b,c) \times C(a,b) \longrightarrow C(a,c)$, called composition functor. This composition functor have to satisfy the associativity law;
- for each object a, a functor $U_a : \mathbf{1} \to C(a,a)$ where $\mathbf{1}$ is the terminal category (i.e. the category with one object, one arrow), called unit functor. This unit functor have to provide a left and right identity for the composition functor.

This set of axioms for a 2-category is exactly like the set of axioms for a category in which the arrows-sets $\mathrm{Hom}(a,b)$ have been replaced by the categories $C(a,b)$. We call the categories $C(a,b)$ (with $a,b \in A$) the **categories of morphisms** of the 2-category $\mathcal{A}$: the objects of $C(a,b)$ are the **1-arrows** of $\mathcal{A}$ and the arrows of $C(a,b)$ are the **2-arrows** of $\mathcal{A}$.

Let $\mathcal{A} = (A, C(a,b), K_{a,b,c}, U_a)_{a,b,c \in A}$ and $\mathcal{A}' = (A', C(a',b'), K_{a',b',c'}, U_{a'})_{a',b',c' \in A'}$ be two 2-categories. A **2-functor** (called also a **morphism of 2-categories**)

$$(F, F_{a,b})_{a,b \in A} : \mathcal{A} \longrightarrow \mathcal{A}'$$

consists of

- an application $F : A \to A'$ between the objects of $\mathcal{A}$ and the objects of $\mathcal{A}'$,
- a family of functors $F_{a,b} : C(a,b) \to C(F(a),F(b))$ (with $a,b \in A$) which are compatible with the composition functors and with the unit functors of $\mathcal{A}$ and $\mathcal{A}'$.

1. The 2-category of Picard stacks

Let $\mathbf{S}$ be a site. For the notions of $\mathbf{S}$-pre-stack, $\mathbf{S}$-stack, morphism of $\mathbf{S}$-stacks and morphism of morphisms of $\mathbf{S}$-stacks we refer to [G71] Chapter II 1.2.

A **strictly commutative Picard S-stack** consists of an $\mathbf{S}$-stack of groupoids $\mathcal{P}$, a morphism of $\mathbf{S}$-stacks $+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ (called the group law of $\mathcal{P}$) with two natural isomorphisms of associativity σ and of commutativity τ, which are described by the functorial isomorphisms

$$\sigma_{a,b,c} \ : \ (a+b)+c \xrightarrow{\cong} a+(b+c) \qquad \forall\ a,b,c \in \mathcal{P}, \tag{1.1}$$

$$\tau_{a,b} \ : \ a+b \xrightarrow{\cong} b+a \qquad \forall\ a,b \in \mathcal{P}, \tag{1.2}$$

a global neutral object e with two natural isomorphisms

$$l_a : e+a \xrightarrow{\cong} a, \qquad r_a : a+e \xrightarrow{\cong} a \qquad \forall\ a \in \mathcal{P} \tag{1.3}$$

which coincide on e ($l_e = r_e$), and finally a morphism of **S**-stacks $- : \mathcal{P} \to \mathcal{P}$ with two natural isomorphisms

$$o_a : a + (-a) \stackrel{\cong}{\longrightarrow} e, \qquad c_{ab} : -(a+b) \stackrel{\cong}{\longrightarrow} (-a) + (-b) \qquad \forall\, a \in \mathcal{P}. \tag{1.4}$$

These data have to satisfy the following conditions:

- the natural isomorphism σ is coherent, i.e. for any a, b, c and $d \in \mathcal{P}$ the following pentagonal diagram commute

$$\begin{array}{ccccc} a + (b + (c + d)) & \xleftarrow{\sigma} & (a + b) + (c + d) & \xleftarrow{\sigma} & ((a + b) + c) + d \\ \uparrow{\scriptstyle id_{\mathcal{P}}+\sigma} & & & & \downarrow{\scriptstyle \sigma+id_{\mathcal{P}}} \\ a + ((b + c) + d) & & \xleftarrow{\qquad\qquad\sigma\qquad\qquad} & & (a + (b + c)) + d \end{array} \tag{1.5}$$

- for any $a \in \mathcal{P}$,

$$\tau_{a,a} : a + a \longrightarrow a + a \tag{1.6}$$

  is the identity; This condition, which is seldom verified, justifies the terminology *strictly* commutative.
- the natural isomorphism τ is coherent, i.e. for any a and $b \in \mathcal{P}$ the following diagram commute

$$\begin{array}{ccc} a + b & \xrightarrow{\tau} & b + a \\ & {\scriptstyle id_{\mathcal{P}}}\searrow & \downarrow{\scriptstyle \tau} \\ & & a + b \end{array} \tag{1.7}$$

- the natural isomorphisms σ and τ are compatible, i.e. for any a, b and $c \in \mathcal{P}$ the following hexagonal diagram commute

$$\begin{array}{ccccc} & & b + (c + a) & & \\ & {\scriptstyle \sigma}\nearrow & & \nwarrow{\scriptstyle id_{\mathcal{P}}+\tau} & \\ (b + c) + a & & & & b + (a + c) \\ \uparrow{\scriptstyle \tau} & & & & \uparrow{\scriptstyle \sigma} \\ a + (b + c) & & & & (b + a) + c \\ & \nwarrow{\scriptstyle \sigma} & & \nearrow{\scriptstyle \tau+id_{\mathcal{P}}} & \\ & & (a + b) + c & & \end{array} \tag{1.8}$$

- the natural isomorphism σ and the neutral object are compatible, i.e. for any a and $b \in \mathcal{P}$ the following diagram commute

$$\begin{array}{ccc} (a + e) + b & \xrightarrow{\sigma} & a + (e + b) \\ & {\scriptstyle r+id_{\mathcal{P}}}\searrow & \downarrow{\scriptstyle id_{\mathcal{P}}+l} \\ & & a + b. \end{array} \tag{1.9}$$

In particular, for any object U of **S**, $(\mathcal{P}(U), +, e, -)$ is a strictly commutative Picard category (see Definition 1.4.2 [D73]). The sheaf of automorphisms of the neutral object $\underline{\mathrm{Aut}}(e)$ is abelian. If $\mathcal{P}$ and $\mathcal{Q}$ are two strictly commutative Picard **S**-stacks, an

additive functor $(F,\sum):\mathcal{P}\to\mathcal{Q}$ is a morphism of **S**-stacks $F:\mathcal{P}\to\mathcal{Q}$ endowed with a natural isomorphism $\sum$ which is described by the functorial isomorphisms

$$\sum_{a,b}: F(a+b)\stackrel{\cong}{\longrightarrow} F(a)+F(b) \qquad \forall\, a,b\in\mathcal{P}$$

and which is compatible with the natural isomorphisms σ and τ of $\mathcal{P}$ and $\mathcal{Q}$. A **morphism of additive functors** $\alpha:(F,\sum)\Rightarrow(F',\sum')$ is a morphism of morphisms of **S**-stacks $\alpha:F\Rightarrow F'$ which is compatible with the natural isomorphisms $\sum$ and $\sum'$ of F and F' respectively. We denote by $\mathbf{Add_S}(\mathcal{P},\mathcal{Q})$ the category whose objects are additive functors from $\mathcal{P}$ to $\mathcal{Q}$ and whose arrows are morphisms of additive functors. The category $\mathbf{Add_S}(\mathcal{P},\mathcal{Q})$ is a groupoid, i.e. any morphism of additive functors is an isomorphism of additive functors.

An **equivalence of strictly commutative Picard S-stacks** between $\mathcal{P}$ and $\mathcal{Q}$ is an additive functor $(F,\sum):\mathcal{P}\to\mathcal{Q}$ with F an equivalence of **S**-stacks. Two strictly commutative Picard **S**-stacks are *equivalent as strictly commutative Picard **S**-stacks* if there exists an equivalence of strictly commutative Picard **S**-stacks between them.

To any strictly commutative Picard **S**-stack $\mathcal{P}$, we associate the sheaffification $\pi_0(\mathcal{P})$ of the pre-sheaf which associates to each object U of **S** the group of isomorphism classes of objects of $\mathcal{P}(U)$, the sheaf $\pi_1(\mathcal{P})$ of automorphisms $\underline{\mathrm{Aut}}(e)$ of the neutral object of $\mathcal{P}$, and an element $\varepsilon(\mathcal{P})$ of $\mathrm{Ext}^2(\pi_0(\mathcal{P}),\pi_1(\mathcal{P}))$. Two strictly commutative Picard **S**-stacks $\mathcal{P}$ and $\mathcal{P}'$ are equivalent as strictly commutative Picard **S**-stacks if and only if $\pi_i(\mathcal{P})$ is isomorphic to $\pi_i(\mathcal{P}')$ for $i=0,1$ and $\varepsilon(\mathcal{P})=\varepsilon(\mathcal{P}')$.

A **strictly commutative Picard S-pre-stack** consists of an **S**-pre-stack of groupoids $\mathcal{P}$, a morphism of **S**-stacks $+:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ with a natural isomorphism of associativity σ (1.1), a global neutral object e with two natural isomorphisms r and l (1.3), and a morphism of **S**-stacks $-:\mathcal{P}\to\mathcal{P}$ with two natural isomorphisms o and c (1.4), such that for any object U of **S**, $(\mathcal{P}(U),+,e,-)$ is a strictly commutative Picard category. If $\mathcal{P}$ is a strictly commutative Picard **S**-pre-stack, there exists modulo a unique equivalence one and only one pair $(a\mathcal{P},j)$ where $a\mathcal{P}$ is a strictly commutative Picard **S**-stack and $j:\mathcal{P}\to a\mathcal{P}$ is an additive functor. $(a\mathcal{P},j)$ is *the strictly commutative Picard **S**-stack generated by* $\mathcal{P}$.

In [D73] §1.4 Deligne associates to each complex K of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ a strictly commutative Picard **S**-stack $st(K)$ and to each morphism of complexes $g:K\to L$ an additive functor $st(g):st(K)\to st(L)$ between the strictly commutative Picard **S**-stacks associated to the complexes K and L. Moreover he proves the following links between strictly commutative Picard **S**-stacks and complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, between additive functors and morphisms of complexes and between morphisms of additive functors and homotopies of complexes:

- for any strictly commutative Picard **S**-stack $\mathcal{P}$ there exists a complex K of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ such that $\mathcal{P}=st(K)$;
- if K,L are two complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then for any additive functor $F:st(K)\to st(L)$ there exists a quasi-isomorphism $k:K'\to K$ and a morphism of complexes $l:K'\to L$ such that F is isomorphic as additive functor to $st(l)\circ st(k)^{-1}$;
- if $f,g:K\to L$ are two morphisms of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then

(1.10)
$$\mathrm{Hom}_{\mathbf{Add_S}(st(K),st(L))}(st(f),st(g))\cong\Big\{\text{homotopies } H:K\to L \mid g-f=dH+Hd\Big\}.$$

Denote by Picard($\mathbf{S}$) the category whose objects are small strictly commutative Picard $\mathbf{S}$-stacks and whose arrows are isomorphism classes of additive functors. The above links between strictly commutative Picard $\mathbf{S}$-stacks and complexes of abelian sheaves on $\mathbf{S}$ furnish the equivalence of category:

$$\begin{array}{rcl} st : \mathcal{D}^{[-1,0]}(\mathbf{S}) & \longrightarrow & \mathrm{Picard}(\mathbf{S}) \\ K & \mapsto & st(K) \\ K \stackrel{f}{\to} L & \mapsto & st(K) \stackrel{st(f)}{\to} st(L). \end{array} \tag{1.11}$$

We denote by [ ] the inverse equivalence of st. Let $\mathcal{P}icard(\mathbf{S})$ be the 2-category of strictly commutative Picard $\mathbf{S}$-stacks whose objects are strictly commutative Picard $\mathbf{S}$-stacks and whose categories of morphisms are the categories $\mathbf{Add_S}(\mathcal{P},\mathcal{Q})$ (i.e. the 1-arrows are additive functors between strictly commutative Picard $\mathbf{S}$-stacks and the 2-arrows are morphisms of additive functors). Via the functor st, there exists a 2-functor between

(a) the 2-category whose objects and 1-arrows are the objects and the arrows of the category $\mathcal{K}^{[-1,0]}(\mathbf{S})$ and whose 2-arrows are the homotopies between 1-arrows (i.e. H such that $g-f = dH + Hd$ with $f,g : K \to L$ 1-arrows),

(b) the 2-category $\mathcal{P}icard(\mathbf{S})$.

Example 1.1. Let $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ be three strictly commutative Picard $\mathbf{S}$-stacks.
I) Let

$$\mathrm{HOM}(\mathcal{P},\mathcal{Q})$$

be the strictly commutative Picard $\mathbf{S}$-stack defined as followed: for any object U of $\mathbf{S}$, the objects of the category $\mathrm{HOM}(\mathcal{P},\mathcal{Q})(U)$ are additive functors from $\mathcal{P}_{|U}$ to $\mathcal{Q}_{|U}$ and its arrows are morphisms of additive functors. By (1.10) and (1.11), we have the equality $[\mathrm{HOM}(\mathcal{P},\mathcal{Q})] = \tau_{\leq 0}\mathrm{RHom}\big([\mathcal{P}],[\mathcal{Q}]\big)$ in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$.
II) A **biadditive functor** $(F,l,r) : \mathcal{P}\times\mathcal{Q} \to \mathcal{G}$ is a morphism of $\mathbf{S}$-stacks $F : \mathcal{P}\times\mathcal{Q} \to \mathcal{G}$ endowed with two natural isomorphisms, which are described by the functorial isomorphisms

$$\begin{array}{rcll} l_{a,b,c} : F(a+b,c) & \stackrel{\cong}{\longrightarrow} & F(a,c)+F(b,c) & \forall\ a,b \in \mathcal{P},\ \forall\ c \in \mathcal{Q} \\ r_{a,c,d} : F(a,c+d) & \stackrel{\cong}{\longrightarrow} & F(a,c)+F(a,d) & \forall\ a \in \mathcal{P},\ \forall\ c,d \in \mathcal{Q}, \end{array}$$

such that

- for any fixed $a \in \mathcal{P}$, $F(a,-)$ is compatible with the natural isomorphisms σ and τ of $\mathcal{P}$ and $\mathcal{G}$,
- for any fixed $c \in \mathcal{Q}$, $F(-,c)$ is compatible with the natural isomorphisms σ and τ of $\mathcal{Q}$ and $\mathcal{G}$,
- for any fixed $a,b \in \mathcal{P}$ and $c,d \in \mathcal{Q}$ is the following diagram commute

$$\begin{array}{ccccc} F(a+b,c+d) & \xrightarrow{r} & F(a+b,c)+F(a+b,d) & \xrightarrow{l+l} & F(a,c)+F(b,c)+F(a,d)+F(b,d) \\ \downarrow{\scriptstyle l} & & & & \uparrow{\scriptstyle id_{\mathcal{G}}+\tau+id_{\mathcal{G}}} \\ F(a,c+d)+F(b,c+d) & & \xrightarrow{r+r} & & F(a,c)+F(a,d)+F(b,c)+F(b,d). \end{array}$$

A **morphism of biadditive functors** $\alpha : (F,l,r) \Rightarrow (F',l',r')$ is a morphism of morphisms of $\mathbf{S}$-stacks $\alpha : F \Rightarrow F'$ which is compatible with the natural isomorphisms l,r and l',r of F and F' respectively. Let

$$\mathrm{HOM}(\mathcal{P},\mathcal{Q};\mathcal{G})$$

be the strictly commutative Picard $\mathbf{S}$-stack defined as followed: for any object U of $\mathbf{S}$, the objects of the category $\mathrm{HOM}(\mathcal{P},\mathcal{Q};\mathcal{G})(U)$ are biadditive functors from $\mathcal{P}_{|U} \times \mathcal{Q}_{|U}$ to $\mathcal{G}_{|U}$ and its arrows are morphisms of biadditive functors.
III) Let

$$\mathcal{P} \otimes \mathcal{Q}$$

be the strictly commutative Picard $\mathbf{S}$-stack endowed with a biadditive functor $\otimes : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \otimes \mathcal{Q}$ such that for any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{G}$, the biadditive functor $\otimes$ defines the following equivalence of strictly commutative Picard $\mathbf{S}$-stacks:

$$\mathrm{HOM}(\mathcal{P} \otimes \mathcal{Q},\mathcal{G}) \cong \mathrm{HOM}(\mathcal{P},\mathcal{Q};\mathcal{G}). \tag{1.12}$$

According to [D73] 1.4.20, in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the equality $[\mathcal{P} \otimes \mathcal{Q}] = \tau_{\geq -1}([\mathcal{P}] \otimes^{\mathbb{L}} [\mathcal{Q}])$.

According to §2 [Be11] we have the following operations on strictly commutative Picard $\mathbf{S}$-stacks:
(1) The **product** of two strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{P}$ and $\mathcal{Q}$ is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} \times \mathcal{Q}$ defined as followed:

- for any object U of $\mathbf{S}$, an object of the category $\mathcal{P} \times \mathcal{Q}(U)$ is a pair (p,q) of objects with p an object of $\mathcal{P}(U)$ and q an object of $\mathcal{Q}(U)$;
- for any object U of $\mathbf{S}$, if (p,q) and (p',q') are two objects of $\mathcal{P} \times \mathcal{Q}(U)$, an arrow of $\mathcal{P} \times \mathcal{Q}(U)$ from (p,q) to (p',q') is a pair (f,g) of arrows with $f : p \to p'$ an arrow of $\mathcal{P}(U)$ and $g : q \to q'$ an arrow of $\mathcal{Q}(U)$.

(2) Let $G : \mathcal{P} \to \mathcal{Q}$ and $F : \mathcal{P}' \to \mathcal{Q}$ be additive functors between strictly commutative Picard $\mathbf{S}$-stacks. The **fibered product** of $\mathcal{P}$ and $\mathcal{P}'$ over $\mathcal{Q}$ via F and G is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ defined as followed:

- for any object U of $\mathbf{S}$, the objects of the category $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$ are triplets (p,p',f) where p is an object of $\mathcal{P}(U)$, p' is an object of $\mathcal{P}'(U)$ and $f : G(p) \stackrel{\cong}{\to} F(p')$ is an isomorphism of $\mathcal{Q}(U)$ between $G(p)$ and $F(p')$;
- for any object U of $\mathbf{S}$, if (p_1,p'_1,f) and (p_2,p'_2,g) are two objects of $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$, an arrow of $(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}')(U)$ from (p_1,p'_1,f) to (p_2,p'_2,g) is a pair (f,g) of arrows with $\alpha : p_1 \to p_2$ of arrow of $\mathcal{P}(U)$ and $\beta : p'_1 \to p'_2$ an arrow of $\mathcal{P}'(U)$ such that $g \circ G(\alpha) = F(\beta) \circ f$.

The fibered product $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ is also called the **pull-back** $F^*\mathcal{P}$ of $\mathcal{P}$ via $F : \mathcal{P}' \to \mathcal{Q}$ or the **pull-back** $G^*\mathcal{P}'$ of $\mathcal{P}'$ via $G : \mathcal{P} \to \mathcal{Q}$.
(3) Let $G : \mathcal{Q} \to \mathcal{P}$ and $F : \mathcal{Q} \to \mathcal{P}'$ be additive functors between strictly commutative Picard $\mathbf{S}$-stacks. The **fibered sum** of $\mathcal{P}$ and $\mathcal{P}'$ under $\mathcal{Q}$ via F and G is the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P} +^{\mathcal{Q}} \mathcal{P}'$ generated by the following strictly commutative Picard $\mathbf{S}$-pre-stack $\mathcal{D}$:

- for any object U of $\mathbf{S}$, the objects of the category $\mathcal{D}(U)$ are the objects of the category $(\mathcal{P} \times \mathcal{P}')(U)$, i.e. pairs (p,p') with p an object of $\mathcal{P}(U)$ and p' an object of $\mathcal{P}'(U)$;

- for any object U of $\mathbf{S}$, if (p_1, p'_1) and (p_2, p'_2) are two objects of $\mathcal{D}(U)$, an arrow of $\mathcal{D}(U)$ from (p_1, p'_1) to (p_2, p'_2) is an equivalence class of triplets (q, α, β) with q an object of $\mathcal{Q}(U)$, $\alpha : p_1 + G(q) \to p_2$ an arrow of $\mathcal{P}(U)$ and $\beta : p'_1 + F(q) \to p'_2$ an arrow of $\mathcal{P}'(U)$. Two triplets (q_1, α_1, β_1) and (q_2, α_2, β_2) are equivalent it there is an arrow $\gamma : q_1 \to q_2$ in $\mathcal{Q}(U)$ such that $\alpha_2 \circ (id + G(\gamma)) = \alpha_1$ and $(F(\gamma) + id) \circ \beta_1 = \beta_2$.

The fibered sum $\mathcal{P} +^{\mathcal{Q}} \mathcal{P}'$ is also called the **push-down** $F_*\mathcal{P}$ of $\mathcal{P}$ via $F : \mathcal{Q} \to \mathcal{P}'$ or the **push-down** $G_*\mathcal{P}'$ of $\mathcal{P}'$ via $G : \mathcal{Q} \to \mathcal{P}$.

We have analogous operations on complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$:
(1) The **product** of two complexes $P = [d^P : P^{-1} \to P^0]$ and $Q = [d^Q : Q^{-1} \to Q^0]$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is the complex $P + Q = [(d^P, d^Q) : P^{-1} + Q^{-1} \to P^0 + Q^0]$. Via the equivalence of category (1.11) we have that $st(P + Q) = st(P) \times st(Q)$.
(2) Let $P = [d^P : P^{-1} \to P^0], Q = [d^Q : Q^{-1} \to Q^0]$ and $G = [d^G : G^{-1} \to G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ and let $f : P \to G$ and $g : Q \to G$ be morphisms of complexes. The **fibered product** $P \times_G Q$ of P and Q over G is the complex $[d_P \times_{d_G} d_Q : P^{-1} \times_{G^{-1}} Q^{-1} \to P^0 \times_{G^0} Q^0]$, where for $i = -1, 0$ the abelian sheaf $P^i \times_{G^i} Q^i$ is the fibered product of P^i and of Q^i over G^i and the morphism of abelian sheaves $d_P \times_{d_G} d_Q$ is given by the universal property of the fibered product $P^0 \times_{G^0} Q^0$. The fibered product $P \times_G Q$ is also called the **pull-back** g^*P of P via $g : Q \to G$ or the **pull-back** f^*Q of Q via $f : P \to G$. Remark that $st(P \times_G Q) = st(P) \times_{st(G)} st(Q)$ via the equivalence of category (1.11).
(3) Let $P = [d^P : P^{-1} \to P^0], Q = [d^Q : Q^{-1} \to Q^0]$ and $G = [d^G : G^{-1} \to G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ and let $f : G \to P$ and $g : G \to Q$ be morphisms of complexes. The **fibered sum** $P +^G Q$ of P and Q under G is the complex $[d_P +^{d_G} d_Q : P^{-1} +^{G^{-1}} Q^{-1} \to P^0 +^{G^0} Q^0]$, where for $i = -1, 0$ the abelian sheaf $P^i +^{G^i} Q^i$ is the fibered sum of P^i and of Q^i under G^i and the morphism of abelian sheaves $d_P +^{d_G} d_Q$ is given by the universal property of the fibered sum $P^{-1} +^{G^{-1}} Q^{-1}$. The fibered sum $P +^G Q$ is also called the **push-down** g_*P of P via $g : G \to Q$ or the **push-down** f_*Q of Q via $f : G \to P$. We have $st(P +^G Q) = st(P) +^{st(G)} st(Q)$ via the equivalence of category (1.11).

2. The 2-category of $\mathcal{G}$-torsors

Let $\mathcal{G}$ be a gr-$\mathbf{S}$-stack, i.e. an $\mathbf{S}$-stack of groupoids $\mathcal{G}$ equipped with the following data: a morphism of $\mathbf{S}$-stacks $+ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ with a natural isomorphism of associativity σ (1.1), a global neutral object e with two natural isomorphisms r and l (1.3), and a morphism of $\mathbf{S}$-stacks $- : \mathcal{P} \to \mathcal{P}$ with two natural isomorphisms o and c (1.4), such that for any object U of $\mathbf{S}$, $(\mathcal{G}(U), +, e, -)$ is a gr-category (i.e. see [B92] §3.1 for more details). Remark that a strictly commutative Picard $\mathbf{S}$-stack is a gr-$\mathbf{S}$-stack endowed with a strict commutative condition τ (1.2) and (1.6).

Definition 2.1. A **left $\mathcal{G}$-torsor** $\mathcal{P} = (\mathcal{P}, M, \mu)$ consists of

- an $\mathbf{S}$-stack of groupoids $\mathcal{P}$,
- a morphism of $\mathbf{S}$-stacks $M : \mathcal{G} \times \mathcal{P} \to \mathcal{P}$, and

- an isomorphism of morphisms of **S**-stacks $\mu : M \circ (+ \times id_{\mathcal{P}}) \Rightarrow M \circ (id_{\mathcal{G}} \times M)$

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \times \mathcal{P} & \xrightarrow{+ \times id_{\mathcal{P}}} & \mathcal{G} \times \mathcal{P} \\ {\scriptstyle id_{\mathcal{G}} \times M} \downarrow & \overset{\mu}{\swarrow} & \downarrow {\scriptstyle M} \\ \mathcal{G} \times \mathcal{P} & \xrightarrow[M]{} & \mathcal{P} \end{array}$$

which is described by the functorial isomorphism $\mu_{g_1,g_2,p} : M(g_1 + g_2, p) \to M(g_1, M(g_2, p))$ for any $g_1, g_2 \in \mathcal{G}$ and $p \in \mathcal{P}$,

such that the following conditions are satisfied:

(i) the natural isomorphism μ is compatible with the natural isomorphism of associativity σ underlying $\mathcal{G}$, i.e. the following diagram commute for any $g_1, g_2, g_3 \in \mathcal{G}$ and $p \in \mathcal{P}$

$$\begin{array}{ccccc} M((g_1+g_2)+g_3,p) & & \xrightarrow{M(\sigma, id_{\mathcal{P}})} & & M(g_1+(g_2+g_3),p) \\ \downarrow {\scriptstyle \mu} & & & & \downarrow {\scriptstyle \mu} \\ M(g_1+g_2, M(g_3,p)) & & & & M(g_1, M(g_2+g_3,p)) \\ & \searrow^{\mu} & & \swarrow^{\mu} & \\ & & M(g_1, M(g_2, M(g_3,p))) & & \end{array}$$

(ii) the restriction of the morphism of **S**-stacks M to $\mathbf{e} \times \mathcal{P}$ is equivalent to the identity, i.e. $M(e,p) \cong p$ for any $p \in \mathcal{P}$ (here $\mathbf{e}$ denotes the gr-**S**-stack such that for any object U of **S**, $\mathbf{e}(U)$ is the category consisting of the neutral object e of $\mathcal{G}$). Moreover we require that this restriction of M is compatible with the natural isomorphism μ, i.e. the following diagrams commute for any $g \in \mathcal{G}$ and $p \in \mathcal{P}$

$$\begin{array}{ccc} M(g+e,p) & \xrightarrow{\mu} & M(g, M(e,p)) \\ & \searrow \quad \swarrow & \\ & M(g,p) & \end{array}$$

$$\begin{array}{ccc} M(e+g,p) & \xrightarrow{\mu} & M(e, M(g,p)) \\ & \searrow \quad \swarrow & \\ & M(g,p) & \end{array}$$

(iii) the morphism of **S**-stacks $(M, Pr_2) : \mathcal{G} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ is an equivalence of **S**-stacks (here $Pr_2 : \mathcal{G} \times \mathcal{P} \to \mathcal{P}$ denotes the second projection),

(iv) it exists a covering sieve R of the site **S** such that for any object U of R the category $\mathcal{P}(U)$ is not empty.

Definition 2.2. A **morphism of left $\mathcal{G}$-torsors**

$$(F, \gamma) : (\mathcal{P}, M, \mu) \to (\mathcal{P}', M', \mu')$$

consists of

- a morphism of **S**-stacks $F : \mathcal{P} \to \mathcal{P}'$ and

- an isomorphism of morphisms of **S**-stacks $\gamma : M' \circ (id_{\mathcal{G}} \times F) \Rightarrow F \circ M$ described by the functorial isomorphism $\gamma_{g,p} : M'(g, F(p)) \to F(M(g,p))$ for any $g \in \mathcal{G}$ and $p \in \mathcal{P}$,

which are compatible with the natural isomorphisms μ and μ', i.e. the following diagram commute for any $g_1, g_2 \in \mathcal{G}$ and $p \in \mathcal{P}$

$$\begin{array}{ccccc}
M'(g_1+g_2, F(p)) & \xrightarrow{\mu'_{g_1,g_2,F(p)}} & M'(g_1, M'(g_2, F(p))) & \xrightarrow{M'(id_{\mathcal{G}}, \gamma_{g_2,p})} & M'(g_1, F(M(g_2,p))) \\
\downarrow{\scriptstyle \gamma_{g_1+g_2,p}} & & & & \downarrow{\scriptstyle \gamma_{g_1, M(g_2,p)}} \\
F(M(g_1+g_2,p)) & & \xrightarrow[F(\mu_{g_1,g_2,p})]{} & & F(M(g_1, M(g_2,p))).
\end{array}$$

Let $(F,\gamma), (\overline{F}, \overline{\gamma}) : (\mathcal{P}, M, \mu) \to (\mathcal{P}', M', \mu')$ be two morphisms of left $\mathcal{G}$-torsors.

Definition 2.3. A **morphism of morphisms of left $\mathcal{G}$-torsors**

$$\varphi : (F,\gamma) \Rightarrow (\overline{F}, \overline{\gamma})$$

consists of a morphism of morphisms of **S**-stacks $\varphi : F \Rightarrow \overline{F}$ which is compatible with the natural isomorphisms γ and $\overline{\gamma}$, i.e. the following diagram commute for any $g \in \mathcal{G}$ and $p \in \mathcal{P}$

$$\begin{array}{ccc}
M'(g, F(p)) & \xrightarrow{\gamma} & F(M(g,p)) \\
\downarrow{\scriptstyle M'(id_{\mathcal{G}}, \varphi(p))} & & \downarrow{\scriptstyle \varphi(M(g,p))} \\
M'(g, \overline{F}(p)) & \xrightarrow[\overline{\gamma}]{} & \overline{F}(M(g,p)).
\end{array}$$

If the gr-**S**-stack $\mathcal{G}$ acts of the right side instead of the left side, we get the definitions of right $\mathcal{G}$-torsor, morphism of right $\mathcal{G}$-torsors and morphism of morphisms of right $\mathcal{G}$-torsors.

Definition 2.4. A **$\mathcal{G}$-torsor** $\mathcal{P} = (\mathcal{P}, M_r, M_l, \mu_r, \mu_l, \kappa)$ consists of an **S**-stack of groupoids $\mathcal{P}$ endowed with a structure of left $\mathcal{G}$-torsor $(\mathcal{P}, M_l, \mu_l)$ and a structure of right $\mathcal{G}$-torsor $(\mathcal{P}, M_r, \mu_r)$ which are compatible one with another. This compatibility is given by the existence of an isomorphism of morphisms of **S**-stacks $\kappa : M_l \circ (id_{\mathcal{G}} \times M_r) \Rightarrow M_r \circ (M_l \times id_{\mathcal{G}})$, described by the functorial isomorphism $\kappa_{g_1,p,g_2} : M_l(g_1, M_r(p, g_2)) \to M_r(M_l(g_1,p), g_2)$ for any $g_1, g_2 \in \mathcal{G}$ and $p \in \mathcal{P}$, such that the following diagrams commute for any $g_1, g_2 \in \mathcal{G}$ and $p \in \mathcal{P}$

$$\begin{array}{ccccc}
M_l(g_1+g_2, M_r(p,g_3)) & & \xrightarrow{\kappa} & & M_r(M_l(g_1+g_2,p), g_3) \\
\downarrow{\scriptstyle \mu_l} & & & & \downarrow{\scriptstyle M_r(\mu_l, id_{\mathcal{G}})} \\
M_l(g_1, M_l(g_2, M_r(p,g_3))) & & & & M_r(M_l(g_1, M_l(g_2,p), g_3) \\
& \searrow{\scriptstyle \kappa} & & \nearrow{\scriptstyle \kappa} & \\
& & M_l(g_1, M_r(M_l(g_2,p), g_3)) & &
\end{array}$$

$$\begin{array}{ccc}
M_l(g_1, M_r(p, g_2+g_3)) & \xrightarrow{\kappa} & M_r(M_l(g_1,p), g_2+g_3) \\
{\scriptstyle M_l(id_{\mathcal{G}},\mu_r)}\downarrow & & \downarrow{\scriptstyle \mu_r} \\
M_l(g_1, M_r(M_r(p,g_2),g_3)) & & M_r(M_r(M_l(g_1,p),g_2),g_3) \\
& \searrow^{\kappa} \quad M_r(M_l(g_1, M_r(p,g_2)),g_3) \quad \nearrow^{\kappa} &
\end{array}$$

Example 2.5. The strictly commutative Picard **S**-stack $\mathcal{G}$ is endowed with a structure of $\mathcal{G}$-torsor: the morphism of **S**-stacks $+ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the natural isomorphism of associativity σ furnish a structure of left $\mathcal{G}$-torsor and a structure of right $\mathcal{G}$-torsor. The natural isomorphism of commutativity τ implies that these two structures are compatible, i.e. $\mathcal{G}$ is in fact a $\mathcal{G}$-torsor. We will call $\mathcal{G}$ the trivial $\mathcal{G}$-torsor.

Definition 2.6. A **morphism of $\mathcal{G}$-torsors**

$$(F, \gamma_r, \gamma_l) : (\mathcal{P}, M_r, M_l, \mu_r, \mu_l, \kappa) \to (\mathcal{P}', M'_r, M'_l, \mu'_r, \mu'_l, \kappa')$$

consists of

- a morphism of **S**-stacks $F : \mathcal{P} \to \mathcal{P}'$,
- two isomorphisms of morphisms of **S**-stacks $(\gamma_l)_{g,p} : M'_l(g, F(p)) \to F(M_l(g,p))$ and $(\gamma_r)_{p,g} : M'_r(F(p), g) \to F(M_r(p,g))$ for any $g \in \mathcal{G}$ and $p \in \mathcal{P}$,

such that $(F, \gamma_r) : (\mathcal{P}, M_r, \mu_r) \to (\mathcal{P}', M'_r, \mu'_r)$ and $(F, \gamma_l) : (\mathcal{P}, M_l, \mu_l) \to (\mathcal{P}', M'_l, \mu'_l)$ are morphisms of right respectively left $\mathcal{G}$-torsors, and such that γ_r and γ_l are compatible with κ and κ', i.e. the following diagram commutate for any $g_1, g_2 \in \mathcal{G}$ and $p \in \mathcal{P}$

$$\begin{array}{ccccc}
M'_l(g_1, F(M_r(p,g_2))) & \xrightarrow{\gamma_l} & F(M_l(g_1, M_r(p,g_2))) & \xrightarrow{F(\kappa)} & F(M_r(M_l(g_1,p),g_2)) \\
{\scriptstyle M'_l(id_{\mathcal{G}},\gamma_r)}\uparrow & & & & \uparrow{\scriptstyle \gamma_r} \\
M'_l(g_1, M'_r(F(p),g_2)) & \xrightarrow{\kappa'} & M'_r(M'_l(g_1,F(p)),g_2) & \xrightarrow{M'_r(\gamma_l, id_{\mathcal{G}})} & M'_r(F(M_l(g_1,p)),g_2).
\end{array}$$

Let $(F, \gamma_r, \gamma_l), (\overline{F}, \overline{\gamma}_r, \overline{\gamma}_l) : (\mathcal{P}, M_r, M_l, \mu_r, \mu_l, \kappa) \to (\mathcal{P}', M'_r, M'_l, \mu'_r, \mu'_l, \kappa')$ be two morphisms of $\mathcal{G}$-torsors.

Definition 2.7. A **morphism of morphisms of $\mathcal{G}$-torsors**

$$\varphi : (F, \gamma_r, \gamma_l) \Rightarrow (\overline{F}, \overline{\gamma}_r, \overline{\gamma}_l)$$

consists of a morphism of morphisms of **S**-stacks $\varphi : F \Rightarrow \overline{F}$ such that $\varphi : (F, \gamma_l) \Rightarrow (\overline{F}, \overline{\gamma}_l)$ and $\varphi : (F, \gamma_r) \Rightarrow (\overline{F}, \overline{\gamma}_r)$ are morphisms of morphisms of left respectively right $\mathcal{G}$-torsors, i.e. such that $\varphi : F \Rightarrow \overline{F}$ is compatible with the natural isomorphisms $\gamma_r, \overline{\gamma}_r$ and with the natural isomorphisms $\gamma_l, \overline{\gamma}_l$.

$\mathcal{G}$-torsors form a 2-category $\mathcal{T}orsor(\mathcal{G})$ where

1. the objects are $\mathcal{G}$-torsors,
2. the 1-arrows are morphisms of $\mathcal{G}$-torsors,
3. the 2-arrows are morphisms of morphisms of $\mathcal{G}$-torsors.

Now we generalize to complexes of sheaves concentrated in degree -1 and 0, the classical notion of "torsor". Let $G = [d^G : G^{-1} \to G^0]$ be a complex of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, i.e. a complex of abelian sheaves on **S** concentrated in degrees -1 and 0:

Definition 2.8. An left G-**torsor** $P = (P, m, \mu)$ consists of

- a complex $P = [d^P : P^{-1} \to P^0]$ of sheaves of sets on $\mathbf{S}$ concentrated in degrees -1 and 0,
- a morphism of complexes $m : G \times P \to P$, i.e. a commutative diagram

$$\begin{array}{ccc} G^{-1} \times P^{-1} & \xrightarrow{m^{-1}} & P^{-1} \\ \downarrow{\scriptstyle d^G \times d^P} & & \downarrow{\scriptstyle d^P} \\ G^0 \times P^0 & \xrightarrow[m^0]{} & P^0 \end{array}$$

- an homotopy μ between the two morphisms of complexes $m \circ (+ \times id_P)$ and $m \circ (id_G \times m)$ from $G \times G \times P$ to P (here $+ : G \times G \to G$ is the group law underlying the complex of abelian sheaves G),

such that the following conditions are satisfied:

(i) the homotopy μ is compatible with the associative law of the complex of abelian sheaves G, i.e. the following diagram commute

$$\begin{array}{ccc} m \circ (+ \circ (+ \times id_G) \times id_P) & = & m \circ (+ \circ (id_G \times +) \times id_P) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ m \circ (+ \times m) & & m \circ (id_G \times m \circ (+ \times id_P)) \\ & \searrow^{\mu} \quad \swarrow^{\mu} & \\ & m \circ (id_G \times m) \circ (id_G \times id_G \times m) & \end{array}$$

(ii) the restriction of the morphism of complexes m to $[id : e_{G^{-1}} \to e_{G^0}] \times \mathcal{P}$ is homotopic to the identity (here $e_{G^{-1}}$ and e_{G^0} denote the neutral sections of the abelian sheaves G^{-1} and G^0 respectively). Moreover we require that this restriction of m is compatible with the homotopy μ, i.e. the following diagram commutes for any $g^i \in G^i$ and $p^i \in P^i$ for $i = -1, 0$

$$\begin{array}{ccc} m^i(g^i + e_{G^i}, p^i) & \xrightarrow{\mu} & m^i(g^i, m^i(e_{G^i}, p^i)) \\ & \searrow^{=} \quad \swarrow & \\ & m^i(g^i, p^i) & \end{array}$$

(iii) the morphism of complexes $(m, pr_2) : G \times P \to P \times P$ is a quasi-isomorphism (here $pr_2 : G \times P \to P$ denotes the second projection),

(iv) it exists a covering sieve R of the site $\mathbf{S}$ such that for any object U of R the sets of sections $P^{-1}(U)$ and $P^0(U)$ are not empty.

Definition 2.9. A **morphism of left** G**-torsors**

$$(f, \gamma) : (P, m, \mu) \to (P', m', \mu')$$

consists of

- a morphism of complexes $f : P \to P'$ and
- an homotopy $\gamma : m' \circ (id_G \times f) \approx f \circ m$,

which are compatible with the homotopies μ and μ', i.e. the following diagram commute

$$\begin{array}{ccccc}
m'\circ(+\times f) & \xrightarrow{\mu'} & m'\circ(id_G\times m'\circ(id_G\times f)) & \xrightarrow{m'(id_G,\gamma)} & m'\circ(id_G\times f\circ m)\\
\downarrow{\scriptstyle\gamma} & & & & \downarrow{\scriptstyle\gamma}\\
f\circ m\circ(+\times id_P) & & \xrightarrow[f(\mu)]{\qquad\qquad\qquad} & & f\circ m\circ(id_G\times m).
\end{array}$$

Let $(f,\gamma),(\overline{f},\overline{\gamma}):(P,m,\mu)\to(P',m',\mu')$ be two morphisms of left $\mathcal{G}$-torsors.

Definition 2.10. A **morphism of morphisms of left G-torsors**

$$\varphi:(f,\gamma)\approx(\overline{f},\overline{\gamma})$$

consists of an homotopy $\varphi: f\approx\overline{f}$ which is compatible with the homotopies γ and $\overline{\gamma}$, i.e. the following diagram commute

$$\begin{array}{ccc}
m'\circ(id_G\times f) & \xrightarrow{\gamma} & f\circ m\\
\downarrow{\scriptstyle m'(id_G,\varphi)} & & \downarrow{\scriptstyle\varphi}\\
m'\circ(id_G\times\overline{f}) & \xrightarrow[\overline{\gamma}]{} & \overline{f}\circ m.
\end{array}$$

If the complex G acts of the right side instead of the left side, we get the definitions of right G-torsor, morphism of right G-torsors and morphism of morphisms of right G-torsors.

Definition 2.11. A **G-torsor** consists of a complex $P=[d^P:P^{-1}\to P^0]$ of sheaves of sets on $\mathbf{S}$ (concentrated in degrees -1 and 0) endowed with a structure of left G-torsor (P,m_l,μ_l) and a structure of right G-torsor (P,m_r,μ_r) which are compatible one with another. This compatibility is given by the existence of an homotopy $\kappa: m_l\circ(id_G\times m_r)\approx m_r\circ(m_l\times id_G)$ such that the following diagrams commute

$$\begin{array}{ccccc}
m_l\circ(+\times m_r) & & \xrightarrow{\kappa} & & m_r\circ(m_l\circ(+\times id_P)\times id_G)\\
\downarrow{\scriptstyle\mu_l} & & & & \downarrow{\scriptstyle\mu_l}\\
m_l\circ(id_G\times m_l\circ(id_G\times m_r)) & & & & m_r\circ(m_l\circ(id_G\times m_l)\times id_G)\\
& \searrow{\scriptstyle\kappa} & & \nearrow{\scriptstyle\kappa} & \\
& & m_l\circ(id_G\times m_r\circ(m_l\times id_G)) & &
\end{array}$$

$$\begin{array}{ccccc}
m_l\circ(id_G\times m_r\circ(id_P\times +)) & & \xrightarrow{\kappa} & & m_r\circ(m_l\times +)\\
\downarrow{\scriptstyle\mu_r} & & & & \downarrow{\scriptstyle\mu_r}\\
m_l\circ(id_G\times m_r\circ(m_r\times id_G)) & & & & m_r\circ(m_r\circ(m_l\times id_G)\times id_G)\\
& \searrow{\scriptstyle\kappa} & & \nearrow{\scriptstyle\kappa} & \\
& & m_r\circ(m_l\circ(id_G\times m_r)\times id_G) & &
\end{array}$$

Remark 2.12. If $G = [G^{-1} \stackrel{0}{\to} G^0]$, then a G-torsor consists of a G^0-torsor and a G^{-1}-torsor.

Example 2.13. The complex $G \in \mathcal{K}^{[-1,0]}(\mathbf{S})$ endowed the morphism of complexes $+ : G \times G \to G$ is a G-torsor. We will call G the trivial G-torsor.

Definition 2.14. A **morphism of G-torsors**

$$(f, \gamma_r, \gamma_l) : (P, m_r, m_l, \mu_r, \mu_l, \kappa) \to (P', m'_r, m'_l, \mu'_r, \mu'_l, \kappa')$$

consists of

- a morphism of complexes $f : P \to P'$, and
- two homotopies $(\gamma_l) : m'_l \circ (id_G \times f) \approx f \circ m_l$ and $(\gamma_r) : m'_r \circ (f \times id_G) \approx f \circ m_r$,

such that $(f, \gamma_r) : (P, m_r, \mu_r) \to (P', m'_r, \mu'_r)$ and $(f, \gamma_l) : (P, m_l, \mu_l) \to (P', m'_l, \mu'_l)$ are morphisms of right respectively left $\mathcal{G}$-torsors, and such that γ_r and γ_l are compatible with κ and κ', i.e. the following diagram commutates

$$\begin{array}{ccccc} m'_l \circ (id_G \times f) \circ (id_G \times m_r) & \xrightarrow{\gamma_l} & f \circ m_l \circ (id_G \times m_r) & \xrightarrow{\kappa} & f \circ m_r \circ (m_l \times id_G) \\ \uparrow{\scriptstyle \gamma_r} & & & & \uparrow{\scriptstyle \gamma_r} \\ m'_l \circ (id_G \times m'_r \circ (f \times id_G)) & \xrightarrow{\kappa'} & m'_r \circ (m'_l \circ (id_G \times f) \times id_G) & \xrightarrow{\gamma_l} & m'_r \circ (f \circ m_l \times id_G). \end{array}$$

Let $(f, \gamma_r, \gamma_l), (\overline{f}, \overline{\gamma}_r, \overline{\gamma}_l) : (P, m_r, m_l, \mu_r, \mu_l, \kappa) \to (P', m'_r, m'_l, \mu'_r, \mu'_l, \kappa')$ be two morphisms of G-torsors.

Definition 2.15. A **morphism of morphisms of G-torsors**

$$\varphi : (f, \gamma_r, \gamma_l) \approx (\overline{f}, \overline{\gamma}_r, \overline{\gamma}_l)$$

consists of an homotopy $\varphi : f \approx \overline{f}$ such that $\varphi : (f, \gamma_l) \approx (\overline{f}, \overline{\gamma}_l)$ and $\varphi : (f, \gamma_r) \approx (\overline{f}, \overline{\gamma}_r)$ are morphisms of morphisms of left respectively right G-torsors, i.e. such that $\varphi : f \approx \overline{f}$ is compatible with the homotopies $\gamma_r, \overline{\gamma}_r$ and with the homotopies $\gamma_l, \overline{\gamma}_l$.

3. The 2-category of extensions of Picard stacks

Let $F : \mathcal{P} \to \mathcal{Q}$ be an additive functor between strictly commutative Picard $\mathbf{S}$-stacks. Denote by $\mathbf{1}$ the strictly commutative Picard $\mathbf{S}$-stack such that for any object U of $\mathbf{S}$, $\mathbf{1}(U)$ is the category with one object and one arrow. By [Be11] §3 the **kernel** of F, $\ker(F)$, is the fibered product $\mathcal{P} \times_{\mathcal{Q}} \mathbf{1}$ of $\mathcal{P}$ and $\mathbf{1}$ over $\mathcal{Q}$ via $F : \mathcal{P} \to \mathcal{Q}$ and $\mathbf{1} : \mathbf{1} \to \mathcal{Q}$, and the **cokernel** of F, $\mathrm{coker}(F)$, is the fibered sum $\mathbf{1} +^{\mathcal{P}} \mathcal{Q}$ of $\mathbf{1}$ and $\mathcal{Q}$ under $\mathcal{P}$ via $F : \mathcal{P} \to \mathcal{Q}$ and $\mathbf{1} : \mathcal{P} \to \mathbf{1}$.

Let $\mathcal{P}$ and $\mathcal{G}$ be two strictly commutative Picard $\mathbf{S}$-stacks.

Definition 3.1. An **extension** $\mathcal{E} = (\mathcal{E}, I, J)$ of $\mathcal{P}$ by $\mathcal{G}$

$$\mathcal{G} \xrightarrow{I} \mathcal{E} \xrightarrow{J} \mathcal{P} \tag{3.1}$$

consists of

- a strictly commutative Picard $\mathbf{S}$-stack $\mathcal{E}$,
- two additive functors $I : \mathcal{G} \to \mathcal{E}$ and $J : \mathcal{E} \to \mathcal{P}$, and
- an isomorphism of additive functors between the composite $J \circ I$ and the trivial additive functor: $J \circ I \cong 0$,

such that the following equivalent conditions are satisfied:

(a) $\pi_0(J) : \pi_0(\mathcal{E}) \to \pi_0(\mathcal{P})$ is surjective and I induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{G}$ and $\ker(J)$;

(b) $\pi_1(I) : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{E})$ is injective and J induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathrm{coker}(I)$ and $\mathcal{P}$.

Let $\mathcal{P}, \mathcal{G}, \mathcal{P}'$ and $\mathcal{G}'$ be strictly commutative Picard $\mathbf{S}$-stacks. Let $\mathcal{E} = (\mathcal{E}, I, J)$ be an extension of $\mathcal{P}$ by $\mathcal{G}$ and let $\mathcal{E}' = (\mathcal{E}', I', J')$ be an extension of $\mathcal{P}'$ by $\mathcal{G}'$.

Definition 3.2. A **morphism of extensions**

$$(F, G, H) : \mathcal{E} \longrightarrow \mathcal{E}'$$

consists of

- three additive functors $F : \mathcal{E} \to \mathcal{E}', G : \mathcal{P} \to \mathcal{P}', H : \mathcal{G} \to \mathcal{G}'$, and
- two isomorphisms of additive functors $J' \circ F \cong G \circ J$ and $F \circ I \cong I' \circ H$,

which are compatible with the isomorphisms of additive functors $J \circ I \cong 0$ and $J' \circ I' \cong 0$ underlying the extensions $\mathcal{E}$ and $\mathcal{E}'$, i.e. the composite

$$0 \overset{\cong}{\longleftrightarrow} G \circ 0 \overset{\cong}{\longleftrightarrow} G \circ J \circ I \overset{\cong}{\longleftrightarrow} J' \circ F \circ I \overset{\cong}{\longleftrightarrow} J' \circ I' \circ H \overset{\cong}{\longleftrightarrow} 0 \circ H \overset{\cong}{\longleftrightarrow} 0$$

should be the identity.

Let $(F, G, H), (\overline{F}, \overline{G}, \overline{H}) : \mathcal{E} \longrightarrow \mathcal{E}'$ be two morphisms of extensions $\mathcal{E} = (\mathcal{E}, I, J)$ of $\mathcal{P}$ by $\mathcal{G}$ and $\mathcal{E}' = (\mathcal{E}', I', J')$ of $\mathcal{P}'$ by $\mathcal{G}'$.

Definition 3.3. A **morphism of morphisms of extensions**

$$(F, G, H) : (F, G, H) \Rightarrow (\overline{F}, \overline{G}, \overline{H})$$

consists of three morphisms of additive functors $\alpha : F \Rightarrow \overline{F}, \beta : G \Rightarrow \overline{G}$ and $\gamma : H \Rightarrow \overline{H}$ which are compatible with the four isomorphisms of additive functors $J' \circ F \cong G \circ J, F \circ I \cong I' \circ H, J' \circ \overline{F} \cong \overline{G} \circ J$ and $\overline{F} \circ I \cong I' \circ \overline{H}$, i.e. the following diagrams commute for any $g \in \mathcal{G}$ and $a \in \mathcal{E}$

$$\begin{array}{ccc} FI(g) & \xrightarrow{\cong} & I'H(g) \\ {\scriptstyle \alpha(I(g))}\downarrow & & \downarrow{\scriptstyle I'(\gamma(g))} \\ \overline{F}I(g) & \xrightarrow[\cong]{} & I'\overline{H}(g). \end{array} \qquad \begin{array}{ccc} J'F(a) & \xrightarrow{\cong} & GJ(a) \\ {\scriptstyle J'(\alpha(a))}\downarrow & & \downarrow{\scriptstyle \beta(J(a))} \\ J'\overline{F}(a) & \xrightarrow[\cong]{} & \overline{G}J(a). \end{array}$$

Extensions of $\mathcal{P}$ by $\mathcal{G}$ form a 2-category $\mathcal{E}xt(\mathcal{P}, \mathcal{G})$ where

1. the objects are extensions of $\mathcal{P}$ by $\mathcal{G}$,
2. the 1-arrows are morphisms of extensions,
3. the 2-arrows are morphisms of morphisms of extensions.

Let $P = [P^{-1} \overset{d^P}{\to} P^0]$ and $G = [G^{-1} \overset{d^G}{\to} G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ and let $F : st(G) \to st(P)$ be an additive functor induced by a morphism of complexes $f = (f^{-1}, f^0) : G \to P$. By [Be11] Lemma 3.4, the strictly commutative Picard $\mathbf{S}$-stacks $\ker(F)$ and $\mathrm{coker}(F)$ correspond via the equivalence of categories (1.11) to the following complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$:

$$\begin{aligned} [\ker(F)] &= \tau_{\leq 0}\big(MC(f)[-1]\big) = \big[G^{-1} \overset{(f^{-1}, -d^G)}{\longrightarrow} \ker(d^P, f^0)\big] \\ [\mathrm{coker}(F)] &= \tau_{\geq -1} MC(f) = \big[\mathrm{coker}(f^{-1}, -d^G) \overset{(d^P, f^0)}{\longrightarrow} P^0\big] \end{aligned}$$

where τ denotes the good truncation and $MC(f)$ is the mapping cone of the morphism f. Therefore we get the following notion of extension for complexes in $\mathcal{K}^{[-1,0]}(\mathbf{S})$: let $P = [P^{-1} \stackrel{d^P}{\to} P^0]$ and $G = [G^{-1} \stackrel{d^G}{\to} G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$.

Definition 3.4. An **extension** $E = (E, i, j)$ of P by G

$$G \stackrel{i}{\longrightarrow} E \stackrel{j}{\longrightarrow} P$$

consists of

- a complex E of $\mathcal{K}^{[-1,0]}(\mathbf{S})$,
- two morphisms of complexes $i : G \to E$ and $j : E \to P$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$,
- an homotopy between $j \circ i$ and 0,

such that the following equivalent conditions are satisfied:

(a) $\mathrm{H}^0(j) : \mathrm{H}^0(E) \to \mathrm{H}^0(P)$ is surjective and i induces a quasi-isomorphism between G and $\tau_{\leq 0}(MC(j)[-1])$;

(b) $\mathrm{H}^{-1}(i) : \mathrm{H}^{-1}(G) \to \mathrm{H}^{-1}(E)$ is injective and j induces a quasi-isomorphism between $\tau_{\geq -1}MC(i)$ and P.

Remark 3.5. If $G = [G^{-1} \stackrel{0}{\to} G^0]$ and $P = [P^{-1} \stackrel{0}{\to} P^0]$, then an extension of P by G consists of an extension of P^0 by G^0 and an extension of P^{-1} by G^{-1}.

Remark 3.6. Consider a short exact sequence of complexes in $\mathcal{K}^{[-1,0]}(\mathbf{S})$

$$0 \longrightarrow K \stackrel{i}{\longrightarrow} L \stackrel{j}{\longrightarrow} M \longrightarrow 0.$$

It exists a distinguished triangle $K \stackrel{i}{\to} L \stackrel{j}{\to} M \to +$ in $\mathcal{D}(\mathbf{S})$, and M is isomorphic to $MC(i)$ in $\mathcal{D}(\mathbf{S})$. Therefore a short exact sequence of complexes in $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is an extension of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ according to the above definition.

Remark 3.7. Let G be a complex of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. If $I = [d^I : I^{-1} \to I^0]$ is a complex of sheaves of sets on $\mathbf{S}$ concentrated in degrees -1 and 0, we denote by $\mathbb{Z}[I] = [\mathbb{Z}[d^I] : \mathbb{Z}[I^{-1}] \to \mathbb{Z}[I^0]]$ the complex of abelian sheaves generated by I, where $\mathbb{Z}[I^i]$ is the abelian sheaf generated by I^i for $i = -1, 0$ (see [D73] Exposé IV 11). By definition of $\mathbb{Z}[I]$, the functor

$$G \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[I], G)$$

is isomorphic to the functor

$$G \longrightarrow G(I) = \mathrm{H}^0(I, G_I),$$

where G_I is the fibered product $G \times_{\mathbf{E}} I$, with $\mathbf{E} = [id_{\mathbf{e}} : \mathbf{e} \to \mathbf{e}]$ and $\mathbf{e}$ the final object of the category of abelian sheaves on the site $\mathbf{S}$ (note that $st(E) = \mathbf{1}$). Taking the respective derived functors, for $i = -1, 0, 1$ we get the isomorphisms

$$\mathrm{Ext}^i(\mathbb{Z}[I], G) \cong \mathrm{H}^i(I, G_I).$$

Hence by [Be11] Theorem 0.1 and by [B90] Proposition 6.2 we can conclude that the equivalence classes of extensions of $\mathbb{Z}[I]$ by G are in bijection with the equivalence classes of G_I-torsors over I.

4. Description of extensions of Picard stacks in terms of torsors

Let $\mathcal{P}$ and $\mathcal{G}$ be two strictly commutative Picard $\mathbf{S}$-stacks. Denote by $\mathbf{1}$ the strictly commutative Picard $\mathbf{S}$-stack such that for any object U of $\mathbf{S}$, $\mathbf{1}(U)$ is the category with one object and one arrow. Let $\wedge$ be the contracted product of $\mathcal{G}$-torsors (see 6.7 [B90]). If K is a subset of a finite set E, $p_K : \mathcal{P}^E \rightarrow \mathcal{P}^K$ is the projection to the factors belonging to K, and $+_K : \mathcal{P}^E \rightarrow \mathcal{P}^{E-K+1}$ is the group law $+ : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ on the factors belonging to K. If ι is a permutation of the set E, $Perm(\iota) : \mathcal{P}^E \rightarrow \mathcal{P}^{\iota(E)}$ is the permutation of the factors according to ι. Moreover let $Sym : \mathcal{P} \wedge \mathcal{G} \rightarrow \mathcal{G} \wedge \mathcal{P}$ be the canonical isomorphism that exchange the factors and let $D : \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$ be the diagonal morphism.

Theorem 4.1. *To have an extension $\mathcal{E}$ of $\mathcal{P}$ by $\mathcal{G}$ is equivalent to have*

(1) *a $\mathcal{G}$-torsor $\mathcal{E}$ over $\mathcal{P}$;*

(2) *a trivialization I of the pull-back $\mathbf{1}^*\mathcal{E}$ of $\mathcal{E}$ via the additive functor $\mathbf{1} : \mathbf{1} \rightarrow \mathcal{P}$, i.e. $I : \mathcal{G} \rightarrow \mathbf{1}^*\mathcal{E}$ is an equivalence of $\mathcal{G}$-torsors between the trivial $\mathcal{G}$-torsor $\mathcal{G}$ and the pull-back $\mathbf{1}^*\mathcal{E}$;*

(3) *a morphism of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P}$*

$$M : p_1^*\,\mathcal{E} \wedge p_2^*\,\mathcal{E} \longrightarrow +^*\,\mathcal{E}$$

whose restriction over $\mathbf{1} \times \mathbf{1}$ is compatible with the trivialization I (i.e. $M(\mathbf{1}^*\mathcal{E}, \mathbf{1}^*\mathcal{E}) = \mathbf{1}^*\mathcal{E}$);

(4) *an isomorphism α of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P} \times \mathcal{P}$*

$$\begin{array}{ccc} p_1^*\,\mathcal{E} \wedge p_2^*\,\mathcal{E} \wedge p_3^*\,\mathcal{E} & \longrightarrow & p_1^*\,\mathcal{E} \wedge +_{23}^*\,\mathcal{E} \\ \downarrow & \overset{\alpha}{\swarrow\!\!\!\swarrow} & \downarrow \\ +_{12}^*\,\mathcal{E} \wedge p_3^*\,\mathcal{E} & \longrightarrow & +_{123}^*\,\mathcal{E} \end{array} \tag{4.1}$$

whose restriction over $\mathbf{1} \times \mathbf{1} \times \mathbf{1}$ is the identity, and whose pull-back over $\mathcal{P}^4$ via the morphisms cited below satisfies the equality

$$p_{123}^*\,\alpha \circ +_{23}^*\,\alpha \circ p_{234}^*\,\alpha = +_{12}^*\,\alpha \circ +_{34}^*\,\alpha; \tag{4.2}$$

(5) *an isomorphism $\chi : M \Rightarrow Sym \circ M$ of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P}$*

$$\begin{array}{ccc} p_1^*\,\mathcal{E} \wedge p_2^*\,\mathcal{E} & \xrightarrow{M} & +^*\,\mathcal{E} \\ {\scriptstyle Sym}\downarrow & \overset{\chi}{\nearrow\!\!\!\nearrow} \quad \nearrow_{M} & \\ p_2^*\,\mathcal{E} \wedge p_1^*\,\mathcal{E} & & \end{array} \tag{4.3}$$

whose pull-back $D^*\chi$ via the diagonal morphism $D : \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$ is the identity, whose composite with itself $\chi \circ \chi$ is the identity, and whose pull-back over $\mathcal{P}^3$ via the morphisms quoted below satisfies the equality

$$Perm(132)^*\,\alpha \circ +_{23}^*\,\chi \circ \alpha = p_{13}^*\,\chi \circ Perm(12)^*\,\alpha \circ p_{12}^*\,\chi. \tag{4.4}$$

Proof. I) Starting from an extension $\mathcal{E} = (\mathcal{E}, I, J)$ of $\mathcal{P}$ by $\mathcal{G}$ we will construct the data $\mathcal{E}, I, M, \alpha, \chi$ given in (1)-(5). Via the additive functor $I : \mathcal{G} \rightarrow \mathcal{E}$, the strictly commutative Picard $\mathbf{S}$-stack $\mathcal{G}$ acts on the left side and on the right side of $\mathcal{E}$, furnishing a structure of $\mathcal{G}$-torsor to $\mathcal{E}$. Since the additive functor $J : \mathcal{E} \rightarrow \mathcal{P}$ induces a surjection $\pi_0(J) : \pi_0(\mathcal{E}) \rightarrow \pi_0(\mathcal{P})$ on the π_0, $\mathcal{E}$ is in fact a $\mathcal{G}$-torsor over $\mathcal{P}$ and so we get (1). By definition, $\ker(J)$ is the pull-back $\mathbf{1}^*\mathcal{E}$ of $\mathcal{E}$ via $\mathbf{1} : \mathbf{1} \rightarrow \mathcal{P}$

and so the condition that I induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{G}$ and $\ker(J)$ is equivalent to (2). The existence for any $g \in \mathcal{G}$ and $a, b \in \mathcal{E}$ of the associative condition $\sigma : (a+g)+b \cong a+(g+b)$, which satisfies the pentagonal axiom (1.5), implies that the morphism of $\mathbf{S}$-stacks $+ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ factorizes through a morphism of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P}$, $M : p_1^* \, \mathcal{E} \wedge p_2^* \, \mathcal{E} \longrightarrow +^* \, \mathcal{E}$. The neutral object e with its two natural isomorphisms (1.3) forces the restriction of M over $\mathbf{1} \times \mathbf{1}$ to be compatible with the trivialization I. Now the existence for any $a, b, c \in \mathcal{E}$ of the isomorphism of associativity $\sigma : (a+b)+c \cong a+(b+c)$ implies the isomorphism α (4). The compatibility of the isomorphism of associativity σ with the neutral object (1.9) forces the restriction of α over $\mathbf{1} \times \mathbf{1} \times \mathbf{1}$ to be the identity. Moreover the pentagonal axiom (1.5) satisfied by σ is equivalent to the equality (4.2). The functorial isomorphism of commutativity $\tau : a+b \cong a+b$ for any $a, b \in \mathcal{E}$ gives the existence of the isomorphism χ (5). The condition that $\tau_{a,a}$ is the identity for any $a \in \mathcal{P}$ (1.6) forces the pull-back $D^*\chi$ to be the identity. The coherence condition for τ (1.7) furnishes that the composite $\chi \circ \chi$ is the identity. Moreover the hexagonal axiom (1.8) satisfied by σ and τ is equivalent to the equality (4.4).

II) Now suppose we have the data $\mathcal{E}, I, M, \alpha, \chi$ given in (1)-(5). We will show that the $\mathcal{G}$-torsor $\mathcal{E}$ over $\mathcal{P}$ is a strictly commutative Picard $\mathbf{S}$-stack endowed with a structure of extension of $\mathcal{P}$ by $\mathcal{G}$. The morphism of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P}$, $M : p_1^* \, \mathcal{E} \wedge p_2^* \, \mathcal{E} \longrightarrow +^* \, \mathcal{E}$ defines a group law $+ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ on the $\mathbf{S}$-stack of groupoids $\mathcal{E}$. The isomorphism α gives the natural isomorphism of associativity σ (1.1). The image of the neutral object of $\mathcal{G}$ via the trivialization $I : \mathcal{G} \to \mathbf{1}^*\mathcal{E}$ furnishes a neutral object in the pull-back $\mathbf{1}^*\mathcal{E}$ and so via the projection $\mathbf{1}^*\mathcal{E} \to \mathcal{E}$ we get a neutral object e in $\mathcal{E}$ (in other words, the neutral object of $\mathcal{E}$ is the composite $\mathcal{G} \to \mathbf{1}^*\mathcal{E} \to \mathcal{E}$). The condition $M(\mathbf{1}^*\mathcal{E}, \mathbf{1}^*\mathcal{E}) = \mathbf{1}^*\mathcal{E}$ implies that $e+e \cong e$. For any $a \in \mathcal{E}$, the restriction of the morphism of $\mathcal{G}$-torsors M to $\mathcal{P} \times \mathbf{1}$ furnishes a $b \in \mathcal{E}$ and an isomorphism $b+e \cong a$. The restriction of the isomorphism α to $\mathcal{P} \times \mathbf{1} \times \mathbf{1}$ determines for each $b \in \mathcal{E}$ an isomorphism of associativity $(b+e)+e \cong b+(e+e)$. Since $e+e \cong e$, for any $a \in \mathcal{E}$ we get the isomorphism $r_a : a+e \cong a$ (1.3). In an analogous way we get the natural isomorphism $l_a : e+a \cong a$. The fact that the restriction of α over $\mathbf{1} \times \mathbf{1} \times \mathbf{1}$ is the identity means that σ is compatible with the neutral object e (1.9). Moreover the equality (4.2) satisfied by α is equivalent to the pentagonal axiom (1.5) satisfied by σ. The isomorphism χ furnishes the natural isomorphism of commutativity τ (1.2). Since the pull-back $D^*\chi$ of χ via the diagonal morphism $D : \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ is the identity, $\tau_{a,a}$ is the identity $\forall a \in \mathcal{P}$ (1.6). The condition $\chi \circ \chi = id$ implies the coherence condition for τ (1.7). Moreover the equality (4.4) satisfied by χ is equivalent to the hexagonal axiom (1.8) satisfied by σ and τ. Now the pull-back $\partial^* M$ of the morphism of $\mathcal{G}$-torsors M via the anti-diagonal morphism $\partial : \mathcal{P} \to \mathcal{P} \times \mathcal{P}, a \mapsto (-a, a)$ furnishes an isomorphisms of $\mathcal{G}$-torsors $-^*\mathcal{E} \wedge \mathcal{E} \cong \mathcal{G}$ (here $- : \mathcal{P} \to \mathcal{P}$ is the morphism of $\mathbf{S}$-stacks underlying $\mathcal{P}$) and therefore we get a morphism of $\mathbf{S}$-stacks $- : \mathcal{E} \to \mathcal{E}, a \mapsto -a$ with a natural isomorphism $o_a : a+(-a) \cong e$ (1.4). The isomorphism α furnishes the second natural isomorphism $c_{ab} : -(a+b) \cong (-a)+(-b)$ of (1.4). Until now we have proved that $\mathcal{E}$ is a strictly commutative Picard $\mathbf{S}$-stack.

If $J : \mathcal{E} \to \mathcal{P}$ denotes the morphism of $\mathbf{S}$-stacks which furnishes to $\mathcal{E}$ the structure of torsor over $\mathcal{P}$, J must be a surjection on the isomorphism classes of objects, i.e. $\pi_0(J) : \pi_0(\mathcal{E}) \to \pi_0(\mathcal{P})$ is surjective. Moreover the compatibility of J with the morphism of $\mathcal{G}$-torsors $M : p_1^* \ \mathcal{E} \wedge p_2^* \ \mathcal{E} \longrightarrow +^* \ \mathcal{E}$ implies that J is an additive functor. As already observed, to have a trivialization I of the pull-back $\mathbf{1}^*\mathcal{E}$ is equivalent to have an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{G}$ and $\ker(J)$. We still denote I the composite $\mathcal{G} \cong \mathbf{1}^*\mathcal{E} \to \mathcal{E}$ where the last arrow is the projection $\mathbf{1}^*\mathcal{E} = \mathcal{E} \times_{\mathcal{P}} \mathbf{1} \to \mathcal{E}$. Clearly I is an additive functor. We can conclude that $(\mathcal{E}, I, J)$ is an extension of $\mathcal{P}$ by $\mathcal{G}$. □

As a corollary we get the following statement whose proof is left to the reader

Corollary 4.2. *With the notations of the above Theorem, it exists an equivalence of 2-categories between the 2-category $\mathcal{E}xt(\mathcal{P},\mathcal{G})$ of extensions of $\mathcal{P}$ by $\mathcal{G}$ and the 2-category consisting of the data $(\mathcal{E}, I, M, \alpha, \chi)$.*

Let $G = [d^G : G^{-1} \to G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. If $\mathbf{e}$ denotes the final object of the category of abelian sheaves on the site $\mathbf{S}$, the complex $\mathbf{E} = [id_{\mathbf{e}} : \mathbf{e} \to \mathbf{e}]$ corresponds to the strictly Picard $\mathbf{S}$-stack $\mathbf{1}$ via the equivalence of category (1.11): $st(\mathbf{E}) = \mathbf{1}$. Let $P = [d^P : P^{-1} \to P^0]$ and $Q = [d^Q : Q^{-1} \to Q^0]$ are two G-torsors the **contracted product** $P \wedge^G Q$ is the G-torsor $[d^P \wedge^{d^G} d^Q : P^{-1} \wedge^{G^{-1}} Q^{-1} \to P^0 \wedge^{G^0} Q^0]$, where $P^i \wedge^{G^i} Q^i$ is the contracted product of P^i and Q^i (for $i = -1, 0$) and $d^P \wedge^{d^G} d^Q$ is induced by $d^P \times d^Q : P^{-1} \times Q^{-1} \to P^0 \times Q^0$ (see 1.3 Chapter III [G71]). If K is a subset of a finite set F, $p_K : P^F \to P^K$ is the projection to the factors belonging to K, and $+_K : P^F \to P^{F-K+1}$ is the group law $+ : P \times P \to P$ on the factors belonging to K. If ι is a permutation of the set E, $Perm(\iota) : P^F \to P^{\iota(F)}$ is the permutation of the factors according to ι. Moreover let $sym : P \wedge P \to P \wedge P$ be the canonical isomorphism that exchange the factors and let $d : P \to P \times P$ be the diagonal morphism. As a consequence of 4.1 we have the following

Corollary 4.3. *To have an extension E of P by G is equivalent to have*

(1) *a G-torsor E over P;*
(2) *a trivialization i of the pull-back $\mathbf{1}^*E$ of E via the morphism of complexes $\mathbf{1} : \mathbf{E} \to P$, i.e. $i : G \to \mathbf{1}^*E$ is a quasi-isomorphism between the trivial G-torsor G and the pull-back $\mathbf{1}^*E$;*
(3) *a morphism of G-torsors over $P \times P$*

$$m : p_1^* \ E \wedge p_2^* \ E \longrightarrow +^* \ E$$

whose restriction over $\mathbf{E} \times \mathbf{E}$ is compatible with the trivialization i (i.e. $m(\mathbf{1}^*E, \mathbf{1}^*E) = \mathbf{1}^*E$);
(4) *an isomorphism α of morphisms of G-torsors over $P \times P \times P$*

$$\begin{array}{ccc} p_1^* \ E \wedge p_2^* \ E \wedge p_3^* \ E & \longrightarrow & p_1^* \ E \wedge +_{23}^* \ E \\ \downarrow & \overset{\alpha}{\swarrow} & \downarrow \\ +_{12}^* \ E \wedge p_3^* \ E & \longrightarrow & +_{123}^* \ E \end{array}$$

whose restriction over $\mathbf{E} \times \mathbf{E} \times \mathbf{E}$ is the identity, and whose pull-back over P^4 via the morphisms cited below satisfies the equality

$$p_{123}^* \ \alpha \circ +_{23}^* \ \alpha \circ p_{234}^* \ \alpha = +_{12}^* \ \alpha \circ +_{34}^* \ \alpha;$$

(5) *an isomorphism $\chi : m \approx sym \circ m$ of morphisms of G-torsors over $P \times P$*

$$\begin{array}{ccc} p_1^* E \wedge p_2^* E & \xrightarrow{\quad m \quad} & +^* E \\ {\scriptstyle sym}\downarrow & \overset{\chi}{\rightsquigarrow} & \nearrow_{m} \\ p_2^* E \wedge p_1^* E & & \end{array}$$

whose pull-back $d^*\chi$ via the diagonal morphism $d : P \to P \times P$ is the identity, whose composite with itself $\chi \circ \chi$ is the identity, and whose pull-back over P^3 via the morphisms quoted below satisfies the equality

$$Perm(132)^* \alpha \circ +_{23}^* \chi \circ \alpha = p_{13}^* \chi \circ Perm(12)^* \alpha \circ p_{12}^* \chi.$$

5. The 2-category of biextensions of Picard stacks

Let $\mathbf{1}$ be the strictly commutative Picard $\mathbf{S}$-stack such that for any object U of $\mathbf{S}$, $\mathbf{1}(U)$ is the category with one object and one arrow. Let $\mathcal{G}, \mathcal{Q}$ and $\mathcal{P}$ be strictly commutative Picard $\mathbf{S}$-stacks. We consider on the fibered product $\mathcal{G} \times_{\mathbf{1}} \mathcal{P}$ the structure of "strictly commutative Picard $\mathbf{S}$-stack over $\mathcal{P}$" of the pull-back $\mathbf{1}^*\mathcal{G}$ of $\mathcal{G}$ via the additive functor $\mathcal{P} \to \mathbf{1}$

$$\begin{array}{ccc} \mathcal{G} \times_{\mathbf{1}} \mathcal{P} & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow {\scriptstyle \mathbf{1}} \\ \mathcal{G} & \xrightarrow[\mathbf{1}]{} & \mathbf{1}. \end{array}$$

In this case we write $\mathcal{G} \times_{\mathbf{1}} \mathcal{P} = \mathcal{G}_{\mathcal{P}}$. On the other hand we can consider on the fibered product $\mathcal{G} \times_{\mathbf{1}} \mathcal{P}$ also the structure of "strictly commutative Picard $\mathbf{S}$-stack over $\mathcal{G}$" of the pull-back $\mathbf{1}^*\mathcal{P}$ of $\mathcal{P}$ via the additive functor $\mathcal{G} \to \mathbf{1}$. In this case we write $\mathcal{G} \times_{\mathbf{1}} \mathcal{P} = \mathcal{P}_{\mathcal{G}}$. In this section, over $\mathcal{P}$ we will consider the two strictly strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{Q}_{\mathcal{P}}$ and over $\mathcal{Q}$ we will consider the two strictly strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{G}_{\mathcal{Q}}$ and $\mathcal{P}_{\mathcal{Q}}$.
We identify $\mathcal{G}_{\mathcal{P} \times_{\mathbf{1}} \mathcal{Q}}$ as the pull-back of $\mathcal{G}_{\mathcal{P}}$ via the projection $Pr_1 : \mathcal{P} \times_{\mathbf{1}} \mathcal{Q} \to \mathcal{P}$, or as the pull-back of $\mathcal{G}_{\mathcal{Q}}$ via the projection $Pr_2 : \mathcal{P} \times_{\mathbf{1}} \mathcal{Q} \to \mathcal{Q}$.

Let $\mathcal{G}, \mathcal{Q}$ and $\mathcal{P}$ be strictly commutative Picard $\mathbf{S}$-stacks.

Definition 5.1. A **biextension** of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$ is a $\mathcal{G}_{\mathcal{P} \times_{\mathbf{1}} \mathcal{Q}}$-torsor $\mathcal{B}$ over $\mathcal{P} \times_{\mathbf{1}} \mathcal{Q}$, endowed with a structure of extension of $\mathcal{Q}_{\mathcal{P}}$ by $\mathcal{G}_{\mathcal{P}}$ and a structure of extension of $\mathcal{P}_{\mathcal{Q}}$ by $\mathcal{G}_{\mathcal{Q}}$, which are compatible one with another.

In oder to explain what it means for two extensions to be compatible we used the description of extensions in term of torsors furnished by Theorem 4.1: denote by $(\mathcal{B}_{\mathcal{P}}, I^{\mathcal{Q}}, M^{\mathcal{Q}}, \alpha^{\mathcal{Q}}, \chi^{\mathcal{Q}})$ and by $(\mathcal{B}_{\mathcal{Q}}, I^{\mathcal{P}}, M^{\mathcal{P}}, \alpha^{\mathcal{P}}, \chi^{\mathcal{P}})$ the data corresponding respectively to the extensions $\mathcal{B}_{\mathcal{P}}$ of $\mathcal{Q}_{\mathcal{P}}$ by $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{B}_{\mathcal{Q}}$ of $\mathcal{P}_{\mathcal{Q}}$ by $\mathcal{G}_{\mathcal{Q}}$ underlying the biextension $\mathcal{B}$. In particular, if $p_i^{\mathcal{Q}} : \mathcal{Q}_{\mathcal{P}} \times \mathcal{Q}_{\mathcal{P}} \to \mathcal{Q}_{\mathcal{P}}$ (resp. $p_i^{\mathcal{P}} : \mathcal{P}_{\mathcal{Q}} \times \mathcal{P}_{\mathcal{Q}} \to \mathcal{P}_{\mathcal{Q}}$) are the projections $(i = 1, 2)$ and $+^{\mathcal{Q}} : \mathcal{Q}_{\mathcal{P}} \times \mathcal{Q}_{\mathcal{P}} \to \mathcal{Q}_{\mathcal{P}}$ (resp. $+^{\mathcal{P}} : \mathcal{P}_{\mathcal{Q}} \times \mathcal{P}_{\mathcal{Q}} \to \mathcal{P}_{\mathcal{Q}}$) is the group law of $\mathcal{Q}_{\mathcal{P}}$ (resp. $\mathcal{P}_{\mathcal{Q}}$),

$$M^{\mathcal{Q}} : p_1^{\mathcal{Q}\,*} \mathcal{B}_{\mathcal{P}} \wedge p_2^{\mathcal{Q}\,*} \mathcal{B}_{\mathcal{P}} \longrightarrow +^{\mathcal{Q}\,*} \mathcal{B}_{\mathcal{P}} \quad (\text{resp. } M^{\mathcal{P}} : p_1^{\mathcal{P}\,*} \mathcal{B}_{\mathcal{Q}} \wedge p_2^{\mathcal{P}\,*} \mathcal{B}_{\mathcal{Q}} \longrightarrow +^{\mathcal{P}\,*} \mathcal{B}_{\mathcal{Q}})$$

is a morphism of $\mathcal{G}_{\mathcal{P}}$-torsors over $\mathcal{Q}_{\mathcal{P}} \times \mathcal{Q}_{\mathcal{P}}$ (resp. of $\mathcal{G}_{\mathcal{Q}}$-torsors over $\mathcal{P}_{\mathcal{Q}} \times \mathcal{P}_{\mathcal{Q}}$).
The two extensions $\mathcal{B}_{\mathcal{P}}$ of $\mathcal{Q}_{\mathcal{P}}$ by $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{B}_{\mathcal{Q}}$ of $\mathcal{P}_{\mathcal{Q}}$ by $\mathcal{G}_{\mathcal{Q}}$ are **compatible** if it

exists an isomorphism β of morphisms of $\mathcal{G}_{\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}}$-torsors over $(\mathcal{P}\times_{\mathbf{1}}\mathcal{Q})\times(\mathcal{P}\times_{\mathbf{1}}\mathcal{Q})$
(5.1)

$$\begin{array}{ccc}
 & +^{\mathcal{P}\,*}p_1^{\mathcal{Q}\,*}\mathcal{B}\wedge +^{\mathcal{P}\,*}p_2^{\mathcal{Q}\,*}\mathcal{B} & \\
\nearrow^{M^{\mathcal{P}}\wedge M^{\mathcal{P}}} & & \searrow^{M^{\mathcal{Q}}} \\
(p_1^{\mathcal{P}},p_1^{\mathcal{Q}})^*\mathcal{B}\wedge(p_1^{\mathcal{Q}},p_2^{\mathcal{P}})^*\mathcal{B}\wedge(p_1^{\mathcal{P}},p_2^{\mathcal{Q}})^*\mathcal{B}\wedge(p_2^{\mathcal{P}},p_2^{\mathcal{Q}})^*\mathcal{B} & \overset{\beta}{\Longrightarrow} & +^{\mathcal{Q}\,*}+^{\mathcal{P}\,*}\mathcal{B} \\
\downarrow{\scriptstyle Sym} & & \nearrow_{M^{\mathcal{P}}} \\
(p_1^{\mathcal{P}},p_1^{\mathcal{Q}})^*\mathcal{B}\wedge(p_1^{\mathcal{P}},p_2^{\mathcal{Q}})^*\mathcal{B}\wedge(p_1^{\mathcal{Q}},p_2^{\mathcal{P}})^*\mathcal{B}\wedge(p_2^{\mathcal{P}},p_2^{\mathcal{Q}})^*\mathcal{B} & & \\
\searrow_{M^{\mathcal{Q}}\wedge M^{\mathcal{Q}}} & & \\
 & +^{\mathcal{Q}\,*}p_1^{\mathcal{P}\,*}\mathcal{B}\wedge +^{\mathcal{Q}\,*}p_2^{\mathcal{P}\,*}\mathcal{B} &
\end{array}$$

Let $\mathcal{G},\mathcal{Q},\mathcal{P},\mathcal{G}',\mathcal{Q}'$ and $\mathcal{P}'$ be strictly commutative Picard $\mathbf{S}$-stacks. Consider a biextension $\mathcal{B}$ of $(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}$ and a biextension $\mathcal{B}'$ of $(\mathcal{P}',\mathcal{Q}')$ by $\mathcal{G}'$.

Definition 5.2. A **morphism of biextensions**

$$(F,U,V,W):\mathcal{B}\longrightarrow\mathcal{B}'$$

consists of

- three additive functors $U:\mathcal{P}\to\mathcal{P}', V:\mathcal{Q}\to\mathcal{Q}', W:\mathcal{G}\to\mathcal{G}'$, and
- a morphism of $\mathbf{S}$-stacks $F:\mathcal{B}\to\mathcal{B}'$,

such that $(F,U\times V,U\times W):\mathcal{B}_{\mathcal{P}}\to\mathcal{B}'_{\mathcal{P}'}$ and $(F,U\times V,V\times W):\mathcal{B}_{\mathcal{Q}}\to\mathcal{B}'_{\mathcal{Q}'}$ are morphisms of extensions.

In the above definition we have used the following notation:
$U\times V:\mathcal{Q}_{\mathcal{P}}\to\mathcal{Q}'_{\mathcal{P}'}, U\times W:\mathcal{G}_{\mathcal{P}}\to\mathcal{G}'_{\mathcal{P}'}, U\times V:\mathcal{P}_{\mathcal{Q}}\to\mathcal{P}'_{\mathcal{Q}'}$ and $V\times W:\mathcal{G}_{\mathcal{Q}}\to\mathcal{G}'_{\mathcal{Q}'}$.

Let $(F,U,V,W),(\overline{F},\overline{U},\overline{V},\overline{W}):\mathcal{B}\longrightarrow\mathcal{B}'$ be two morphisms of biextensions.

Definition 5.3. A **morphism of morphisms of biextensions**

$$(\varphi,\alpha,\beta,\gamma):(F,U,V,W)\Rightarrow(\overline{F},\overline{U},\overline{V},\overline{W})$$

consists of

- three morphisms of additive functors $\alpha:U\times V\Rightarrow\overline{U}\times\overline{V}, \beta:U\times W\Rightarrow\overline{U}\times\overline{W}$ and $\gamma:V\times W\Rightarrow\overline{V}\times\overline{W}$,
- a morphism of morphisms of $\mathbf{S}$-stacks $\varphi:F\Rightarrow\overline{F}$,

such that $(\varphi,\alpha,\beta):(F,U\times V,U\times W)\Rightarrow(\overline{F},\overline{U}\times\overline{V},\overline{U}\times\overline{W})$ and $(\varphi,\alpha,\gamma):(F,U\times V,V\times W)\Rightarrow(\overline{F},\overline{U}\times\overline{V},\overline{V}\times\overline{W})$ are morphisms of morphisms of extensions.

Biextensions of $(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}$ form a 2-category $\mathcal{B}iext(\mathcal{P},\mathcal{Q};\mathcal{G})$ where

(1) the objects are biextensions of $(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}$,
(2) the 1-arrows are morphisms of biextensions,
(3) the 2-arrows are morphisms of morphisms of biextensions.

We have the following equivalence of 2-categories

$$\mathcal{B}iext(\mathcal{P},\mathbf{1};\mathcal{G})\cong\mathcal{B}iext(\mathbf{1},\mathcal{P};\mathcal{G})\cong\mathcal{E}xt(\mathcal{P},\mathcal{G}).$$

Let $P=[d^P:P^{-1}\to P^0], Q=[d^Q:Q^{-1}\to Q^0]$ and $G=[d^G:G^{-1}\to G^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. If $\mathbf{e}$ denotes the final object of the category of abelian sheaves on the site $\mathbf{S}$, the complex $\mathbf{E}=[id_{\mathbf{e}}:\mathbf{e}\to\mathbf{e}]$ corresponds to the strictly

Picard **S**-stack **1** via the equivalence of category (1.11): $st(\mathbf{E}) = \mathbf{1}$. We denote by G_P (resp. $P_Q, G_Q, G_{P\times_{\mathbf{E}}Q}$) the fibered product $G \times_{\mathbf{E}} P$ (resp. $P \times_{\mathbf{E}} Q, G \times_{\mathbf{E}} Q, G \times_{\mathbf{E}} P \times_{\mathbf{E}} Q$).

Definition 5.4. An **biextension** of (P, Q) by G is a $G_{P\times_{\mathbf{E}}Q}$-torsor B over $P \times_{\mathbf{E}} Q$, endowed with a structure of extension of Q_P by G_P and a structure of extension of P_Q by G_Q, which are compatible one with another.

Remark 5.5. Because of Remarks (2.12) and (3.5), if $G = [G^{-1} \stackrel{0}{\to} G^0]$, $P = [P^{-1} \stackrel{0}{\to} P^0]$ and $Q = [Q^{-1} \stackrel{0}{\to} Q^0]$, then a biextension of (P, Q) by G consists of a biextension of (P^0, Q^0) by G^0 and a biextension of (P^{-1}, Q^{-1}) by G^{-1}.

In oder to explain what it means for two extensions to be compatible we used the description of extensions in term of torsors furnished by Corollary 4.3: denote by $(B_P, i^Q, m^Q, \alpha^Q, \chi^Q)$ and by $(B_Q, i^P, m^P, \alpha^P, \chi^P)$ the data corresponding respectively to the extensions B_P of Q_P by G_P and B_Q of P_Q by G_Q underlying the biextension B. In particular, if $p_i^Q : Q_P \times Q_P \to Q_P$ (resp. $p_i^P : P_Q \times P_Q \to P_Q$) are the projections $(i = 1, 2)$ and $+^Q : Q_P \times Q_P \to Q_P$ (resp. $+^P : P_Q \times P_Q \to P_Q$) is the group law of Q_P (resp. P_Q),

$$m^Q : p_1^{Q\,*} B_P \wedge p_2^{Q\,*} B_P \longrightarrow +^{Q\,*} B_P \quad (\text{resp. } m^P : p_1^{P\,*} B_Q \wedge p_2^{P\,*} B_Q \longrightarrow +^{P\,*} B_Q)$$

is a morphism of G_P-torsors over $Q_P \times Q_P$ (resp. of G_Q-torsors over $P_Q \times P_Q$). The two extensions B_P of Q_P by G_P and B_Q of P_Q by G_Q are **compatible** if it exists an isomorphism β of morphisms of $G_{P\times_{\mathbf{E}}Q}$-torsors over $(P \times_{\mathbf{E}} Q) \times (P \times_{\mathbf{E}} Q)$ (5.2)

$$\begin{array}{ccc}
 & & +^{P\,*}p_1^{Q\,*}B \wedge +^{P\,*}p_2^{Q\,*}B \\
 & \nearrow^{m^P\wedge m^P} & \searrow^{m^Q} \\
(p_1^P, p_1^Q)^*B \wedge (p_1^Q, p_2^P)^*B \wedge (p_1^P, p_2^Q)^*B \wedge (p_2^P, p_2^Q)^*B & \stackrel{\beta}{\rightsquigarrow} & +^{Q\,*} +^{P\,*} B \\
\downarrow{\scriptstyle sym} & & \nearrow^{m^P} \\
(p_1^P, p_1^Q)^*B \wedge (p_1^P, p_2^Q)^*B \wedge (p_1^Q, p_2^P)^*B \wedge (p_2^P, p_2^Q)^*B & \searrow^{m^Q\wedge m^Q} & \\
 & & +^{Q\,*}p_1^{P\,*}B \wedge +^{Q\,*}p_2^{P\,*}B
\end{array}$$

6. Operations on biextensions of strictly commutative Picard stacks

Let $U : \mathcal{P}' \to \mathcal{P}, V : \mathcal{Q}' \to \mathcal{Q}, W : \mathcal{G} \to \mathcal{G}'$ be three additive functors. Consider a biextension $\mathcal{B}$ of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$.

Definition 6.1. The *pull-back* $(U \times V)^*\mathcal{E}$ *of the biextension* $\mathcal{B}$ *via the additive functors* $U \times V : \mathcal{P}' \times_{\mathbf{1}} \mathcal{Q}' \to \mathcal{P} \times_{\mathbf{1}} \mathcal{Q}$ is the fibered product $\mathcal{B} \times_{\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}} (\mathcal{P}' \times_{\mathbf{1}} \mathcal{Q}')$ of $\mathcal{B}$ and $\mathcal{P}' \times_{\mathbf{1}} \mathcal{Q}'$ over $\mathcal{P} \times_{\mathbf{1}} \mathcal{Q}$ via $U \times V$.

By [Be11] Lemma 4.2 the pull-back $(U \times V)^*\mathcal{B}$ is a biextension of $(\mathcal{P}', \mathcal{Q}')$ by $\mathcal{G}$.

Definition 6.2. The *push-down* $W_*\mathcal{B}$ *of the biextension* $\mathcal{B}$ *via the additive functor* $W : \mathcal{G} \to \mathcal{G}'$ is the fibered sum $\mathcal{B} +^{\mathcal{G}} \mathcal{G}'$ of $\mathcal{B}$ and $\mathcal{G}'$ under $\mathcal{G}$ via W.

By [Be11] Lemma 4.4 the push-down $W_*\mathcal{B}$ is a biextension of $(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}'$.

Now let $\mathcal{B}'$ be another biextension of $(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}$. According to [Be11] Lemma 4.5, the product $\mathcal{B}\times\mathcal{B}'$ is a biextension of $(\mathcal{P}\times\mathcal{P},\mathcal{Q}\times\mathcal{Q})$ by $\mathcal{G}\times\mathcal{G}$.

Definition 6.3. The *sum* $\mathcal{B}+\mathcal{B}'$ *of the biextensions* $\mathcal{B}$ *and* $\mathcal{B}'$ is the following biextension of $(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}$

$$D^* +_* (\mathcal{B}\times\mathcal{B}') \tag{6.1}$$

where $+ : \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is the group law of $\mathcal{G}$ and $D=(D_{\mathcal{P}},D_{\mathcal{Q}}) : \mathcal{P}\times\mathcal{Q}\to(\mathcal{P}\times\mathcal{P})\times(\mathcal{Q}\times\mathcal{Q})$ with $D_{\mathcal{P}}$ (resp. $D_{\mathcal{Q}}$) the diagonal functor of $\mathcal{P}$ (resp. $\mathcal{Q}$).

As a consequence of [Be11] Lemma 4.7 we have the following

Lemma 6.4. *The above notion of sum of biextensions defines on the set of equivalence classes of biextensions of* $(\mathcal{P},\mathcal{Q})$ *by* $\mathcal{G}$ *an associative, commutative group law with neutral object, that we denote* $\mathcal{G}\times_{\mathbf{1}}\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}$.

Remark that the neutral object is the trivial $\mathcal{G}_{\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}}$-torsor over $\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}$.

7. Proof of theorem 0.1 (b) and (c)

Let $\mathcal{P},\mathcal{Q}$ and $\mathcal{G}$ be three strictly commutative Picard $\mathbf{S}$-stacks. According to Lemma 6.4, the set of equivalence classes of objects of $\mathcal{B}iext(\mathcal{P},\mathcal{Q};\mathcal{G})$ is a commutative group with neutral object $\mathcal{B}_0=\mathcal{G}\times_{\mathbf{1}}\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}$. We denote this group by

$$\mathcal{B}iext^1(\mathcal{P},\mathcal{Q};\mathcal{G}).$$

The monoid of isomorphism classes of arrows from an object $\mathcal{B}$ of $\mathcal{B}iext(\mathcal{P},\mathcal{Q};\mathcal{G})$ to itself is canonically isomorphic to the monoid of isomorphism classes of arrows from $\mathcal{B}_0$ to itself: to an isomorphism class of an arrow $F:\mathcal{B}_0\to\mathcal{B}_0$ the canonical isomorphism associates the isomorphism class of the arrow $F+Id_{\mathcal{B}}$ from $\mathcal{B}_0+\mathcal{B}\cong\mathcal{B}$ to itself. The monoid of isomorphism classes of arrows from $\mathcal{B}_0$ to itself is a commutative group via the composition law $(\overline{F},\overline{G})\mapsto\overline{F+G}$ (here $\overline{F+G}$ is the isomorphism class of the arrow $F+G$ from $\mathcal{B}_0+\mathcal{B}_0\cong\mathcal{B}_0$ to itself). Hence we can conclude that the set of isomorphism classes of arrows from an object of $\mathcal{B}iext(\mathcal{P},\mathcal{Q};\mathcal{G})$ to itself is a commutative group that we denote by

$$\mathcal{B}iext^0(\mathcal{P},\mathcal{Q};\mathcal{G}).$$

The monoid of automorphisms of arrows from an object $\mathcal{B}$ of $\mathcal{B}iext(\mathcal{P},\mathcal{Q};\mathcal{G})$ to itself is canonically isomorphic to the monoid of automorphisms of arrows from $\mathcal{B}_0$ to itself: to an automorphism $\alpha:F\Rightarrow F$ of an arrow $F:\mathcal{B}_0\to\mathcal{B}_0$ the canonical isomorphism associates the automorphism $\alpha+id_{Id_{\mathcal{B}}}:F+Id_{\mathcal{B}}\Rightarrow F+Id_{\mathcal{B}}$ of the arrow $F+Id_{\mathcal{B}}$ from $\mathcal{B}_0+\mathcal{B}\cong\mathcal{B}$ to itself. The monoid of automorphisms of arrows from $\mathcal{B}_0$ to itself is a commutative group via the following composition law: if $\alpha:F\Rightarrow F$ and $\beta:G\Rightarrow G$, then $\alpha+\beta:F+G\Rightarrow F+G$, with $F+G$ an arrow from $\mathcal{B}_0+\mathcal{B}_0\cong\mathcal{B}_0$ to itself. Hence we can conclude that the set of automorphisms of an arrow from an object of $\mathcal{B}iext(\mathcal{P},\mathcal{Q};\mathcal{G})$ to itself is a commutative group that we denote by

$$\mathcal{B}iext^{-1}(\mathcal{P},\mathcal{Q};\mathcal{G}).$$

Proof of Theorem 0.1 (**b**) and (**c**). As we have observed at the beginning of this section, in order to prove (**b**) and (**c**) we can work with the biextension $\mathcal{B}_0=\mathcal{G}\times_{\mathbf{1}}\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}$ of $(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}$. In particular $\mathcal{B}_0$ is a strictly commutative Picard $\mathbf{S}$-stack and so the group of isomorphism classes of arrows from $\mathcal{B}_0$ to itself is the

cohomology group $\mathrm{H}^0([\mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)])$ and the group of automorphisms of arrows from $\mathcal{B}_0$ to itself is the cohomology group $\mathrm{H}^{-1}([\mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)])$. Therefore, in order to conclude it is enough to compute the complex $[\mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)]$.
Let $F:\mathcal{B}_0\to\mathcal{B}_0$ be an additive functor. Since F is first of all an arrow from the $\mathcal{G}_{\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}}$-torsor over $\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}$ underlying $\mathcal{B}_0$ to itself, F is given by the formula

$$F(b)=b+IF'J(b)\qquad \forall\, b\in\mathcal{B}_0$$

where $F':\mathcal{P}\times\mathcal{Q}\to\mathcal{G}$ is an additive functor and $J:\mathcal{B}_0\to\mathcal{P}\times\mathcal{Q}$ and $I:\mathcal{G}\to\mathcal{B}_0$ the additive functors underlying the structure of $\mathcal{G}_{\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}}$-torsor over $\mathcal{P}\times_{\mathbf{1}}\mathcal{Q}$ of $\mathcal{B}_0$. Now $F:\mathcal{B}_0\to\mathcal{B}_0$ must be compatible with the structures of extension of $\mathcal{Q}_{\mathcal{P}}$ by $\mathcal{G}_{\mathcal{P}}$ and of extension of $\mathcal{P}_{\mathcal{Q}}$ by $\mathcal{G}_{\mathcal{Q}}$ underlying $\mathcal{B}_0$, and so $F':\mathcal{P}\times\mathcal{Q}\to\mathcal{G}$ must be a biadditive functor. Hence we get that $\mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)$ is equivalent as strictly commutative Picard $\mathbf{S}$-stack to $\mathrm{HOM}(\mathcal{P},\mathcal{Q};\mathcal{G})$ via the following additive functor

$$\begin{array}{ccc}\mathrm{HOM}(\mathcal{P},\mathcal{Q};\mathcal{G}) & \longrightarrow & \mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)\\ F' & \mapsto & \big(b\mapsto b+IF'J(b)\big).\end{array}$$

In the example 1.1 we have observed that the strictly commutative Picard $\mathbf{S}$-stacks $\mathrm{HOM}(\mathcal{P},\mathcal{Q};\mathcal{G})$ and $\mathrm{HOM}(\mathcal{P}\otimes\mathcal{Q},\mathcal{G})$ are equivalent as strictly commutative Picard $\mathbf{S}$-stacks and so

$$[\mathrm{HOM}(\mathcal{B}_0,\mathcal{B}_0)]=\tau_{\leq 0}\mathrm{RHom}\Big(\tau_{\geq -1}([\mathcal{P}]\otimes^{\mathbb{L}}[\mathcal{Q}]),[\mathcal{G}]\Big),$$

i.e. the group of isomorphism classes of additive functors from $\mathcal{B}_0$ to itself is isomorphic to the group $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}]\otimes^{\mathbb{L}}[\mathcal{Q}],[\mathcal{G}])$, and the group of automorphisms of an additive functor from $\mathcal{B}_0$ to itself is isomorphic to the group $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}]\otimes^{\mathbb{L}}[\mathcal{Q}],[\mathcal{G}][-1])$.

In Section 10 we gives another proof of Theorem 0.1 **b** and **c**.

8. The 2-category $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ and its homological interpretation

A cochain complex of strictly commutative Picard $\mathbf{S}$-stacks

$$\longrightarrow\mathcal{L}^{-1}\stackrel{D^{-1}}{\longrightarrow}\mathcal{L}^{0}\stackrel{D^{0}}{\longrightarrow}\mathcal{L}^{1}\stackrel{D^{1}}{\longrightarrow}$$

consists of

- strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{L}^i$ for $i\in\mathbb{Z}$,
- additive functors D^i for $i\in\mathbb{Z}$,
- isomorphisms of additive functors between the composite $D^{i+1}\circ D^i$ and the trivial additive functor: $D^{i+1}\circ D^i\cong 0$ for $i\in\mathbb{Z}$.

Let $\mathcal{G}$ be a strictly commutative Picard $\mathbf{S}$-stack and let

$$\mathcal{L}^{\cdot}:\qquad \mathcal{R}\stackrel{D^{\mathcal{R}}}{\longrightarrow}\mathcal{Q}\stackrel{D^{\mathcal{Q}}}{\longrightarrow}\mathcal{P}\stackrel{D^{\mathcal{P}}}{\longrightarrow}0$$

be a complex of strictly commutative Picard $\mathbf{S}$-stacks with $\mathcal{P},\mathcal{Q}$ and $\mathcal{R}$ in degrees 0,-1 and -2 respectivelly.

Definition 8.1. Denote by $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$ the 2-category

(1) whose objects are pairs $(\mathcal{E},I)$ with $\mathcal{E}$ an extension of $\mathcal{P}$ by $\mathcal{G}$ and I a trivialization of the extension $(D^{\mathcal{Q}})^*\mathcal{E}$ of $\mathcal{Q}$ by $\mathcal{G}$ obtained as pull-back of $\mathcal{E}$ by $D^{\mathcal{Q}}$. Moreover we require that the corresponding trivialization $(D^{\mathcal{R}})^*I$ of $(D^{\mathcal{R}})^*(D^{\mathcal{Q}})^*\mathcal{E}$ is the trivialization arising from the isomorphism

of transitivity $(D^{\mathcal{R}})^*(D^{\mathcal{Q}})^*\mathcal{E} \cong (D^{\mathcal{Q}} \circ D^{\mathcal{R}})^*\mathcal{E}$ and the relation $D^{\mathcal{Q}} \circ D^{\mathcal{R}} \cong 0$. Note that to have such a trivialization I is the same thing as to have a lifting $I : \mathcal{Q} \to \mathcal{E}$ of $D^{\mathcal{Q}} : \mathcal{Q} \to \mathcal{P}$ such that $I \circ D^{\mathcal{R}} \cong 0$;

(2) whose 1-arrows $F : (\mathcal{E}, I) \to (\mathcal{E}', I')$ are morphisms of extensions $F : \mathcal{E} \to \mathcal{E}'$ compatible with the trivializations I, I', i.e. we have an isomorphism of additive functors $F \circ I \cong I'$,

(3) whose 2-arrows $\alpha : F \Rightarrow \overline{F}$ are morphisms of morphisms of extensions which are compatible with the isomorphisms of additive functors $F \circ I \cong I'$ and $\overline{F} \circ I \cong I'$, i.e. the following diagram commutes for any $q \in \mathcal{Q}$

$$\begin{array}{ccc} FI(q) & \xrightarrow{\cong} & I'(q) \\ {\scriptstyle \alpha(I(q))}\downarrow & \nearrow_{\cong} & \\ \overline{F}I(q). & & \end{array}$$

We can summarize the data $(\mathcal{E}, I)$ with the following diagram:

$$\begin{array}{ccccccc} \mathcal{G} & = & \mathcal{G} & = & \mathcal{G} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ (D^{\mathcal{R}})^*(D^{\mathcal{Q}})^*\mathcal{E} & \to & (D^{\mathcal{Q}})^*\mathcal{E} & \to & \mathcal{E} & & \\ \downarrow & & I \uparrow\downarrow & & \downarrow & & \\ \mathcal{R} & \overset{D^{\mathcal{R}}}{\to} & \mathcal{Q} & \overset{D^{\mathcal{Q}}}{\to} & \mathcal{P} & \to & 0 \end{array}$$

The sum of extensions of strictly commutative Picard **S**-stacks defined in [Be11] 4.6 furnishes a group law on the set of equivalence classes of objects of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$. We denote this group by $\Psi^1_{\mathcal{L}^\cdot}(\mathcal{G})$. The neutral object of $\Psi^1_{\mathcal{L}^\cdot}(\mathcal{G})$ is the object $(\mathcal{E}_0, I_0)$ where $\mathcal{E}_0$ is the extension $\mathcal{G} \times_{\mathbf{1}} \mathcal{P}$ of $\mathcal{P}$ by $\mathcal{G}$ and I_0 is the trivialization $(Id_{\mathcal{Q}}, 0)$ of the extension $(D^{\mathcal{Q}})^*\mathcal{E}_0 = \mathcal{G} \times_{\mathbf{1}} \mathcal{Q}$ of $\mathcal{Q}$ by $\mathcal{G}$. We can consider I_0 as the lifting $(D^{\mathcal{Q}}, 0)$ of $D^{\mathcal{Q}} : \mathcal{Q} \to \mathcal{P}$.

The monoid of isomorphism classes of arrows from an object $(\mathcal{E}, I)$ of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$ to itself is canonically isomorphic to the monoid of isomorphism classes of arrows from $(\mathcal{E}_0, I_0)$ to itself: to an isomorphism class of an arrow $F : (\mathcal{E}_0, I_0) \to (\mathcal{E}_0, I_0)$ the canonical isomorphism associates the isomorphism class of the arrow $F + Id_{(\mathcal{B},I)}$ from $(\mathcal{E}_0, I_0) + (\mathcal{E}, I) \cong (\mathcal{E}, I)$ to itself. The monoid of isomorphism classes of arrows from $(\mathcal{E}_0, I_0)$ to itself is a commutative group via the composition law $(\overline{F}, \overline{G}) \mapsto \overline{F + G}$ (here $\overline{F + G}$ is the isomorphism class of the arrow $F + G$ from $(\mathcal{E}_0, I_0) + (\mathcal{E}_0, I_0) \cong (\mathcal{E}_0, I_0)$ to itself). Hence we can conclude that the set of isomorphism classes of arrows from an object of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$ to itself is a commutative group that we denote by $\Psi^0_{\mathcal{L}^\cdot}(\mathcal{G})$.

The monoid of automorphisms of arrows from an object $(\mathcal{E}, I)$ of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$ to itself is canonically isomorphic to the monoid of automorphisms of arrows from $(\mathcal{E}_0, I_0)$ to itself: to an automorphism $\alpha : F \Rightarrow F$ of an arrow $F : (\mathcal{E}_0, I_0) \to (\mathcal{E}_0, I_0)$ the canonical isomorphism associates the automorphism $\alpha + id_{Id_{(\mathcal{E},I)}} : F + Id_{(\mathcal{E},I)} \Rightarrow F + Id_{(\mathcal{E},I)}$ of the arrow $F + Id_{(\mathcal{E},I)}$ from $(\mathcal{E}_0, I_0) + (\mathcal{E}, I) \cong (\mathcal{E}, I)$ to itself. The monoid of automorphisms of arrows from $(\mathcal{E}_0, I_0)$ to itself is a commutative group via the following composition law: if $\alpha : F \Rightarrow F$ and $\beta : G \Rightarrow G$, then $\alpha + \beta : F + G \Rightarrow F + G$, with $F + G$ an arrow from $(\mathcal{E}_0, I_0) + (\mathcal{E}_0, I_0) \cong (\mathcal{E}_0, I_0)$ to itself. Hence we can conclude that the set of automorphisms of an arrow from an object of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$ to itself is a commutative group that we denote by $\Psi^{-1}_{\mathcal{L}^\cdot}(\mathcal{G})$.

If $[\mathcal{R}] = [d^R : R^{-1} \to R^0], [\mathcal{P}] = [d^P : P^{-1} \to P^0]$ and $[\mathcal{Q}] = [d^Q : Q^{-1} \to Q^0]$, the complex $\mathcal{L}^\cdot$ of strictly commutative Picard **S**-stacks furnishes, modulo quasi-isomorphisms, a diagram in the category $\mathcal{K}(\mathbf{S})$ of complexes of abelian sheaves

$$[\mathcal{L}^\cdot] : \qquad R \xrightarrow{D^R} Q \xrightarrow{D^Q} P \longrightarrow 0$$

where $D^R = (d^{R,-1}, d^{R,0}), D^Q = (d^{Q,-1}, d^{Q,0})$ and $D^R \circ D^Q$ is homotopic to zero. We can consider $[\mathcal{L}^\cdot]$ as a bicomplex of abelian sheaves,

$$\begin{array}{ccccccc} R^{-1} & \xrightarrow{d^{R,-1}} & Q^{-1} & \xrightarrow{d^{Q,-1}} & P^{-1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle d^R} & & \downarrow{\scriptstyle d^Q} & & \downarrow{\scriptstyle d^P} & & \\ R^0 & \xrightarrow{d^{R,0}} & Q^0 & \xrightarrow{d^{Q,0}} & P^0 & \longrightarrow & 0 \end{array}$$

where $P^0, P^{-1}, Q^0, Q^{-1}, R^0, R^{-1}$ are respectively in degrees $(0,0), (0,-1), (-1,0), (-1,-1), (-2,0), (-2,-1)$. Denote by $\mathrm{Tot}([\mathcal{L}^\cdot])$ the total complex of this bicomplex. We have the following homological interpretation of the groups $\Psi^i_{\mathcal{L}^\cdot}(\mathcal{G})$.

Theorem 8.2.

$$\Psi^i_{\mathcal{L}^\cdot}(\mathcal{G}) \cong \mathrm{Ext}^i\big(\mathrm{Tot}([\mathcal{L}^\cdot]), [\mathcal{G}]\big) = \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}\big(\mathrm{Tot}([\mathcal{L}^\cdot]), [\mathcal{G}][i]\big) \qquad i = -1, 0, 1.$$

Proof of the cases i=-1 and 0. As observed above, $\Psi^0_{\mathcal{L}^\cdot}(\mathcal{G})$ is canonically isomorphic to the group of isomorphism classes of arrows from $(\mathcal{E}_0, I_0)$ to itself, and $\Psi^{-1}_{\mathcal{L}^\cdot}(\mathcal{G})$ is canonically isomorphic to the group of automorphisms of arrows from $(\mathcal{E}_0, I_0)$ to itself. This implies that in order to prove the cases $i = -1, 0$ we can work with the neutral object $(\mathcal{E}_0, I_0)$. By definition of 1-arrows in the 2-category $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$, the additive functor $F : \mathcal{E}_0 \to \mathcal{E}_0$ is a 1-arrow from $(\mathcal{E}_0, I_0)$ to itself if we have an isomorphism of additive functors $F \circ D^{\mathcal{Q}} \cong 0$, i.e. if F is an object of the strictly commutative Picard **S**-stack

$$\mathcal{K} = \ker\big(\mathrm{HOM}(\mathcal{P}, \mathcal{G}) \overset{D^{\mathcal{Q}}}{\to} \mathrm{HOM}(\mathcal{Q}, \mathcal{G})\big).$$

Therefore we have the equalities

$$\Psi^i_{\mathcal{L}^\cdot}(\mathcal{G}) = \mathrm{H}^i\big([\mathcal{K}]\big) \qquad i = -1, 0 \tag{8.1}$$

and in order to conclude, it is enough to compute the complex $[\mathcal{K}]$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. By [Be11] Lemma 3.4 we have

$$[\mathcal{K}] = \tau_{\leq 0}\Big(MC\big(\tau_{\leq 0}\mathrm{RHom}([\mathcal{P}], [\mathcal{G}]) \overset{(d^{R,-1}, d^{R,0})}{\longrightarrow} \tau_{\leq 0}\mathrm{RHom}([\mathcal{Q}], [\mathcal{G}])\big)[-1]\Big).$$

Explicitly, if $[\mathcal{G}] = [d^G : G^{-1} \to G^0]$ we get

$$[\mathcal{K}] = \big[\mathrm{Hom}(P^0, G^{-1}) \overset{((d^G, d^P), d^{Q,0})}{\longrightarrow} K_1 + K_2\big] \tag{8.2}$$

where

$$K_1 = \ker\big(\mathrm{Hom}(P^0, G^0) + \mathrm{Hom}(P^{-1}, G^{-1}) \overset{(d^{Q,0}, d^{Q,-1})}{\to} \mathrm{Hom}(Q^0, G^0) + \mathrm{Hom}(Q^{-1}, G^{-1})\big)$$

$$K_2 = \ker\big(\mathrm{Hom}(Q^0, G^{-1}) \overset{(d^G, d^Q)}{\to} \mathrm{Hom}(Q^0, G^0) + \mathrm{Hom}(Q^{-1}, G^{-1})\big).$$

In order to simplify notation let $L^\cdot : L^{-3} \to L^{-2} \to L^{-1} \to L^0$ be the total complex $\mathrm{Tot}([\mathcal{L}^\cdot])$. In particular $L^0 = P^0, L^{-1} = P^{-1} + Q^0$ and $L^{-2} = Q^{-1} + R^0$. The stupid filtration of the complexes $L^\cdot$ and G furnishes the spectral sequence

$$\mathrm{E}_1^{pq} = \bigoplus_{p_2-p_1=p} \mathrm{Ext}^q(L^{p_1}, G^{p_2}) \Longrightarrow \mathrm{Ext}^*(L^\cdot, G). \tag{8.3}$$

This spectral sequence is concentrated in the region of the plane defined by $-1 \leq p \leq 3$ and $q \geq 0$. We are interested on the total degrees -1 and 0. The rows $q = 1$ and $q = 0$ are

$$\mathrm{Ext}^1(L^0, G^{-1}) \to \mathrm{Ext}^1(L^0, G^0) \oplus \mathrm{Ext}^1(L^{-1}, G^{-1}) \to \mathrm{Ext}^1(L^{-1}, G^0) \oplus \mathrm{Ext}^1(L^{-2}, G^{-1}) \to ...$$

$$\mathrm{Hom}(L^0, G^{-1}) \stackrel{d_1^{-10}}{\to} \mathrm{Hom}(L^0, G^0) \oplus \mathrm{Hom}(L^{-1}, G^{-1}) \stackrel{d_1^{00}}{\to} \mathrm{Hom}(L^{-1}, G^0) \oplus \mathrm{Hom}(L^{-2}, G^{-1}) \to ...$$

Since $\mathrm{Ext}^1(L^0, G^{-1}) = 0$, i.e. the only extension of $[G^{-1} \to 0]$ by $[0 \to L^0]$ is the trivial one, we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(L^\cdot, G[-1]) &= \mathrm{Ext}^{-1}(L^\cdot, G) = \mathrm{E}_2^{-10} = \ker(d_1^{-10}), \\ \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(L^\cdot, G) &= \mathrm{Ext}^0(L^\cdot, G) = \mathrm{E}_2^{00} = \ker(d_1^{00})/\mathrm{im}(d_1^{-10}). \end{aligned} \tag{8.4}$$

Comparing the above equalities with the explicit computation (8.2) of the complex $[\mathcal{K}]$, we get

$$\mathrm{Ext}^i(L^\cdot, G) = \mathrm{H}^i\big([\mathcal{K}]\big) \qquad\qquad i = -1, 0.$$

These equalities together with equalities (8.1) give the expected statement.

Remark 8.3. In the computation (8.2) the term $\mathrm{Hom}(P^{-1}, G^0)$ does not appear because we work with the good truncation $\tau_{\leq 0}\mathrm{RHom}([\mathcal{P}], [\mathcal{G}])$. In the spectral sequence (8.3) this term appear but we are interested in elements which become zero in $\mathrm{Hom}(P^{-1}, G^0)$.

Remark 8.4. If $\mathcal{H}(\mathbf{S})$ denotes the category of complexes of abelian sheaves on $\mathbf{S}$ modulo homotopy, by equality (8.4) we have $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(L^\cdot, G) = \mathrm{Hom}_{\mathcal{H}(\mathbf{S})}(L^\cdot, G)$.

Proof of the case i=1. First we show how an object $(\mathcal{E}, I)$ of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$ defines a morphism $\mathrm{Tot}([\mathcal{L}^\cdot]) \to [\mathcal{G}][1]$ in the derived category $\mathcal{D}(\mathbf{S})$. Recall that $\mathcal{E}$ is an extension of $\mathcal{P}$ by $\mathcal{G}$. Denote $J : \mathcal{E} \to \mathcal{P}$ the additive functor underlying the extension $\mathcal{E}$. Since the trivialization I can be seen as a lifting $\mathcal{Q} \to \mathcal{E}$ of $D^{\mathcal{Q}} : \mathcal{Q} \to \mathcal{P}$ such that $I \circ D^{\mathcal{R}} \cong 0$, the diagram of additive functors

$$\begin{array}{ccccccc} \mathcal{R} & \xrightarrow{D^{\mathcal{R}}} & \mathcal{Q} & \xrightarrow{D^{\mathcal{Q}}} & \mathcal{P} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle Id_{\mathcal{P}}} & & \\ 0 & \longrightarrow & \mathcal{E} & \xrightarrow{J} & \mathcal{P} & \longrightarrow & 0 \end{array}$$

commutes. It furnishes, modulo quasi-isomorphisms, a diagram in the category $\mathcal{K}(\mathbf{S})$ of complexes of abelian sheaves on $\mathbf{S}$

$$\begin{array}{rccccccc} [\mathcal{L}^\cdot] : & R & \xrightarrow{D^R} & Q & \xrightarrow{D^Q} & P & \longrightarrow & 0 \\ & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id_P} & & \\ MC(j) : & 0 & \longrightarrow & E & \xrightarrow{j} & P & \longrightarrow & 0 \end{array} \tag{8.5}$$

where $E = [\mathcal{E}] \in \mathcal{K}^{[-1,0]}(\mathbf{S})$, $D^R = (d^{R,-1}, d^{R,0}), D^Q = (d^{Q,-1}, d^{Q,0})$, $i \circ D^R$ is homotopic to zero and $j \circ i$ is homotopic to $id_P \circ D^Q$. Putting the complex P in degree 0, the above diagram gives an arrow

$$c(\mathcal{E}, I) : \mathrm{Tot}([\mathcal{L}^\cdot]) \longrightarrow MC(j)$$

in the derived category $\mathcal{D}(\mathbf{S})$. Since $\mathcal{G}$ is equivalent as strictly commutative Picard $\mathbf{S}$-stack to $\ker(J)$, i.e. $[\mathcal{G}]$ is quasi-isomorphic to $\tau_{\leq 0}(MC(j)[-1])$, we have constructed a canonical arrow

$$\begin{array}{rcl} c : \Psi^1_{\mathcal{L}^\cdot}(\mathcal{G}) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}\big(\mathrm{Tot}([\mathcal{L}^\cdot]), [\mathcal{G}][1]\big) \\ (\mathcal{E}, I) & \mapsto & c(\mathcal{E}, I). \end{array} \tag{8.6}$$

Now we will show that this arrow is bijective. The proof that this bijection is additive, i.e. that c is an isomorphism of groups, is left to the reader. From now on let $[\mathcal{G}] = G = [d^G : G^{-1} \to G^0] \in \mathcal{K}^{[-1,0]}(\mathbf{S})$.

Injectivity: Let $(\mathcal{E}, I)$ be an object of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$ such that the morphism $c(\mathcal{E}, I)$ that it defines in $\mathcal{D}(\mathbf{S})$ is the zero morphism. The hypothesis that $c(\mathcal{E}, I)$ is zero in $\mathcal{D}(\mathbf{S})$ implies that there exists a resolution of G

$$V^0 \longrightarrow V^1 \longrightarrow V^2 \longrightarrow ...$$

and a quasi isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{j} & P & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle v^0} & & \downarrow{\scriptstyle v^1} & & & & \\ 0 & \longrightarrow & V^0 & \xrightarrow{k} & V^1 & \longrightarrow & V^2 & \longrightarrow & \cdots \end{array} \tag{8.7}$$

such that the composite

$$\begin{array}{ccccccccc} R & \xrightarrow{D^R} & Q & \xrightarrow{D^Q} & P & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id_P} & & & & \\ 0 & \longrightarrow & E & \xrightarrow{j} & P & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle v^0} & & \downarrow{\scriptstyle v^1} & & & & \\ 0 & \longrightarrow & V^0 & \xrightarrow{k} & V^1 & \longrightarrow & V^2 & \longrightarrow & \cdots \end{array}$$

is homotopic to zero. We can assume $V^i \in \mathcal{K}^{[-1,0]}(\mathbf{S})$ for all i and $V^i = 0$ for $i \geq 2$ (instead of the complex of complexes $(V^i)_i$ consider its good truncation in degree 1). Since the complex of complexes $(V^i)_i$ is a resolution of G, the short sequence of complexes

$$0 \longrightarrow G \longrightarrow V^0 \longrightarrow V^1 \longrightarrow 0$$

is exact, i.e. V^0 is an extension of W by G (see Definition 3.4). Since the quasi-isomorphism (8.7) induces the identity on G, the extension E is the fibred product $P \times_{V^1} V^0$ of P and V^0 over V^1. Therefore, the morphism $s : P \to V^0$ inducing the homotopy $(v^0, v^1) \circ c(\mathcal{E}, I) \sim 0$, i.e. satisfying $k \circ s = v^1 \circ id_P$, factorizes through a morphism

$$h : P \longrightarrow E = P \times_{V^1} V^0$$

satisfying

$$j \circ h = id_P \qquad\qquad h \circ D^Q = i.$$

These two equalities mean that $st(h)$ splits the extension $\mathcal{E}$, which is therefore the trivial extension of $\mathcal{P}$ by $\mathcal{G}$, and that $st(h)$ is compatible with the trivializations I. Hence we can conclude that the object $(\mathcal{E}, I)$ lies in the equivalence class of the zero object of $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$.

Surjectivity: Now we show that for any morphism f of $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(\mathrm{Tot}([\mathcal{L}^{\cdot}]), G[1])$, there is an element of $\Psi^1_{\mathcal{L}^{\cdot}}(\mathcal{G})$ whose image via c is f. The hypothesis that f is an element of $\mathcal{D}(\mathbf{S})$ implies that there exists a resolution of G

$$V^0 \longrightarrow V^1 \longrightarrow V^2 \longrightarrow ...$$

such that the morphism f can be described in the category $\mathcal{H}(\mathbf{S})$ via the following diagram

$$\begin{array}{ccccccccc} R & \xrightarrow{D^R} & Q & \xrightarrow{D^Q} & P & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle v^0} & & \downarrow{\scriptstyle v^1} & & & & \\ 0 & \longrightarrow & V^0 & \xrightarrow{k} & V^1 & \longrightarrow & V^2 & \longrightarrow & \cdots \end{array} \tag{8.8}$$

We can assume $V^i \in \mathcal{K}^{[-1,0]}(\mathbf{S})$ for all i and $V^i = 0$ for $i \geq 2$ (instead of the complex of complexes $(V^i)_i$ consider its good truncation in degree 1). Since the complex of complexes $(V^i)_i$ is a resolution of G, the short sequence of complexes

$$0 \longrightarrow G \longrightarrow V^0 \longrightarrow V^1 \longrightarrow 0$$

is exact, i.e. V^0 is an extension of V^1 by G (see Definition 3.4). Consider the extension of P by G

$$Z = (v^1)^* V^0 = V^0 \times_{V^1} P$$

obtained as pull-back of V^0 via $w : P \to V^1$. The condition $v^1 \circ D^Q = k \circ v^0$ implies that $v^0 : Q \to V^0$ factories through a morphism

$$z : Q \to Z$$

satisfying $l \circ z = D^Q$, with $l : Z \to P$ the canonical surjection of the extension Z. Moreover the conditions that $v^0 \circ D^R$ and $D^Q \circ D^R$ are homotopic to zero furnish that also $z \circ D^R$ is homotopic to zero. Therefore the datum $(st(Z), st(z))$ is an object of the category $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$. Consider now the morphism $c(st(Z), st(z)) : \mathrm{Tot}([\mathcal{L}^{\cdot}]) \to G[1]$ associated to $(st(Z), st(z))$. By construction, the morphism f (8.8) is the composite of the morphism $c(st(Z), st(z))$

$$\begin{array}{ccccccc} R & \xrightarrow{D^R} & Q & \xrightarrow{D^Q} & P & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle id_P} & & \\ 0 & \longrightarrow & Z & \xrightarrow{l} & P & \longrightarrow & 0 \end{array}$$

with the morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z & \xrightarrow{l} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle v^1} & & \\ 0 & \longrightarrow & V^0 & \xrightarrow{k} & V^1 & \longrightarrow & 0, \end{array}$$

where $h : Z = (v^1)^* V^0 \to V^0$ is the canonical projection underlying the pull-back Z. Since this last morphism is a morphism of resolutions of G (inducing the identity on G), we can conclude that in the derived category $\mathcal{D}(\mathbf{S})$ the morphism $f : \mathrm{Tot}([\mathcal{L}^{\cdot}]) \to G[1]$ (8.8) is the morphism $c(st(Z), st(z))$.

Using the above homological description of the groups $\Psi^i_{\mathcal{L}^\cdot}(\mathcal{G})$ for $i=-1,0,1$ we can study how the 2-category $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$ varies with respect to the complex $\mathcal{L}^\cdot$. Consider another complex $\mathcal{L}'^\cdot : \mathcal{R}' \to \mathcal{Q}' \to \mathcal{P}' \to 0$ and a morphism of complexes

$$F^\cdot : \mathcal{L}'^\cdot \longrightarrow \mathcal{L}^\cdot$$

given by the following commutative diagram (modulo isomorphisms of additive functors)

$$\begin{array}{ccccccc} \mathcal{R}' & \xrightarrow{D^{\mathcal{R}'}} & \mathcal{Q}' & \xrightarrow{D^{\mathcal{Q}'}} & \mathcal{P}' & \longrightarrow & 0 \\ \downarrow{\scriptstyle F^{-2}} & & \downarrow{\scriptstyle F^{-1}} & & \downarrow{\scriptstyle F^{0}} & & \\ \mathcal{R} & \xrightarrow[D^{\mathcal{R}}]{} & \mathcal{Q} & \xrightarrow[D^{\mathcal{Q}}]{} & \mathcal{P} & \longrightarrow & 0. \end{array} \tag{8.9}$$

The morphism $F^\cdot$ defines a canonical 2-functor

$$(F^\cdot)^* : \Psi_{\mathcal{L}^\cdot}(\mathcal{G}) \longrightarrow \Psi_{\mathcal{L}'^\cdot}(\mathcal{G})$$

as follows: if $(\mathcal{E}, I)$ is an object of $\Psi_{\mathcal{L}^\cdot}(\mathcal{G})$, $(F^\cdot)^*(\mathcal{E}, I)$ is the object $(\mathcal{E}', I')$ where

- $\mathcal{E}'$ is the extension $(F^0)^*\mathcal{E}$ of $\mathcal{P}'$ by $\mathcal{G}$ obtained as pull-back of $\mathcal{E}$ via $F^0 : \mathcal{P}' \to \mathcal{P}$;
- I' is the trivialization $(F^{-1})^* I$ of $(D^{\mathcal{Q}'})^*\mathcal{E}'$ induced by the trivialization I of $(D^{\mathcal{Q}})^*\mathcal{E}$ via the commutativity of the first square of (8.9).

The commutativity of the diagram (8.9) implies that $(\mathcal{E}', I')$ is in fact an object of $\Psi_{\mathcal{L}'^\cdot}(\mathcal{G})$ (the condition $I' \circ D^{\mathcal{Q}'} \cong 0$ is easily deducible from the corresponding conditions on I and from the commutativity of the diagram (8.9)).

Proposition 8.5. *Let $F^\cdot : \mathcal{L}'^\cdot \to \mathcal{L}^\cdot$ be morphism of complexes. The corresponding 2-functor $(F^\cdot)^* : \Psi_{\mathcal{L}^\cdot}(\mathcal{G}) \to \Psi_{\mathcal{L}'^\cdot}(\mathcal{G})$ is an equivalence of 2-categories if and only if the homomorphisms*

$$\mathrm{H}^i\big(\mathrm{Tot}(F^\cdot)\big) : \mathrm{H}^i\big(\mathrm{Tot}([\mathcal{L}'^\cdot])\big) \longrightarrow \mathrm{H}^i\big(\mathrm{Tot}([\mathcal{L}^\cdot])\big) \qquad i=-1,0,1$$

are isomorphisms.

Proof. The 2-functor $(F^\cdot)^* : \Psi_{\mathcal{L}^\cdot}(\mathcal{G}) \to \Psi_{\mathcal{L}'^\cdot}(\mathcal{G})$ defines the following homomorphisms

$$((F^\cdot)^*)^i : \Psi^i_{\mathcal{L}^\cdot}(\mathcal{G}) \longrightarrow \Psi^i_{\mathcal{L}'^\cdot}(\mathcal{G}) \qquad i=-1,0,1. \tag{8.10}$$

On the other hand the morphism of complexes $F^\cdot : \mathcal{L}'^\cdot \to \mathcal{L}^\cdot$ defines the following homomorphisms

$$(\mathrm{Tot}(F^\cdot))^i : \mathrm{Ext}^i\big(\mathrm{Tot}([\mathcal{L}^\cdot]), -\big) \longrightarrow \mathrm{Ext}^i\big(\mathrm{Tot}([\mathcal{L}'^\cdot]), -\big) \qquad i \in \mathbb{Z}. \tag{8.11}$$

Since the homomorphisms (8.10) and (8.11) are compatible with the canonical isomorphisms obtained in Theorem 8.2, the following diagrams (with $i=-1,0,1$) are commutative:

$$\begin{array}{ccc} \Psi^i_{\mathcal{L}^\cdot}(\mathcal{G}) & \to & \mathrm{Ext}^i\big(\mathrm{Tot}([\mathcal{L}^\cdot]), [\mathcal{G}]\big) \\ \downarrow & & \downarrow \\ \Psi^i_{\mathcal{L}'^\cdot}(\mathcal{G}) & \to & \mathrm{Ext}^i\big(\mathrm{Tot}([\mathcal{L}'^\cdot]), [\mathcal{G}]\big). \end{array}$$

The 2-functor $(F^\cdot)^* : \Psi_{\mathcal{L}^\cdot}(\mathcal{G}) \to \Psi_{\mathcal{L}'^\cdot}(\mathcal{G})$ is an equivalence of 2-categories if and only if the homomorphisms (8.10) are isomorphisms, and so using the above commutative diagrams we are reduced to prove that the homomorphisms (8.11) are isomorphisms if and only if the homomorphisms $\mathrm{H}^i\big(\mathrm{Tot}(F^\cdot)\big) : \mathrm{H}^i\big(\mathrm{Tot}([\mathcal{L}'^\cdot])\big) \to \mathrm{H}^i\big(\mathrm{Tot}([\mathcal{L}^\cdot])\big)$ are isomorphisms. This last assertion is clearly true. □

9. Geometrical description of $\Psi_{\mathcal{L}^{\cdot}}(\mathcal{G})$

In this section we switch from cohomological notation to homological.

Let $\mathcal{P}$ be a strictly commutative Picard **S**-stack. Because of the new homological notations the complex $[\mathcal{P}] = P = [d_P : P_1 \to P_0]$ has P_1 in degree 1 and P_0 in degree 0. We start constructing **a canonical flat partial resolution** for the complex $[\mathcal{P}]$. We introduce the following notations: if A is an abelian sheaf on **S**, we denote by $[a]$ the element of $\mathbb{Z}[A](U)$ defined by the point a of $A(U)$ with U a object of **S**. In an analogous way, if a, b and c are points of $A(U)$ we denote by $[a, b]$, $[a, b, c]$ the elements of $\mathbb{Z}[A \times A](U)$ and $\mathbb{Z}[A \times A \times A](U)$ respectively. Denote by $\mathbb{Z}[P] = [\mathbb{Z}[d_P] : \mathbb{Z}[P_1] \to \mathbb{Z}[P_0]]$ the complex of abelian sheaves generated by P, where $\mathbb{Z}[P_i]$ is the abelian sheaf generated by P_i for $i = 1, 0$ (see [D73] Exposé IV 11). Moreover let $\mathbb{Z}[\mathcal{P}]$ the strictly commutative Picard **S**-stack $st(\mathbb{Z}[P])$ corresponding to the complex $\mathbb{Z}[P]$ via (1.11).

Consider the following complexes of strictly commutative Picard **S**-stacks

$$\mathcal{L}.(\mathcal{P}) : \qquad \mathbb{Z}[\mathcal{P}\times\mathcal{P}] + \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{P}] \xrightarrow{D_1} \mathbb{Z}[\mathcal{P}\times\mathcal{P}] \xrightarrow{D_0} \mathbb{Z}[\mathcal{P}] \longrightarrow 0$$

with $\mathcal{L}_0(\mathcal{P}) = \mathbb{Z}[\mathcal{P}], \mathcal{L}_1(\mathcal{P}) = \mathbb{Z}[\mathcal{P}\times\mathcal{P}]$ and $\mathcal{L}_2(\mathcal{P}) = \mathbb{Z}[\mathcal{P}\times\mathcal{P}] + \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{P}]$ in degrees 0,1 and 2 respectively. The differential operators are defined as follows: if $p_1, p_2, p_3 \in \mathbb{Z}[\mathcal{P}]$, we set

$$\begin{aligned} D_0[p_1,p_2] &= [p_1+p_2] - [p_1] - [p_2] \\ D_1[p_1,p_2] &= [p_1,p_2] - [p_2,p_1] \\ D_1[p_1,p_2,p_3] &= [p_1+p_2,p_3] - [p_1,p_2+p_3] + [p_1,p_2] - [p_2,p_3]. \end{aligned} \tag{9.1}$$

Consider also the additive functor $\epsilon : \mathbb{Z}[\mathcal{P}] \to \mathcal{P}$ defined by $\epsilon([p]) = p$ for any $p \in \mathcal{P}$. This additive functor is an augmentation map for the complex $\mathcal{L}.(\mathcal{P})$. Note that the relation $\epsilon \circ D_0 = 0$ is just the group law $+ : \mathcal{P}\times\mathcal{P} \to \mathcal{P}$ on $\mathcal{P}$, and the relation $D_0 \circ D_1 = 0$ decomposes in two relations which express the commutativity τ (1.2) and the associativity σ (1.1) of the group law on $\mathcal{P}$. This augmented complex $\mathcal{L}.(\mathcal{P})$ depends functorially on $\mathcal{P}$: in fact, any additive functor $F : \mathcal{P} \to \mathcal{P}'$ furnishes a commutative diagram

$$\begin{array}{ccc} \mathcal{L}.(\mathcal{P}) & \stackrel{\mathcal{L}.(F)}{\longrightarrow} & \mathcal{L}.(\mathcal{P}') \\ \epsilon\downarrow & & \downarrow\epsilon \\ \mathcal{P} & \stackrel{F}{\longrightarrow} & \mathcal{P}'. \end{array}$$

Moreover the components of the complex $\mathcal{L}.(\mathcal{P})$ are flat since they are free $\mathbb{Z}$-modules. In order to conclude that $\mathcal{L}.(\mathcal{P})$ is a canonical flat partial resolution of $\mathcal{P}$ we need the following Lemma. Let $\mathcal{G}$ be a strictly commutative Picard **S**-stack.

Lemma 9.1. *The 2-category $\mathcal{E}xt(\mathcal{P},\mathcal{G})$ of extensions of $\mathcal{P}$ by $\mathcal{G}$ is equivalent to the 2-category $\Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G})$:*

$$\mathcal{E}xt(\mathcal{P},\mathcal{G}) \cong \Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G}).$$

Proof. In order to describe explicitly the objects of the category $\Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G})$ we use the description (3.7) in terms of torsors, of the extensions of complexes whose entries are free commutative groups:

- an extension of $\mathbb{Z}[\mathcal{P}]$ by $\mathcal{G}$ is a $(\mathcal{G})_{\mathcal{P}}$-torsor,
- an extension of $\mathbb{Z}[\mathcal{P}\times\mathcal{P}]$ by $\mathcal{G}$ is a $(\mathcal{G})_{\mathcal{P}\times\mathcal{P}}$-torsor, and finally
- an extension of $\mathbb{Z}[\mathcal{P}\times\mathcal{P}] + \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{P}]$ by $\mathcal{G}$ consists of a couple of a $(\mathcal{G})_{\mathcal{P}\times\mathcal{P}}$-torsor and a $(\mathcal{G})_{\mathcal{P}\times\mathcal{P}\times\mathcal{P}}$-torsor.

According to these considerations an object $(\mathcal{E}, I)$ of $\Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G})$ consists of

(1) an extension $\mathcal{E}$ of $\mathbb{Z}[\mathcal{P}]$ by $\mathcal{G}$, i.e. a $\mathcal{G}$-torsor $\mathcal{E}$ over $\mathcal{P}$. Since $\mathrm{Ext}^1(\mathbb{Z}[\mathbf{1}], \mathcal{G}) = 0$, it exists a trivialization T of the pull-back $\mathbf{1}^*\mathcal{E}$ of the $\mathcal{G}$-torsor $\mathcal{E}$ via the additive functor $\mathbf{1} : \mathbf{1} \to \mathcal{P}$;

(2) a trivialization I of the extension $D_0^*\mathcal{E}$ of $\mathbb{Z}[\mathcal{P}\times\mathcal{P}]$ by $\mathcal{G}$ obtained as pull-back of $\mathcal{E}$ via $D_0 : \mathbb{Z}[\mathcal{P} \times \mathcal{P}] \to \mathbb{Z}[\mathcal{P}]$, i.e. a trivialization I of the $\mathcal{G}$-torsor $D_0^*\mathcal{E}$ over $\mathcal{P} \times \mathcal{P}$ obtained as pull-back of $\mathcal{E}$ via D_0. This trivialization can be interpreted as a morphism of $\mathcal{G}$-torsors $\mathcal{E}$:

$$M : p_1^*\, \mathcal{E} \wedge p_2^*\, \mathcal{E} \longrightarrow +^*\, \mathcal{E}$$

where $p_i : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ are the projections and $+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is the group law of $\mathcal{P}$. The restriction of M over $\mathbf{1} \times \mathbf{1}$ is compatible with the trivialization T.
The compatibility of I with the relation $D_0 \circ D_1 = 0$ imposes on the datum $(\mathcal{E}, T, M)$ two relations through the two torsors over $\mathcal{P}\times\mathcal{P}$ and $\mathcal{P}\times\mathcal{P}\times\mathcal{P}$. These two relations are the isomorphism α of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P} \times \mathcal{P}$ described in (4.1) and the isomorphism χ of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P}$ described in (4.3), which satisfy the equalities (4.2) and (4.4). Moreover, the restriction of α over $\mathbf{1} \times \mathbf{1} \times \mathbf{1}$ is the identity and since we are dealing with extensions of strictly commutative Picard stacks, the pull-back $D^*\chi$ of χ via the diagonal $D : \mathcal{P} \to \mathcal{P}\times\mathcal{P}$ is the identity and the composite of χ with itself is the identity.

Hence by Theorem 4.1 the object $(\mathcal{E}, T, M, \alpha, \chi)$ of $\Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G})$ is an extension of $\mathcal{P}$ by $\mathcal{G}$ and we can conclude that the 2-category $\Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G})$ is equivalent to the 2-category $\mathcal{E}xt(\mathcal{P}, \mathcal{P}')$. □

Proposition 9.2. *The augmentation map* $\epsilon : \mathcal{L}.(\mathcal{P}) \to \mathcal{P}$ *induces the isomorphisms* $\mathrm{H}_i(\mathrm{Tot}(\mathcal{L}.(\mathcal{P}))) \cong \mathrm{H}_i([\mathcal{P}])$ *for* $i = 1, 0, -1$.

Proof. Applying Proposition 8.5 to the augmentation map $\epsilon : \mathcal{L}.(\mathcal{P}) \to \mathcal{P}$, we just have to prove that for any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{G}$ the 2-functor

$$\epsilon^* : \Psi_{\mathcal{P}}(\mathcal{G}) \to \Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G})$$

is an equivalence of 2-categories (in the symbol $\Psi_{\mathcal{P}}(\mathcal{G})$, $\mathcal{P}$ is seen as a complex whose only non trivial entry is $\mathcal{P}$ in degree 0). According to Definition 8.1, it is clear that the 2-category $\Psi_{\mathcal{P}}(\mathcal{G})$ is the 2-category $\mathcal{E}xt(\mathcal{P}, \mathcal{G})$ of extensions of $\mathcal{P}$ by $\mathcal{G}$. On the other hand, by Lemma 9.1 also the 2-category $\Psi_{\mathcal{L}.(\mathcal{P})}(\mathcal{G})$ is equivalent to the 2-category $\mathcal{E}xt(\mathcal{P}, \mathcal{G})$. Hence we can conclude. □

Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{G}$ be three strictly commutative Picard $\mathbf{S}$-stacks and let $\mathcal{L}.(\mathcal{P}), \mathcal{L}.(\mathcal{Q})$ be the canonical flat partial resolutions of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Denote by $\mathcal{L}.(\mathcal{P}, \mathcal{Q})$ the complex $\mathcal{L}.(\mathcal{P}) \otimes \mathcal{L}.(\mathcal{Q})$.

Theorem 9.3. *The 2-category* $\mathcal{B}iext(\mathcal{P}, \mathcal{Q}; \mathcal{G})$ *of biextensions of* $(\mathcal{P}, \mathcal{Q})$ *by* $\mathcal{G}$ *is equivalent to the 2-category* $\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G})$:

$$\mathcal{B}iext(\mathcal{P}, \mathcal{Q}; \mathcal{G}) \cong \Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G})$$

Proof. Explicitly, the non trivial components of $\mathcal{L}.(\mathcal{P},\mathcal{Q})$ are

$$\begin{aligned}
\mathcal{L}_0(\mathcal{P},\mathcal{Q}) &= \mathcal{L}_0(\mathcal{P})\otimes\mathcal{L}_0(\mathcal{Q})\\
&= \mathbb{Z}[\mathcal{P}\times\mathcal{Q}]\\
\mathcal{L}_1(\mathcal{P},\mathcal{Q}) &= \mathcal{L}_0(\mathcal{P})\otimes\mathcal{L}_1(\mathcal{Q})+\mathcal{L}_1(\mathcal{P})\otimes\mathcal{L}_0(\mathcal{Q})\\
&= \mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}]+\mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}]\\
\mathcal{L}_2(\mathcal{P},\mathcal{Q}) &= \mathcal{L}_0(\mathcal{P})\otimes\mathcal{L}_2(\mathcal{Q})+\mathcal{L}_2(\mathcal{P})\otimes\mathcal{L}_0(\mathcal{Q})+\mathcal{L}_1(\mathcal{P})\otimes\mathcal{L}_1(\mathcal{Q})\\
&= \mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}]+\mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}]+\\
&\quad\ \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}]+\mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{P}\times\mathcal{Q}]+\\
&\quad\ \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}]
\end{aligned}$$

The differential operators of the complex $\mathcal{L}.(\mathcal{P},\mathcal{Q})$ have to satisfy the following conditions: the sequences

$$(9.2)\quad \mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}]+\mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}] \stackrel{id_{\mathcal{P}}\times D_1^{\mathcal{Q}}}{\longrightarrow} \mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}] \stackrel{id_{\mathcal{P}}\times D_0^{\mathcal{Q}}}{\longrightarrow} \mathbb{Z}[\mathcal{P}\times\mathcal{Q}]$$

$$(9.3)\quad \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}]+\mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{P}\times\mathcal{Q}] \stackrel{D_1^{\mathcal{P}}\times id_{\mathcal{Q}}}{\longrightarrow} \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}] \stackrel{D_0^{\mathcal{P}}\times id_{\mathcal{Q}}}{\longrightarrow} \mathbb{Z}[\mathcal{P}\times\mathcal{Q}]$$

are exact and the diagram

$$(9.4)\qquad \begin{array}{ccc}
\mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}] & \stackrel{id_{\mathcal{P}\times\mathcal{P}}\times D_0^{\mathcal{Q}}}{\longrightarrow} & \mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}]\\
{\scriptstyle D_0^{\mathcal{P}}\times id_{\mathcal{Q}\times\mathcal{Q}}}\downarrow & & \downarrow{\scriptstyle D_0^{\mathcal{P}}\times id_{\mathcal{Q}}}\\
\mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}] & \stackrel{id_{\mathcal{P}}\times D_0^{\mathcal{Q}}}{\longrightarrow} & \mathbb{Z}[\mathcal{P}\times\mathcal{Q}]
\end{array}$$

is anticommutative.

In order to describe explicitly the objects of $\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G})$ we use the description (3.7) in terms of torsors, of the extensions of complexes whose entries are free commutative groups:

- an extension of $\mathcal{L}_0(\mathcal{P},\mathcal{Q})$ by $\mathcal{G}$ is a $(\mathcal{G})_{\mathcal{P}\times\mathcal{Q}}$-torsor,
- an extension of $\mathcal{L}_1(\mathcal{P},\mathcal{Q}))$ by $\mathcal{G}$ consists of a $(\mathcal{G})_{\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}}$-torsor and a $(\mathcal{G})_{\mathcal{P}\times\mathcal{P}\times\mathcal{Q}}$-torsor,
- an extension of $\mathcal{L}_2(\mathcal{P},\mathcal{Q}))$ by $\mathcal{G}$ consists of a system of 5 torsors under the groups deduced from $\mathcal{G}$ by base change over the bases $\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}$, $\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}$, $\mathcal{P}\times\mathcal{P}\times\mathcal{Q}$, $\mathcal{P}\times\mathcal{P}\times\mathcal{P}\times\mathcal{Q}$, $\mathcal{P}\times\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}$.

By these considerations an object $(\mathcal{E},I)$ of $\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G})$ consists of

(1) an extension $\mathcal{E}$ of $\mathbb{Z}[\mathcal{P}\times\mathcal{Q}]$ by $\mathcal{G}$, i.e. a $\mathcal{G}$-torsor $\mathcal{E}$ over $\mathcal{P}\times\mathcal{Q}$. Since $\mathrm{Ext}^1(\mathbb{Z}[\mathbf{1}\times\mathbf{1}],\mathcal{G})=0$, it exists two trivializations $T^{\mathcal{P}}$ and $T^{\mathcal{Q}}$ of the pull-back $(\mathbf{1}\times\mathbf{1})^*\mathcal{E}$ of the $\mathcal{G}$-torsor $\mathcal{E}$ via the additive functor $\mathbf{1}\times\mathbf{1}\to\mathcal{P}\times\mathcal{Q}$;

(2) a trivialization I of the extension $(id_{\mathcal{P}}\times D_0^{\mathcal{Q}}+D_0^{\mathcal{P}}\times id_{\mathcal{Q}})^*\mathcal{E}$ of $\mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}]+\mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}]$ by $\mathcal{G}$ obtained as pull-back of $\mathcal{E}$ via

$$(id_{\mathcal{P}}\times D_0^{\mathcal{Q}}+D_0^{\mathcal{P}}\times id_{\mathcal{Q}}):\mathbb{Z}[\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}]+\mathbb{Z}[\mathcal{P}\times\mathcal{P}\times\mathcal{Q}]\longrightarrow\mathbb{Z}[\mathcal{P}\times\mathcal{Q}],$$

i.e. a couple of trivializations of the couple of $\mathcal{G}$-torsors over $\mathcal{P}\times\mathcal{Q}\times\mathcal{Q}$ and $\mathcal{P}\times\mathcal{P}\times\mathcal{Q}$ which are the pull-back of $\mathcal{E}$ via $(id_{\mathcal{P}}\times D_0^{\mathcal{Q}}+D_0^{\mathcal{P}}\times id_{\mathcal{Q}})$. These trivializations can be interpreted as a morphism of $\mathcal{G}_{\mathcal{P}}$-torsors over $\mathcal{Q}_{\mathcal{P}}\times\mathcal{Q}_{\mathcal{P}}$ and a morphism of $\mathcal{G}_{\mathcal{Q}}$-torsors over $\mathcal{P}_{\mathcal{Q}}\times\mathcal{P}_{\mathcal{Q}}$

$$M^{\mathcal{Q}}:p_1^{\mathcal{Q}\,*}\,\mathcal{B}_{\mathcal{P}}\wedge p_2^{\mathcal{Q}\,*}\,\mathcal{B}_{\mathcal{P}}\longrightarrow +^{\mathcal{Q}\,*}\,\mathcal{B}_{\mathcal{P}},\qquad M^{\mathcal{P}}:p_1^{\mathcal{P}\,*}\,\mathcal{B}_{\mathcal{Q}}\wedge p_2^{\mathcal{P}\,*}\,\mathcal{B}_{\mathcal{Q}}\longrightarrow +^{\mathcal{P}\,*}\,\mathcal{B}_{\mathcal{Q}}$$

where $p_i^{\mathcal{Q}} : \mathcal{Q}_{\mathcal{P}} \times \mathcal{Q}_{\mathcal{P}} \to \mathcal{Q}_{\mathcal{P}}$, $p_i^{\mathcal{P}} : \mathcal{P}_{\mathcal{Q}} \times \mathcal{P}_{\mathcal{Q}} \to \mathcal{P}_{\mathcal{Q}}$ are the projections $(i = 1, 2)$ and $+^{\mathcal{Q}} : \mathcal{Q}_{\mathcal{P}} \times \mathcal{Q}_{\mathcal{P}} \to \mathcal{Q}_{\mathcal{P}}$, $+^{\mathcal{P}} : \mathcal{P}_{\mathcal{Q}} \times \mathcal{P}_{\mathcal{Q}} \to \mathcal{P}_{\mathcal{Q}}$ are the group laws of $\mathcal{Q}_{\mathcal{P}}$ and of $\mathcal{P}_{\mathcal{Q}}$ respectively. Remark that the restriction of $M^{\mathcal{P}}$ over $\mathbf{1} \times \mathbf{1}$ is compatible with the trivialization $T^{\mathcal{P}}$ (idem for $M^{\mathcal{Q}}$).
Finally, the compatibility of I with the relation

$$\big(id_{\mathcal{P}} \times D_0^{\mathcal{Q}} + D_0^{\mathcal{P}} \times id_{\mathcal{Q}}\big) \circ \big(id_{\mathcal{P}} \times D_1^{\mathcal{Q}} + D_1^{\mathcal{P}} \times id_{\mathcal{Q}} + (D_0^{\mathcal{P}} \times id_{\mathcal{Q}\times\mathcal{Q}}, id_{\mathcal{P}\times\mathcal{P}} \times D_0^{\mathcal{Q}})\big) = 0$$

imposes on the datum $(\mathcal{E}, T^{\mathcal{P}}, T^{\mathcal{Q}}, M^{\mathcal{P}}, M^{\mathcal{Q}})$ 5 relations of compatibility through the system of 5 torsors over $\mathcal{P} \times \mathcal{Q} \times \mathcal{Q}$, $\mathcal{P} \times \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q}$, $\mathcal{P} \times \mathcal{P} \times \mathcal{Q}$, $\mathcal{P} \times \mathcal{P} \times \mathcal{P} \times \mathcal{Q}$, $\mathcal{P} \times \mathcal{P} \times \mathcal{Q} \times \mathcal{Q}$ arising from $\mathcal{L}_2(\mathcal{P}, \mathcal{Q})$:

- the exact sequence (9.2) furnishes two relations through the two torsors over $\mathcal{P} \times \mathcal{Q} \times \mathcal{Q}$ and $\mathcal{P} \times \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q}$. These two relations are the isomorphism $\alpha^{\mathcal{Q}}$ of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q}$ described in (4.1) and the isomorphism $\chi^{\mathcal{Q}}$ of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{Q} \times \mathcal{Q}$ described in (4.3), which satisfy the equalities (4.2) and (4.4). Moreover, the restriction of $\alpha^{\mathcal{Q}}$ over $\mathbf{1} \times \mathbf{1} \times \mathbf{1}$ is the identity and since we are dealing with extensions of strictly commutative Picard stacks, the pull-back $D^*\chi^{\mathcal{Q}}$ of $\chi^{\mathcal{Q}}$ via the diagonal morphism is the identity and the composite of $\chi^{\mathcal{Q}}$ with itself is the identity. Hence by Theorem 4.1 the $\mathcal{G}$-torsor $\mathcal{E}$ is endowed with a structure of extension of $(\mathcal{Q})_{\mathcal{P}}$ by $(\mathcal{G})_{\mathcal{P}}$;
- the exact sequence (9.3) expresses two relations through the two torsors over $\mathcal{P} \times \mathcal{P} \times \mathcal{Q}$ and $\mathcal{P} \times \mathcal{P} \times \mathcal{P} \times \mathcal{Q}$. These two relations are the isomorphism $\alpha^{\mathcal{P}}$ of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P} \times \mathcal{P} \times \mathcal{Q}$ described in (4.1) and the isomorphism $\chi^{\mathcal{P}}$ of morphisms of $\mathcal{G}$-torsors over $\mathcal{P} \times \mathcal{P} \times \mathcal{Q}$ described in (4.3), which satisfy the equalities (4.2) and (4.4). Moreover, the restriction of $\alpha^{\mathcal{P}}$ over $\mathbf{1} \times \mathbf{1} \times \mathbf{1}$ is the identity and since we are dealing with extensions of strictly commutative Picard stacks, the pull-back $D^*\chi^{\mathcal{P}}$ of $\chi^{\mathcal{P}}$ via the diagonal morphism is the identity and the composite of $\chi^{\mathcal{P}}$ with itself is the identity. Hence by Theorem 4.1 the $\mathcal{G}$-torsor $\mathcal{E}$ is endowed with a structure of extension of $(\mathcal{P})_{\mathcal{Q}}$ by $(\mathcal{G})_{\mathcal{Q}}$;
- the anticommutative diagram (9.4) furnishes a relations through the torsor over $\mathcal{P} \times \mathcal{P} \times \mathcal{Q} \times \mathcal{Q}$. This relation is the isomorphism β of morphisms of $\mathcal{G}_{\mathcal{P}\times\mathcal{Q}}$-torsors over $(\mathcal{P} \times \mathcal{Q}) \times (\mathcal{P} \times \mathcal{Q})$ described in (5.1). This means that the two structures of extension of $(\mathcal{Q})_{\mathcal{P}}$ by $(\mathcal{G})_{\mathcal{P}}$ and of extension of $(\mathcal{P})_{\mathcal{Q}}$ by $(\mathcal{G})_{\mathcal{Q}}$ that we have on the $\mathcal{G}$-torsor $\mathcal{E}$ are compatible.

The object $(\mathcal{E}, T^{\mathcal{P}}, T^{\mathcal{Q}}, M^{\mathcal{P}}, M^{\mathcal{Q}}, \alpha^{\mathcal{P}}, \alpha^{\mathcal{Q}}, \chi^{\mathcal{P}}, \chi^{\mathcal{Q}}, \beta)$ of $\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G})$ is therefore a biextension of $(\mathcal{P}, \mathcal{Q})$ by $\mathcal{G}$. We can then conclude that the 2-category $\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G})$ is equivalent to the 2-category $\mathcal{B}iext(\mathcal{P}, \mathcal{Q}, \mathcal{G})$. □

10. Proof of Theorem 0.1 (a)

Let $\mathcal{P}, \mathcal{Q}$ and $\mathcal{G}$ be three strictly commutative Picard **S**-stacks.

Denote respectively by $\mathcal{L}.(\mathcal{P})$ and $\mathcal{L}.(\mathcal{Q})$ the canonical flat partial resolutions of $\mathcal{P}$ and $\mathcal{Q}$ introduced in §9. According to Proposition 9.2, there exists arbitrary flat resolutions $\mathcal{L}.'(\mathcal{P}), \mathcal{L}.'(\mathcal{Q})$ of $\mathcal{P}$ and $\mathcal{Q}$ such that we have the following isomorphisms for $j = -1, 0, 1$

$$\mathrm{Tot}(\mathcal{L}.(\mathcal{P}))_j \cong \mathrm{Tot}(\mathcal{L}.'(\mathcal{P}))_j \qquad \mathrm{Tot}(\mathcal{L}.(\mathcal{Q}))_j \cong \mathrm{Tot}(\mathcal{L}.'(\mathcal{Q}))_j.$$

Hence it exists two canonical morphisms of complexes

$$\mathcal{L}.(\mathcal{P}) \longrightarrow \mathcal{L}.'(\mathcal{P}) \qquad \mathcal{L}.(\mathcal{Q}) \longrightarrow \mathcal{L}.'(\mathcal{Q})$$

inducing a canonical morphism between the corresponding total complexes

$$\mathrm{Tot}([\mathcal{L}.(\mathcal{P}) \otimes \mathcal{L}.(\mathcal{Q})]) \longrightarrow \mathrm{Tot}([\mathcal{L}.'(\mathcal{P}) \otimes \mathcal{L}.'(\mathcal{Q})])$$

which is an isomorphism in degrees -1, 0 and 1. Denote by $\mathcal{L}.(\mathcal{P},\mathcal{Q})$ (resp. $\mathcal{L}.'(\mathcal{P},\mathcal{Q})$) the complex $\mathcal{L}.(\mathcal{P}) \otimes \mathcal{L}.(\mathcal{Q})$ (resp. $\mathcal{L}.'(\mathcal{P}) \otimes \mathcal{L}.'(\mathcal{Q})$). Remark that $\mathrm{Tot}([\mathcal{L}.'(\mathcal{P},\mathcal{Q})])$ represents $[\mathcal{P}]\overset{\mathbb{L}}{\otimes}[\mathcal{Q}]$ in the derived category $\mathcal{D}(\mathbf{S})$:

$$\mathrm{Tot}([\mathcal{L}.'(\mathcal{P},\mathcal{Q})]) = [\mathcal{P}]\overset{\mathbb{L}}{\otimes}[\mathcal{Q}].$$

By Proposition 8.5 we have the equivalence of categories

$$\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G}) \cong \Psi_{\mathcal{L}.'(\mathcal{P},\mathcal{Q})}(\mathcal{G}).$$

Hence applying Theorem 9.3, which furnishes the following geometrical description of the category $\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G})$:

$$\Psi_{\mathcal{L}.(\mathcal{P},\mathcal{Q})}(\mathcal{G}) \cong \mathbf{Biext}(\mathcal{P},\mathcal{Q};\mathcal{G}),$$

and applying Theorem 8.2, which furnishes the following homological description of the groups $\Psi^i_{\mathcal{L}.'(\mathcal{P},\mathcal{Q})}(\mathcal{G})$ for $i=-1,0,1$:

$$\Psi^i_{\mathcal{L}.'(\mathcal{P},\mathcal{Q})}(\mathcal{G}) \cong \mathrm{Ext}^i\big(\mathrm{Tot}\big([\mathcal{L}.'(\mathcal{P},\mathcal{Q})]\big),[\mathcal{G}]\big) \cong \mathrm{Ext}^i([\mathcal{P}]\overset{\mathbb{L}}{\otimes}[\mathcal{Q}],[\mathcal{G}]),$$

we get Theorem 0.1, i.e. $\mathrm{Biext}^i(\mathcal{P},\mathcal{Q};\mathcal{G}) \cong \mathrm{Ext}^i([\mathcal{P}]\overset{\mathbb{L}}{\otimes}[\mathcal{Q}],[\mathcal{G}])$ for $i=-1,0,1$.

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